

# Real and Functional Analysis

Notes from the lectures of Prof. Soave

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# Real Analysis

## 1 Set Theory

### 1.1 Collections and sequences of sets

Let  $X$  be a set. Then

$$\mathcal{P}(X) = \{Y \mid Y \subseteq X\} \quad (\text{Power Set})$$

Let  $I \subseteq \mathbb{R}$  be a set of indexes. A family of sets induced by  $I$  is

$$\{E_i\}_{i \in I}, \quad E_i \subseteq X \quad (\text{Family/Collection})$$

If  $I = \mathbb{N}$  is called a

$$\{E_n\}_{n \in \mathbb{N}} \quad (\text{Sequence})$$

#### Definition 1.1

$\{E_n\} \subseteq \mathcal{P}(X)$  is monotone increasing (resp. decreasing) if

$$E_n \subseteq E_{n+1} \quad \forall n \quad (\text{resp. } E_n \supseteq E_{n+1} \quad \forall n)$$

and is written as

$$\{E_n\} \nearrow \quad (\text{resp. } \{E_n\} \searrow)$$

Given a family of sets  $\{E_i\}_{i \in I} \subseteq \mathcal{P}(X)$ , will often be considered

$$\bigcup_{i \in I} E_i = \{x \in X : \exists i \in I \text{ s.t. } x \in E_i\}$$

$$\bigcap_{i \in I} E_i = \{x \in X : x \in E_i, \forall i \in I\}$$

$\{E_i\}$  is said to be **disjoint** if  $E_i \cap E_j = \emptyset \quad \forall i \neq j$ .

#### Example 1.1

$$[a, b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right)$$
$$(a, b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$$

### 1.2 lim inf, lim sup

#### Definition 1.2

$\{E_n\} \subseteq \mathcal{P}(X)$ . We define

$$\limsup_n E_n := \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} E_k\right) \quad \liminf_n E_n := \bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} E_k\right)$$

If these two sets are equal, then

$$\lim_n E_n = \limsup_n E_n = \liminf_n E_n$$

which is the limit of the succession.

### 1.3 lim of sequences of sets

#### Proposition 1.1

Some limits are:

- $\limsup_n E_n = \{x \in X : x \in E_n \text{ for } \infty - \text{many indexes } n\}$
- $\liminf_n E_n = \{x \in X : x \in E_n \text{ for all but finitely many indexes } n\}$
- $\liminf_n E_n \subseteq \limsup_n E_n$
- $(\liminf_n E_n)^C = \limsup_n E_n^C$

**Proof.** We can define:

$$\begin{aligned} x \in \limsup_n E_n &\Leftrightarrow x \in \bigcap_{k=1}^{\infty} \left( \bigcup_{n=k}^{\infty} E_n \right) \\ &\Leftrightarrow \forall k \in \mathbb{N} : x \in \bigcup_{n=k}^{\infty} E_n \\ &\Leftrightarrow \forall k \in \mathbb{N} \exists n_k \geq k \text{ s.t. } x \in E_{n_k} \end{aligned}$$

So  $x \in \limsup_n E_n \Rightarrow \exists m_1 = n_1 \text{ s.t. } x \in E_{n_1}$   
 $\exists m_2 := n_{m_1+1} \geq m_1 + 1 \text{ s.t. } x \in E_{n_2}$   
 $\vdots$   
 $\exists m_k := n_{m_{k-1}+1} \geq m_{k-1} + 1 \text{ s.t. } x \in E_{n_k}$   
 $\vdots$   
 $x \in E_{m_1}, \dots, E_{m_k}, \dots$

On the other hand, assume that  $x \in E_n$  for  $\infty$ -many indexes. We claim that  $\forall k \in \mathbb{N}, \exists n_k \geq k$  s.t.  $x \in E_{n_k} \Leftrightarrow x \in \limsup_n E_n$ . If that claim is not true, then  $\exists \bar{k} \text{ s.t. } x \notin E_n \quad \forall n > \bar{k} \Rightarrow x$  belongs at most to  $E_1, \dots, E_{\bar{k}}$ , a contradiction. ★

### 1.4 Cover and subcover of a set

#### Definition 1.3

$\{E_i\}_{i \in I}$  is a **covering** of  $X$  if

$$X \subseteq \bigcup_{i \in I} E_i$$

A subfamily of  $E_i$  that is still a covering is called a **subcovering**

### 1.5 Characteristic function of a set

#### Definition 1.4

Let  $E \subseteq X$ . The function  $\chi_E : X \rightarrow \mathbb{R}$

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in X \setminus E \end{cases}$$

is called **characteristic function** of  $E$

Let  $E_1, E_2$  be sets:

$$\chi_{E_1 \cap E_2} = \chi_{E_1} \cdot \chi_{E_2}$$

$$\chi_{E_1 \cup E_2} = \chi_{E_1} + \chi_{E_2} - \chi_{E_1 \cap E_2}$$

$$\{E_n\} \subseteq \mathcal{P}(X), \text{ disjoint}, E = \bigcup_{n=1}^{\infty} E_n \Rightarrow \chi_E = \sum_{n=1}^{\infty} \chi_{E_n}$$

$$\{E_n\} \subseteq \mathcal{P}, P = \liminf_n E_n, Q = \limsup_n E_n \Rightarrow \chi_P = \liminf_n \chi_{E_n}, \chi_Q = \limsup_n \chi_{E_n}$$

Recall that  $\limsup_n a_n = \lim_{k \rightarrow \infty} \sup_{n \geq k} a_n$  and  $\liminf_n a_n = \lim_{k \rightarrow \infty} \inf_{n \geq k} a_n$

Let's also check that  $\chi_Q = \limsup_n \chi_{E_n}$

$$\begin{aligned} x \in \limsup_n E_n &\Leftrightarrow \chi_Q(x) = 1 \\ &\Leftrightarrow \forall k \in \mathbb{N} \exists n_k \geq k \text{ s.t. } x \in E_{n_k} \end{aligned}$$

If we fix  $k$  then

$$\begin{aligned} \sup_{n \geq k} \chi_{E_n}(x) &= \chi_{E_{n_k}}(x) = 1 \\ \lim_{k \rightarrow \infty} \sup_{n \geq k} \chi_{E_n}(x) &= \limsup_n \chi_{E_n}(x) = 1 \end{aligned}$$

Let now  $x \notin \limsup_n E_n \Leftrightarrow \chi_Q(x) = 0$ . Then  $x$  belongs at most to finitely many  $E_n \Rightarrow \exists \bar{k} \text{ s.t. } x \notin E_n, \forall n \geq \bar{k}$

If  $k \geq \bar{k}$ , then  $\sup_{n \geq k} \chi_{E_n}(x) = 0 \Rightarrow \lim_{k \rightarrow \infty} \sup_{n \geq k} \chi_{E_n}(x) = \limsup_n \chi_{E_n}(x) = 0$

## 1.6 Equivalence relations

Given  $X, Y$  sets, is called a **relation** of  $X$  and  $Y$  a subset of  $X \times Y$

$$R \subseteq X \times Y \quad R = \{(x, y) : x \in X, y \in Y\}$$

$$(x, y) \in R \Leftrightarrow xRy$$

$$X = \{0, 1, 2, 3\} \quad R = \{(0, 1), (1, 2), (2, 1)\} \text{ is a relation in } X$$

### Definition 1.5

A **function** from  $X$  to  $Y$  is a relation  $R$  s.t. for any element  $x$  of  $X$   $\exists!$  element  $y$  of  $Y$  s.t.  $xRy$

### Definition 1.6

$R$  on  $X$  is an **equivalence relation** if

- (1)  $xRx \forall x \in X$  ( $R$  is **reflexive**)
- (2)  $xRy \Rightarrow yRx$  ( $R$  is **symmetric**)
- (3)  $xRy, yRz \Rightarrow xRz$  ( $R$  is **transitive**)

If  $R$  is an equivalence relation, the set  $E_x := \{y \in X : yRx\}$ ,  $x \in X$  is called the **equivalence class** of  $x$

### Definition 1.7

$\frac{X}{R} := \{E_x : x \in X\}$  is the **quotient set**

### Example 1.2

$X = \mathbb{Z}$ , let's say that  $nRm$  if  $n - m$  is even. This is an equivalence relation.

$$E_n = \{\dots, n-4, n-2, n, n+2, n+4, \dots\}$$

in this case if  $n$  is even,  $E_n = \{\text{even numbers}\}$  and if  $n$  is odd,  $E_n = \{\text{odd numbers}\}$

## 2 Measure Spaces

### 2.1 $\sigma$ -algebra

#### Definition 2.1

A family  $\mathcal{M} \subseteq \mathcal{P}(X)$  is called a  **$\sigma$ -algebra** if

- (1)  $X \in \mathcal{M}$
- (2)  $E \in \mathcal{M} \Rightarrow E^C = X \setminus E \in \mathcal{M}$
- (3) If  $E = \bigcup_{n \in \mathbb{N}} E_n$  and  $E_n \in \mathcal{M} \forall n$ , then  $E \in \mathcal{M}$

### 2.2 Measurable space and sets

If  $\mathcal{M}$  is a  $\sigma$ -algebra,  $(X, \mathcal{M})$  is called **measurable space** and the sets in  $\mathcal{M}$  are called **measurable**.

**Example 2.1** •  $(X, \mathcal{P}(X))$  is a measurable space

- Let  $X$  be a set, then  $\{\emptyset, X\}$  is a  $\sigma$ -algebra

#### Remark 2.1

$\sigma$  is often used to denote the closure with respect to countably many operators. If we replace the countable unions with finite unions in the definition of  $\sigma$ -algebra, we obtain an **algebra**.

Some **basic properties** of a measurable space  $(X, \mathcal{M})$ :

- (1)  $\emptyset \in \mathcal{M}$ :  $\emptyset = X^C$  and  $X \in \mathcal{M}$
- (2)  $\mathcal{M}$  is an algebra, and  $E_1, \dots, E_n \in \mathcal{M}$

$$E_1 \cup \dots \cup E_n = E_1 \cup \dots \cup E_n \cup \underbrace{\emptyset}_{\in \mathcal{M}} \cup \emptyset \dots \in \mathcal{M}$$

- (3)  $E_n \in \mathcal{M}, \bigcap_{n \in \mathbb{N}} E_n \in \mathcal{M}$

$$\bigcap_{n \in \mathbb{N}} E_n = \left( \bigcup_{n \in \mathbb{N}} \underbrace{E_n^C}_{\in \mathcal{M}} \right)^C \quad (\mathcal{M} \text{ is also closed under finite intersection})$$

- $E, F \in \mathcal{M} \Rightarrow E \setminus F \in \mathcal{M} = E \setminus F = E \cap F^C \in \mathcal{M}$
- If  $\Omega \subset X$ , then the **restriction** of  $\mathcal{M}$  to  $\Omega$ , written as

$$\mathcal{M}|_{\Omega} := \{F \subseteq \Omega : F = E \cap \Omega, \text{ with } E \in \mathcal{M}\}$$

is a  $\sigma$ -algebra on  $\Omega$

### 2.3 $\sigma$ -algebra generated by a set

#### Theorem 2.1

$\mathcal{S} \subseteq \mathcal{P}(X)$ . Then it is well-defined the smallest  $\sigma$ -algebra containing  $\mathcal{S}$ , the  $\sigma$ -algebra generated by  $\mathcal{S} := \sigma_0(\mathcal{S})$ :

- $\mathcal{S} \subseteq \sigma_0(\mathcal{S})$  and thus is a  $\sigma$ -algebra
- $\forall \sigma(\mathcal{M})$  s.t.  $\mathcal{M} \supseteq \mathcal{S}$ , we have  $\mathcal{M} \supseteq \sigma_0(\mathcal{S})$

*Proof idea.*

$$\mathcal{V} = \{\mathcal{M} \subseteq \mathcal{P}(X) : \mathcal{M} \text{ is a } \sigma\text{-algebra and } \mathcal{S} \subseteq \mathcal{M}\} \neq \emptyset \text{ since } \mathcal{P}(X) \in \mathcal{V}$$

We define  $\sigma_0(\mathcal{S}) = \bigcap \{\mathcal{M} : \mathcal{M} \in \mathcal{V}\}$ , so it can be proved that this is the desired  $\sigma$ -algebra

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## 2.4 Borel sets

Given  $(X, d)$  metric space, the  $\sigma$ -algebra generated by the open sets is called **Borel**  $\sigma$ -algebra, written as  $\mathcal{B}(X)$ . The sets in  $\mathcal{B}(X)$  are called **Borel sets**. The following sets are Borel sets:

- open sets
- closed sets
- countable intersections of open sets:  $G_\delta$  sets
- countable unions of closed sets:  $F_\sigma$  sets

## 2.5 Generation of $\mathcal{B}(\mathbb{R})$

### Remark 2.2

$\mathcal{B}(\mathbb{R})$  can be equivalently defined as the  $\sigma$ -algebra generated by

$$\{(a, b) : a, b \in \mathbb{R}, a < b\}$$

$$\{(-\infty, b) : b \in \mathbb{R}\}$$

$$\{(a, +\infty) : a \in \mathbb{R}\}$$

$$\{[a, b) : a, b \in \mathbb{R}, a < b\}$$

$\vdots$

Question: What is  $\mathcal{B}(\mathbb{R})$ ? Is  $\mathcal{B}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$ ? No.

## 2.6 Measure

### Definition 2.2

$(X, \mathcal{M})$  measurable space. A function  $\mu : \mathcal{M} \rightarrow [0, +\infty]$  is called a **positive measure** if  $\mu(\emptyset) = 0$  and if  $\mu$  is countably additive, that is

$$\forall \{E_n\} \subseteq \mathcal{M} \text{ disjoint}$$

we have that

$$\mu \left( \bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \mu(E_n) \quad \sigma\text{-additivity}$$

### Remark 2.3

A set  $A$  is countable if  $\exists f : A \rightarrow \mathbb{N}$  s.t.  $f$  is 1-1

### Example 2.2

$\mathbb{Z}, \mathbb{Q}$  are countable, while  $\mathbb{R}$  is not, also  $(0, 1)$  is uncountable.

We always assume that  $\exists E \neq \emptyset, E \in \mathcal{M}$  s.t.  $\mu(E) \neq \infty$ .

## 2.7 Measure space

If  $(X, \mathcal{M})$  is a measurable space, and  $\mu$  is a measure on it, then  $(X, \mathcal{M}, \mu)$  is a measure space. Then:

(1)  $\mu$  is **finitely additive**:

$$\forall E, F \in \mathcal{M}, \text{ with } E \cap F = \emptyset \Rightarrow \mu(E \cup F) = \mu(E) + \mu(F)$$

(2) the **excision property**

$$\forall E, F \in \mathcal{M}, \text{ with } E \subset F \text{ and } \mu(E) < +\infty \Rightarrow \mu(F \setminus E) = \mu(F) - \mu(E)$$

(3) **monotonicity**

$$\forall E, F \in \mathcal{M}, \text{ with } E \subset F \Rightarrow \mu(E) \leq \mu(F)$$

(4) if  $\Omega \in \mathcal{M}$  then  $(\Omega, \mathcal{M}|_{\Omega}, \mu|_{\mathcal{M}|_{\Omega}})$  is a measure space

**Proof.** (1)  $E_1 = E, E_2 = F, E_3 = \dots = E_n = \dots = \emptyset$  This is a disjoint sequence  $\Rightarrow$  by  $\sigma$ -additivity.

$$\mu(E \cup F) = \mu\left(\bigcup_n E_n\right) = \sum_n \mu(E_n) = \mu(E) + \mu(F) + \underbrace{\mu(E_k)}_{=\mu(\emptyset)}$$

(2)  $E \subset F$ , so  $F = E \cup (F \setminus E)$  and this is disjoint  $\stackrel{(i)}{\Rightarrow} \mu(F) = \mu(E) + \mu(F \setminus E)$ , and since  $\mu(E) < \infty$ , the property follows.

(3)  $E \subset F \Rightarrow \mu(F) = \mu(E) + \underbrace{\mu(F \setminus E)}_{\geq 0} \geq \mu(E)$

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## 2.8 Basic properties of a measure

### Definition 2.3

$(X, \mathcal{M}, \mu)$  measure space.

- If  $\mu(X) < +\infty$ , we say that  $\mu$  is **finite**.
- If  $\mu(X) = +\infty$ , and  $\exists \{E_n\} \subset \mathcal{M}$  s.t.  $X = \bigcup_n E_n$  and each  $E_n$  has finite measure, then we say that  $\mu$  is  $\sigma$ -finite.
- If  $\mu(X) = 1$  we say that  $\mu$  is a **probability measure**.

**Example 2.3** • Trivial Measure:  $(X, \mathcal{M})$  measurable space.  $\mu : \mathcal{M} \rightarrow [0, \infty]$  defined by  $\mu(E) = 0 \quad \forall E \in \mathcal{M}$

- Counting Measure:  $(X, \mathcal{P}(X))$  measurable space. We define

$$\mu_C : \mathcal{P}(X) \rightarrow [0, \infty], \quad \mu_C(E) = \begin{cases} n & \text{if } E \text{ has } n \text{ elements} \\ \infty & \text{if } E \text{ has } \infty\text{-many elements} \end{cases}$$

- Dirac Measure:  $(X, \mathcal{P}(X))$  measurable space,  $t \in X$ . We define

$$\delta_t : \mathcal{P}(X) \rightarrow [0, +\infty], \quad \delta_t(E) = \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{otherwise} \end{cases}$$

## 2.9 Continuity of the measure along monotone sequences

$(X, \mathcal{M}, \mu)$  measure space

- (1)  $\{E_i\} \subset \mathcal{M}$ ,  $E_i \subseteq E_{i+1} \forall i$  and let

$$E = \bigcup_{i=1}^{\infty} E_i = \lim_i E_i$$

Then:

$$\mu(E) = \lim_i \mu(E_i)$$

- (2)  $\{E_i\} \subset \mathcal{M}$ ,  $E_{i+1} \subseteq E_i \forall i$  and let  $E = \bigcap_{i=1}^{\infty} E_i = \lim_i E_i$ .

**Proof.** (1) If  $\exists i$  s.t.  $\mu(E_i) = +\infty$ , then is trivial. Assume then that every  $E_i$  has a finite measure, so that  $E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=0}^{\infty} (E_{i+1} \setminus E_i)$  with  $E_0 = \emptyset$ .

So, by  $\sigma$ -additivity

$$\begin{aligned} \mu(E) &= \mu\left(\bigcup_{i=0}^{\infty} (E_{i+1} \setminus E_i)\right) = \\ &= \sum_{i=0}^{\infty} \mu(E_{i+1} \setminus E_i) \stackrel{(excision)}{=} \sum_{i=0}^{\infty} (\mu(E_{i+1}) - \mu(E_i)) = \\ &\stackrel{(telescopic series)}{=} \lim_n \mu(E_n) - \underbrace{\mu(E_0)}_{=0} = \lim_n \mu(E_n) \end{aligned}$$

- (2) For simplicity, suppose  $\tau = 1$ , and define  $F_k = E_i \setminus E_k$

$$\{E_k\} \searrow \Rightarrow \{F_k\} \nearrow$$

$$\mu(E_i) = \mu(E_k) + \mu(F_k) \text{ and } \bigcup_k F_k = E_i \setminus \left(\bigcap_k E_k\right)$$

$$\mu(E_i) = \mu\left(\bigcup_k F_k\right) + \underbrace{\mu\left(\bigcap_k E_k\right)}_{\mu(E)} =$$

$$\stackrel{(i)}{=} \lim_k \mu(F_k) + \mu(E) = \lim_k (\mu(E_i) - \mu(E_k)) + \mu(E)$$

Since  $\mu(E_i) < \infty$  we can subtract it from both sides

$$0 = -\lim_k \mu(E_k) + \mu(E)$$





**Example 2.4** (Counterexample)

Given  $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_C)$  measure space. Let  $E_n = \{n, n+1, n+2, \dots\}$ . In this case  $\mu_C(E_n) = +\infty, E_{n+1} \subseteq E_n \forall n$ , but  $\bigcap_n E_n = \emptyset \Rightarrow \mu(\bigcap_n E_n) = 0$

**2.10  $\sigma$ -subadditivity of the measure****Theorem 2.2** ( $\sigma$ -subadditivity of the measure)

$(X, \mathcal{M}, \mu)$  is a measure space.  $\forall \{E_n\} \subseteq \mathcal{M}$  (not necessarily disjoint):  $\mu(\bigcup_n E_n) \leq \sum_n \mu(E_n)$

**Proof.**  $E_1, E_2 \in \mathcal{M}$  and also  $E_1 \cup E_2 = E_1 \cup (E_2 \setminus E_1)$  disjoint sets.

$$\mu(E_1 \cup E_2) = \mu(E_1) + \underbrace{\mu(E_2 \setminus E_1)}_{\subseteq E_2} \stackrel{(\text{monotonicity})}{\leq} \mu(E_1) + \mu(E_2)$$

that means that we have the subadditivity for finitely many sets.

Given  $A = \bigcup_{n=1}^{\infty} E_n, A_k = \bigcup_{n=1}^k E_n, \{A_k\} \nearrow, A_{k+1} \supseteq A_k, \lim_k A_k = A$  :

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \stackrel{(\text{continuity})}{=} \lim_k \mu(A_k) = \lim_k \mu\left(\bigcup_{n=1}^k E_n\right) \leq \lim_k \sum_{n=1}^k \mu(E_n) = \sum_{n=1}^{\infty} \mu(E_n)$$

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Exercise:  $(X, \mathcal{M})$  measurable space.  $\mu : \mathcal{M} \rightarrow [0, +\infty]$  s.t.  $\mu$  is finitely additive,  $\sigma$ -subadditive and  $\mu(\emptyset) = 0 \Rightarrow \mu$  is  $\sigma$ -additive, and hence is a measure.

**2.11 Borel-Cantelli Lemma****Lemma 2.1**

Given  $(X, \mathcal{M}, \mu)$  and  $\{E_n\} \subseteq \mathcal{M}$ . Then

$$\sum_{n=0}^{\infty} \mu(E_n) < \infty \Rightarrow \mu(\limsup_n E_n) = 0$$

It can be phrased as:

If the series of the probability of the events  $E_n$  is convergent, then the probability that  $\infty$ -many events occur is 0

**Proof.** The thesis is:

$$\mu(\limsup_n E_n) = \mu\left(\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k \geq n} E_k}_{A_n := \bigcup_{k \geq n} E_k}\right)$$

Is it true that  $\{A_n\} \searrow$ ? Yes.

$$A_{n+1} = \bigcup_{k \geq n+1} E_k \subseteq \bigcup_{k \geq n} E_k = A_n$$

Does some  $A_n$  have a finite measure?

$$\mu(A_n) = \mu\left(\bigcup_{k \geq n} E_k\right) \leq \sum_{k \geq n} \mu(E_k) < \infty$$

by assumption. Therefore, we can use the continuity along decreasing sequences:

$$\mu(\limsup_n E_n) = \lim_n \mu(A_n) = \lim_n \mu\left(\bigcup_{k \geq n} E_k\right) \stackrel{\sigma\text{-sub.}}{\leq} \lim_n \sum_{k=n}^{\infty} \mu(E_k) = 0$$

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## 2.12 Sets of 0 measure, negligible sets

$(X, \mathcal{M}, \mu)$  measure space.

- $N \subseteq X$  is a set of 0 measure if  $N \in \mathcal{M}$  and  $\mu(N) = 0$
- $E \subseteq X$  is called **negligible set** if  $\exists N \in \mathcal{M}$  with 0 measure s.t.  $E \subseteq N$  ( $E$  does not necessarily stay in  $\mathcal{M}$ )

## 2.13 Complete measure space

### Definition 2.4

$(X, \mathcal{M}, \mu)$  measure space s.t. every negligible set is measurable (and hence of 0 measure), then  $(X, \mathcal{M}, \mu)$  is said to be a **complete measure space**.

A measure space may not be complete. However, let

$$\overline{\mathcal{M}} := \{E \subseteq X : \exists F, G \in \mathcal{M} \text{ with } F \subseteq E \subseteq G \text{ and } \mu(G \setminus F) = 0\}$$

Clearly  $\mathcal{M} \subseteq \overline{\mathcal{M}}$ . For  $E \in \overline{\mathcal{M}}$ , take  $F$  and  $G$  as above and let  $\bar{\mu}(E) = \bar{\mu}(F)$  then  $\bar{\mu}|_{\mathcal{M}} = \mu$ , and moreover:

### Theorem 2.3

$(X, \mathcal{M}, \mu)$  is a complete measure space. Let's observe that  $\bar{\mu}$  is well-defined: let  $E \subseteq X$  and  $F_1, F_2, G_1, G_2$  s.t.  $F_i \subset E \subset G_i$   $i = 1, 2$ . Then  $\mu(G_i \setminus F_i) = 0$ . Now we have to check that  $\mu(F_1) = \mu(F_2)$ .

Since

$$F_1 \setminus F_2 \subseteq E \setminus F_2 \subseteq G_2 \setminus F_2$$

and  $G_2 \setminus F_2$  has 0 measure  $\Rightarrow \mu(F_1 \setminus F_2) = 0$ . Then  $F_1 = (F_1 \setminus F_2) \cup (F_1 \cap F_2) \Rightarrow \mu(F_1) = \mu(F_1 \cap F_2)$ . In the same way,  $\mu(F_2) = \mu(F_1 \cap F_2)$

The elements of  $\overline{\mathcal{M}}$  are sets of the type  $E \cup N$ , with  $E \in \mathcal{M}$  and  $\mu(N) = 0$ .

## 2.14 Outer measure

We wish to define a measure  $\lambda$  "on  $\mathbb{R}$ " with the following properties:

- (1)  $\lambda((a, b)) = b - a$
- (2)  $\lambda(E + t)^\dagger = \lambda(E)$  for every measurable set  $E \subset \mathbb{R}$  and  $t \in \mathbb{R}$

It would be nice to define such a measure on  $\mathcal{P}(\mathbb{R})$ . In such case, note that  $\lambda(\{x\}) = 0, \forall x \in \mathbb{R}$   
But then

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$^\dagger \{x \in \mathbb{R} : x = y + t, \text{ with } y \in E\}$

**Theorem 2.4** (Ulam)

The only measure on  $\mathcal{P}(\mathbb{R})$  s.t.  $\lambda(\{x\}) = 0 \quad \forall x$  is the trivial measure. Thus, a measure satisfying the two properties of the outer measure cannot be defined on  $\mathcal{P}(\mathbb{R})$

We'll learn in what follows how to create a measure space on  $\mathbb{R}$ , with a  $\sigma$ -algebra including all the Borel sets, and a measure satisfying properties of the outer measure. This is the so-called **Lebesgue measure**.

**Definition 2.5**

Given a set  $X$ . An **outer measure** is a function  $\mu^* : \mathcal{P}(X) \rightarrow [0, +\infty]$  s.t.

- $\mu^*(\emptyset) = 0$
- $\mu^*(A) \leq \mu^*(B)$  if  $A \subseteq B$  (Monotonicity)
- $\mu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$  ( $\sigma$ -subadditivity)

**2.15 Generation of an outer measure**

The common way to define an outer measure is to start with a family of elementary sets  $\mathcal{E}$  on which a notion of measure is defined (e.g. intervals on  $\mathbb{R}$ , rectangles on  $\mathbb{R}^2, \dots$ ) and then to approximate arbitrary sets from outside by **countable** unions of members of  $\mathcal{E}$ .

**Proposition 2.1**

Let  $\mathcal{E} \subset \mathcal{P}(\mathbb{R})$  and  $\rho : \mathcal{E} \rightarrow [0, +\infty]$  be such that  $\emptyset \in \mathcal{E}, X \in \mathcal{E}$  and  $\rho(\emptyset) = 0$ . For any  $A \in \mathcal{P}(X)$ , let

$$\mu^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \rho(E_n) : E_n \in \mathcal{E} \text{ and } A \subset \bigcup_{n=1}^{\infty} E_n \right\}$$

Then  $\mu^*$  is an outer measure, the outer measure generated by  $(\mathcal{E}, \rho)$ .

**Proof.**  $\forall A \subset X \exists \{E_n\} \subset \mathcal{E}$  s.t.  $A \subset \bigcup_n E_n$ .

Now take  $E_n = X \quad \forall n$ , then  $\mu^*$  is well-defined. Obviously,  $\mu^*(\emptyset) = 0$  (with  $E_n = \emptyset \quad \forall n$ ), and  $\mu^*(A) \leq \mu^*(B)$  for  $A \subset B$  (any covering of  $B$  with elements of  $\mathcal{E}$  is also a covering of  $A$ .)

We have to prove the  $\sigma$ -subadditivity.

Let  $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}(X)$  and  $\varepsilon > 0$ . For each  $n$ ,  $\exists \{E_{n,j}\}_{j \in \mathbb{N}} \subset \mathcal{E}$  s.t.  $A_n \subset \bigcup_{j=1}^{\infty} E_{n,j}$  and  $\sum_{j=1}^{\infty} \rho(E_{n,j}) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}$ . But then, if  $A = \bigcup_{n=1}^{\infty} A_n$ , we have that  $A \subset \bigcup_{n,j \in \mathbb{N}^2} E_{n,j}$  and

$$\mu^*(A) \leq \sum_{n,j} \rho(E_{n,j}) \leq \sum_n \left( \mu^*(A_n) + \frac{\varepsilon}{2^n} \right) = \sum_n \mu^*(A_n) + \varepsilon$$

Since  $\varepsilon$  is arbitrary, we are done. ★

**Example 2.5** (1)  $X \in \mathbb{R}, \mathcal{E} = \{(a, b) : a \leq b, a, b \in \mathbb{R}\}$  family of open intervals:

$$\rho((a, b)) = b - a$$

(2)  $X = \mathbb{R}^n, \mathcal{E} = \{(a_1, b_1) \times \dots \times (a_n, b_n) : a_i \leq b_i, a_i, b_i \in \mathbb{R}\}$ :

$$\rho((a_1, b_1) \times \dots \times (a_n, b_n)) = (b_1 - a_1) \cdot \dots \cdot (b_n - a_n)$$

**Remark 2.4**

$E \in \mathcal{E} \Rightarrow \mu^*(E) = \rho(E)$ .

In examples 1 and 2, we have in fact

$$\mu^*((a, b)) = b - a, \mu^*((a_1, b_1) \times \dots \times (a_n, b_n)) = \prod_{i=1}^n (b_i - a_i)$$

## 2.16 Caratheodory condition

To pass from the outer measure to a measure there is a condition:

**Definition 2.6** (Caratheodory condition)

If  $\mu^*$  is an outer measure on  $X$ , a set  $A \subset X$  is called  $\mu^*$ -**measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall E \subset X$$

### Remark 2.5

If  $E$  is a “nice” set containing  $A$ , then the above equality says that the outer measure of  $A$ ,  $\mu^*(E \cap A)$ , is equal to  $\mu^*(E) - \mu^*(E \cap A^C)$ , which can be thought as an “inner measure”. So basically we are saying that  $A$  is measurable if the outer and inner measure coincide. (Like the definition of Riemann integration with lower and upper sum)

### Remark 2.6

$\mu^*$  is subadditive by definition  $\Rightarrow \mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall E, A \subset X$ . So, to prove that a set is  $\mu^*$ -measurable it is enough to prove the reverse inequality,  $\forall E \subset X$ . In fact, if  $\mu^*(E) = +\infty$ , then  $+\infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^C)$ , and hence  $A$  is  $\mu^*$ -measurable iff

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall E \subset X \text{ with } \mu^*(E) < +\infty$$

## 2.17 Measure induced by an outer measure

Their relevance to the notion of  $\mu^*$ -measurability is clarified by the following

**Theorem 2.5** (Caratheodory)

If  $\mu^*$  is an outer measure on  $X$ , the family

$$\mathcal{M} = \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\}$$

is a  $\sigma$ -algebra and  $\mu^*|_{\mathcal{M}}$  is a complete measure.

### Lemma 2.2

If  $A \subset X$  and  $\mu^*(A) = 0$ , then  $A$  is  $\mu^*$ -measurable.

**Proof.** Let  $E \subset X$  with  $\mu^*(E) < +\infty$ . Then

$$\mu^*(E) \geq \mu^*(E) + \mu^*(A) \geq \mu^*(E \cap A) + \mu^*(E \cap A^C)$$

This implies that  $A$  is  $\mu^*$ -measurable. ★

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<sup>a</sup> $E \cap A^C \subseteq E$  and  $E \cap A \subseteq A$  + monotonicity

To sum up:  $X$  set,  $(\mathcal{E}, \rho)$  elementary and measurable sets, so  $\mu^*$  is an outer measure. Then given  $\mu^*$  and the Caratheodory condition, we have  $(X, \mathcal{M}, \mu)$  that is a complete measure space.

### Remark 2.7

So far we did not prove that  $\mathcal{E} \subseteq \mathcal{M}$ . We will do it in a particular case.

## 2.18 Lebesgue's measure on $\mathbb{R}^n$

- $X = \mathbb{R}$ ,  $\mathcal{E}$  family of open intervals,  $\rho((a, b)) = b - a = \lambda((a, b))$ , the complete measure space is  $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$  with  $\mathcal{L}(\mathbb{R})$  the Lebesgue-measurable sets on  $\mathbb{R}$  and  $\lambda$  the Lebesgue measure on  $\mathbb{R}$ .
- $X = \mathbb{R}^n$ ,  $\mathcal{E} = \{\prod_{k=1}^n (a_k, b_k) : a_k \leq b_k \quad \forall k = 1, \dots, n\}$ ,  $\rho(\prod_{k=1}^n (a_k, b_k)) = \prod_{k=1}^n (b_k - a_k)$  and this is a complete measure space  $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$

### Notation 2.1

$\mathcal{E}$  = family of open intervals  $(a, b)$ ,  $a, b \in \mathbb{R}^*$ ,  $a < b$ .  $\rho = \text{length } l$ .  $\rho((a, b)) = b - a$ .

Open interval  $I$  with length  $l(I)$   $E \subset \mathbb{R}$ . The outer measure of  $E$  is

$$\lambda^*(E) = \inf \left\{ \sum_{n=1}^{+\infty} l(I_n) \mid I_n \text{ is an open interval, } E \subset \bigcup_{n=1}^{\infty} I_n \right\}$$

$A \subset \mathbb{R}$  is  $\lambda^*$ -measurable if

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \cap A^C) \quad \forall E \subset \mathbb{R}$$

$$\{A \subset \mathbb{R} : A \text{ is } \lambda^*\text{-measurable}\} =: \mathcal{L}(\mathbb{R}) \quad (\text{Lebesgue } \sigma\text{-algebra})$$

$$\lambda := \lambda^*|_{\mathcal{L}(\mathbb{R})} \quad (\text{Lebesgue measure on } \mathbb{R})$$

Then,  $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$  is a complete measure space. In particular,  $\lambda^*(A) = 0 \Rightarrow A \in \mathcal{L}(\mathbb{R})$  and  $\lambda(A) = 0$ .

### Remark 2.8 (CC-Criterion for measurability)

To check that  $A$  is  $\lambda^*$ -measurable, it is sufficient to check that

$$\lambda^*(E) \geq \lambda^*(E \cap A) + \lambda^*(E \cap A^C)$$

for every  $E \subset \mathbb{R}$  with  $\lambda^*(E) < +\infty$

## 2.19 Every countable set is Lebesgue-measurable

### Proposition 2.2

Any countable set is measurable, with 0 Lebesgue measure.

**Proof.** Let  $a \in \mathbb{R}$ ,

$$\{a\} \subseteq (a - \varepsilon, a + \varepsilon), \forall \varepsilon > 0 \xRightarrow{\text{by def.}} \lambda^*(\{a\}) \leq 2\varepsilon \xRightarrow{\lim} \lambda^*(\{a\}) = 0$$

$\{a\}$  is measurable with  $\lambda(\{a\}) = 0, \forall a \in \mathbb{R}$ . Now if a set  $A$  is countable,  $A = \{a_n\}_{n \in \mathbb{N}} = \bigcup_n \{a_n\}$  (disjoint)  $\Rightarrow \lambda(A) \stackrel{\sigma\text{-add}}{=} \sum_n \lambda(\{a_n\}) = 0$  ★

### Remark 2.9

$\lambda(\mathbb{Q} = 0)$ .  $\mathbb{Q}$  is dense on  $\mathbb{R}$ ,  $\bar{\mathbb{Q}} = \mathbb{R}$ . In general, measure theoretical info and topological info cannot be compared.

## 2.20 Relation between $\mathcal{B}(\mathbb{R})$ and $\mathcal{L}(\mathbb{R})$

### Proposition 2.3

$\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$

#### Remark 2.10

So far we didn't prove the fact that open intervals are  $\mathcal{L}$ -measurable.

**Proof.** We know that  $\mathcal{B}(\mathbb{R})$  is generated by  $\{(a, +\infty) : a \in \mathbb{R}\}$ . Then, we can directly show that  $(a, +\infty) \in \mathcal{L}(\mathbb{R}) \quad \forall a \in \mathbb{R}$ . Let  $a \in \mathbb{R}$  be fixed. We use the criterion for measurability, and we check that

$$\lambda^*(E) \geq \underbrace{\lambda^*(E \cap (a, +\infty))}_{=: E_1} + \underbrace{\lambda^*(E \cap (-\infty, a])}_{=: E_2} \quad \forall E \subset \mathbb{R}, \lambda^*(E) < +\infty$$

Since  $\lambda^*(E) < +\infty$ ,  $\exists$  a countable union  $\bigcup_n I_n \supset E$ , where  $I_n$  is an open interval  $\forall n$  and

$$\sum_n l(I_n) \leq \lambda^*(E) + \varepsilon$$

Let  $I_n^1 := I_n \cap E_1$ ,  $I_n^2 := I_n \cap (-\infty, a + \frac{\varepsilon}{2^n})$ . These are open intervals:

$$E_1 \subset \bigcup_n I_n^1 \quad E_2 \subset \bigcup_n I_n^2 \quad \text{countable unions}$$

and moreover

$$l(I_n) \geq l(I_n^1) + l(I_n^2) - \frac{\varepsilon}{2^n}$$

By definition of  $\lambda^*$ ,  $\lambda^*(E_1) \leq \sum_n l(I_n^1)$  and  $\lambda^*(E_2) \leq \sum_n l(I_n^2)$ , therefore

$$\lambda^*(E_1) + \lambda^*(E_2) \leq \sum_n l(I_n^1) + \sum_n l(I_n^2) \leq \sum_n \left( l(I_n) + \frac{\varepsilon}{2^n} \right) = \left( \sum_n l(I_n) \right) + 2\varepsilon \leq \lambda^*(E) + 3\varepsilon$$

Since  $\varepsilon$  was arbitrarily chosen, we have

$$\lambda^*(E) \geq \lambda^*(E_1) + \lambda^*(E_2)$$

which is the thesis. ★

So, the Lebesgue measure measures all the open, closed  $G_\delta$ ,  $F_\sigma$  sets. Clearly

$$\lambda((a, b)) = b - a$$

One can also show that  $\lambda$  is invariant under translation.

Questions:  $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R}) \subseteq \mathcal{P}(\mathbb{R})$ , is it a strict inclusion or not?

- By Ulam's theorem, if a measure is such that  $\lambda(\{a\}) = 0, \forall a$  and all the sets in  $\mathcal{P}(\mathbb{R})$  are measurable, then  $\lambda \equiv 0$ . This and the fact that  $\lambda((a, b)) \neq 0$  simply that  $\mathcal{L}(\mathbb{R}) \subsetneq \mathcal{P}(\mathbb{R})$ :  $\exists$  non-measurable sets called Vitali sets. Every measurable set with positive measure contains a Vitali set. (Explanation)
- $\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{L}(\mathbb{R})$ . The construction of an  $\mathcal{L}$ -measurable set which is not a Borel set will be done during exercise classes.

The relation between  $\mathcal{B}(\mathbb{R})$  and  $\mathcal{L}(\mathbb{R})$  is clarified by

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<sup>†</sup>I had no choice

## 2.21 Regularity of Lebesgue measure

### Theorem 2.6 (Regularity of $\lambda$ )

The following sentences are equivalent:

- |  |  |
|--|--|
| (1) $E \in \mathcal{L}(\mathbb{R})$                                    |  |
| (2) $\forall \varepsilon > 0 \exists A \supset E, A \text{ open s.t.}$ | $\lambda(A \setminus E) < \varepsilon$ |
| (3) $\exists G \supset E, G \text{ of class } G_\delta, \text{ s.t.}$  | $\lambda(G \setminus E) = 0$           |
| (4) $\exists C \subset E, C \text{ closed, s.t.}$                      | $\lambda(E \setminus C) = 0$           |
| (5) $\exists F \subset E, F \text{ of class } F_\sigma, \text{ s.t.}$  | $\lambda(E \setminus F) = 0$           |

**Consequence:**  $E \in \mathcal{L}(\mathbb{R}) \Rightarrow E = F \cup N$ , where  $F$  is of class  $F_\sigma$ , and  $\lambda(N) = 0$ .

*Partial proof.* For simplicity, we will consider only sets with finite measure.

- (1)  $\Rightarrow$  (2)  $E \in \mathcal{L}(\mathbb{R})$ . By definition of  $\lambda^*$ ,  $\forall \varepsilon > 0 \exists \bigcup_n I_n \supset E$  s.t. each  $I_n$  is an open interval, and

$$\lambda(E) = \lambda^*(E) \geq \sum_n l(I_n) - \varepsilon$$

We define  $A = \bigcup_n I_n$ , which is open. Also,  $A \supset E$  and

$$\lambda(A) = \lambda\left(\bigcup_n I_n\right) \stackrel{\sigma\text{-sub.}}{\leq} \sum_n l(I_n) \leq \lambda(E) + \varepsilon$$

Then, by excision

$$\lambda(A \setminus E) = \lambda(A) - \lambda(E) \leq \varepsilon$$

- (2)  $\Rightarrow$  (3) Define, for every  $K \in \mathbb{N}$ , an open set  $A_k$  s.t.  $A_k \supset E$  and  $\lambda(A_k \setminus E) < \frac{1}{k}$ . Let  $A = \bigcap_k A_k$ . This is a  $G_\delta$  set, it contains  $E$  (since each  $A_k$  contains  $E$ ) and

$$\lambda(A \setminus E) \stackrel{(A \subset A_k \forall k)}{\leq} \lambda(A_k \setminus E) < \frac{1}{k} \Rightarrow \lambda(A \setminus E) = 0 \quad \forall k$$

- (3)  $\Rightarrow$  (1) If  $E \subset \mathbb{R}$  and  $\exists G \supset E$ , with  $G$  of class  $G_\delta$ , s.t.  $\lambda(G \setminus E) = 0$ , then

$$E = G \setminus (G \setminus E) \text{ is measurable}$$

since  $G$  is a Borel set and  $(G \setminus E)$  has 0 measure, then both are in  $\mathcal{L}$

★

### Remark 2.11

Any countable set has 0 measure. The inverse is false. An example is given by the **Cantor set**.

Let  $T_0 = [0, 1]$ . Then we define  $T_{n+1}$  starting from  $T_n$  in the following way: given  $T_n$ , finite

union of closed disjoint intervals of length  $l_n(\frac{1}{3})^n$ ,  $T_{n+1}$  is obtained by removing from each interval of  $T_n$ , the open central subinterval of length  $\frac{l_n}{3}$ .

The Cantor set is  $T := \bigcap_{k=0}^{+\infty} T_k$ . It can be proved that  $T$  is compact,  $\lambda(T) = 0$  and  $T$  is uncountable.

If, instead of removing intervals of size  $\frac{1}{3}, \frac{1}{9}, \dots, \frac{1}{3^k}$ , we remove sets of size  $(\frac{\varepsilon}{3})^k$ , with  $\varepsilon \in (0, 1)$ , we obtain the **generalized Cantor set** (or **fat Cantor set**)  $T_\varepsilon$ .  $T_\varepsilon$  is uncountable, compact and has no interior points (it contains no intervals). However,  $\lambda(T_\varepsilon) = \frac{3(1-\varepsilon)}{3-2\varepsilon} > 0$

### Remark 2.12

We worked on  $\mathbb{R}$ , but everything can be adapted to  $\mathbb{R}^n$



### 3 Measurable functions

#### 3.1 Definition of measurable functions

**Definition 3.1**

$f : X \rightarrow Y$ , then it is well-defined the counterimage

$$f^{-1} : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$$

$$E \mapsto f^{-1}(E) = \{x \in X : f(x) \in E\}$$

**Definition 3.2**

$(X, \mathcal{M}), (Y, \mathcal{N})$  measurable spaces.  $f : X \rightarrow Y$  is called **measurable** or  $(\mathcal{M}, \mathcal{N})$ -measurable if

$$f^{-1}(E) \in \mathcal{M} \text{ for every } E \in \mathcal{N}$$

so, the counterimage of measurable sets in  $Y$  is a measurable set on  $X$ .

To check if a function is measurable or not, it is often used the following proposition

**Proposition 3.1**

$(X, \mathcal{M}), (Y, \mathcal{N})$  measurable spaces. Let  $\mathcal{F} \subseteq \mathcal{P}(Y)$  be s.t.  $\mathcal{N} = \sigma_0(\mathcal{F})$ . Then

$$f : X \rightarrow Y \text{ is } (\mathcal{M}, \mathcal{N}) - \text{measurable} \Leftrightarrow f^{-1}(E) \in \mathcal{M} \text{ for every } E \in \mathcal{F}$$

We will mainly focus on 2 situations:

- (1)  $(X, \mathcal{M})$  is a measurable space obtained by means of an outer measure.  
Ex:  $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n)), (Y, d_y)$  metric space  $\rightarrow (Y, \mathcal{B}(Y))$ .  
If  $X \rightarrow Y$  is (Lebesgue) measurable  $\Leftrightarrow (\mathcal{M}, \mathcal{B}(Y))$  is measurable
- (2)  $(X, d_X), (Y, d_Y)$  are metric spaces  $\rightarrow (X, \mathcal{B}(X)), (Y, \mathcal{B}(Y))$   
 $f : X \rightarrow Y$  is Borel measurable  $\Leftrightarrow (\mathcal{M}(X), \mathcal{B}(Y))$ -measurable.

**Remark 3.1**

$f$  is Lebesgue measurable if the continuity of the Borel set is a Lebesgue-measurable set.

#### 3.2 Continuous functions

**Proposition 3.2**

There are two parts:

- (1)  $(X, d_X), (Y, d_Y)$  metric spaces. If  $f : X \rightarrow Y$  is continuous, then is Borel measurable
- (2)  $(Y, d_Y)$  metric space. If  $f : \mathbb{R}^n \rightarrow Y$  is continuous, then it is Lebesgue measurable.

**Proof.** The proof is divided in:

- (1) This holds true:

$$f \text{ is continuous} \Leftrightarrow f^{-1}(A) \text{ is open } \forall A \text{ open } \subset Y \Rightarrow f^{-1}(A) \in \mathcal{B}(Y) \forall A \text{ open } \subset Y.$$

Since  $\mathcal{B}(Y) = \sigma_0(\text{open sets})$  by proposition 3.1 this implies that  $f$  is Borel measurable

- (2)  $f$  is continuous  $\stackrel{(1)}{\Rightarrow} f$  is Borel measurable.

$$f^{-1}(A) \in \mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{L}(\mathbb{R}^n) \forall A \in \mathcal{B}(Y).$$

Namely,  $f$  is Lebesgue measurable

★

### Proposition 3.3

$(X, \mathcal{M})$  measurable space,  $(X, d_X), (Y, d_Y)$  metric spaces. If  $f : X \rightarrow Y$  is  $(\mathcal{M}, \mathcal{B}(Y))$ -measurable and  $g : Y \rightarrow Z$  is continuous  $\Rightarrow g \circ f : x \rightarrow Z$  is  $(\mathcal{M}, \mathcal{B}(Z))$ -measurable

### Proposition 3.4

$(X, \mathcal{M})$  measurable space,  $u, v : X \rightarrow \mathbb{R}$  measurable functions. Let  $\Phi : \mathbb{R}^2 \rightarrow Y$  be continuous where  $(Y, d_Y)$  is a metric space. Then  $h : X \rightarrow Y$  defined by  $h(x) = \Phi(u(x), v(x))$  is  $(\mathcal{M}, \mathcal{B}(Y))$ -measurable.

## 3.3 Measurability of composition

### Definition 3.3

$u, v$  measurable  $\Rightarrow u + v$  is measurable.

**Proof.** Define  $f : X \rightarrow \mathbb{R}^2$ ,  $f(x) = (u(x), v(x))$ . By definition  $h = \Phi \circ f$  by proposition 3.3. If we show that  $f$  is  $(\mathcal{M}, \mathcal{B}(\mathbb{R}^2))$ -measurable, then  $h$  is measurable. It can be proved that

$$\mathcal{B}(\mathbb{R}^2) = \sigma_0(\underbrace{\{(a_1, b_1) \times (a_2, b_2) : a, b \in \mathbb{R}\}}_{\text{open rectangle}})$$

Thanks to proposition 3.1, to check that  $f$  is measurable we can simply check that  $f^{-1}(R) \in \mathcal{M} \forall$  open rectangle in  $\mathbb{R}^2$  and  $R = I \times J$ , with  $I$  and  $J$  open intervals:

$$\begin{aligned} f^{-1}(R) &= \{x \in X : (u(x), v(x)) \in R\} \\ &\quad \Updownarrow \\ &= \{x \in X : u(x) \in I \text{ and } v(x) \in J\} \\ &= \underbrace{u^{-1}(I)}_{\in \mathcal{M}} \cap \underbrace{v^{-1}(J)}_{\in \mathcal{M}} \in \mathcal{M} \\ &\quad \text{since both } u, v \text{ are measurable} \end{aligned}$$

This completes the proof

★

Consequences: by proposition 3.3 and 3.4, if  $u$  and  $v$  are measurable, then also  $u + v$ ,  $u \cdot v$ . Other measurable functions include  $u^+ = \max\{u, 0\}$ ,  $u^- = -\min\{u, 0\}$ ,  $|u| = u^+ + u^-$ ,  $u^2, \dots$ . Recall that  $u = u^+ - u^-$

### Remark 3.2

$u^+$  is measurable since  $u^+ = g \circ u$ , where:

$$g(x) = \begin{cases} x & \text{where } x \geq 0 \\ 0 & \text{where } x < 0 \end{cases}$$

Most of the time we will work with functions  $f : X \rightarrow \mathbb{R}$  or  $f : X \rightarrow \underbrace{\overline{\mathbb{R}}}_{\mathbb{R} \cup \{\pm\infty\}}$   $(X, \mathcal{M})$  measurable space, then such a function  $f$  is measurable iff

$$f^{-1}((a, +\infty))^\dagger \in \mathcal{M} \quad \forall a \in \mathbb{R}$$

<sup>†</sup>We use  $)$  if  $f$  takes values in  $\mathbb{R}$  and  $] \text{ if } f \text{ takes values in } \overline{\mathbb{R}}$

or equivalently

$$f^{-1}([a, +\infty)) \in \mathcal{M} \quad \forall a \in \mathbb{R}$$

Let now  $\{f_n\}$  be a sequence of measurable functions from  $X$  to  $\overline{\mathbb{R}}$ . Then we define

$$\begin{aligned} (\inf_n f_n)(x) &= \inf_n f_n(x) \\ (\sup_n f_n)(x) &= \sup_n f_n(x) \\ (\liminf_n f_n)(x) &= \liminf_n f_n(x) \\ (\limsup_n f_n)(x) &= \limsup_n f_n(x) \\ (\lim_n f_n)(x) &= \lim_n f_n(x) \quad \text{if the limit exists} \end{aligned}$$

### 3.4 Measurability of limits for real valued functions

#### Proposition 3.5

$(X, \mathcal{M})$  measurable space,  $f_n : X \rightarrow \overline{\mathbb{R}}$  measurable, then

$$\sup_n f_n \quad \inf_n f_n \quad \liminf_n f_n \quad \limsup_n f_n$$

are measurable, in particular if  $\lim_n f_n$  is well-defined, then  $f$  is measurable

**Proof.**  $(\sup_n f_n)^{-1}((a, \infty]) = \{x \in X : \sup_n f_n(x) > a\}$

$\Downarrow$   
 $\exists \text{ some indexes } n \text{ s.t. } f_n(x) > a$

$$\bigcup_n \{x \in X : f_n(x) > a\} = \bigcup_n \underbrace{f_n^{-1}((a, +\infty))}_{\in \mathcal{M}}$$

Then  $(\sup_n f_n)^{-1}((a, \infty])$  is measurable, since it is the countable union of measurable sets.

Now we check that the  $\limsup_n f_n$  is measurable

$$\limsup_n f_n(x) = \lim_n \underbrace{(\sup_{k \geq n} f_k(x))}_{\text{is decreasing on } n} = \inf_n (\sup_{k \geq n} f_k(x))$$

If we write  $g_n(x) = \sup_{k \geq n} f_k(x)$ , then

- $g_n$  is measurable, by what we proved previously
- $\limsup_n f_n = \inf_n g_n$  is measurable

★

### 3.5 Simple functions and step functions

#### Definition 3.4

$(X, \mathcal{M})$  measurable space. A measurable function  $s : X \rightarrow \overline{\mathbb{R}}$  is said to be simple if  $s(X)$  is a finite set.

$$s(X) = \{a_1, \dots, a_n\} \text{ for some } n \in \mathbb{N}, a_i \neq a_j$$

Then

$$s(x) = \sum_{n=1}^N a_n \chi_{E_n}(x)$$

where  $E_n$  is a measurable set,  $E_n = \{x \in X : s(x) = a_n\}$ , and  $E_i \cap E_j = \emptyset$  for  $i \neq j$ , and  $\bigcup_{n=1}^N E_n = X$ .

### Example 3.1

If  $s : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ , and each  $E_n$  is a finite union of intervals, then  $s$  is said to be a **step** function.

## 3.6 Simple approximation theorem

The idea is to approximate arbitrary measurable functions with simple functions.

### Theorem 3.1

$(X, \mathcal{M})$  measurable space,  $f : X \rightarrow [0, \infty]$  measurable. Then  $\exists$  a sequence  $\{s_n\}$  of simple functions s.t.

$$0 \leq s_1 \leq \dots \leq s_n \leq \dots \leq f \quad (\text{pointwise})$$

$\forall x \in X$

and  $s_n(x) \rightarrow f(x) \quad \forall x \in X$  as  $n \rightarrow \infty$ .

Moreover, if  $f$  is bounded then  $s_n \rightarrow f$  uniformly on  $X$  as  $n \rightarrow \infty$

*Proof - For  $f$  bounded.* Fix  $n \in \mathbb{N}$  and divide  $[0, n]$  in  $n \cdot 2^n$  intervals called  $I_j = [a_j, b_j)$  with length  $\frac{1}{2^n}$ .

Let  $E_0 = f^{-1}([n, +\infty))$ ,  $E_j = f^{-1}([a_j, b_j))$  for  $j = 1, \dots, n \cdot 2^n$ . We let

$$\begin{aligned} s_n(x) &= a_j \quad \text{for } x \in E_j \\ s_n(x) &= n \quad \text{for } x \in E_0 \end{aligned}$$

Namely, we define the simple function  $s_n$  as

$$s_n(x) = n \chi_{E_0}(x) + \sum_{j=1}^{n \cdot 2^n} a_j \chi_{E_j}(x)$$

Then  $s_n \leq s_{n+1}$  by construction, and, since  $f$  is bounded,  $E_0 = \emptyset$  for  $n$  sufficiently large ( $n > \sup f$ ).

Then any  $x \in X$  stays in  $f^{-1}([a_j, b_j))$  for some  $j$

$$\begin{aligned} \Rightarrow a_j &\leq f(x) < b_j \\ &\parallel \\ &s_n(x) \\ \Rightarrow 0 &\leq f(x) - s_n(x) < \underbrace{b_j - a_j}_{= \frac{1}{2^n}} \\ \Rightarrow \sup_{x \in X} |f(x) - s_n(x)| &< \frac{1}{2^n} \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

Namely,  $s_n \rightarrow f$  uniformly on  $X$ .



### Remark 3.3

On the relation between  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$  and  $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$  ( $\lambda = \text{Lebesgue measure}$ )  
 $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$  is not complete. In fact,  $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$  is the completion of  $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ .

Note that,  $\forall E \in \mathcal{L}(\mathbb{R}) \exists$  a  $G_\delta$  - set  $A$  and an  $F_\sigma$  - set  $B$  s.t.

$$\begin{aligned} A \supset E \text{ and } \lambda(A \setminus E) &= 0 \\ B \subset E \text{ and } \lambda(E \setminus B) &= 0 \end{aligned}$$

$(X, \mathcal{M}, \mu)$  complete measure space.

### Definition 3.5

Let  $P(x)$  be a proposition depending on  $x \in X$ . We say that  $P(x)$  is true  $(\mu-)$  almost everywhere if

$$\mu(\{x \in X : P(x) \text{ is false}\}) = 0$$

$P(x)$  is true  $\underset{(\mu\text{-a.e.})}{\text{a.e.}}$  on  $X$ .

### Example 3.2

$(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ ,  $f(x) = x^2$ . Then  $f(x) > 0$  a.e. on  $\mathbb{R}$  (for a.e.  $x$ ):

$$\{f(x) \leq 0\} = \{0\}, \text{ and } \lambda(\{0\}) = 0$$

$(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mu_C)$  with  $\mu_C$  counting measure. Then it is not true that  $f(x) > 0$   $\mu_C$ -a.e.

$$\mu_C(\{0\}) = 1$$

It will be useful to consider sequences converging a.e.:

$$f_n \rightarrow f \quad \text{a.e. on } X$$

if  $\mu(\{x \in X : \lim_n f_n(x) \neq f(x), \text{ or does not exist}\}) = 0$

### Proposition 3.6

$(X, \mathcal{M}, \mu)$  complete measure space.

- (1)  $f : X \rightarrow \mathbb{R}$  is measurable, and  $g = f$  a.e. on  $X$ , then  $g$  is measurable
- (2)  $f_n \rightarrow f$  a.e. on  $X$ ,  $f_n : X \rightarrow \mathbb{R}$  measurable for all  $n$ , then  $f$  is measurable

## 4 The Lebesgue integral

### Notation 4.1

$$\begin{aligned}\{x \in X : f(x) \geq 0\} &= \{f \geq 0\} \\ \{x \in X : f(x) > 0\} &= \{f > 0\} \\ &\vdots\end{aligned}$$

$(X, \mathcal{M}, \mu)$  complete measure space. We consider measurable functions  $f : X \rightarrow [0, +\infty]$

Convention: we define

$$\begin{aligned}a + \infty &= +\infty \quad \forall a \in \mathbb{R} \\ a \cdot (+\infty) &= \begin{cases} +\infty & \text{if } a \neq 0, a > 0 \\ 0 & \text{if } a = 0 \end{cases}\end{aligned}$$

With this convention,  $+$  and  $\cdot$  of measurable functions are measurable functions.

### 4.1 Integral of non-negative simple functions

#### Definition 4.1

Let  $s : X \rightarrow [0, +\infty]$  be a measurable simple function,

$$s(x) = \sum_{n=1}^m a_n \chi_{D_n}(\bar{x})$$

where  $D_1, \dots, D_m$  are measurable, disjoint, and  $\bigcup_{n=1}^m D_n = X$ . Let also  $E \in \mathcal{M}$ . Then we define

$$\int_E s \, d\mu := \sum_{n=1}^m a_n \mu(D_n \cap E)$$

#### Remark 4.1

Given a simple function  $s$ :

$$s : [a, b] \rightarrow \mathbb{R}, \lambda = \mu \Rightarrow \int_E s \, d\mu \text{ is the area under the curve}$$

#### Remark 4.2

There are several points:

- In the definition we have already used the convention  $\mu(D_n \cap E = +\infty)$  for some  $n$
- $E \in \mathcal{M} \Rightarrow \chi_E$  is a simple function

$$\chi_E(x) = 1 \cdot \chi_E + 0 \cdot \chi_{X \setminus E}(x)$$

In this case

$$\int_X \chi_E \, d\mu = 1 \cdot \mu(E) + 0 \cdot \mu(X \setminus E) = \mu(E)$$

- $s\chi_E = \sum_{n=1}^m a_n \chi_{D_n \cap E} \Rightarrow \int_E s \, d\mu = \int_X s\chi_E \, d\mu$

## 4.2 Integral of non-negative measurable functions

### Definition 4.2

$f : X \rightarrow [0, +\infty]$  measurable,  $E \in \mathcal{M}$ . The **Lebesgue integral** of  $f$  on  $E$ , with respect to (w.r.t.)  $\mu$ , is

$$\int_E f d\mu = \sup \left\{ \int_E s d\mu \mid \begin{array}{l} s \text{ is simple} \\ 0 \leq s \leq f \end{array} \right\}$$

- (1) If  $f$  is simple, the definitions are consistent
- (2) Also for  $f$  measurable:  $\int_E f d\mu = \int_X f \chi_E d\mu$
- (3)  $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_C)$ .  $f : \mathbb{N} \rightarrow \mathbb{R}$  is a sequence  $\{a_n\}_{n \in \mathbb{N}}$

$$\int_{\mathbb{N}} \{a_n\} d\mu_C = \sum_{n=0}^{\infty} a_n$$

## 4.3 Basic properties of Lebesgue integral

Let  $f, g : X \rightarrow [0, \infty]$  measurable.  $E, F \in \mathcal{M}$ ,  $\alpha \geq 0$ . Then:

- (1)  $\mu(E) = 0 \Rightarrow \int_E f d\mu = 0$
- (2)  $f \leq g$  on  $E \Rightarrow \int_E f d\mu \leq \int_E g d\mu$
- (3)  $E \subset F \Rightarrow \int_E f d\mu \leq \int_F f d\mu$
- (4)  $\alpha \geq 0 \Rightarrow \int_E \alpha f d\mu = \alpha \int_E f d\mu$

### Remark 4.3

$([0, 1], \mathcal{L}([0, 1]), \lambda)$

Consider  $\chi_{\mathbb{Q}}$ , it is the Dirichlet function on  $[0, 1]$ . This is not Riemann integrable.

However,  $\int_{[0,1]} \chi_{\mathbb{Q}} d\lambda = \lambda(\mathbb{Q} \cap [0, 1]) = 0$

## 4.4 Chebychev's inequality

### Theorem 4.1 (Chebychev's inequality)

$f : X \rightarrow [0, \infty]$  measurable,  $c > 0$ . Then

$$\mu(\{f \geq c\}) \leq \frac{1}{c} \int \{f \geq c\} f d\mu \leq \frac{1}{c} \int_X f d\mu$$

**Proof.**

$$\int_X f d\mu \stackrel{X \supset \{f \geq c\}}{\geq} \int_{\{f \geq c\}} f d\mu \geq \int_{\{f \geq c\}} c d\mu = c \int_{\{f \geq c\}} d\mu = c \mu(\{f \geq c\})$$

★

## 4.5 Measure defined by the integral

### Theorem 4.2

$s : X \rightarrow [0, \infty]$  simple. Define  $\varphi : \mathcal{M} \rightarrow [0, \infty]$ ,  $\varphi(E) = \int_E s d\mu \Rightarrow \varphi$  is a measure.

### Definition 4.3

$\{E_n \subset \mathcal{M}\}$  disjoint, and let  $E = \bigcup_{n=1}^{\infty} E_n \Rightarrow s = \sum_{k=1}^m a_k \chi_{D_k}$   $D_k \in \mathcal{M}$

**Proof.**  $\mu(\emptyset) = 0 \Rightarrow \varphi(\emptyset) = 0$  by definition of the integral.

Then, by definition 4.3 and since  $\mu$  is a measure and  $E \cap D_k = \bigcup_n (E_n \cap D_k)$

$$\begin{aligned} \varphi(E) &= \sum_{k=1}^m a_k \mu(D_k \cap E) = \sum_{k=1}^m a_k \sum_{n=1}^{\infty} \mu(E_n \cap D_k) = \\ &= \sum_{n=1}^{\infty} \left( \sum_{k=1}^m a_k \mu(E_n \cap D_k) \right) = \sum_{n=1}^{\infty} \int_{E_n} s d\mu = \sum_{n=1}^{\infty} \varphi(E_n) \end{aligned}$$

★

## 4.6 Vanishing Lemma

### Theorem 4.3 (Vanishing Lemma)

$f : X \rightarrow [0, \infty]$  measurable.  $E \subset X$  measurable

$$\int_E f d\mu = 0 \Leftrightarrow f = 0 \text{ a.e. on } E$$

**Proof.**  $\Leftarrow$  easy

$\Rightarrow$  Consider  $E \cap \{f > 0\} = \bigcup_{n=1}^{\infty} \underbrace{\left( E \cap \left\{ f \geq \frac{1}{n} \right\} \right)}_{=: E_n}$

Then  $\{E_n\}$  is an increasing sequence. By Chebychev

$$\mu(E_n) \leq \frac{1}{\frac{1}{n}} \int_{E_n} f d\mu = 0 \quad \forall n \Rightarrow \mu(E_n) = 0 \quad \forall n$$

$\mu(E \cap \{f > 0\}) \stackrel{\text{continuity}}{=} \lim_n \mu(E_n) = 0$ , namely  $f = 0$  a.e. on  $E$

★

The  $\int$  does not see sets with 0 measure.

### Definition 4.4

If  $f : X \rightarrow [0, \infty]$  is measurable, and  $\int_X f d\mu < \infty$  then we say that  $f$  is integrable.

## 4.7 Monotone Convergence Theorem

### Theorem 4.4 (Monotone Convergence - Beppo Levi)

$f_n : X \rightarrow [0, \infty]$  measurable. Suppose that

- $f_n(x) \leq f_{n+1}(x)$  for a.e.  $x \in X$  for every  $n$
- $f_n \rightarrow f$  a.e. on  $X$



Then

$$\int_X f d\mu = \lim_n \int_X f_n d\mu$$

**Proof.** Part 1.

Assume that the two hypothesis hold everywhere. First, if  $f$  is measurable

$$\int_X f_n d\mu \nearrow \Rightarrow \exists \alpha = \lim_n \int_X f_n d\mu$$

Also,  $f_n \leq f$  everywhere  $\Rightarrow \int_X f_n d\mu \leq \int_X f d\mu \quad \forall n$

$$\Rightarrow \alpha \leq \int_X f d\mu$$

We want to show that also  $\geq$  is true. Let  $s$  be a simple function s.t.  $0 \leq s \leq f$  and  $c \in (0, 1)$

Let  $E_n = \{f_n \geq cs\} \in \mathcal{M}$

- $E_n \subset E_{n+1} \quad \forall n$  :  
if  $x \in E_n$ , then  $f_n(x) \geq cs(x) \Rightarrow f_{n+1}(x) \geq cs(x)$   
 $\Rightarrow f_{n+1}(x) \geq f_n(x) \geq cs(x) \Rightarrow x \in E_{n+1}$
- Moreover,  $X = \bigcup_{n=1}^{\infty} E_n$ . Indeed:  
- if  $f(x) = 0$ , then  $s(x) = 0 \Rightarrow f_1(x) = 0 = cs(x)$ ,  $x \in E_1$   
- if  $f(x) > 0$ , then  $cs(x) < f(x) = \lim_n f_n(x)$  since  $s \leq f$  and  $c < 1$   
 $\Rightarrow cs(x) < f_n(x)$  for  $n$  sufficiently large, namely  $x \in E_n$  for  $n$  sufficiently large.

Therefore,

$$\alpha \geq \int_X f_n d\mu \geq \int_{E_n} f_n d\mu \geq c \int_{E_n} s d\mu = c\varphi(E_n)$$

$\forall n, \forall 0 \leq s \leq f, \forall c \in [0, 1] \quad \varphi(E_n) = \int_{E_n} s d\mu$ .  $\varphi$  is a measure, and  $\{E_n\} \nearrow$

Therefore, taking the limit when  $n \rightarrow \infty$  by continuity

$$\alpha \geq \lim_n c \int_{E_n} s d\mu = c \int_X s d\mu \quad \forall c \in [0, 1]$$

Take the limit when  $c \rightarrow 1^-$  :  $\alpha \geq \int_X s d\mu \quad \forall 0 \leq s \leq f$

Take the sup over  $s$ :  $\alpha \geq \int_X f d\mu$ . We proved both inequalities, so the thesis holds.

Part 2.

Note that  $\{x \in X : \text{assumptions of the theorem do not hold}\}$  is a set of zero measure. Take  $F$ .  $X = E \cup F$  since we have the assumption on  $E$  and  $\mu(F) = 0$ .

Then, by the Vanishing Lemma, since  $(f - f\chi_E) = 0$  a.e. and  $(f_n - f_n\chi_E) = 0$  we have that

$$\int_X f d\mu = \int_E f d\mu = \lim_n \int_E f_n d\mu = \lim_n \int_X f_n d\mu$$

★

**Corollary 4.1** (Monotone convergence for series)

$f_n : X \rightarrow [0, +\infty]$  measurable, then

$$\int_X \left( \sum_{n=0}^{\infty} f_n \right) d\mu = \sum_{n=0}^{\infty} \int_X f_n d\mu$$

**Theorem 4.5** (Approximation with simple functions)

Given  $(X, \mathcal{M})$  measure space,  $f : X \rightarrow [0, +\infty]$  measurable, then  $\exists$  a sequence  $\{s_n\}$  of simple functions s.t.

$$0 \leq s_1 \leq \dots \leq s_n \leq \dots \leq f \quad \text{pointwise } \forall x \in X$$

and

$$s_n(x) \rightarrow f(x) \quad \forall x \in X \text{ as } n \rightarrow \infty$$

Moreover, if  $f$  is bounded, then  $s_n \rightarrow f$  uniformly on  $X$  as  $n \rightarrow \infty$ .

**Remark 4.4**

There is also

$$\int_X f d\mu = \sup \left\{ \int_X s d\mu \mid \begin{array}{l} s \text{ is simple} \\ 0 \leq s \leq f \end{array} \right\}$$

But let  $\{s_n\}$  be the sequence given by the simple approximation theorem. By monotone convergence

$$\int_X f d\mu = \lim_n \int_X s_n d\mu$$

**Example 4.1**

$f, g : X \rightarrow [0, +\infty]$ . Then

$$\int_X (f + g) d\mu = \int_X f d\mu + \int_X g d\mu$$

**4.8 Fatou's Lemma****Lemma 4.1** (Fatou's Lemma)

Given  $f_n : X \rightarrow [0, +\infty]$  measurable  $\forall n$ . Then

$$\int_X (\liminf_n f_n) d\mu \leq \liminf_n \int_X f_n d\mu$$

In particular, if  $f_n \rightarrow f$  a.e. on  $X$ , then

$$\int_X f d\mu \leq \liminf_n \int_X f_n d\mu$$

**Proof.** Given that  $(\liminf_n f_n)(x) = \lim_n (\underbrace{\inf_{k \geq n} f_k(x)}_{=g_n(x)})$ . Now, for every  $x \in X$ ,  $\{g_n(x)\} \nearrow$

$$g_{n+1}(x) = \inf_{k \geq n+1} f_k(x) \geq \inf_{k \geq n} f_k(x) = g_n(x)$$

Also,  $g_n \geq 0$  on  $X$ . Thus, by monotone convergence

$$\int_X \liminf_n f_n d\mu = \int_X \lim_n g_n d\mu = \lim_n \int_X g_n d\mu = \liminf_n \int_X g_n d\mu$$

Now, note that

$$g_n(x) = \inf_{k \geq n} f_k(x) \leq f_n(x) \leq \liminf_n \int_X f_n d\mu$$

★

## 4.9 Integration of series of non-negative functions

### Theorem 4.6 ( $\sigma$ -additivity of $\int$ )

Given  $(X, \mathcal{M}, \mu)$  measure space,  $\phi : X \rightarrow [0, +\infty]$ . Define  $\nu(E) = \int_E \phi d\mu$ , with  $E \in \mathcal{M}$ .  $\nu : \mathcal{M} \rightarrow [0, +\infty]$  is a measure. Moreover, let  $f : X \rightarrow [0, +\infty]$  measurable

$$\int_X f d\nu = \int_X f \phi d\mu \quad *$$

**Proof.**  $\nu$  is a measure:

$\nu(\emptyset) = 0$ , since  $\mu(\emptyset) = 0$ . Now, let  $E = \bigcup_{k=1}^{\infty} E_k$ ,  $\{E_k\}$  disjoint. Then

$$\nu(E) = \int_X \phi \chi_E d\mu = \int_X \phi \sum_n \chi_{E_n} d\mu \underset{\text{monot. conv. for } \sum}{=} \sum_n \int_X \phi \chi_{E_n} d\mu = \sum_n \int_{E_n} \phi d\mu = \sum_n \nu(E_n)$$

We have proven  $\sigma$  additivity, so  $\nu$  is a measure.

(\*) holds: Let  $E \in \mathcal{M}$ . Then

$$\int_X \chi_E d\nu = \int_E 1 d\nu = \nu(E) = \int_E \phi d\mu = \int_X \phi \chi_E d\mu$$

This shows that (\*) holds for  $\chi_E$ ,  $\forall E$ . Then it holds for simple functions.

Let now  $f$  be any measurable function, positive. Then we can take  $\{s_n\}$  given by the simple approximation theorem

$$\int_X f d\nu \overset{\text{monot}}{=} \lim_n \int_X s_n d\nu = \lim_n \int_X s_n \phi d\mu \overset{\text{monot}}{=} \int_X f \phi d\mu$$

which is (\*) ★

### Remark 4.5

$X, \mathcal{M}, \mu$  complete measure space. Then, by the vanishing lemma, it is not difficult to deduce that

$$f = g \text{ a.e. on } X \Leftrightarrow \int_E f d\mu = \int_E g d\mu \quad \forall E \in \mathcal{M}$$

The  $\int$  does not see differences of sets with 0 measure. As a consequence, in all the theorems, it is sufficient to assume that the assumptions are satisfied a.e.

## 4.10 Integrable functions

$X, \mathcal{M}, \mu$  complete measure space.

$f : X \rightarrow \overline{\mathbb{R}} = [-\infty, \infty]$  measurable. Recall  $f = f^+ - f^-$  where  $f^+ = \max\{f, 0\}$ ,  $f^- = -\min\{f, 0\}$  and  $|f| = f^+ + f^-$ . Note that both are positive and measurable.

### Definition 4.5

We say that  $f : X \rightarrow \overline{\mathbb{R}}$  measurable is integrable on  $X$  if

$$\int_X |f| d\mu < \infty$$

If  $f$  is integrable, we define  $\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu$

## 4.11 The set $\mathcal{L}^1$

The set of integrable functions is denoted by

$$\mathcal{L}^1(X, \mathcal{M}, \mu) := \{f : X \rightarrow \overline{\mathbb{R}} \text{ integrable}\}$$

$$\mathcal{L}^1(X, \mathcal{M}, \mu) = \mathcal{L}^1(X) = \mathcal{L}^1$$

If  $E \in \mathcal{M}$ , we define

$$\int_E f d\mu = \int_X f \chi_E d\mu$$

### Remark 4.6

$f \in \mathcal{L}^1(X) \Rightarrow \int_X f d\mu \in \mathbb{R}$ . Indeed,  $0 \leq f^\pm \leq |f|$

$$\Rightarrow 0 \leq \int_X f^+ d\mu, \int_X f^- d\mu \leq \int_X |f| d\mu < \infty$$

$$\Rightarrow \int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu \in \mathbb{R}$$

## 4.12 Triangle inequality

### Proposition 4.1

$f : X \rightarrow \overline{\mathbb{R}}$  measurable. Then

$$(1) f \in \mathcal{L}^1 \Leftrightarrow |f| \in \mathcal{L}^1 \Leftrightarrow \text{both } f^+, f^- \in \mathcal{L}^1$$

$$(2) f \in \mathcal{L}^1, \text{ then}$$

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu \quad (\text{triangle inequality})$$

**Proof.** Of the second part.

$$\left| \int_X f d\mu \right| = \left| \int_X f^+ d\mu + \int_X f^- d\mu \right| \leq \int_X f^+ d\mu + \int_X f^- d\mu = \int_X |f| d\mu$$

★

## 4.13 $\mathcal{L}^1$ is a vector space

### Proposition 4.2

$\mathcal{L}^1(X, \mathcal{M}, \mu)$  is a vector space, and  $f, g \in \mathcal{L}^1, \alpha \in \mathbb{R}$

$$\Rightarrow \int_X (\alpha f + g) d\mu = \alpha \int_X f d\mu + \int_X g d\mu$$

by linearity of the integrals.

Many results can be extended from non-negative functions to general functions.

### Theorem 4.7

$(X, \mathcal{M}, \mu)$  complete measure space.  $f, g \in \mathcal{L}^1$ . Then

$$f = g \text{ a.e. on } X \Leftrightarrow \int_X |f - g| d\mu = 0 \Leftrightarrow \int_E f d\mu = \int_E g d\mu \quad \forall E \in \mathcal{M}$$

## 4.14 Dominated convergence theorem

The most relevant theorem for convergence is the following

**Theorem 4.8** (Dominated convergence theorem)

$\{f_n\}$  sequence of measurable functions  $X \rightarrow \overline{\mathbb{R}}$ . Suppose that

- (1)  $f_n \rightarrow f$  a.e. on  $X$
- (2)  $\exists g : X \rightarrow \overline{\mathbb{R}}, g \in \mathcal{L}^1(X)$ , such that  $|f_n(x)| \leq g(x)$  a.e. on  $X \forall n \in \mathbb{N}$

Then  $f \in \mathcal{L}^1$  and

$$\lim_n \int_X |f_n - f| d\mu = 0 \quad \left( \Rightarrow \int_X f d\mu = \lim_n \int_X f_n d\mu \right)$$

**Proof.** Note that  $f_n \in \mathcal{L}^1 \forall n$ , since  $|f_n| \leq g$  and we have the monotonicity of  $\int$  for non-negative functions

$$\begin{aligned} |f_n(x)| \leq g(x) \quad n \rightarrow \infty \quad |f(x)| \leq g(x) \text{ a.e. on } X \\ \Rightarrow f \in \mathcal{L}^1(X) \end{aligned}$$

Now, consider  $\phi_n = 2g - |f_n - f|$ . We have

$$|f_n - f| \leq |f_n| + |f| \leq 2g \quad \text{a.e. on } X \quad \phi_n \geq 0 \quad \text{a.e. on } X$$

We can use Fatou's lemma:

$$\begin{aligned} \int_X \underbrace{(\liminf_n \phi_n)}_{= 2g \text{ a.e.}} d\mu &\leq \liminf_n \int_X \phi_n d\mu = \liminf_n \int_X (2g - |f_n - f|) d\mu = \\ &\stackrel{||}{=} \int_X 2g d\mu \\ &= \int_X 2g d\mu + \liminf_n \left( - \int_X |f_n - f| d\mu \right) = \int_X 2g d\mu - \limsup_n \int_X |f_n - f| d\mu \end{aligned}$$

Subtracting  $\int_X 2g d\mu$  from both sides

$$0 \leq - \limsup_n \int_X |f_n - f| d\mu \Rightarrow 0 \leq \liminf_n \int_X |f_n - f| d\mu \leq \limsup_n \int_X |f_n - f| d\mu \leq 0$$

★

**Remark 4.7**

If  $\mu(X) < +\infty$ , and  $\exists M > 0$  s.t.  $|f_n| \leq M$  a.e. on  $X, \forall n$ , then we can take  $g = M$  as dominating function.

## 4.15 Comparison between Riemann and Lebesgue integrals

Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ ,  $I$  interval, be bounded. Assume also that  $I$  is closed and bounded.

**Theorem 4.9**

Let  $f$  be Riemann-integrable on  $I$  ( $f \in R(I)$ ). Then

$$f \in \mathcal{L}^1(I, \mathcal{L}(I), \lambda)$$

and

$$\int_I f d\lambda = \int_I f(x) dx$$

**Theorem 4.10**

$f \in R(I) \Leftrightarrow f$  is continuous on  $x$ , for a.e.  $x \in I$ .

**Example 4.2**

$\chi_{\mathbb{Q}}$  on  $[0, 1]$  is not Riemann integrable, because it is discontinuous at any point. Note that, instead,  $\chi_{\mathbb{Q}} = 0$  a.e. on  $[0, 1] \Rightarrow \int_{[0,1]} \chi_{\mathbb{Q}} d\lambda = 0$ .

Let  $f \notin R(I)$ . Is it true that  $\exists g \in R(I)$  s.t.  $g = f$  a.e. on  $I$ ? No.

For instance, consider  $T_{\mathcal{E}}$ , the generalized Cantor set ( $\lambda(T_{\mathcal{E}}) = 0$ ) and then consider  $\chi_{T_{\mathcal{E}}}$ .

In general,  $\chi_A$  is discontinuous on  $\delta A$ . But  $T_{\mathcal{E}}$  has no interior parts  $\Rightarrow T_{\mathcal{E}} = \delta T_{\mathcal{E}} \Rightarrow \chi_{T_{\mathcal{E}}}$  is discontinuous on  $T_{\mathcal{E}}$ , which has positive measure  $\Rightarrow$  by the last theorem,  $\chi_{T_{\mathcal{E}}}$  is not  $R(I)$

Clearly

$$\int_{[0,1]} \chi_{T_{\mathcal{E}}} d\lambda = \lambda(T_{\mathcal{E}})$$

so  $\chi_{T_{\mathcal{E}}} \in \mathcal{L}^1([0, 1])$ .

If  $g = \chi_{T_{\mathcal{E}}}$  a.e., then  $g$  is discontinuous at almost every part of  $T_{\mathcal{E}} \Rightarrow g$  is discontinuous on a set of positive measure  $\Rightarrow g \notin R(I)$ . So, the Lebesgue integral is a true extension of the Riemann one.

Regarding generalized integrals we have

**Theorem 4.11**

$-\infty \leq a < b \leq +\infty$ ,  $f \in R^g([a, b])$  where

$$R^g([a, b]) = \{\text{Riemann-int functions on } [a, b] \text{ in the generalized sense}\}$$

Then,  $f$  is  $([a, b], \mathcal{L}([a, b]))$ -measurable. Moreover,

$$(1) f \geq 0 \text{ on } [a, b] \Rightarrow f \in \mathcal{L}^1([a, b])$$

$$(2) |f| \in R^g([a, b]) \Rightarrow f \in \mathcal{L}^1([a, b])$$

and in both cases

$$\int_{[a,b]} f d\lambda = \int_a^b f(x) dx$$

If  $f$  is in  $R^g([a, b])$ , but  $|f| \notin R^g([a, b])$ , then the two notions of  $\int$  are not really related

**Example 4.3**

$$f(x) = \frac{\sin x}{x}, x \in [1, \infty]$$

$$\int_1^{\infty} |f(x)| dx = +\infty \Rightarrow f \notin \mathcal{L}^1([1, +\infty])$$

But on the other hand

$$\int_1^{\infty} \frac{\sin x}{x} dx = \lim_{\omega \rightarrow \infty} \int_1^{\omega} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

**Proposition 4.3**

$(X, \mathcal{M}, \mu)$  complete measure space. Let  $\{f_n\} \subseteq \mathcal{L}^1(X, \mathcal{M}, \mu)$ .

Suppose that  $\sum_{n=1}^{\infty} \int_X |f_n| d\mu < \infty$  Then the series  $\sum_{n=1}^{\infty} f_n$  converges a.e. on  $X$ , it is in  $\mathcal{L}^1(X)$  and

$$\int_X \left( \sum_{n=1}^{\infty} f_n \right) d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$$

## 4.16 The spaces $\mathcal{L}^1$ and $\mathcal{L}^\infty$

$(X, \mathcal{M}, \mu)$  complete measure space.

$$\mathcal{L}^1 = \{f : X \rightarrow \overline{\mathbb{R}} : f \text{ is integrable}\}$$

$\mathcal{L}^1$  is a vector space. On  $\mathcal{L}^1$  we can introduce  $d : \mathcal{L}^1 \times \mathcal{L}^1 \rightarrow [0, +\infty)$  defined by

$$d_1(f, g) = \int_X |f - g|$$

It is immediate to check that

$$d_1(f, g) = d_1(g, f) \quad (\text{symmetry})$$

$$d_1(f, g) \leq d_1(f, h) + d_1(h, g) \quad \forall f, g, h \in \mathcal{L}^1(X) \quad (\text{triangular inequality})$$

However,  $d_1$  is not a distance on  $\mathcal{L}^1(X)$ , since

$$d_1(f, g) = 0 \Rightarrow f = g \quad \text{a.e. on } X \quad (\text{pseudo-distance})$$

To overcome this problem, we introduce an equivalent relation in  $\mathcal{L}^1(X)$ : we say that

$$f \sim g \Leftrightarrow f = g \quad \text{a.e. on } X$$

If  $f \in \mathcal{L}^1(X)$ , we can consider the equivalence class

$$[f] = \{g \in \mathcal{L}^1(X) : g = f \text{ a.e. on } X\}$$

We define

$$L^1(X) = \frac{\mathcal{L}^1(X)}{\sim} = \{[f] : f \in \mathcal{L}^1(X)\}$$

$L^1(X)$  is a vector space, and on  $L^1(X)$  the function  $d_1$  is a distance:

$$d_1([f], [g]) = 0 \Leftrightarrow \int_X |[f] - [g]| d\mu = 0 \Leftrightarrow [f] = [g] \text{ a.e.} \Leftrightarrow f = g \text{ a.e.}$$

To simplify the notations, the elements of  $L^1(X)$  are called functions, and one writes  $f \in L^1(X)$ . With this, we mean that we choose a representative in  $[f]$ , and  $f$  denotes both the representative and the equivalence class. The representative can be arbitrarily modified on any set with 0 measure.

Another relevant space of measurable functions is the space of **essentially bounded** functions.

### Definition 4.6

$f : X \rightarrow \overline{\mathbb{R}}$  measurable is called essentially bounded if  $\exists M > 0$  s.t.

$$\mu(\{x \in X : |f(x)| \geq M\}) = 0$$

#### Example 4.4

$$f(x) = \begin{cases} 1 & x > 0 \\ +\infty & x = 0 \\ 0 & x < 0 \end{cases}$$

For  $M > 1$ ,  $\lambda(\{x \in \mathbb{R} : |f(x)| > M\}) = \lambda(\{0\}) = 0 \Rightarrow f$  is essentially bounded.

If  $f$  is essentially bounded, it is well-defined the **essential supremum** of  $f$ .

$$\operatorname{ess\,sup}_X f := \inf \{M > 0 \text{ s.t. } f \leq M \text{ a.e. on } X\} = \inf \{M > 0 \text{ s.t. } \mu(\{f \geq M\}) = 0\}$$

It can also be defined an essential inf.

#### Remark 4.8

Note that, by definition of  $\inf$ ,  $\forall \varepsilon > 0$  we have

$$f \leq (\operatorname{ess\,sup}_X f) + \varepsilon \quad \text{a.e. on } X$$

We define

$$L^\infty(X, \mathcal{M}, \mu) = \frac{\mathcal{L}^\infty(X, \mathcal{M}, \mu)}{\sim}$$

$L^\infty(X)$  is a vector space, and it is also a metric space for  $d_\infty(f, g) = \operatorname{ess\,sup}_X |f - g|$



## 5 Types of convergence

### 5.1 Various types of convergence

$\{f_n\}$  sequence of measurable functions  $X \rightarrow \overline{\mathbb{R}}$

- $f_n \rightarrow f$  pointwise (everywhere) on  $X$  if  $f_n(x) \xrightarrow{n \rightarrow \infty} f(x) \forall x \in X$
- $f_n \rightarrow f$  uniformly on  $X$  if  $\sup_{x \in X} |f_n(x) - f(x)| \xrightarrow{n \rightarrow \infty} 0$
- $f_n \rightarrow f$  a.e. on  $X$  if

$$\mu \left( \left\{ x \in X : \lim_n f_n(x) \neq f(x) \text{ or does not exist} \right\} \right) = 0$$

$$\Downarrow$$

$$f_n(x) \rightarrow f(x) \text{ for a.e. } x \in X$$

- Convergence in  $L^1(X)$ :  $f_n \rightarrow f$  in  $L^1(X)$  if

$$\int_X |f_n - f| d\mu \xrightarrow{n \rightarrow \infty} 0$$

$$\parallel$$

$$d_1(f_n, f)$$

- Convergence in measure/probability:  $f_n \rightarrow f$  in measure if  $\forall \alpha > 0$

$$\lim_{n \rightarrow \infty} \mu(\{|f_n - f| \geq \alpha\}) = 0$$

Basic facts: uniformly convergence  $\Rightarrow$  pointwise  $\Rightarrow$  a.e. convergence.

#### Example 5.1

$$f_n(x) = \exp\{-nx\}, x \in [0, 1]$$

$$f(x) = 0, \quad g(x) = \begin{cases} 0 & x \in (0, 1] \\ 1 & x = 0 \end{cases}$$

Then  $f_n \rightarrow g$  pointwise,  $g = f$  a.e.  $\Rightarrow f_n \rightarrow f$  a.e. on  $[0, 1]$ . But  $f(0) \neq g(0) \Rightarrow f_n \rightarrow f$  pointwise.

$f_n \not\rightarrow g$  uniformly on  $[0, 1] \mid \begin{cases} f_n \in \mathcal{C}([0, 1]) \\ f_n \rightarrow g \Rightarrow g \in \mathcal{C}([0, 1]) \end{cases}$   
a.e.  $\Rightarrow$  uniform, but not all is lost...

### 5.2 Egorov's theorem

#### Theorem 5.1 (Egorov)

Let  $\mu(X) < +\infty$ , and suppose that  $f_n \rightarrow f$  a.e. on  $X$ . Then,  $\forall \varepsilon > 0, \exists X_\varepsilon \subset X$ , measurable, s.t.

$$\mu(X \setminus X_\varepsilon) < \varepsilon$$

and  $f_n \rightarrow f$  uniformly on  $X_\varepsilon$

Ex: in an example  $f_n \rightarrow 0$  a.e.,  $f_n \rightarrow 0$  uniformly on  $[0, 1]$ , but  $f_n \rightarrow 0$  uniformly on  $[\varepsilon, 1]$ .  
Regarding a.e. convergence and in measure convergence there is the following theorem

**Theorem 5.2**

If  $\mu(X) < +\infty$  and  $f_n \rightarrow f$  a.e. on  $X \Rightarrow f_n \rightarrow f$  in measure on  $X$

**Proof.** Let  $\alpha > 0$ . We want to show that  $\forall \varepsilon > 0 \exists \bar{n} \in \mathbb{N}$  s.t.

$$n > \bar{n} \Rightarrow \mu(\{|f_n - f| \geq \alpha\}) < \varepsilon$$

$f_n \rightarrow f$  a.e. on  $X$ ,  $\mu(X) < +\infty \xrightarrow{\text{Egorov}} \exists X_\varepsilon \subseteq X$  s.t.  $\mu(X \setminus X_\varepsilon) < \varepsilon$  and  $f_n \rightarrow f$  uniformly on  $X_\varepsilon \Leftrightarrow \sup_{X_\varepsilon} |f_n - f| \xrightarrow{n \rightarrow \infty} 0$ .

In particular, this means that  $\exists \bar{n} \in \mathbb{N}$  s.t.  $n > \bar{n} \Rightarrow |f_n - f| < \alpha$  on  $X_\varepsilon$ .

Therefore,

$$\{|f_n - f| \geq \alpha\} \cap X_\varepsilon = \emptyset \Rightarrow \{|f_n - f| \geq \alpha\} \subseteq X \setminus X_\varepsilon \quad \text{for } n > \bar{n}$$

By monotonicity of  $\mu$ , we deduce that

$$\mu(\{|f_n - f| \geq \alpha\}) \leq \mu(X \setminus X_\varepsilon) < \varepsilon \quad \text{for } n > \bar{n}$$

Namely,  $f_n \rightarrow f$  in measure. ★

**Remark 5.1**

$\mu(X) < +\infty$  is essential

For example, in  $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$  consider

$$f_n(x) = \chi_{[n, n+1)}(x)$$

$f_n(x) \rightarrow 0$  for every  $x \in \mathbb{R}$ . However,  $\lambda(\{|f_n| \geq \frac{1}{2}\}) = \lambda([n, n+1)) = 1$  not 0

**5.3 The typewriter sequence****Remark 5.2**

Convergence in measure  $\Rightarrow$  a.e convergence?

No, not even if  $\mu(X) < +\infty$ .

Consider  $\chi_{n,k} = \chi_{[\frac{k-1}{n}, \frac{k}{n}]}$  with  $n \in \mathbb{N}, k = 1, \dots, n$

$$\begin{aligned} \chi_{1,1}(x) &= \chi_{[0,1]}(x) \\ \chi_{2,1}(x) &= \chi_{[0, \frac{1}{2}]}(x) & \chi_{2,2}(x) &= \chi_{[\frac{1}{2}, 1]}(x) \\ \chi_{3,1}(x) &= \chi_{[0, \frac{1}{3}]}(x) & \chi_{3,2}(x) &= \chi_{[\frac{1}{3}, \frac{2}{3}]}(x) & \chi_{3,3}(x) &= \chi_{[\frac{2}{3}, 1]}(x) \end{aligned}$$

For  $n$  fixed and  $k$  variable, we move the  $\chi$  from the left to right. When the  $\chi$  reaches 1, we switch  $n$ , and  $\chi$  reappear from the left, being thinner.

$$\int_{[0,1]} \chi_{n,k} d\lambda = \frac{1}{n} \quad \int_{[0,1]} \chi_{n+1,k} d\lambda = \frac{1}{n+1}$$

We can order the elements of  $\chi_{n,k}$  in a sequence, letting  $f_p = \chi_{n,k}$  for  $p = 1+2+\dots+(n-1)+k$ . We will prove that  $\{f_p\}$  converges in measure, but not a.e.

This is the **typewriter sequence**  $(p(n, k))$ . For every  $x \in [0, 1]$  there are  $\infty$  many indexes s.t.  $f_p(x) = 1$  and  $\infty$  many indexes s.t.  $f_p(x) = 0$ , meaning that  $\nexists \lim_{p \rightarrow \infty} f_p(x)$   $f_p \not\rightarrow 0$  a.e. on  $[0, 1]$ .

But we do have convergence in measure to 0:  $\alpha \in (0, 1)$

$$\lambda(\{|f_{p(n,k)}| \geq \alpha\}) = \lambda\left(\left[\frac{k-1}{n}, \frac{k}{n}\right]\right) = \frac{1}{n} \rightarrow 0 \text{ as } \begin{matrix} n \rightarrow \infty \\ \updownarrow \\ p \rightarrow \infty \end{matrix}$$

**Remark 5.3**

So,  $f_p \not\rightarrow 0$  a.e. on  $[0, 1]$ . But consider  $\{f_{p(n,1)} : n \in \mathbb{N}\}$ . This is a subsequence and, by definition

$$f_{p(n,1)}(x) = \chi_{n,1}(x) = \chi_{[0, \frac{1}{n}]}(x)$$

For this subsequence, we have  $f_{p(n,1)}(x) \rightarrow 0$  as  $n \rightarrow \infty \forall x \in (0, 1]$ , then a.e. on  $[0, 1]$   
This is not random!

**Proposition 5.1**

If  $\mu(X) < \infty$  and  $f_n \rightarrow f$  in measure, then  $\exists$  a subsequence  $\{f_{n_k}\}$  s.t.  $f_{n_k} \rightarrow f$  a.e. on  $X$ .

Now we analyze the relation between convergence in  $L^1(X)$  and the other convergences.

**Theorem 5.3**

$\{f_n\} \subset L^1(X), f \in L^1(X)$ . If  $f_n \rightarrow f$  in  $L^1(X)$  then  $f_n \rightarrow f$  in measure on  $X$

**Proof.** By contradiction. Suppose that  $f_n \not\rightarrow f$  in measure on  $X$ :  $\exists \bar{\alpha} > 0$  s.t.

$$\limsup_{n \rightarrow \infty} \mu(\{|f_n - f| \geq \bar{\alpha}\}) > 0$$

$\Rightarrow \exists \bar{\varepsilon}$  and a subsequence  $\{f_{n_k}\}$  s.t.

$$\mu(\{|f_{n_k} - f| \geq \bar{\alpha}\}) > \bar{\varepsilon}$$

Consider then

$$\begin{aligned} d_1(f_{n_k}, f) &= \int_X |f_{n_k} - f| d\mu \stackrel{\text{monot.}}{\geq} \int_{\{|f_{n_k} - f| \geq \bar{\alpha}\}} |f_{n_k} - f| d\mu \geq \\ &\geq \int_{\{|f_{n_k} - f| \geq \bar{\alpha}\}} \bar{\alpha} d\mu = \bar{\alpha} \mu(\{|f_{n_k} - f| \geq \bar{\alpha}\}) > \bar{\alpha} \bar{\varepsilon} \end{aligned}$$

But, by assumption,  $d_1(f_n, f) \rightarrow 0$

$$\Rightarrow d_1(f_{n_k}, f) \rightarrow 0$$

Contradiction. ★

**Remark 5.4**

The convergence in measure doesn't imply the convergence in  $L^1$ .

For example, consider

$$f_n(x) = n\chi_{[0, \frac{1}{n}]}(x)$$

$$\underbrace{\mu(\{|f_n| \geq \alpha\})}_{=\frac{1}{n}} \rightarrow 0 \text{ for every } \alpha$$

On the other hand

$$\int_{[0,1]} n\chi_{[0, \frac{1}{n}]} d\lambda = \int_{[0, \frac{1}{n}]} n d\lambda = n \frac{1}{n} = 1$$

$f_n \not\rightarrow 0$  in  $L^1$

Convergence a.e.  $\not\Rightarrow$  convergence in  $L^1$ :

Use the same example above,  $f_n \rightarrow 0$  a.e. on  $[0, 1] \not\Rightarrow f_n \rightarrow 0$  in  $L^1$

Convergence in  $L^1 \not\Rightarrow$  convergence a.e.:

Consider the typewriter sequence:  $d_1(f_{p(n,k)}, 0) \rightarrow 0$  when  $p \rightarrow \infty$

But we don't have a.e. convergence.

However, recall the dominated convergence theorem: (DOM)

$$f_n \rightarrow f \text{ a.e.} + \exists \text{ of a dominating function} \Rightarrow d(f_n, f) \rightarrow 0$$

It is also possible to show a reverse DOM:

If  $f_n \rightarrow f$  in  $L^1(X)$ , then  $\exists$  a subsequence  $\{f_{n_k}\}$  and  $w \in L^1(X)$  s.t.

- (1)  $f_{n_k} \rightarrow f$  a.e. on  $X$
- (2)  $|f_{n_k}(x)| \leq w(x)$  for a.e.  $x \in X$

## 6 Derivative of a measure

### 6.1 Radon-Nikodym derivative

$(X, \mathcal{M}, \mu)$  measure space,  $\phi : X \rightarrow [0, \infty]$  measurable.

We learned that  $\nu : \mathcal{M} \rightarrow [0, \infty]$  by

$$\nu(E) = \int_E \phi d\mu \text{ is a measure on } (X, \mathcal{M})$$

If the equation above holds, then we say that  $\phi$  is the **Radon Nikodym derivative** of  $\nu$  with respect to  $\mu$ , and we write

$$\phi = \frac{d\nu}{d\mu}$$

#### Definition 6.1

$\mu, \nu$  measures on  $(X, \mathcal{M})$ . We say that  $\nu$  is absolutely continuous with respect to  $\mu$ ,  $\nu \ll \mu$  if

$$\mu(E) = 0 \Rightarrow \nu(E) = 0$$

#### Lemma 6.1

There is a necessary condition:

$$\exists \frac{d\nu}{d\mu} \Rightarrow \nu \ll \mu$$

**Proof.**

$$\nu(E) = \int_E \left( \frac{d\nu}{d\mu} \right) d\mu = 0$$

if  $\mu(E) = 0$  by basic properties of  $\int$

★

### 6.2 Radon-Nikodym theorem

#### Theorem 6.1 (Radon Nikodym Theorem)

$(X, \mathcal{M})$  measurable space,  $\mu, \nu$  measures.

If  $\nu \ll \mu$  and moreover  $\mu$  is  $\sigma$ -finite, then  $\phi : X \rightarrow [0, \infty]$  measurable s.t.

$$\phi = \frac{d\nu}{d\mu} \quad \text{namely } \nu(E) = \int_E \phi d\mu \quad \forall E \in \mathcal{M}$$

#### Remark 6.1

If  $\mu$  is not sigma finite the theorem may fail.

In  $([0, 1], \mathcal{L}([0, 1]))$  consider the counting measure  $\mu = \mu_C$  and the Lebesgue measure  $\nu = \lambda$   
 $\nu \ll \mu$  since  $\mu(E) = 0 \Leftrightarrow E = \emptyset \Rightarrow \lambda(E) = \nu(E) = 0$

But we can check that  $\nexists \phi : [0, 1] \rightarrow [0, \infty]$  measurable s.t.  $\lambda(E) = \int_E \phi d\mu_C$

Check by contradiction: assume that  $\phi$  does exist, and take  $x_0 \in [0, 1]$

$$0 = \lambda(\{x_0\}) = \int_{\{x_0\}} \phi d\mu_C = \phi(x_0) \overbrace{\mu_C(\{x_0\})}^{=1} = \phi(x_0)$$

$\Rightarrow \phi(x_0) = 0 \forall x_0 \in [0, 1]$ .

But then  $1 = \lambda([0, 1]) = \int_{[0, 1]} 0 d\mu_C = 0$ . Contradiction

Note that  $\mu_C([0, 1]) = \infty$  and  $([0, 1], \mathcal{L}([0, 1]), \mu_C)$  is not  $\sigma$ -finite ( $[0, 1]$  is uncountable)

## 7 Product measures space

### 7.1 Construction of product measure spaces

$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$  measure spaces. The goal is to define a measure space on  $X \times Y$

#### Definition 7.1

We call **measurable rectangle** in  $X \times Y$  a set of type  $A \times B$  where  $A \in \mathcal{M}, B \in \mathcal{N}$

$$R = \{A \times B \subset X \times Y \text{ s.t. } A \in \mathcal{M}, B \in \mathcal{N}\}$$

We define the product  $\sigma$ -algebra  $\mathcal{M} \otimes \mathcal{N}$  as  $\sigma_0(R)$ .

This is a  $\sigma$ -algebra in  $X \times Y$

#### Definition 7.2

Let  $E \subset X \times Y$ . For  $\bar{x} \in X$  and  $\bar{y} \in Y$  we define

$$\begin{aligned} E_{\bar{x}} &= \{y \in Y : (\bar{x}, y) \in E\} \subseteq Y & \bar{x}\text{-section of } E \\ E_{\bar{y}} &= \{x \in X : (x, \bar{y}) \in E\} \subseteq X & \bar{y}\text{-section of } E \end{aligned}$$

#### Proposition 7.1

$(X, \mathcal{M}), (Y, \mathcal{N})$  measurable spaces.  $E \in \mathcal{M} \otimes \mathcal{N}$

Then  $E_x \in \mathcal{M}$  and  $E_y \in \mathcal{N} \Rightarrow$  we can define

$$\begin{aligned} \varphi : X &\rightarrow [0, \infty] & \psi : Y &\rightarrow [0, \infty] \\ x &\mapsto \nu(E_x) & y &\mapsto \mu(E_y) \end{aligned}$$

#### Theorem 7.1

If  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$  are  $\sigma$  finite spaces, then:

- (1)  $\varphi$  is  $\mathcal{M}$ -measurable and  $\psi$  is  $\mathcal{N}$ -measurable
- (2) we have that  $\int_X \nu(E_x) d\mu = \int_Y \mu(E_y) d\nu$

Using the fact that  $\mu$  and  $\nu$  are measures, and that  $\int$  of non-negative function is a measure, we deduce the following

#### Theorem 7.2 (Iterated integrals for characteristic functions)

$\mu \otimes \nu : \mathcal{M} \otimes \mathcal{N} \rightarrow \mathbb{R}$  defined by

$$(\mu \otimes \nu)(E) = \int_X \nu(E_x) d\mu = \int_Y \mu(E_y) d\nu$$

is a measure, the product measure.

#### Remark 7.1 (On the completion of product measure spaces)

$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$  complete measures spaces.

In general, it is not true that  $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \otimes \nu)$  is complete.

#### Example 7.1

$X = Y = \mathbb{R}, \mathcal{M} = \mathcal{N} = \mathcal{L}(\mathbb{R}), \mu = \nu = \lambda.$

Given  $A$  non meas. set,  $A \subseteq [0, 1], B = \{y_0\}, E = A \times B$ . If  $E$  were measurable, then its sections must be measurable. But  $E_{y_0} = A$  which is not measurable.

However,  $E$  is negligible:

$$E \subseteq [0, 1] \times \{y_0\}, \text{ and } (\lambda \otimes \lambda)([0, 1] \times \{y_0\}) = 0$$

Then  $(\lambda \otimes \lambda)$  is not complete

$$\Rightarrow (\mathbb{R}^2, \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R}), \lambda \otimes \lambda) \neq (\mathbb{R}^2, \mathcal{L}(\mathbb{R}^2), \lambda_2)$$

### Theorem 7.3

Let  $\lambda_n$  be the Lebesgue measure in  $\mathbb{R}^n$ . If  $n = K + m$ , then  $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$  is the completion of  $(\mathbb{R}^k \times \mathbb{R}^m, \mathcal{L}(\mathbb{R}^k) \otimes \mathcal{L}(\mathbb{R}^m), \lambda_k \otimes \lambda_m)$

## 7.2 Integration on product spaces

$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$  measure spaces.  $f : X \times Y \rightarrow \overline{\mathbb{R}}$  measurable.

If  $f \geq 0$ , then

$$\iint_{X \times Y} f d\mu \otimes d\nu$$

Goal: obtain a formula of iterated integral like the one in Analysis 2.

$\forall \bar{x} \in X$  and  $\bar{y} \in Y$ , we define

$$\begin{aligned} f_{\bar{x}} : Y &\rightarrow \overline{\mathbb{R}} & f_{\bar{y}} : X &\rightarrow \overline{\mathbb{R}} \\ y &\mapsto f(\bar{x}, y) & x &\mapsto f(x, \bar{y}) \end{aligned}$$

### Proposition 7.2

If  $f$  is measurable  $\Rightarrow f_{\bar{x}}$  is  $(\mathcal{N}, \mathcal{B}(\overline{\mathbb{R}}))$ -measurable and  $f_{\bar{y}}$  is  $(\mathcal{M}, \mathcal{B}(\overline{\mathbb{R}}))$ -measurable. Then we can consider

$$\begin{aligned} \varphi : X &\rightarrow \overline{\mathbb{R}} & \varphi(x) &= \int_Y f_x d\nu = \int_Y f(x, y) \underbrace{d\nu(y)}_{d_y} \\ \psi : Y &\rightarrow \overline{\mathbb{R}} & \psi(y) &= \int_X f_y d\mu = \int_X f(x, y) d\mu(x) \end{aligned}$$

## 7.3 Tonelli and Fubini's theorem

Questions: what is the solution of  $\iint_{X \times Y}$ ,  $\varphi$  and  $\psi$ ?

### Theorem 7.4 (Tonelli's theorem)

$(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$  complete measure spaces and  $\sigma$ -finite.

Suppose that  $f$  is  $(\mathcal{M} \otimes \mathcal{N}, \mathcal{B}(\overline{\mathbb{R}}))$ -measurable and that  $f > 0$  a.e. on  $X \times Y$ . Then  $\psi$  and  $\varphi$  are measurable and

$$\iint_{X \times Y} f d\mu \otimes d\nu = \int_X \varphi(x) d\mu(x) = \int_Y \psi(y) d\nu(y) \quad \text{Integration formula}$$

Equally holds also if one of the integrals is  $\infty$ .

$$\begin{aligned} \int_X \varphi(x) d\mu(x) &= \int_X \left( \int_Y f(x, y) d\nu(y) \right) d\mu(x) \\ \int_Y \psi(y) d\nu(y) &= \int_Y \left( \int_X f(x, y) d\mu(x) \right) d\nu(y) \end{aligned}$$

**Remark 7.2**

The double integral can be reduced to single integrals, iterated. Moreover, we can always change the order of the integrals. For sign changing functions the situation is more involved.

**Theorem 7.5** (Fubini's theorem)

$(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$  complete measure spaces and  $\sigma$ -finite. If  $f \in L^1(X \times Y)$ , then  $\psi$  and  $\varphi$  defined above are measurable, the integration formula holds, and all the integrals are finite.

Question: how to check if  $f \in L^1(X \times Y)$ ? Typically, to check that  $f \in L^1(X \times Y)$  one uses Tonelli:

$$f \in L^1(X \times Y) \Leftrightarrow \iint_{X \times Y} |f| d\mu \otimes d\nu$$

We use Tonelli to check that this is finite. If  $\iint_{X \times Y} |f| d\mu \otimes d\nu < \infty$  then we can apply Fubini for  $\iint_{X \times Y} f d\mu \otimes d\nu$

**Remark 7.3**

the proof of Fubini's and Tonelli's theorems is based for the iterated integrals for characteristic functions. Note that

$$(\mu \otimes \nu)(E) = \int_X \varphi(x) d\mu(x) = \int_X \left( \int_Y f(x, y) d\nu(y) \right) d\mu(x) \\ = \int_Y \psi(y) d\nu(y) = \int_Y \left( \int_X f(x, y) d\mu(x) \right) d\nu(y)$$

**Remark 7.4**

Sometimes double integrals are very useful to compute single integrals.

**Example 7.2**

$$\int_{-\infty}^{+\infty} e^{-x^2} = \sqrt{\pi}$$



## 8 AC and BV functions

### 8.1 Lebesgue points

Consider  $f \in L^1([a, b])$ . We can define the **integral function**

$$F(x) = \int_{[a, b]} f d\lambda = \int_a^b f(t) dt, \quad x \in [a, b]$$

If  $f \in \mathcal{C}([a, b])$ , then  $F$  is differentiable on  $[a, b]$ , and  $F'(x) = f(x)$

What happens if  $f \in L^1([a, b])$ ?

#### Definition 8.1

Given  $f \in L^1([a, b])$ . We say that  $x \in [a, b]$  is a **Lebesgue point** for  $f$  if

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| dt = 0$$

If  $x = a$  or  $x = b$ , this is the left/right lim.

#### Remark 8.1

A point  $x$  is called a Lebesgue point for  $f$  if  $f$  ‘does not oscillate too much’ close to  $x$ :

- $f \in \mathcal{C}([a, b]) \rightarrow$  every  $x \in [a, b]$  is a Lebesgue point.

- 

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_0^h |f(t) - f(0)| dt = \lim_{h \rightarrow 0} \frac{1}{|h|} \int_0^h |0 - 1| dt = 0$$

#### Theorem 8.1 (Lebesgue)

If  $f \in L^1([a, b])$  then a.e.  $x \in [a, b]$  is a Lebesgue point for  $f$

#### Remark 8.2

In the definition of Lebesgue point, the pointwise values of  $f$  are relevant

$$f = g \in L^1 \Leftrightarrow f = g \text{ a.e.}$$

Then the Lebesgue point of  $f$  could be different from the one of  $g$ . This is not a big problem if  $f = g$  a.e. on  $[a, b] \Rightarrow f = g \in [a, b] \setminus N$  where  $\lambda(N) = 0$ ;  $x$  is a Lebesgue point for  $f$ ,  $\forall x \in [a, b] \setminus M$ ,  $\lambda(M) = 0$

$\Rightarrow x$  is a Lebesgue point for  $g$ ,  $\forall x \in [a, b] \setminus (M \cup N)$

$[a, b] \setminus (M \cup N)$  is a set of full measure of Lebesgue points for  $f$  and  $g$ .

To speak about Lebesgue points, one has to choose a specific representative  $f \in L^1([a, b])$ . If you change representative, you obtain the same set of Lebesgue points up to sets with 0-measure.

## 8.2 First fundamental theorem of calculus

### Theorem 8.2 (First fundamental theorem of calculus)

Given  $f \in L^1([a, b])$ ,  $F(x) = \int_a^x f(t) dt$

Then  $f$  is differentiable a.e. on  $[a, b]$  and  $F'(x) = f(x)$  a.e. in  $[a, b]$

**Proof.** Let  $x \in [a, b]$  for any Lebesgue point for  $f$  (a.e.  $x \in [a, b]$  is fine). Consider

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| = \left| \frac{1}{h} \int_x^{x+h} (f(t) - f(x)) dt \right| \leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| dt \rightarrow 0$$

Since  $x$  is a Lebesgue point. ★

## 8.3 AC functions

### Definition 8.2

Given  $f : I \rightarrow \mathbb{R}$  is called **absolutely continuous** (AC) in  $I$ ,  $f \in AC(I)$ , if  $\forall \varepsilon > 0 \exists \delta > 0$  s.t.

$$\begin{aligned} \bigcup_{k=1}^n [a_k, b_k] \subseteq I \text{ with disjoint interiors} &\Rightarrow \sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon \\ \lambda\left(\bigcup_{k=1}^n [a_k, b_k]\right) = \sum_{k=1}^n (b_k - a_k) < \delta & \end{aligned}$$

### Remark 8.3

$f$  is uniformly continuous on  $[a, b]$  if  $\forall \varepsilon > 0 \exists \delta > 0$  s.t.

$$|t - \tau| < \delta \Rightarrow |f(t) - f(\tau)| < \varepsilon$$

An absolutely continuous function is also uniformly continuous.

But the converse is false.

- If  $f$  is Lipschitz on  $[a, b] \Rightarrow f \in AC([a, b])$

Recall that  $f \in \text{Lip}([a, b])$  if  $\exists L > 0$  s.t.

$$|f(x) - f(y)| \leq L|x - y| \quad \forall x, y \in [a, b]$$

Check: For any  $\varepsilon > 0$ , and consider

$$\sum_{k=1}^n |f(b_k) - f(a_k)| \leq \sum_{k=1}^n L(b_k - a_k) = L \sum_{k=1}^n (b_k - a_k)$$

If we take  $\delta = \delta(\varepsilon) = \frac{\varepsilon}{L}$ , then

$$\sum_{k=1}^n (b_k - a_k) < \delta \Rightarrow \sum_{k=1}^n |f(b_k) - f(a_k)| \leq L \sum_{k=1}^n (b_k - a_k)$$

★

$$\text{Lip}([a, b]) \subsetneq AC([a, b]) \subsetneq UC([a, b])$$

## 8.4 Absolute continuity of the integral

**Theorem 8.3** (Absolute continuity of the integral)

Given  $f \in L^1(X, \mathcal{M}, \mu)$ . Then  $\forall \varepsilon > 0 \exists \delta > 0$  s.t.

$$E \in \mathcal{M} \quad \mu(E) < \delta \quad \Rightarrow \quad \int_E |f| d\mu < \varepsilon$$

**Proof.** We fix  $\varepsilon > 0$ . Let  $F_n := \{|f| < n\}$ ,  $n \in \mathbb{N}$ . Also,  $F_n \in \mathcal{M} \forall n$ ,  $F_n \subseteq F_{n+1}$  and

$$\bigcup_{n=1}^{\infty} F_n = \{|f| < \infty\} =: F$$

$f \in L^1 \Rightarrow |f|$  is finite a.e.:  $\mu(X \setminus F) = 0$ . Therefore:

$$\int_X |f| d\mu = \int_{X \setminus F} |f| d\mu + \int_F |f| d\mu = \lim_{n \rightarrow \infty} \int_{F_n} |f| d\mu$$

$$\lim_{n \rightarrow \infty} \int_X |f| (\chi_{F_n^C}) d\mu = 0$$

$\forall \varepsilon > 0 \exists \bar{n} \in \mathbb{N}$  s.t.

$$n > \bar{n} \Rightarrow \left| \int_X |f| \chi_{F_n^C} d\mu \right| < \frac{\varepsilon}{2}$$

Now, fix  $\varepsilon > 0$ , and take  $n > \bar{n}(\varepsilon)$ . If  $E \in \mathcal{M}$ , then

$$\int_E |f| d\mu = \int_{E \cap F_n} |f| d\mu + \int_{E \cap F_n^C} |f| d\mu \leq n \int_E 1 d\mu + \int_{F_n^C} |f| d\mu$$

If we suppose that  $\mu(E) < \frac{\varepsilon}{2n} =: \delta(\varepsilon)$ , we deduce that

$$n \int_E 1 d\mu = n\mu(E) < \frac{\varepsilon}{2}$$

Also, since  $n > \bar{n}$

$$\begin{aligned} \int_{F_n^C} |f| d\mu &< \frac{\varepsilon}{2} \\ \Rightarrow \int_E |f| d\mu &< \varepsilon \end{aligned}$$

★

**Theorem 8.4** (Regularity of integral functions)

Given  $f \in L^1([a, b])$ ,  $F(x) = \int_a^x f(t) dt$ , then  $F \in \text{AC}([a, b])$

**Proof.** Let  $\varepsilon > 0$ , and  $\delta = \delta(\varepsilon) > 0$  be the value given by the absolute continuity of  $\int |f| d\mu$ . Take

$$E = \bigcup_{k=1}^n [a_k, b_k] \quad E \subseteq [a, b]$$

If  $\lambda(E) < \delta$ , then

$$\sum_{k=1}^n |F(b_k) - F(a_k)| = \sum_{k=1}^n \left| \int_{a_k}^{b_k} f(t) dt \right| \leq \sum_{k=1}^n \int_{a_k}^{b_k} |f(t)| dt = \int_E |f| d\lambda < \varepsilon$$

by absolute continuity of  $\int$

★

**Remark 8.4**

$\sqrt{x}$  is AC( $[0, 1]$ ), but is not Lip( $[0, 1]$ ).

$$\sqrt{x} = \int_0^x \frac{1}{2\sqrt{t}} dt$$

$\Rightarrow \sqrt{x}$  is the  $\int$  function of an  $L^1$  function

$\Rightarrow \sqrt{x} \in \text{AC}([0, 1])$

To sum up: the  $\int$  function of an  $L^1$  function is AC, it is differentiable a.e., and

$$F(x) - F(a) = \int_a^x F'(t) dt \quad \text{FC}$$

Suppose  $G$  is differentiable a.e. on  $[a, b]$  and FC holds for  $G$ :

$$G(x) - G(a) = \int_a^x G'(t) dt$$

What can we say about  $G$ ?

**Remark 8.5**

If  $G \in \mathcal{C}^1([a, b]) \Rightarrow$  FC holds.

If FC holds, then  $G' \in L^1([a, b])$  (necessary condition). Is the necessary condition also sufficient? In general not. Take  $v(x)$ , the Vital Cantor function:  $v \in \mathcal{C}([0, 1])$ ,  $v(0) = 0$ ,  $v(1) = 1$ .  $v$  is differentiable a.e. on  $[0, 1]$ , but the calculus formula doesn't hold!

**Remark 8.6**

A function which is differentiable a.e. on an interval can behave very badly

**Theorem 8.5**

$G \in \text{AC}([a, b])$ . Then  $G$  is differentiable a.e. on  $[a, b]$ ,  $G' \in L^1([a, b])$ , and FC holds.

**Remark 8.7**

These theorems say that AC function are precisely the ones for which FC holds:

- $G \in \text{AC} \Rightarrow$  FC holds.
- If FC holds, then  $G' \in L^1([a, b])$

$$\Rightarrow \int_a^x G'(t) dt \in \text{AC}$$

$$\Rightarrow G(x) - G(a) = \int_a^x G'(t) dt \in \text{AC}$$

**Remark 8.8**

$v \in \text{UC}([0, 1])$  by continuity and Heine Cantor, but  $v \notin \text{AC}([0, 1])$  because FC does not hold.

The proof of the second fundamental theorem of calculus is divided into two steps.

**Lemma 8.1**

The second fundamental theorem hold under the additional assumption that  $G$  is monotone.

Second step: to get rid of the monotonicity.

## 8.5 Functions of bounded variation

For step 2, is it useful to give the

**Definition 8.3**

$[a, b] \subset \mathbb{R}$ . Let

$$\mathcal{P}_{[a,b]} := \{(x_0, x_1, \dots, x_n) : n \in \mathbb{N} \text{ and } a = x_0 < x_1 < x_2 < \dots < x_n = b\}$$

For  $P \in \mathcal{P}_{[a,b]}$  and  $f : [a, b] \rightarrow \overline{\mathbb{R}}$ , define

$$v_a^b(f, P) := \sum_{k=1}^n |f(x_k) - f(x_{k-1})|$$

The total variation of  $f$  on  $[a, b]$  is

$$V_a^b(f) := \sup_{P \in \mathcal{P}_{[a,b]}} v_a^b(f, P) = \sup \left\{ \sum_{k=1}^n |f(x_k) - f(x_{k-1})| : n \in \mathbb{N}, a = x_0 < x_1 < \dots < x_n = b \right\}$$

If  $V_a^b(f) < \infty$ , we say that  $f$  is a function with **bounded variation**,  $f \in \text{BV}([a, b])$

**8.6 The 2<sup>nd</sup> fundamental theorem of calculus**

**Theorem 8.6** (The 2<sup>nd</sup> fundamental theorem of calculus.)

$G \in \text{AC}([a, b]) \Leftrightarrow G$  is differentiable a.e. on  $[a, b]$ ,  $G' \in L^1([a, b])$ , and (FC) holds.

Example and comments:

- If  $f$  is bounded and monotone  $\Rightarrow f \in \text{BV}$

$$V_a^b(f) = |f(b) - f(a)|$$

Note that  $f$  may not be continuous

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \Rightarrow f \in \text{BV}([-1, 1])$$

- $f \in \text{BV}([a, b]) \Rightarrow f$  is bounded. Indeed,

$$\sup_{x \in [a,b]} |f(x)| \leq |f(x)| + V_a^b(f) \stackrel{f \in \text{BV}}{<} +\infty$$

- $f$  is continuous on  $[a, b]$ , or even if  $f$  is differentiable everywhere in  $[a, b] \nRightarrow f \in \text{BV}([a, b])$

$$f(x) = \begin{cases} x^2 \cos \frac{2\pi}{x^2} & x \in (0, 1] \\ 0 & x = 0 \end{cases}$$

It is continuous in  $[0, 1]$ , but  $f \notin \text{BV}([0, 1])$

- $f \in \text{BV}([a, b]) \cap \text{UC}([a, b]) \nRightarrow f \in \text{AC}([a, b])$

$v$  a Vitali-Cantor function

$v$  is bounded and monotone  $\Rightarrow v \in \text{BV}([0, 1])$

$v \in \text{UC}([0, 1])$

But  $v \notin \text{AC}([0, 1])$

- If  $f \in \text{BV}([a, b]) \Rightarrow f$  is differentiable a.e. on  $[a, b]$ , and  $f' \in L^1([a, b])$

We can now come back to the proof of Lemma 1 of the last lesson.

Preliminary result:  $A \in \mathbb{R}$  open. Then

$$A = \bigcup_{n=1}^{\infty} (a_n, b_n) \text{ disjoint}$$

any open set of  $\mathbb{R}$  is the (at most) countable union of open disjoint intervals.

Preliminary result (equivalent definition for AC):  $f \in \text{AC}([a, b]) \Leftrightarrow \forall \varepsilon > 0 \exists \delta > 0$  depending on  $\varepsilon$  s.t.

$$\begin{aligned} \forall \bigcup_{n=1}^{\infty} [a_n, b_n], \quad [a_n, b_n] \text{ have disjoint interiors} \\ \sum_{n=1}^{\infty} (b_n - a_n) < \delta \Rightarrow \sum_{n=1}^{\infty} |f(b_n) - f(a_n)| < \varepsilon \end{aligned}$$

**Proof.** We defined  $\lambda$  starting from two properties

- invariance under translations
- $\lambda((x, y)) = y - x \quad \forall a \leq x < y \leq b$

Now,  $G$  is monotone, say  $G$  increasing (if  $G \searrow$ , take  $-G$ ). We can repeat the construction of  $\lambda$  in order to obtain a measure  $\mu$  s.t.

- $\mu$  is invariant under translations
- $\mu((x, y)) = \underbrace{G(y) - G(x)}_{\geq 0} \quad \forall a \leq x < y \leq b$  (for  $\lambda$ , take  $G(t) = t$ )

It can be proved that we obtain a measure on  $(\mathbb{R}, \mathcal{L}(\mathbb{R}))$ , complete.

On  $(\mathbb{R}, \mathcal{L}(\mathbb{R}))$  we have two measures:  $\lambda$  and  $\mu$ .

Idea: We take these measures on  $([a, b], \mathcal{L}([a, b]))$ , and we want to show that  $\exists \frac{d\mu}{d\lambda}$  (Radon-Nikodym)

We can check the hypothesis of the Radon-Nikodym theorem:

- $\lambda$  is  $\sigma$ -finite:  $\lambda([a, b]) = b - a < +\infty$
- $\mu \ll \lambda$ :  $E \in \mathcal{L}([a, b]), \lambda(E) = 0 \Rightarrow \mu(E) = 0$

Assume  $\lambda(E) = 0$ .  $G$  is AC( $[a, b]$ ): then  $\forall \varepsilon > 0 \exists \delta = \delta(\varepsilon) > 0$  s.t.

$$\begin{aligned} \forall \bigcup_{n=1}^{\infty} [a_n, b_n], \quad [a_n, b_n] \text{ have disjoint interiors} \\ \lambda \left( \bigcup_{n=1}^{\infty} [a_n, b_n] \right) < \delta \Rightarrow \sum_{n=1}^{\infty} |G(b_n) - G(a_n)| < \varepsilon \end{aligned}$$

Take this  $\delta$ . By regularity of  $\lambda$ ,  $\exists A$  open set of  $[a, b]$  s.t.  $A \supset E$  and  $\lambda(A) < \delta$

$$A \text{ is open} \Rightarrow A = \left( \bigcup_{n=1}^{\infty} I_n^+ \right) \cap [a, b], \text{ disjoint}$$

it is a countable union of open intervals (maybe two of them contains  $a$  or  $b$ )

$$\lambda(A) < \delta \Leftrightarrow \sum_{n=1}^{\infty} (y_n - x_n) < \delta$$

But then, since  $\mu$  is a measure it is countably additive

$$\mu(E) \leq \mu(A) = \sum_n \mu(I_n) = \sum_n |G(y_n) - G(x_n)| < \varepsilon$$

by the choice of  $\delta$  and the fact that  $G \in \text{AC}$ . We proved that

$$\lambda(E) = 0 \Rightarrow \forall \varepsilon > 0 : \mu(E) < \varepsilon \Rightarrow \mu(E) = 0$$

So  $\mu \ll \lambda$ . We can apply Radon Nikodym  $\exists \phi : [a, b] \rightarrow [0, \infty]$  s.t.

$$G(x) - G(a) = \int_a^x \phi d\lambda$$

Since  $G$  is bounded, then  $\phi \in L^1([a, b])$

$$G(x) = G(a) + \int_a^x \phi(t) dt$$

By the first fundamental theorem of calculus, this is differentiable a.e.

$$\Rightarrow G'(x) = \phi(x) \text{ a.e. on } [a, b]$$

$$\Rightarrow G(x) = G(a) + \int_a^x G'(t) dt$$

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<sup>b</sup>open intervals =  $(x_n, y_n)$

Now we want to get rid of the additional assumption (monotonicity).

Preliminary result:  $f \in \text{BV}([a, b])$ . Then

$$\varphi(x) = V_a^x(f), \quad \forall x \in [a, b]$$

is an increasing function.

**Proof.** By  $a \leq x < y \leq b$ . Then

$$V_a^y(f) = V_a^x(f) + \underbrace{V_x^y(f)}_{\geq 0} \geq V_a^x(f)$$

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Preliminary result: If  $G \in \text{AC}([a, b])$ , then  $G \in \text{BV}([a, b])$ , and moreover

$$\varphi(x) = V_a^x(G) \text{ is in } \text{AC}([a, b])$$

*Proof of the second fundamental theorem of calculus in the general case.*  $G \in \text{AC}([a, b])$

We want to write  $G = G_1 + G_2$  where  $G_1 \nearrow$  and  $G_2 \searrow$ , both AC.

Then the second fundamental theorem holds for  $G_1$  and  $G_2$ , so it holds for  $G$  by linearity of the integral.

We pose:

$$G_1(x) = \frac{G(x) + V_a^x(G)}{2}$$

$$G_2(x) = \frac{G(x) - V_a^x(G)}{2}$$

Clearly,  $G_1 + G_2 = G$ ,  $G_1, G_2$  are AC, by the last preliminary result.

$G_1 \nearrow$ : Let  $a \leq x < y \leq b$

$$|G(y) - G(x)| \leq V_x^y(G)$$

Therefore,

$$\begin{aligned} G_1(y) - G_1(x) &= \frac{1}{2} \left( \underbrace{G(y) - G(x)}_{\geq -|G(y) - G(x)|} + V_a^y(G) + V_a^x(G) \right) \geq \frac{1}{2} (-V_x^y(G) + V_x^y(G)) = 0 \\ &\geq -V_x^y(G) \end{aligned}$$

So  $G_1$  is decreasing. In an analogue way, we can prove that  $G_2$  is decreasing. ★



# Functional Analysis

## 9 Review on metric spaces

### 9.1 Distance function

#### Proposition 9.1

$(X, \|\cdot\|)$  normed space. Then  $(X, d)$  is a metric space for

$$d(x, y) = \|x - y\|$$

#### Remark 9.1

Normed space  $\xrightarrow{\text{not}} \text{metric space}$

#### Example 9.1 • $\mathbb{R}^N$

$$\|x\|_p := \left( \sum_{i=1}^N |x_i|^p \right)^{\frac{1}{p}} \quad \forall p \in [1, +\infty) \quad \|x\|_\infty := \max_{i=1, \dots, N} |x_i|$$

- $\mathcal{C}^0([a, b])$

$$\|f\|_\infty := \max_{x \in [a, b]} |f(x)|$$

- $L^1(X, \mathcal{M}, \mu)$

$$\|f\|_1 := \int_X |f| d\mu$$

This is a norm in  $L^1$ , but not on  $\mathcal{L}^1$  ( $\int_X |f| d\mu = 0 \Rightarrow f = 0$  a.e.)

- $L^\infty(X, \mathcal{M}, \mu)$

$$\|f\|_\infty := \operatorname{ess\,sup}_{[a, b]} |f|$$

$(X, \|\cdot\|)$  normed space  $\rightarrow (X, d)$  metric space  $\rightarrow$  convergent sequences on  $X$ :  $\{x_n\} \subset X$  is convergent in  $X$  iff

$$d(x_n, x) \rightarrow 0 \Leftrightarrow \|x_n - x\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

#### Example 9.2

$x_n \rightarrow x$  in  $X$ , then  $\|x_n\| \rightarrow \|x\|$  (the norm is a continuous function on  $X$ )

### 9.2 Cauchy sequences

#### Definition 9.1

$\{x_n\}$  is a **Cauchy sequence** in  $(X, \|\cdot\|)$  if  $\forall \varepsilon > 0 \exists \bar{n} \in \mathbb{N}$  s.t.

$$n, m > \bar{n} \Rightarrow \|x_n - x_m\| < \varepsilon$$

## 10 Banach Spaces

### 10.1 Normed spaces

#### Definition 10.1

Given  $X$  vector space, a norm on  $X$  is a function  $\|\cdot\| : X \rightarrow [0, \infty)$  s.t.

- $\|x\| = 0 \Leftrightarrow x = 0$
- $\forall \alpha \in \mathbb{R}, \forall x \in X :$   
$$\|\alpha x\| = |\alpha| \|x\|$$
(positive homogeneity)
- $\forall x, y \in X :$   
$$\|x + y\| \leq \|x\| + \|y\|$$
(triangle inequality)

Then,  $(X, \|\cdot\|)$  is called a **normed space**

#### Example 10.1

$$|||x| - |y|| \leq \|x - y\| \quad \forall x, y \in X$$

### 10.2 Banach spaces

#### Definition 10.2

$(X, \|\cdot\|)$  is called a **Banach space** if  $(X, d)$  is complete, namely if any Cauchy sequence in  $(X, d)$  is convergent.

If  $(X, \|\cdot\|)$  is a normed space, we can speak about series in  $X$ . Let  $\{x_n\} \subset X$  and  $s_n = x_0 + x_1 + \dots + x_n$ , then  $\sum_{n=0}^{+\infty} x_n = \{s_n\}$ .

Then  $\sum x_n$  is convergent if  $\{s_n\}$  is convergent. If  $\sum x_n$  is convergent, we write

$$s = \sum_{n=0}^{+\infty} x_n \Leftrightarrow s_n \rightarrow s$$

For numerical series

$$\sum_{n=1}^{\infty} |a_n| < +\infty \Rightarrow \sum a_n \text{ is convergent}$$

In general, in normed spaces

$$\sum_{n=1}^{\infty} \|x_n\| < +\infty \not\Rightarrow \sum_{n=1}^{\infty} x_n \text{ is convergent}$$

### 10.3 Equivalent norms

$(X, \|\cdot\|)$  is a Banach space  $\Leftrightarrow$  every series s.t.  $\sum \|x_n\| < +\infty$  is also s.t.  $\sum x_n$  is convergent.

$(X, \|\cdot\|) \rightarrow (X, d) \rightarrow$  open sets, closed sets, bounded sets....

In  $\mathbb{R}^n$  we are used to working with  $\|\cdot\|_2$ , but we could have many norms.

#### Definition 10.3

Let  $\|\cdot\|$  and  $\|\cdot\|_2$  be two norms on the same vector space  $X$ . We say that these norms are **equivalent** if  $\exists m, M > 0$  s.t.

$$m\|x\| \leq \|x\|_2 \leq M\|x\| \quad \forall x \in X$$

It can be proved that if two norms are equivalent they lead to different metric spaces, but to the same open sets, closed sets, convergent sequences, compact sets ...

**Theorem 10.1**

If  $X$  is any finite dimension vector space, then all the norms on  $X$  are equivalent.

**Remark 10.1**

This is why in  $\mathbb{R}^n$  usually one does not specify the choice of the norm. One choose the Euclidean norm, since it comes from a scalar product. (ref. Hilbert spaces)

Preliminary fact: The set  $S_1 = \{s \in \mathbb{R}^n : \|s\|_1 = 1\}$  is compact in  $(\mathbb{R}^n, d)$

**Proof.** We show that any norm is equivalent to  $\|\cdot\|_1 = \sum_{i=1}^n |x_i|$

$$x = \sum_{i=1}^n x_i e_i \quad \{e_i\}_{i=1, \dots, n} \text{ canonical basis}$$

Let's introduce the norm star

$$\|x\|_* = \left\| \sum_{i=1}^n x_i e_i \right\|_* \leq \sum_{i=1}^n \|x_i e_i\|_* = \sum_{i=1}^n |x_i| \|e_i\|_* \leq \left( \max_{1 \leq i \leq n} \|e_i\|_* \right) \sum_{i=1}^n |x_i| = M \|x\|_1$$

We proved that  $\exists M > 0$  s.t.

$$\|x\|_* \leq M \|x\|_1 \quad \forall x \in X \quad (1)$$

Note that this proves that  $\varphi(x) = \|x\|_*$  is continuous in  $(X, d)$ . Indeed,

$$x_n \rightarrow x \Leftrightarrow d_1(x_n, x) \rightarrow 0$$

then

$$|\varphi(x_n) - \varphi(x)| = \left| \|x_n\|_* - \|x\|_* \right| \leq \|x_n - x\|_* \stackrel{(1)}{\leq} M \|x_n - x\|_1 \rightarrow 0$$

Therefore, by the Weierstrass theorem,  $\exists$  a minimum point  $x_0 \in S_1$  s.t.

$$\varphi(x) \geq \varphi(x_0) = m \quad \forall x \in S_1$$

(recall that  $S_1$  is compact)

$$\|x\|_* \geq m \quad \forall x \in S_1$$

We claim that  $m > 0$ . If  $m = 0$  then  $\|x_0\|_* = 0 \Rightarrow x_0 = 0$  that is impossible, since  $x_0 \in S_1$ . Thus,  $m > 0$ . Let now  $y \in \mathbb{R}^n, y \neq 0$ . Then

$$\frac{y}{\|y\|_1} \in S_1 \Rightarrow \left\| \frac{y}{\|y\|_1} \right\|_* \geq m \Rightarrow \frac{1}{\|y\|_1} \|y\|_* \geq m \Rightarrow \|y\|_* \geq m \|y\|_1 \quad \forall y \in \mathbb{R}^n$$

★

If  $\dim X = +\infty$ , then there are many non-equivalent norms.

**Example 10.2**

In  $\mathcal{C}^0([a, b])$ , we can define  $\|\cdot\|_\infty$  and  $\|f\|_1 = \int_a^b |f(t)| dt$ . This is a norm in  $\mathcal{C}^0$ , but these norms are not equivalent.

**10.4 Separable spaces**

$(X, d)$  metric space.

**Definition 10.4**

We say that  $X$  is separable if  $\exists A \subset X$  which is dense ( $\bar{A} = X$ ) and countable

In  $\mathbb{R}^n$ ,  $\mathbb{Q}^n$  is dense and countable, while in  $\infty$  – dim we can have separable spaces or not.

## 10.5 Compactness

In finite dimension (in  $\mathbb{R}^n$ ), one has that

$$E \subset X \text{ is compact} \Leftrightarrow E \text{ is closed and bounded}$$

If  $\dim X = \infty$ , then only ' $\Rightarrow$ ' is true. In finite dimension, we know that the closed unit ball is compact

$$\bar{B}_1(0) = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$$

## 10.6 Riesz's theorem and quasi-orthogonality lemma

What happens now if  $(X, \|\cdot\|)$  is on  $\infty$  – dim normed space? The proof of the Riesz's theorem is based on the Riesz's **quasi-orthogonality lemma**.

**Lemma 10.1** (Riesz Quasi-Orthogonality Lemma)

Let  $X$  be a normed space,  $E \subsetneq X$  a closed subspace. Then  $\forall \varepsilon \in (0, 1) \exists x \in X$  s.t.

$$\|x\| = 1 \text{ and } \text{dist}(x, E) = \inf_{y \in E} \|x - y\| \geq 1 - \varepsilon$$

**Proof.** Let  $y \in X \setminus E$ , and  $d := \text{dist}(y, E) > 0$ , since  $E$  is closed.

$\forall \rho > 0 \exists z \in E$  s.t.

$$\|y - z\| \leq (1 + \rho)d = \frac{d}{1 - \varepsilon} \quad (1)$$

since we choose  $\rho$  s.t.  $1 + \rho = \frac{1}{1 - \varepsilon}$ . Now we set  $x = \frac{y - z}{\|y - z\|}$ .

Clearly  $\|x\| = 1$ . Moreover,  $\forall u \in E$ , we have that

$$\begin{aligned} \|x - u\| &= \left\| \frac{y - z}{\|y - z\|} - u \right\| = \left\| \frac{y - z - \|y - z\|u}{\|y - z\|} \right\| = \frac{1}{\|y - z\|} \|y - (z + \|y - z\|u)\| = \\ &= \frac{1}{\|y - z\|} \|y - w\| \geq \frac{1}{\|y - z\|} \text{dist}(y, E) \stackrel{(1)}{\geq} \frac{1 - \varepsilon}{d} d = 1 - \varepsilon \end{aligned}$$

Since this is true  $\forall u \in E$ , we deduce that

$$\text{dist}(x, E) \geq 1 - \varepsilon$$

★

**Theorem 10.2** (Riesz's theorem)

$X$  normed space,  $\dim X = +\infty \Rightarrow \overline{B_1(0)}$  is not compact

**Remark 10.2**

It is well known that if  $E \subset \mathbb{R}^n$  is compact, then  $\forall \{x_n\} \in E \exists \{x_{n_k}\}$  subsequence s.t.  $x_{n_k} \rightarrow x \in E$ . This proposition is much harder to prove in  $\infty$  – dim.

**Proof.** Assume that  $\overline{B_1(0)}$  is compact, and  $X$  has infinite dimension.

$\exists$  a sequence  $\{E_n\}$  of finite dimensional subspaces (hence closed) of  $X$  s.t.

$$E_{n-1} \subset E_n \text{ and } E_{n-1} \neq E_n$$

$E_{n-1}$  is a proper closed subspace of  $E_n \forall n$

We can apply the Riesz Lemma with  $X = E_n$ ,  $E = E_{n-1}$ ,  $\varepsilon = \frac{1}{2}$ . Then  $\forall n \exists u_n \in E_n$  s.t.  $\|u_n\| = 1$  and  $\text{dist}(u_n, E_{n-1}) \geq \frac{1}{2} \forall n$

Therefore, we have a sequence  $\{u_n\}$  with the following properties

$$\|u_n\| = 1 \quad \forall n$$

$$\|u_n - u_m\| \geq \frac{1}{2} \quad \forall n \neq m$$

$\Rightarrow$  this sequence cannot have any convergent subsequence. But then  $\overline{B_1(0)} \supseteq \{u_n\}$ , this implies that  $\overline{B_1(0)}$  is not compact. Contradiction.

(In any  $(X, \|\cdot\|)$  normed space, if  $E$  is compact, then  $\forall \{x_n\} \subset E \exists \{x_{n_k}\}$  s.t.  $x_{n_k} \rightarrow x \in E$ ) ★

$(X, d)$  metric space.

### Definition 10.5

$E \subset X$  is compact if, for any open covering  $\{A_i\}_{i \in I}$ , has a finite subcover.

### Definition 10.6

$E \subset X$  is sequentially compact if  $\forall \{x_n\} \subset E$  there exists  $\{x_{n_k}\}$  subsequence convergent to some limit  $x \in E$

Well known fact: if  $(X, d)$  is a metric space, then  $E$  is compact  $\Leftrightarrow E$  is sequentially compact.

### Remark 10.3

Also:

- $E \in X$  closed. Then  $\text{dist}(x, E) = 0 \Leftrightarrow x \in E$
- By definition of infimum, if  $d = \text{dist}(x, E)$ , then  $\forall \rho > 0 \exists z \in E$  s.t.

$$\|x - z\| < (1 + \rho)d$$

## 11 The space $\mathcal{C}^0$

### Definition 11.1

$\{f_n\}$  sequence in  $\mathcal{C}^0([a, b])$ . We say that  $\{f_n\}$  is **uniformly equicontinuous** in  $[a, b]$  if  $\forall \varepsilon > 0 \exists \delta > 0$  depending only on  $\varepsilon$  s.t.

$$|t - \tau| < \delta \Rightarrow \|f_n(t) - f_n(\tau)\| < \varepsilon \quad \forall n$$

### Remark 11.1

With respect to the uniform continuity, in this case  $\delta$  does not depend on  $f$ .  $\delta$  is the same for all the  $f_n$  s

### 11.1 Ascoli-Arzelà theorem

#### Theorem 11.1 (Ascoli Arzelà)

$\{f_n\} \subseteq \mathcal{C}^0([a, b])$ . Suppose that:

- $\{f_n\}$  is uniformly equicontinuous
- $\{f_n\}$  is equibounded:  $\exists M > 0$  s.t.  $\|f_n\|_\infty < M \quad \forall n$

Then  $\exists$  a subsequence  $\{f_{n_k}\}$  and  $f \in \mathcal{C}^0([a, b])$  s.t.  $f_{n_k} \rightarrow f$  uniformly.

### 11.2 $\mathcal{C}^0$ is a separable space

For instance,  $(L^\infty, \|\cdot\|_\infty)$  is not separable. Instead,  $(\mathcal{C}^0([a, b]), \|\cdot\|_\infty)$  is a separable space.

*Sketch of the proof.* We will use the **Stone-Weierstrass theorem**.

The set of polynomials is dense on  $\mathcal{C}^0([a, b])$  and is an uncountable set. However, it can be proved that the set of polynomials with coefficients in  $\mathbb{Q}$  is dense in the set of all polynomials. Moreover, this set is countable. Then, by Stone-Weierstrass this is a countable dense set in  $\mathcal{C}^0([a, b])$  ★

### Remark 11.2

One can show that  $\mathcal{C}^0(K)$  is separable whenever  $K$  is a compact set of a metric space  $(X, d)$

## 12 Lebesgue spaces

### 12.1 The sets $\mathcal{L}^p$ and $L^p$

$(X, \mathcal{M}, \mu)$  measure space,  $p \in [1, \infty]$ . We defined  $L^1(X)$  and  $L^\infty(X)$ . Similarly, we define  $L^p(X) \forall p \in [1, \infty]$

$$\mathcal{L}^p(X, \mathcal{M}, \mu) := \{f : X \rightarrow \overline{\mathbb{R}} \text{ measurable s.t. } \int_X |f|^p d\mu < \infty\}$$

On  $\mathcal{L}^p$  we introduce the equivalent relation

$$f \sim g \text{ in } \mathcal{L}^p \Leftrightarrow f = g \text{ a.e. on } X$$

and define

$$L^p(X, \mathcal{M}, \mu) := \frac{\mathcal{L}^p(X, \mathcal{M}, \mu)}{\sim}$$

We want to show that this is a normed space with

$$\|f\|_p := \begin{cases} \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}} & p \in [1, \infty) \\ \text{ess sup}_X |f| & p = \infty \end{cases}$$

The fact that  $L^p$  is a vector space is easy to prove. The only non-trivial part is that  $f, g \in L^p \Rightarrow f + g \in L^p$ .

### 12.2 $L^p$ is a vector space

This comes directly from the

#### Lemma 12.1

$p \in [1, \infty)$ ,  $a, b \geq 0$ . Then

$$(a + b)^p \leq 2^{p-1} (a^p + b^p)$$

**Proof.**  $f, g \in L^p$ ,  $p \in [1, \infty)$

$$\begin{aligned} \int_X |f + g|^p d\mu &\leq \int_X (|f| + |g|)^p d\mu \leq 2^{p-1} \int_X (|f|^p + |g|^p) d\mu \\ &= 2^{p-1} \int_X |f|^p d\mu + 2^{p-1} \int_X |g|^p d\mu < \infty \end{aligned}$$

$L^p$  is a vector space,  $\forall p \in [1, \infty)$ .

$f, g \in L^\infty$ . Then a.e.

$$\Rightarrow |f + g| \leq |f| + |g| \leq \|f\|_\infty + \|g\|_\infty < \infty \Rightarrow f + g \in L^\infty$$

$L^\infty$  is a vector space. ★

### 12.3 $\ell^p$ spaces

#### Remark 12.1

$\ell^p := L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_c)$ .  $\ell^p$  is a particular case of  $L^p$

$$\ell^p = \{x = (x^{(k)})_{k \in \mathbb{N}} : \sum_{k=1}^{\infty} |x^{(k)}|^p < \infty\} \quad \|x\|_p = \left( \sum_{k=1}^{\infty} |x^{(k)}|^p \right)^{\frac{1}{p}} \quad p \in [1, \infty)$$

$$\ell^\infty = \{x = (x^{(k)})_{k \in \mathbb{N}} : \sup_{k \in \mathbb{N}} |x^{(k)}| < \infty\} \quad \|x\|_\infty = \sup_{k \in \mathbb{N}} |x^{(k)}|$$

Now we prove that  $\|\cdot\|_p$  is actually a norm in  $L^p$ . We will concentrate on  $p < \infty$  ( $p = \infty$  is the easy case)

Properties 1 and 2 of the norm are immediate to check:

- (1)  $\|f\|_p = 0 \Leftrightarrow \int_X |f|^p d\mu = 0 \Leftrightarrow f = 0$  a.e. on  $X \Leftrightarrow f = 0 \in L^p$
- (2) Obvious, by linearity
- (3) About triangle inequality? We need some preliminaries

## 12.4 Young's inequality

**Theorem 12.1** (Young's Inequality)

Let  $p \in (1, \infty)$ ,  $a, b \geq 0$ . We say that  $q$  is the conjugate exponent of  $p$  if

$$\frac{1}{p} + \frac{1}{q} = 1 \Leftrightarrow q = \frac{p}{p-1}$$

Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

**Remark 12.2**

$p \in (1, \infty) \Rightarrow q \in (1, \infty)$ . Moreover, we say that 1 and  $\infty$  are conjugate

**Proof.**  $\varphi(x) = e^x$  is convex:

$$\varphi((1-t)x + ty) \leq (1-t)\varphi(x) + t\varphi(y) \quad \forall x, y \in \mathbb{R} \quad \forall t \in [0, 1]$$

If  $a = 0$  or  $b = 0$ , then the thesis holds.

If  $a, b > 0$

$$ab = e^{\log a} e^{\log b} = e^{\log a \frac{p}{p}} e^{\log b \frac{q}{q}} = e^{\frac{1}{p} \log a^p} e^{\frac{1}{q} \log b^q}$$

Since  $\varphi$  is convex

$$\frac{1}{p} e^{\log a^p} + \frac{1}{q} e^{\log b^q} = \frac{1}{p} a^p + \frac{1}{q} b^q$$

$$x = \log a^p, y = \log b^q \quad 1-t = \frac{1}{p}, t = \frac{1}{q}$$

★

## 12.5 Holders's inequality

**Theorem 12.2** (Holder's Inequality)

$(X, \mathcal{M}, \mu)$  measure space.  $f, g$  measurable functions.  $p, q \in [1, \infty]$  conjugate exponents.

Then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$



**Proof.** Case  $p, q \in (1, \infty)$ . Obvious if  $\|f\|_p \|g\|_q = \infty$ .

If  $\|f\|_p \|g\|_q = 0 \Rightarrow$  either  $f = 0$  a.e. on  $X$  or  $g = 0$  a.e. on  $X \Rightarrow fg = 0$  a.e. on  $X \Rightarrow \|fg\|_1 = 0$ . Let then  $\|f\|_p, \|g\|_q \in (0, \infty)$ .

For  $x \in X$ , we set

$$a := \frac{|f(x)|}{\|f\|_p}, \quad b := \frac{|g(x)|}{\|g\|_q}$$

and use Young:

$$\frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \leq \frac{1}{p} \frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q} \frac{|g(x)|^q}{\|g\|_q^q}$$

$\forall x \in X$ . By integrating, we obtain

$$\begin{aligned} \frac{1}{\|f\|_p \|g\|_q} \int_X |fg| d\mu &\leq \frac{1}{p \|f\|_p^p} \int_X |f|^p d\mu + \frac{1}{q \|g\|_q^q} \int_X |g|^q d\mu = \frac{1}{p} + \frac{1}{q} = 1 \\ &\Rightarrow \|fg\|_1 \leq \|f\|_p \|g\|_q \end{aligned}$$

Case  $p = 1, q = \infty$ . Exercise

★

## 12.6 Minkowski's inequality

**Theorem 12.3** (Minkowski Inequality)

$f, g \in L^p(X, \mathcal{M}, \mu)$ ,  $p \in [1, \infty]$ . Then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

**Proof.**  $p \in (1, \infty)$

$$\begin{aligned} \|f + g\|_p^p &= \int_X |f + g|^p d\mu = \int_X |f + g| |f + g|^{p-1} d\mu \\ &\leq \int_X (|f| + |g|) |f + g|^{p-1} d\mu = \int_X |f| |f + g|^{p-1} d\mu + \int_X |g| |f + g|^{p-1} d\mu \end{aligned}$$

Using Holder with  $p, q = \frac{p}{p-1}$

$$\begin{aligned} &\leq \|f\|_p \left( \int_X (|f + g|^{p-1})^{\frac{p}{p-1}} d\mu \right)^{\frac{p-1}{p}} + \|g\|_p \left( \int_X (|f + g|^{p-1})^{\frac{p}{p-1}} d\mu \right)^{\frac{p-1}{p}} \\ &= \|f\|_p \|f + g\|_p^{p-1} + \|g\|_p \|f + g\|_p^{p-1} \end{aligned}$$

We divide left-hand side and right-hand side by  $\|f + g\|_p^{p-1}$ :

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

★

We introduced  $L^p(X, \mathcal{M}, \mu)$ , and we proved that this is a normed space with

$$\|f\|_p := \begin{cases} \left( \int_X |f|^p d\mu \right)^{\frac{1}{p}} & \text{if } p \in [1, +\infty) \\ \text{ess sup}_X |f| & \text{if } p = +\infty \end{cases}$$

## 12.7 Inclusion of $L^p$ spaces

### Theorem 12.4

Suppose that  $\mu(X) < +\infty$ . Then

$$1 \leq p \leq q \leq \infty \Rightarrow L^q(X) \subseteq L^p(X)$$

Meaning that any  $f \in L^q$  is also in  $L^p$ . More precisely,  $\exists C > 0$  depending on  $\mu(X), p, q$  s.t.

$$\|f\|_p \leq C \|f\|_q \quad f \in L^q(X)$$

**Proof.** If  $q = +\infty$

$f \in L^\infty(X)$ : then  $|f(x)| \leq \operatorname{ess\,sup}_X |f| = \|f\|_\infty$  for a.e.  $x \in X$ , say  $\forall x \in X \setminus A$ , with  $\mu(A) = 0$ .

Then

$$\int_X |f|^p d\mu = \int_{X \setminus A} |f|^p d\mu \leq \|f\|_\infty^p \int_{X \setminus A} 1 d\mu = \|f\|_\infty^p \underbrace{\mu(X)}_{=\mu(X \setminus A)}$$

If  $q < +\infty$

Then  $\frac{q}{p} > 1$ , and we can use Hölder  $\left(\frac{q}{p}, \left(\frac{q}{p}\right)'\right)$ , where  $\left(\frac{q}{p}\right)' = \frac{\frac{q}{p}}{\frac{q}{p}-1} = \frac{q}{q-p}$

$$\begin{aligned} \|f\|_p^p &= \int_X |f|^p d\mu \stackrel{\text{Hölder}}{\leq} \left( \int_X \left(|f|^p\right)^{\frac{q}{q-p}} d\mu \right)^{\frac{q-p}{q}} \cdot \left( \int_X 1 d\mu \right)^{\frac{q}{q-p}} = \left( \int_X |f|^q d\mu \right)^{\frac{p}{q}} \cdot (\mu(X))^{\frac{q-p}{q}} \\ &\Rightarrow \|f\|_p \leq \mu(X)^{\frac{q-p}{qp}} \|f\|_q \end{aligned}$$

★

The assumption  $\mu(X) < \infty$  is essential. For example, in  $X = [1, \infty]$

$$\frac{1}{x} \in L^2([1, \infty]) \Leftrightarrow \int_1^\infty \frac{dx}{x^2} < \infty$$

$$\frac{1}{x} \notin L^1([1, \infty]) \Leftrightarrow \int_1^\infty \frac{dx}{x} = \infty$$

In particular, the previous theorem is false for  $\ell^p$ -spaces

$$\ell^p = L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_C)$$

$$1 \leq p \leq q \leq \infty \Rightarrow \ell^p \subseteq \ell^q, \text{ and } \exists C > 0 \text{ s.t. } \|x\|_q \leq C \|x\|_p \quad \forall x \in \ell^p$$

## 12.8 Interpolation inequality

Without assumptions on  $\mu(X)$ , in general one has the interpolation inequality.

### Theorem 12.5

$(X, \mathcal{M}, \mu)$  measure space. Let  $1 \leq p \leq q \leq \infty$ . If  $f \in L^p(X) \cap L^q(X)$ , then

$$f \in L^r(X) \quad \forall r \in (p, q)$$

and moreover

$$\|f\|_r \leq \|f\|_p^\alpha \|f\|_q^{1-\alpha}$$

where  $\alpha$  is such that  $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$

**Proof.** For exercise. Use Holder

★

## 12.9 $L^p$ is a Banach space

### Theorem 12.6

For  $1 \leq p \leq \infty$ ,  $L^p(X, \mathcal{M}, \mu)$  is a Banach space (with reference to  $\|\cdot\|_p$ )

**Proof.**

$p < \infty$ .

By using the characterization of completeness with the series, we want to show that if  $\{f_n\} \subseteq L^p(X)$ , and  $\sum_{k=1}^{\infty} \|f_k\|_p < \infty \Rightarrow \sum_{k=1}^{\infty} f_k$  is convergent in  $L^p$ , namely  $s_n = \sum_{k=1}^n f_k$  has a limit in  $L^p$ :  $\|s_n - s\|_p \rightarrow 0$  as  $n \rightarrow \infty$ .

Let then  $\{f_n\} \subseteq L^p(X)$  s.t.

$$\sum_{k=1}^{\infty} \|f_k\|_p = M < \infty$$

Define

$$g_n(x) = \sum_{k=1}^n |f_k(x)|$$

By Minkowski,  $\|g_n\|_p \leq \sum_{k=1}^n \|f_k\|_p \leq M < \infty$ . Moreover, for every  $x \in X$  fixed,  $\{g_n(x)\}$  is increasing  $\Rightarrow g_n(x) \rightarrow g(x)$  as  $n \rightarrow \infty$ ,  $\forall x \in X$

$$\int_X |g|^p d\mu \stackrel{\text{Monot.} = \text{conv.}}{=} \lim_n \int_X |g_n|^p \leq M^p < \infty \Rightarrow g \in L^p(X)$$

$\Rightarrow |g|^p$  is finite a.e.:

$$\sum_{k=1}^{\infty} |f_k(x)| < \infty \text{ for a.e. } x \in X$$

$$\Rightarrow \sum_{k=1}^{\infty} f_k(x) \text{ is convergent a.e. to a limit } s(x)$$

Thus, we proved that  $s_n(x) = \sum_{k=1}^n f_k(x) \rightarrow s(x)$  a.e. in  $X$ . Namely,  $|s_n - s|^p \rightarrow 0$  a.e. in  $X$ . To find a dominating function for  $|s_n - s|^p$ , we start by observing that

$$|s_n(x)| = \left| \sum_{k=1}^n f_k(x) \right| \leq \sum_{k=1}^n |f_k(x)| = g_n(x) \leq g(x) \text{ for a.e. } x \in X$$

Therefore

$$|s_n - s|^p \leq 2^{p-1}(|s_n|^p + |s|^p) \leq 2^{p-1}(g^p + g^p) = 2^p g^p \in L^1(X)$$

By the dominated convergence theorem

$$\int_X |s_n - s|^p d\mu \rightarrow 0 \Leftrightarrow \|s_n - s\|_p \rightarrow 0$$

Thus  $L^p$  is complete.

$p = \infty$  exercise

★

To speak about separability, we give a

**Definition 12.1**

$g : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ . The support of  $g$  is

$$\text{supp } g = \overline{\{x \in \Omega : g(x) \neq 0\}}$$

Also

$$\mathcal{C}_C^0 = \{f \in \mathcal{C}^0(\Omega) : \text{supp } f \text{ is compact in } \Omega\} = \mathcal{C}_O^0(\Omega) = \mathcal{C}_C(\Omega)$$

**12.10 Lusin's theorem****Theorem 12.7** (Lusin Theorem)

$\Omega \in \mathcal{L}(\mathbb{R})$ ,  $\lambda(\Omega) < +\infty$ . Let also  $f : \mathbb{R} \rightarrow \mathbb{R}$  measurable, s.t.  $f \equiv 0$  in  $\Omega^C$ .  
Then  $\forall \varepsilon > 0 \exists g \in \mathcal{C}_C^0(\mathbb{R})$  s.t.

$$\lambda(\{x \in \mathbb{R} : g(x) \neq f(x)\}) < \varepsilon$$

and

$$\sup_{\mathbb{R}} |g| \leq \sup_{\mathbb{R}} |f|$$

**Definition 12.2**

Given  $s$  simple function  $= \sum_{k=1}^n a_k \chi_{E_k}$ , where  $E_1, \dots, E_n$  are  $\mathcal{L}$ -measurable sets,  $a_1, \dots, a_n \in \mathbb{R}$ .

$$E_1 \cup E_2 \cup \dots \cup E_n = \mathbb{R}$$

We consider

$$\tilde{\mathcal{S}}(\mathbb{R}) = \{s \text{ simple in } \mathbb{R} \text{ s.t. } \lambda(\{s \neq 0\}) < +\infty\}$$

What does it mean for a simple function to be in  $L^p(\mathbb{R})$ ?

$$\int_{\mathbb{R}} |s|^p d\mu = \sum_{k=1}^n a_k^p \lambda(E_k) < +\infty \quad 1 \leq p \leq +\infty$$

iff  $s \equiv 0$  outside a set of finite measure  $\Leftrightarrow s \in \tilde{\mathcal{S}}(\mathbb{R})$ .

$\tilde{\mathcal{S}}(\mathbb{R})$  is the set of integrable simple functions.

**12.11  $\tilde{\mathcal{S}}(\mathbb{R})$  is dense in  $L^p$** **Theorem 12.8**

$\tilde{\mathcal{S}}(\mathbb{R})$  is dense in  $L^p$ ,  $\forall p \in (1, +\infty)$

**Proof.**  $f \in L^p(\mathbb{R})$ ,  $f \geq 0$  a.e. in  $\mathbb{R}$ .

We want to show that  $\exists \{s_n\} \subseteq \tilde{\mathcal{S}}(\mathbb{R})$  s.t.  $\|s_n - f\|_p \rightarrow 0$ .

By the simple approximation theorem,  $\exists \{s_n\}$  of simple functions s.t.  $\{s_n(x)\}$  is increasing, for every  $x$ , and  $s_n \rightarrow f$  pointwise in  $\mathbb{R}$ .

Since  $|s_n|^p \leq f^p \Rightarrow s_n \in L^p$  for every  $n \Rightarrow \{s_n\} \subseteq \tilde{\mathcal{S}}(\mathbb{R})$ . Moreover,

$$|s_n - f|^p \rightarrow 0 \quad \text{a.e. in } \mathbb{R}$$

$$|s_n - f|^p \leq 2^{p-1}(|s_n|^p + |f|^p) \leq 2^p |f|^p \in L^1$$

$\Rightarrow$  by dominated convergence

$$\int_{\mathbb{R}} |s_n - f|^p d\lambda \rightarrow 0, \text{ namely } \|s_n - f\|_p \rightarrow 0$$

If  $f$  is sign changing, then  $f = f^+ - f^-$  and argue as before on  $f^+$  and  $f^-$  ★

## 12.12 Separability of $L^p$

### Theorem 12.9

$\forall p \in [1, \infty)$ , the space  $L^p(\mathbb{R})$  is separable.

*Sketch of the proof.* Here we'll outline a sketch of the proof:

- Step 1:  $\mathcal{C}_c^0(\mathbb{R})$  is dense in  $L^p(\mathbb{R})$ ,  $\forall 1 \leq p \leq \infty$ .

Take  $s \in \tilde{\mathcal{S}}(\mathbb{R})$ . Then, by Lusin theorem,  $\exists \{f_n\} \subseteq \mathcal{C}_c^0(\mathbb{R})$  s.t.  $\|f_n - s\|_p \rightarrow 0$ . Then, since any  $f \in L^p$  can be approximated by simple integrable functions, we have that  $f$  can be approximated by functions in  $\mathcal{C}_c^0(\mathbb{R})$ .

- Step 2:

By Stone Weierstrass, the set of polynomials  $\mathcal{P}(\mathbb{R})$  is dense in  $\mathcal{C}_c^0(\mathbb{R})$  with the  $\|\cdot\|_\infty$  norm. Since we work with functions with compact support, this implies that  $\mathcal{P}(\mathbb{R})$  is dense in  $\mathcal{C}_c^0(\mathbb{R})$  also with respect to  $\|\cdot\|_p$

$$\int_{-M}^M |f - p_n|^p d\lambda \leq \|f - p_n\|_\infty^p 2M \rightarrow 0$$

if  $\|f - p_n\|_\infty \rightarrow 0$ ,  $\Rightarrow \mathcal{P}(\mathbb{R})$  is dense in  $L^p(\mathbb{R})$ .

$\tilde{\mathcal{P}}(\mathbb{R}) = \{\text{polynomials with rational coefficients}\}$ . This is countable, and is dense in  $(\mathcal{P}(\mathbb{R}), \|\cdot\|_p)$ .  $\Rightarrow$  is dense in  $L^p$

★

What about  $L^\infty(\mathbb{R})$ ? In this case  $\mathcal{C}(\mathbb{R})$  are not dense in  $L^\infty(\mathbb{R})$ . For example, consider

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

If  $g \in L^\infty$  s.t.  $\|g - f\|_\infty < \frac{1}{3}$ , then  $g$  cannot be continuous. Assume by contradiction that  $\exists g \in \mathcal{C}(\mathbb{R})$  s.t.  $\|g - f\|_\infty < \frac{1}{3}$ . Then

$$\text{ess sup}_{\mathbb{R}} |g(x) - f(x)| < \frac{1}{3}$$

In particular,  $g(x) < \frac{1}{3} \quad \forall x < 0$

$$\Rightarrow \lim_{x \rightarrow 0^-} g(x) \leq \frac{1}{3}$$

On the other hand,  $g(x) > \frac{2}{3} \quad \forall x > 0$

$$\Rightarrow g(0) = \lim_{x \rightarrow 0^+} g(x) \geq \frac{2}{3}$$

Quick recap about the ‘delirium’ on the separability

The thing that you need to know, in  $\rightarrow L^p(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ , are:

- (1)  $L^p$  is separable  $\forall p \in [1, \infty)$

(2)  $\tilde{S}(\mathbb{R})$  is dense in  $L^p(\mathbb{R}) \forall p \in [1, \infty)$ , namely  $\forall p \in L^p(\mathbb{R})$  and  $\forall \varepsilon > 0 \exists s \in \tilde{S}(\mathbb{R})$  s.t.

$$\|f - s\|_p < \varepsilon$$

(3)  $\mathcal{C}_c^0(\mathbb{R})$  is dense in  $L^p$ , namely  $\forall p \in L^p(\mathbb{R})$  and  $\forall \varepsilon > 0 \exists g \in \mathcal{C}_c^0(\mathbb{R})$  s.t.

$$\|f - g\|_p < \varepsilon$$

Everything remains true if you replace  $\mathbb{R}$  with  $X$  open or closed, or with  $X \in L(\mathbb{R}^n)$ , and consider  $(X, L(X), \lambda)$ .

What happens for  $L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ ?

$\mathcal{C}(\mathbb{R})$  is not dense in  $L^\infty$ .

By the simple approximation theorem, we have that simple functions are dense in  $L^\infty$ .

### Theorem 12.10

$L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$  is not separable.

**Proof.**  $\{\chi_{[-\alpha, \alpha]} : \alpha > 0\} \subseteq L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$   $\chi_\alpha = \chi_{[-\alpha, \alpha]}$

This is an uncountable family of functions.  $\|\chi_\alpha - \chi_{\alpha'}\|_\infty = 1 \forall \alpha \neq \alpha'$ , indeed

$$|\chi_\alpha(x) - \chi_{\alpha'}(x)| = \begin{cases} 0 & \text{if } x \in [-\alpha, \alpha] \cup (\alpha', \infty) \cup (-\infty, -\alpha') \\ 1 & \text{if } x \in (\alpha, \alpha'] \cup [-\alpha', \alpha) \end{cases}$$

In particular,  $B_{\frac{1}{2}}(\chi_\alpha) \cap B_{\frac{1}{2}}(\chi_{\alpha'}) = \emptyset \forall \alpha \neq \alpha'$

Assume by contradiction that  $L^\infty(\mathbb{R})$  is separable:  $\exists Z \subset L^\infty$  which is countable and dense.

In particular,  $\forall f \in L^\infty \exists g \in Z$  s.t.

$$\|g - f\|_\infty < \frac{1}{2}$$

Therefore,  $\forall \alpha, \exists g_\alpha \in B_{\frac{1}{2}}(\chi_\alpha) \cap Z$ . But  $B_{\frac{1}{2}}(\chi_\alpha) \cap B_{\frac{1}{2}}(\chi_{\alpha'}) = \emptyset$

$$\Rightarrow \alpha \neq \alpha', \text{ we have } g_\alpha \neq g_{\alpha'}$$

$Z \supseteq \{g_\alpha : \alpha > 0\}$ , which is uncountable. This is not possible, since  $Z$  is countable. ★

### Remark 12.3

The same is true if  $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$  is swapped with  $(X, \mathcal{L}(X), \lambda)$ ,  $X$  is open or closed on  $\mathbb{R}$  or  $\mathbb{R}^n$

## 13 Linear operators

### 13.1 Linear and bounded operator

$(X, \|\cdot\|_X)$ ,  $(Y, \|\cdot\|_Y)$  normed spaces.

#### Definition 13.1

$T : D(T) \subseteq X \rightarrow Y$  is a **linear operator** (or map) if

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2) \quad \forall x_1, x_2 \in D(T) \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}$$

$D(T)$  is a linear subspace of  $X$ , and is called the domain of  $T$ . When  $D(T) = X$  and  $Y = \mathbb{R}$ ,  $T$  is called linear functional.

#### Definition 13.2

A linear operator  $T : D(T) \subseteq X \rightarrow Y$  is bounded if  $D(T) = X$  and  $\exists M > 0$  s.t.

$$\|Tx\|_Y \leq M\|x\|_X \quad \forall x \in X$$

Recall that  $T$  is continuous in  $x_0 \in X$  iff

$$\forall \{x_n\} \subset X, x_n \xrightarrow{X} x_0 \Rightarrow Tx_n \xrightarrow{Y} Tx_0$$

**Example 13.1** •  $L : \mathbb{R}^n \rightarrow \mathbb{R}$  is a linear functional. Then  $\exists y \in \mathbb{R}^n$  s.t.

$$Lx = \langle y, x \rangle = (y, x) = y \cdot x$$

In particular, then  $L$  is continuous on  $\mathbb{R}^n$  and bounded:

$$|Lx| = |\langle y, x \rangle| \stackrel{\text{Cauchy-Schwarz}}{\leq} \|y\| \|x\| \quad \forall x \in \mathbb{R}^n$$

So  $L$  is bounded with  $M = \|y\|$ .

- Linear operators in  $\infty$ -dim may not be defined everywhere, and many may not be continuous:  $(X, \|\cdot\|_X) = (Y, \|\cdot\|_Y) = (\mathcal{C}([0, 1]), \|\cdot\|_\infty)$ .

Consider

$$\begin{array}{ccc} \frac{d}{dx} : \mathcal{C}'([0, 1]) \subseteq X \rightarrow Y & \frac{d}{dx}(\alpha f + \beta g) = \alpha \frac{d}{dx} f + \beta \frac{d}{dx} g \\ f \mapsto f' & \end{array}$$

This is not continuous or bounded. For example, take  $f_n(x) = \frac{1}{n} \sin 2\pi n x$ .  $\|f_n\|_\infty \rightarrow 0$  but  $\|f'_n\|_\infty = 1$

In this case  $f_n \rightarrow 0 \not\Rightarrow \frac{d}{dx} f_n \rightarrow 0$ , then  $\frac{d}{dx}$  is not bounded as well.

- Let  $(X, \|\cdot\|_X)$  be a normed space. If  $\dim X = \infty$ , is it possible to find linear functionals which are not bounded? Yes.

### 13.2 Hamel basis

#### Definition 13.3

A subset  $\{e_i\}_{i \in I}$  is called **Hamel basis** of  $X$  if

$$\|e_i\|_X = 1 \quad \forall i$$

and if every  $x \in X$  can be written in a unique way as

$$x = \sum_{k=1}^n x_k e_{i_k}, \quad x_k \in \mathbb{R}, \quad n \in \mathbb{N}$$

Every  $x$  can be written uniquely as a finite linear combination of element of the basis. If  $\dim X = \infty$  is not immediate that the Hamel basis exists. This can be proved using the axiom of choice. (Zorn's lemma).

Any normed space has a Hamel basis  $\dim X = \infty \Rightarrow \{e_i\}_{i \in I}$  has  $\infty$  many elements.

Let then  $(X, \|\cdot\|_X)$  be  $\infty$  - dim, with Hamel basis  $\{e_i\}_{i \in I}$ .  $I$  is infinite  $\Rightarrow I \supseteq \mathbb{N}$ .

We define  $L : X \rightarrow \mathbb{R}$  in the following way

$$\begin{aligned} Le_0 &= 0 & Le_1 &= 1 & \dots & Le_n &= n & \dots \\ Le_i &= 0 \quad \forall i \in I \setminus \mathbb{N} \end{aligned}$$

Then, for  $x \in X$  we set

$$Lx = L\left(\sum_{k=1}^n x_k e_{i_k}\right) = \sum_{k=1}^n x_k Le_{i_k}$$

$L$  is linear by contradiction, and it is not bounded:

$$\begin{aligned} |Le_n| &= n \rightarrow \infty & \|e_n\|_X &= 1 \quad \forall n \\ \frac{|Le_n|}{\|e_n\|_X} &\rightarrow \infty \Rightarrow L \text{ is not bounded} \end{aligned}$$

### Remark 13.1

In practice, Hamel basis are hard to use. They differ from Hilbertian basis.

For linear operators, boundedness and continuity are equivalent.

### Theorem 13.1

$T : X \rightarrow Y$  linear map. Then the following are equivalent

- (1)  $T$  is continuous in  $0 \in X$
- (2)  $T$  is continuous everywhere in  $X$
- (3)  $T$  is bounded

### Remark 13.2

$T$  linear  $\Rightarrow T0 = 0$ . Indeed,

$$T0 = T(0x) = 0Tx = 0$$

**Proof.** • (2)  $\Rightarrow$  (1) obvious.

- (1)  $\Rightarrow$  (3) Suppose by contradiction that  $T$  is not bounded.

Then  $\exists \{x_n\} \subset X$ ,  $x_n \neq 0$ , s.t.

$$\frac{\|Tx_n\|_Y}{\|x_n\|_X} \geq n \quad \forall n$$

Define

$$z_n := \frac{x_n}{n\|x_n\|_X}$$



Then  $\|z_n\|_X = \frac{1}{n\|x_n\|_X} \|x_n\|_X \rightarrow 0$ , namely  $z_n \rightarrow 0$  in  $X \Rightarrow (T \text{ is continuous in } 0)$   
 $Tz_n \rightarrow T0 = 0$ . However,

$$\|Tz_n\|_Y = \left\| T \left( \frac{x_n}{n\|x_n\|_X} \right) \right\|_Y = \frac{1}{n\|x_n\|_X} \|Tx_n\|_Y \geq 1 \quad \forall n$$

Contradiction.

- (3)  $\Rightarrow$  (2) We observe that

$$\|Tx_1 - Tx_2\|_Y = \|T(x_1 - x_2)\|_Y \leq M\|x_1 - x_2\|_X \quad \forall x_1, x_2 \in X$$

Then, let  $x \in X$  and let  $x_n \rightarrow x$  in  $X$ :  $\|x_n - x\|_X \rightarrow 0$ . But then

$$\|Tx_n - Tx\|_Y \leq M\|x_n - x\|_X \rightarrow 0$$

namely  $Tx_n \rightarrow Tx$  in  $Y$ . This is the continuity.

★

### 13.3 $\mathcal{L}(X, Y)$

#### Definition 13.4

The set of linear operators  $T : X \rightarrow Y$  which are also bounded (continuous) is denoted by  $\mathcal{L}(X, Y)$ . If  $Y = X$ , one simply writes  $\mathcal{L}(X)$

This is a vector space.  $\forall T, S \in \mathcal{L}(X, Y), \forall \alpha, \beta \in \mathbb{R}$ :

$$(\alpha T + \beta S)(x) = \alpha Tx + \beta Sx \quad \in \mathcal{L}(X, Y)$$

We can also introduce a norm:

$$\|T\|_{\mathcal{L}(X, Y)} = \|T\|_{\mathcal{L}} := \sup_{\|x\|_X \leq 1} \|Tx\|_Y$$

Also,

$$\|T\|_{\mathcal{L}(X, Y)} = \sup_{\|x\|_X = 1} \|Tx\|_Y = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \inf M > 0 \text{ s.t. } \|Tx\|_Y \leq M\|x\|_X \quad \forall x \in X$$

### 13.4 When $\mathcal{L}(X, Y)$ is a Banach space

#### Theorem 13.2

$X$  normed space,  $Y$  Banach space. Then  $(\mathcal{L}(X, Y), \|\cdot\|_{\mathcal{L}(X, Y)})$  is a Banach space.

**Proof.** Let  $\{T_n\}$  be a Cauchy sequence in  $\mathcal{L}(X, Y)$ . We want to show that  $\exists T \in \mathcal{L}(X, Y)$  s.t.

$$\|T_n - T\|_{\mathcal{L}} \rightarrow 0$$

$\{T_n\}$  Cauchy:  $\forall \varepsilon > 0 \exists \bar{n} \in \mathbb{N}$  s.t.

$$n, m > \bar{n} \Rightarrow \|T_n - T_m\|_{\mathcal{L}} < \varepsilon$$

Consider then  $\{T_n x\}, x \in X$

$$\|T_n x - T_m x\|_Y = \|(T_n - T_m)x\|_Y \leq \|T_n - T_m\|_{\mathcal{L}} \|x\|_X \leq \varepsilon \|x\|_X \quad (*)$$

This means that  $\{T_n x\}$  is a Cauchy sequence in  $Y$ , which is complete: then  $\forall x \in X \exists$  a vector  $y_x \in Y$  s.t.  $T_n x \rightarrow y_x$  in  $Y$ .

Define

$$T : X \rightarrow Y \quad x \mapsto y_x = Tx$$

$T$  is linear: indeed,  $\forall x_1, x_2 \in X$  and  $\alpha_1, \alpha_2 \in \mathbb{R}$ :

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \lim_{n \rightarrow \infty} T_n(\alpha_1 x_1 + \alpha_2 x_2) = \lim_{n \rightarrow \infty} (\alpha_1 T_n x_1 + \alpha_2 T_n x_2) = \alpha_1 T x_1 + \alpha_2 T x_2$$

So  $T$  is linear. It remains to show that  $T$  is bounded, and that  $\|T_n - T\|_{\mathcal{L}} \rightarrow 0$ . To show that  $T$  is bounded, note that, by (\*),  $\forall \varepsilon > 0 \exists \bar{n}$  s.t.

$$n, m > \bar{n} \Rightarrow \|T_n x - T_m x\|_Y \leq \varepsilon \|x\|_X \quad \forall x$$

Take the limit for  $m \rightarrow \infty$ :

$$\|T_n x - Tx\|_Y \leq \varepsilon \|x\|_X$$

But then, since  $T_n$  is bounded,

$$\|Tx\|_Y = \|Tx \pm T_n x\|_Y \leq \|T_n x\|_Y + \|Tx - T_n x\|_Y \leq M_n \|x\|_X + \varepsilon \|x\|_X = (M_n + \varepsilon) \|x\|_X$$

and  $T$  is bounded. To show that  $\|T_n - T\|_{\mathcal{L}} \rightarrow 0$ , observe that  $\forall \varepsilon > 0 \exists \bar{n}$  s.t.  $n > \bar{n}$

$$\|T_n x - Tx\|_Y \leq \varepsilon \|x\|_X \Leftrightarrow \frac{\|(T_n - T)x\|_Y}{\|x\|_X} \leq \varepsilon \quad \forall x \in X \setminus 0 \xrightarrow{\text{take sup over } x \neq 0} \|T_n - T\|_{\mathcal{L}} < \varepsilon$$

namely,  $T_n \rightarrow T$  in  $\mathcal{L}$

★

Let  $T$  be a linear operator from  $X$  to  $Y$ .

### Definition 13.5

The **kernel** of  $T$  is the set

$$\ker(T) = \{x \in X : Tx = 0\} \subset X$$

This is a vector subspace of  $X$ .

$T$  is injective  $\Leftrightarrow \ker(T) = \{0\}$ . If  $T$  is continuous,  $\ker(T)$  is closed

$$\ker(T) = T^{-1}(\{0\})$$

### Definition 13.6

$X, Y$  normed spaces.  $X$  and  $Y$  are isomorphic if  $\exists T \in \mathcal{L}(X, Y)$  bijective, and such that  $T^{-1} \in \mathcal{L}(X, Y)$

### Definition 13.7

$T \in \mathcal{L}(X, Y)$  is an isometry if

$$\|Tx\|_Y = \|x\|_X \quad \forall x \in X$$

### Definition 13.8

If  $X \subseteq Y$  is a vector subspace, and  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  are normed space, then we can consider

$$J : X \rightarrow Y \quad (\text{inclusion map}) \\ x \mapsto x$$

If  $J \in \mathcal{L}(X, Y)$  (namely, if  $\exists M > 0$  s.t.  $\|x\|_Y \leq M \|x\|_X \quad \forall x \in X$ ), then we say that  $J$  is an embedding of  $X$  into  $Y$ , and we write  $X \hookrightarrow Y$

**Example 13.2**

$\mu(X) < \infty, 1 \leq p < q \leq \infty$

$$L^q(X) \hookrightarrow L^p(X)$$

(inclusion of  $L^p$  spaces)

**Definition 13.9**

$(X, d)$  metric space.  $A \subset X$ .  $x \in X$  is an **adherence point** of  $A$  if  $\forall r > 0 : B_r(x) \cap A \neq \emptyset$

$$\bar{A} = \{x \in X : x \text{ is an adherence point of } A\} = A \cup \partial A$$

**Definition 13.10**

$A \subset X$  is **dense** in  $X$  if  $\bar{A} = X$ .

For example,  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , and  $(a, b)$  is dense in  $[a, b]$ .

**13.5 Nowhere dense sets****Definition 13.11**

$A \subset X$  is **nowhere dense** if the interior of the closure of  $A$  is empty, namely

$$\text{int}(\bar{A}) = \bar{A}^\circ = \emptyset$$

**Example 13.3** •  $\{\bar{x}\}^\circ = \{x\}^\circ = \emptyset$

- $\mathbb{Z} \subset \mathbb{R}$ :  $\bar{\mathbb{Z}}^\circ = \mathbb{Z}^\circ = \emptyset$
- $\mathbb{Q}$  is not nowhere dense:  $(\bar{\mathbb{Q}})^\circ = (\mathbb{R})^\circ = \mathbb{R}$

**Definition 13.12**

$A \subset X$  is called **of first category** (or **meager set**) in  $X$  if  $A$  is the (at most) countable union of nowhere dense sets.

Ex:  $\mathbb{Q}$  is of first category in  $\mathbb{R}$ : countable union of nowhere dense sets

$$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$$

**Definition 13.13**

$A \subset X$  is of second category if it is not of first category.

**13.6 Baire's category theorem****Theorem 13.3** (Baire's category theory)

$(X, d)$  complete metric space. Then

- $\{U_n\}_{n=0}^\infty$  is a sequence of open and dense sets in  $X \Rightarrow \bigcap_{n=0}^\infty U_n$  is dense in  $X$ .
- $X$  is of second category in itself:  $X$  cannot be the countable union of nowhere dense sets.

Preliminaries:

- $A \subset X$  is dense  $\Leftrightarrow \forall W \subset X, W$  open,  $W \neq \emptyset$ , we have that  $A \cap W \neq \emptyset$
- $A$  is nowhere dense  $\Leftrightarrow (\bar{A})^c$  is open and dense

**Proof.** Here's the proof of the two parts of the theorem:

- (a) Thanks to the first preliminary, we show that  $\forall W \subset X$  open and non-empty we have  $(\cap_n U_n) \cap W \neq \emptyset$

$$\begin{aligned} U_0 \text{ is open and dense: } & \xRightarrow{1^{st} \text{ prel.}} \underbrace{U_0 \cap W}_{\text{is open}} \neq \emptyset \\ & \Rightarrow \text{it contains an open ball} \\ & \Rightarrow (U_0 \cap W) \supset B_{r_0}(x_0) \text{ for some } x_0 \in X \text{ and } r_0 > 0 \end{aligned}$$

For  $n > 0$ , we choose  $x_n \in X$  and  $r_n > 0$  inductively in the following way: we have

$$\begin{aligned} U_n \cap B_{r_{n-1}}(x_{n-1}) & \neq \emptyset & (1^{st} \text{ prel.} + U_n \text{ is dense}) \\ \Rightarrow \overline{B_{r_n}(x_n)} & \subset (U_n \cap B_{r_{n-1}}(x_{n-1})) \\ & \text{all these balls are included in } B_{r_0}(x_0) \end{aligned}$$

with  $x_n \in X$  and  $0 < r_n < \frac{1}{2^n}$

By the condition on  $r_n$ , we see that

$$x_n, x_m \in B_{r_N}(x_N) \quad \forall n, m > N$$

$\Rightarrow \{x_n\}$  is a Cauchy sequence in  $X$

$$d(x_n, x_m) \leq \frac{1}{2^N} \quad \forall n, m > N$$

$X$  is complete:  $x_n \xrightarrow{d} x \in X$ .

Since

$$\begin{aligned} x_n & \in B_{r_N}(x_N) & \forall n > N \\ \Rightarrow x = \lim_n x_n & \in \overline{B_{r_N}(x_N)} \subset (U_N \cap B_{r_0}(x_0)) \subset (U_N \cap W) & \forall n \in \mathbb{N} \\ \Rightarrow x_n & \in \bigcap_n (U_n \cap W) = \left( \bigcap_n U_n \right) \cap W \end{aligned}$$

This means that  $\bigcap_n U_n$  is dense.

- (b) It follows from (a):

If  $\{E_n\}$  is a sequence of nowhere dense sets in  $X$ , then, by the second preliminary  $\{(E_n)^C\}$  is a sequence of open and dense sets. By (a)

$$\begin{aligned} \bigcap_n (\overline{E_n})^C & \neq \emptyset \\ \Rightarrow \bigcup_n E_n & \subset \bigcup_n \overline{E_n} = X \setminus \left( \bigcap_n (\overline{E_n})^C \right) \stackrel{=\emptyset}{=} \neq X \end{aligned}$$

★

#### Example 13.4

$(X, \|\cdot\|)$   $\infty$ -dim Banach space.  $\{e_i\}_{i \in I}$  Hamel basis.

Then  $I$  is uncountable.

### 13.7 Banach-Steinhaus's theorem

#### Theorem 13.4 (Banach-Steinhaus)

$X$  Banach space,  $Y$  normed space,  $\mathcal{F} \subseteq \mathcal{L}(X, Y)$  family. Suppose that  $\mathcal{F}$  is pointwise bounded:

$$\forall x \in X \quad \exists M_x > 0 \text{ s.t. } \sup_{T \in \mathcal{F}} \|Tx\|_Y \leq M_x \quad (\text{PB})$$

Then  $\mathcal{F}$  is uniformly bounded:

$$\exists M \geq 0 \text{ s.t. } \sup_{T \in \mathcal{F}} \|T\|_{\mathcal{L}(X, Y)} \leq M \quad (\text{UB})$$

**Proof.**  $\forall n \in \mathbb{N}$ , let

$$C_n := \{x \in X : \|Tx\|_Y \leq n \quad \forall T \in \mathcal{F}\} = \bigcap_{T \in \mathcal{F}} \{x \in X : \|Tx\|_Y \leq n\}$$

$C_n$  is a closed set  $\forall n$ , since  $T$  is continuous. (also  $\varphi : X \rightarrow \mathbb{R} \quad \varphi(x) = \|Tx\|_Y$  is continuous)  
By (PB), every  $x \in X$  stays in some  $C_n$ :  $X = \bigcup_{n=1}^{\infty} C_n$ . Since  $X$  is Banach, by the Baire theorem it is necessary that  $\exists n_0 \in \mathbb{N}$  s.t.  $C_{n_0}^\circ \neq \emptyset \Rightarrow$  a ball  $\overline{B_r(x_0)} \subset C_{n_0}$ : then

$$\|T(x_0 + rz)\|_Y \leq n_0 \quad \forall z \in \overline{B_1(0)}$$

$$\|T(x_0 + rz)\|_Y \stackrel{\text{linearity}}{=} \|Tx_0 + rTz\|_Y \stackrel{\text{triangle ineq.}}{\leq} r\|Tz\|_Y + \|Tx_0\|_Y \quad \forall T \in \mathcal{F}$$

To sum up:  $\forall T \in \mathcal{F}, \forall z \in \overline{B_1(0)}$  we have

$$r\|Tz\|_Y + \|Tx_0\|_Y \leq n_0 \Rightarrow \|Tz\|_Y \leq \frac{1}{r}(n_0 + M_{x_0})$$

We take sup over  $T \in \mathcal{F}$ :

$$\sup_{T \in \mathcal{F}} \|T\|_{\mathcal{L}(X, Y)} \leq \frac{1}{r}(n_0 + M_{x_0}) =: M$$

★

#### Corollary 13.1

$X$  Banach space,  $Y$  normed space.  $\{T_n\} \subseteq \mathcal{L}(X, Y)$  s.t.  $\{T_n x\}$  has a limit, denoted by  $Tx$ ,  $\forall x \in X$  (pointwise convergence). Then  $T \in \mathcal{L}(X, Y)$

**Proof.**  $T$  is linear:

$$\begin{aligned} T_n(\alpha_1 x_1 + \alpha_2 x_2) &= \alpha_1 T_n x_1 + \alpha_2 T_n x_2 \\ \downarrow &\quad \quad \quad \downarrow \\ T(\alpha_1 x_1 + \alpha_2 x_2) &= \alpha_1 T x_1 + \alpha_2 T x_2 \end{aligned}$$

Now we observe that we have (PB): if  $\{T_n x\}$  is convergent  $\Rightarrow \{T_n x\}$  is bounded  $\Rightarrow$  by Banach Steinhaus,  $\{T_n\}$  is uniformly bounded:

$$\exists M > 0 \text{ s.t. } \sup_n \|T_n\|_{\mathcal{L}(X, Y)} \leq M$$

Therefore,  $\forall x \in X$ :

$$\|Tx\|_Y = \left\| \lim_n (T_n x) \right\|_Y = \lim_n \|T_n x\|_Y \leq \lim_n \|T_n\|_{\mathcal{L}} \|x\|_X \leq \lim_n M \|x\|_X = M \|x\|_X$$

Thus,  $T$  is bounded:  $T \in \mathcal{L}(X, Y)$  ★

Let  $X, Y$  be normed spaces.

#### Definition 13.14

$T : X \rightarrow Y$  is called **open map** if,  $\forall A \subset X$  open, the set  $T(A) \subset Y$  is open.

#### Remark 13.3

Recall that  $T$  is continuous on  $X$  if  $T^{-1}(O)$  is open on  $X$ ,  $\forall O$  open in  $Y$ .

#### Example 13.5

$f(x) : \text{constant}$  is continuous, but not open.  $f((a, b)) = \{\text{const}\}$

## 13.8 Open map theorem

#### Theorem 13.5 (Open map theorem)

$X, Y$  Banach spaces.  $T \in \mathcal{L}(X, Y)$  is surjective. Then  $T$  is an open map.

#### Corollary 13.2

$X, Y$  Banach spaces,  $T \in \mathcal{L}(X, Y)$  is bijective. Then  $T$  is an isomorphism:  $T^{-1} \in \mathcal{L}(X, Y)$

**Proof.** •  $T : Y \rightarrow X$  is linear. (Exercise. Hint: Use  $T^{-1} \circ T = \text{Id}$  + linearity of  $T$ )

- We want now to check that  $T^{-1}$  is continuous on  $Y$ :  $(T^{-1})^{-1}(O)$  is open in  $Y$ ,  $\forall O$  open in  $X$ . We know that  $T$  is an open map thanks to the open map theorem.

$$\begin{aligned} (T^{-1})^{-1}(O) &= \{y \in Y, T^{-1}(y) \in O\} = \{y \in Y, T^{-1}(y) = x, \text{ for some } x \in O\} = \\ &= \{y \in Y, y = Tx, \text{ for some } x \in O\} = T(O) \text{ is open} \end{aligned}$$

Since  $T$  is an open map,  $\forall O \subset X$ , open.

★

#### Corollary 13.3

$X$  vector space,  $\|\cdot\|, \|\cdot\|_*$  norms on  $X$ . Assume  $(X, \|\cdot\|), (X, \|\cdot\|_*)$  are Banach spaces. Assume that  $\exists C_1 > 0$  s.t.

$$\|x\|_* \leq C_1 \|x\| \quad \forall x \in X$$

Then  $\|\cdot\|$  and  $\|\cdot\|_*$  are equivalent, namely  $\exists C_2 > 0$  s.t.

$$\|x\| \leq C_2 \|x\|_*$$

**Proof.** Consider

$$\begin{aligned} I : (X, \|\cdot\|) &\rightarrow (X, \|\cdot\|_*) \\ x &\mapsto x \end{aligned}$$

By assumption,  $I$  is bounded:  $\exists C_1 > 0$  s.t.

$$\|Ix\|_* = \|x\|_* \leq C_1 \|x\|$$

$I$  is bijective.

Thus, by the corollary before

$$I^{-1} = I \in \mathcal{L}((X, \|\cdot\|_*), (X, \|\cdot\|))$$

namely  $\exists C_2 > 0$  s.t.

$$\|Ix\| \leq C_2 \|x\|_*$$

$$\parallel\parallel$$

$$\|x\|$$

★

### Definition 13.15

$T : D(T) \subset X \rightarrow Y$  linear operator. We say that  $T$  is **closed** if  $\forall \{x_n\} \subset D(T)$ .

$$\left. \begin{array}{l} x_n \rightarrow x \quad \text{in } X \\ Tx_n \rightarrow y \quad \text{in } Y \end{array} \right\} \Rightarrow x \in D(T) \text{ and } Tx = y$$

### Example 13.6

$X = Y = \mathcal{C}^0([0, 1])$  with the supremum norm.

$$T = \frac{d}{dx}$$

$T$  is not continuous. But it is closed: it can be proved that if  $\{f_n\} \subset \mathcal{C}^1([0, 1])$  is s.t.

$$\left. \begin{array}{l} f_n \rightarrow f \quad \text{uniformly} \\ f'_n \rightarrow g \quad \text{uniformly} \end{array} \right\} \Rightarrow f \text{ is } \mathcal{C}^1([0, 1]) \text{ and } f' = g$$

### Example 13.7

$T \in \mathcal{L}(X, Y) \Rightarrow T$  is closed

### Remark 13.4

$T$  is a closed operator  $\Leftrightarrow$  the graph of  $T$  is closed.

$$\text{graph}(T) = \{(x, Tx) : x \in X\}$$

## 13.9 Closed graph theorem

### Theorem 13.6 (Closed graph theorem)

$X, Y$  Banach spaces.

$T : X \rightarrow Y$  linear closed operator ( $D(T) = X$ ).

Then  $T \in \mathcal{L}(X, Y)$ .

### Remark 13.5

In general, it is easier to prove that an operator is closed, rather than it is continuous.

**Proof.** Define on  $X$  the graph-norm of  $T$

$$\|x\|_* = \|x\|_X + \|Tx\|_Y$$

Then is a norm on  $X$ . If  $\{x_n\} \in X$  is a Cauchy sequence for  $\|\cdot\|_*$ , then  $\{x_n\}$  is a Cauchy sequence in  $(X, \|\cdot\|_X)$  and  $\{Tx_n\}$  is a Cauchy sequence on  $(Y, \|\cdot\|_Y)$

$$\Rightarrow \left. \begin{array}{l} x_n \rightarrow x \quad \text{in } X \\ Tx_n \rightarrow y \quad \text{in } Y \end{array} \right\} \text{ since } T \text{ is closed, we deduce that } y = Tx$$

Thus

$$\|x_n - x\|_X + \|Tx_n - Tx\|_Y \rightarrow 0$$

This proves that  $(X, \|\cdot\|_*)$  is a Banach space. Also, we know that

$$\|x\|_X \leq \|x\|_X + \|Tx\|_Y = \|x\|_*$$

By the last corollary of the open map theorem,  $\exists C_2$  s.t.

$$\|x\|_* \leq C_2 \|x\|_X$$

$$\|Tx\|_Y \leq \|x\|_* \leq C_2 \|x\|_X \quad \forall x \in X$$

This means that  $T$  is bounded.

★



## 14 Duality and reflexivity

### 14.1 Dual space of a normed space

$X$  normed space:

$X^* = \mathcal{L}(X, \mathbb{R})$  is called **dual space of  $X$**

$X$  normed space,  $Y$  Banach space  $\Rightarrow \mathcal{L}(X, Y)$  is a Banach space with  $\|\cdot\|_{\mathcal{L}}$ .

Since  $\mathbb{R}$  is a Banach space, the dual space  $X^*$  is a Banach space with

$$\|L\|_* = \sup_{\|x\|_X \leq 1} |Lx|$$

**Example 14.1** • In  $\mathbb{R}^n$ , any linear functional is represented by a scalar product:

$$L : \mathbb{R}^n \rightarrow \mathbb{R} \text{ is linear} \Rightarrow \exists! y \in \mathbb{R}^n \text{ s.t. } Lx = \langle y, x \rangle$$

It can be proved that

$$L \in (\mathbb{R}^n)^* \mapsto y \in \mathbb{R}^n$$

is an isometric isomorphism

$$(\mathbb{R}^n)^* \cong \mathbb{R}^n$$

Then  $X^*$  is very complicated.

- Dual of  $L^p$ ?

$(X, \mathcal{M}, \mu)$  measure space.  $p \in [1, \infty]$ ,  $q$  conjugate exponent.

$$\frac{1}{p} + \frac{1}{q} = 1 \Leftrightarrow \begin{cases} q = \frac{p}{p-1} & p \in (1, \infty) \\ q = \infty & p = 1 \\ q = 1 & p = \infty \end{cases}$$

For  $g \in L^q(X)$ , define  $L_g : L^p(X) \rightarrow \mathbb{R}$  by

$$L_g f := \int_X fg \, d\mu \quad \forall f \in L^p(X)$$

This is well-defined, by the Holder inequality:

$$\left| \int_X fg \, d\mu \right| \leq \int_X |fg| \, d\mu = \|fg\|_1 \leq \|g\|_q \|f\|_p$$

\*

If  $g \in L^q$ , this shows that  $L_g f \in \mathbb{R} \quad \forall f \in L^p$

#### Proposition 14.1

If  $p \in [1, \infty]$  then  $L_g \in (L^p(X))^*$ . Moreover,

- if  $p > 1$ , then  $\|L_g\|_* = \|g\|_q$
- if  $p = 1$  then  $\|L_g\|_* = \|g\|_\infty$  with more assumptions (they are satisfied in  $(X, \mathcal{L}(X), \lambda)$ )

#### Remark 14.1

We are saying that  $L^q$  can be identified with a subspace of the dual space  $(L^p)^*$  and this identification is an isometry.

Question: are there functional in  $(L^p)^*$ ?

**Proof.** (of the proposition)

- Case  $p = \infty$  ex
- Case  $p = 1$  but difficult it's ok if you don't do it
- Case  $p \in (1, \infty)$

$L_g$  is clearly linear, by linearity of  $\int$ , indeed:  $\forall \alpha \beta \in \mathbb{R}, f_1 f_2 \in L^p(X)$ . Then

$$L_g(\alpha f_1 + \beta f_2) = \int_X g(\alpha f_1 + \beta f_2) d\mu = \alpha \int_X g f_1 d\mu + \beta \int_X g f_2 d\mu = \alpha L_g f_1 + \beta L_g f_2$$

We want to show now that  $L_g$  is bounded. We proved in (\*) that

$$|L_g f| \leq \|g\|_q \|f\|_p \quad \forall f \in L^p(X)$$

This shows that  $L_g$  is bounded, with norm  $\|L_g\|_* \leq \|g\|_q$  (remember that  $\|T\|_{\mathcal{L}} = \inf\{M > 0 : \|Tx\|_Y \leq M\|x\|_X \quad \forall x \in X\}$ )

We want to show that  $\|L_g\|_* = \|g\|_q$ . If  $\|L_g\|_* < \|g\|_q$ , then  $\exists M < \|g\|_q$  s.t.

$$|L_g f| \leq M \|f\|_p \quad \forall f \in L^p(X)$$

We rule out this possibility by choosing an explicit  $\tilde{f} \in L^p$  s.t.

$$|L_g \tilde{f}| = \|g\|_q \|\tilde{f}\|_p$$

We take

$$\tilde{f} = \frac{|g|^{q-1}}{\|g\|_q^{q-1}} \frac{g}{|g|}$$

Now,

$$\|\tilde{f}\|_p^p = \int_X |\tilde{f}|^p d\mu = \int_X \frac{|g|^{p(q-1)}}{\|g\|_q^{p(q-1)}} d\mu = (*)$$

$$(q)' = p \Rightarrow p = \frac{q}{q-1} \Rightarrow p(q-1) = q$$

$$(*) = \frac{1}{\|g\|_q^q} \int_X |g|^q d\mu = \frac{\|g\|_q^q}{\|g\|_q^q} = 1$$

$$|L_g \tilde{f}| = \left| \int_X \frac{|g|^{q-1}}{\|g\|_q^{q-1}} |g| d\mu \right| = \left| \int_X \frac{|g|^q}{\|g\|_q^{q-1}} d\mu \right| = \frac{1}{\|g\|_q^{q-1}} \|g\|_q^q = \|g\|_q = \|g\|_q \|\tilde{f}\|_p$$

★

### 14.1.1 Hahn-Banach theorem

#### Definition 14.1

$X$  vector space. A map  $p : X \rightarrow \mathbb{R}$  is called **sublinear functional** if

- $p(\alpha x) = \alpha p(x) \quad \forall x \in X, \alpha > 0$
- $p(x + y) \leq p(x) + p(y) \quad \forall x, y \in X$

**Theorem 14.1** (Hahn Banach)

$X$  real vector space,  $p : X \rightarrow \mathbb{R}$  sublinear functional.  $Y$  subspace of  $X$  and suppose that  $\exists f : Y \rightarrow \mathbb{R}$  linear on  $Y$  s.t.

$$f(y) \leq p(y) \quad \forall y \in Y$$

Then  $\exists$  a linear functional  $F : X \rightarrow \mathbb{R}$  s.t.

$$F(y) = f(y) \quad \forall y \in Y \quad \quad \quad F \text{ is an extension of } f$$

Moreover,

$$F(x) \leq p(x) \quad \forall x \in X$$

**Theorem 14.2** (Hahn-Banach regarding continuous extension)

$X$  (real) normed space.  $Y$  subspace of  $X$ ,  $f \in Y^* = \mathcal{L}(Y, \mathbb{R})$

Then  $\exists F \in X^* = \mathcal{L}(X, \mathbb{R})$  s.t.

$$\begin{aligned} F(y) &= f(y) \quad \forall y \in Y \\ \|F\|_{X^*} &= \|f\|_{Y^*} \end{aligned}$$

**Proof.** Define  $p : X \rightarrow \mathbb{R}$ ,  $p(x) = \|f\|_{Y^*} \|x\|_X \quad \forall x \in X$ . Then  $p$  is sublinear (from the properties of  $\|\cdot\|_X$ ).

Moreover,  $f(y) \leq |f(y)| \leq \|f\|_{Y^*} \|y\|_X = p(y) \quad \forall y \in Y$ . Then, by Hahn-Banach theorem (general version),  $\exists F : X \rightarrow \mathbb{R}$  s.t.  $F$  is an extension of  $f$  and  $F(x) \leq p(x) \quad \forall x \in X$ .

Now, if  $F(x) \geq 0$

$$|F(x)| = F(x) \leq p(x) = \|f\|_{Y^*} \|x\|_X$$

If  $F(x) < 0$

$$|F(x)| = -F(x) = F(-x) \leq p(-x) = \|f\|_{Y^*} \|-x\|_X = \|f\|_{Y^*} \|x\|_X$$

$\forall x \in X$

$$|F(x)| \leq \|f\|_{Y^*} \|x\|_X$$

namely,  $F \in X^*$  (it is bounded), and

$$\|F\|_{X^*} \leq \|f\|_{Y^*}$$

Also,  $\|F\|_{X^*} \geq \|f\|_{Y^*}$  since  $F$  extends  $f$ :

$$\|F\|_{X^*} = \sup_{\|x\|_X \leq 1} |F(x)| \geq \sup_{\|y\|_Y \leq 1} |F(y)| = \sup_{\|y\|_X \leq 1, y \in Y} |f(y)| = \|f\|_{Y^*}$$

★

Consequence 1**Theorem 14.3**

$(L^\infty(X))^*$  ‘strictly contains’  $L^1(X)$

**Proof.** We must show that  $\exists L \in (L^\infty(X))^*$  s.t.  $\nexists g \in L^1(X)$  s.t.

$$Lf = \int_X fg d\mu \quad \forall f \in L^\infty(X)$$

For simplicity, we consider  $(X, \mathcal{M}, \mu) = ([-1, 1], \mathcal{L}([-1, 1]), \lambda)$ . Let  $Y$  be the subspace of

$L^\infty([-1, 1])$  of the bounded continuous functions  $\mathcal{C}^0([-1, 1])$ . On  $Y$  we define

$$\Lambda f = f(0) \quad \forall f \in Y$$

We can do it since  $f \in \mathcal{C}^0([-1, 1])$  (for elements in  $L^\infty$  we cannot speak about pointwise values!).

$\Lambda$  is linear:

$$\Lambda(\alpha f + \beta g) = \alpha \Lambda f + \beta \Lambda g$$

Moreover,  $\Lambda$  is in  $Y^*$ :

$$|\Lambda f| = |f(0)| < \max_{[-1, 1]} |f| = \|f\|_\infty$$

This proves that  $\Lambda \in Y^*$ ,  $\|\Lambda\|_{Y^*} \leq 1$ . By Hahn-Banach,  $\exists L \in (L^\infty(X))^*$  which is an extension of  $\Lambda$ , and is s.t.

$$\|L\|_{(L^\infty)^*}$$

Can we have

$$Lf = \int_{-1}^1 fg d\mu \quad \text{for some } g \in L^1(X)?$$

Suppose by contradiction that this is true, take

$$f_n \in \mathcal{C}^0([-1, 1])$$

defined in this way:

$$f_n(x) = \varphi(nx)$$

where  $\varphi$  is continuous,  $\text{supp } \varphi \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]$

$$\varphi(0) = 1, \varphi(nx) = 0 \quad \forall x \text{ s.t. } |nx| > \frac{1}{2} \Leftrightarrow |x| > \frac{1}{2n}$$

By contradiction,

$$\text{supp } f_n \subseteq \left[-\frac{1}{2n}, \frac{1}{2n}\right] \Rightarrow f_n(x) \rightarrow 0$$

Therefore, if  $g \in L^1([-1, 1])$  is s.t.

$$\int_{-1}^1 f_n g d\lambda = Lf_n$$

Then, on one side

$$\int_{-1}^1 f_n g d\mu = Lf_n = f_n(0) = 1 \quad \forall n \tag{1}$$

But on the other side

- $f_n(x)g(x) \rightarrow 0$  a.e. in  $[-1, 1]$
- $|f_n(x)g(x)| \leq g(x) \in L^1([-1, 1])$

$$\stackrel{\text{DQM}}{\Rightarrow} \int_{-1}^1 f_n g d\lambda \rightarrow 0 \tag{2}$$

But (1) and (2) are in contradiction. In conclusion, there is no  $g \in L^1([-1, 1])$  s.t.

$$\int_{-1}^1 fg d\lambda = Lf \quad \forall f \in L^\infty([-1, 1])$$

★

**Corollary 14.1**

$X$  (real) normed space,  $x_0 \in X \setminus \{0\}$ . Then  $\exists L_{x_0} \in X^*$  s.t.

$$\|L_{x_0}\|_{X^*} = 1 \text{ and } L_{x_0}(x_0) = \|x_0\|_X$$

**Proof.** Take  $Y = \{\lambda x_0 : \lambda \in \mathbb{R}\}$  (1-d vector space generated by  $x_0$ )

$$\begin{aligned} L_0 : Y &\rightarrow \mathbb{R} \\ \lambda x_0 &\mapsto \lambda \|x_0\|_X \end{aligned}$$

This is linear and continuous on  $Y \Rightarrow$  by Hahn-Banach (continuous extension)  $\exists \tilde{L}_0 \in X^*$  s.t.  $\tilde{L}_0$  extends  $L_0$  and

$$\|\tilde{L}_0\|_{X^*} = \|L_0\|_{Y^*} = \sup_{\substack{\lambda x_0 \in Y \\ \|\lambda x_0\|=1}} |L_0(\lambda x_0)| = \sup |\lambda| \|x_0\|_X = 1$$

Thus  $\tilde{L}_0$  is precisely the desired functional.

$$\tilde{L}_0(x_0) = L_0(x_0) = \|x_0\|_X$$

and

$$\|\tilde{L}_0\|_{X^*} = 1$$

★

**Corollary 14.2** (The bounded linear functionals separate points)

If  $x, y \in X$  and  $Lx = Ly \forall L \in X^* \Rightarrow x = y$  (if  $x \neq y, \exists L \in X^*$  s.t.  $Lx \neq Ly$ )

**Proof.** Assume  $x - y \neq 0$ . Then, by the previous corollary,  $\exists L \in X^*$  s.t.

$$\|L\|_{X^*} \text{ and } L(x - y) = \|x - y\|_X \Rightarrow Lx - Ly = L(x - y) = \|x - y\|_X \neq 0$$

★

**Corollary 14.3**

$X$  normed space,  $Y$  closed subspace of  $X$ ,  $x_0 \in X \setminus Y$ .

Then  $\exists L \in X^*$  s.t.  $L|_Y = 0$  and  $Lx_0 \neq 0$

## 14.2 Bidual spaces

$X$  Banach space,  $X^*$  dual space.

**Notation 14.1**

$$L \in X^* : Lx = L(x) = \langle L, x \rangle = {}_{X^*} \langle L, x \rangle_X$$

$(X^*)^*$  dual space of  $X^*$  is called the **bidual** of  $X$ , denoted by  $X^{**}$

$$X^{**} = \mathcal{L}(X^*, \mathbb{R})$$

We can describe many elements of  $X^{**}$  in the following way: for  $x \in X$ , define

$$\begin{aligned} \Lambda : X^* &\rightarrow \mathbb{R} \\ L &\mapsto Lx = {}_{X^*} \langle L, x \rangle_X \end{aligned}$$

( $\Lambda_x$  evaluates functionals in  $X^*$  in the point  $x$ ).

$\Lambda_x$  is linear:

$$\Lambda_x(\alpha L_1 + \beta L_2) = (\alpha L_1 + \beta L_2)(x) = \alpha L_1 x + \beta L_2 x = \alpha \Lambda_x L_1 + \beta \Lambda_x L_2$$

Moreover, it is bounded

$$|\Lambda_x(L)| = |Lx| \leq \|L\|_{X^*} \|x\|_X \quad \forall L \in X^*$$

Moreover,

$$\|\Lambda_x\|_{\mathcal{L}(X^*, \mathbb{R})} = \sup_{L \neq 0} \frac{|\Lambda_x L|}{\|L\|_{X^*}}$$

We claim that  $\|\Lambda_x\|_{\mathcal{L}} = \|x\|_X$ . Indeed, by the first corollary of Hahn-Banach, given any  $x \in X \setminus \{0\} \exists Lx \in X^*$

$$\exists L_x \in X^* \text{ s.t } |L_x x| = \|x\|_X, \text{ and } \|L_x\|_{X^*} = 1$$

$$\begin{aligned} \Rightarrow \sup_{L \neq 0} \frac{|\Lambda_x L|}{\|L\|_{X^*}} &= \sup_{L \neq 0} \frac{|Lx|}{\|L\|_{X^*}} \geq \frac{|L_x x|}{\|L_x\|_{X^*}} = \|x\|_X \\ &\Rightarrow \|\Lambda_x\|_{X^{**}} = \|x\|_X \end{aligned}$$

### 14.3 Canonical Map

#### Theorem 14.4

$\exists$  a map

$$\begin{aligned} \tau : X &\rightarrow X^{**} \\ x &\mapsto \Lambda_x \end{aligned} \quad (\text{Canonical Map})$$

which is linear, continuous and an isometry. Namely, the canonical map is an isometric isomorphism from  $X$  into  $\tau(X) \subseteq X^{**}$

### 14.4 Reflexive spaces

Question: are there other elements in  $X^{**}$ ?

#### Definition 14.2

If the canonical map is surjective, then we say that  $X$  is **reflexive**,  $X \cong X^{**}$ . Otherwise,  $\tau(X)$  will be a strict close subspace of  $X$ .

#### Remark 14.2

$X$  reflexive  $\Leftrightarrow X$  and  $X^{**}$  are isometrically isomorphic.

#### Theorem 14.5

$X$  reflexive space. Then every closed subspace of  $X$  is reflexive.

#### Theorem 14.6

$X$  Banach.

$$X \text{ reflexive} \Leftrightarrow X^* \text{ reflexive}$$

**Theorem 14.7**

$X$  Banach.

- If  $X^*$  is separable  $\Rightarrow X$  is separable
- If  $X$  is separable and reflexive  $\Rightarrow X^*$  is separable

To show that a space is reflexive, it is convenient to introduce the following notion.

**Definition 14.3**

$X$  Banach space.  $X$  is called **uniformly convex** if  $\forall \varepsilon > 0 \exists \delta > 0$  s.t.

$$\forall x, y \in X \text{ with } \|x\| \leq 1, \|y\| \leq 1, \|x - y\| > \varepsilon$$

then we have

$$\left\| \frac{x + y}{2} \right\| < 1 - \delta$$

This is a quantitative version of the strict convexity.

**Definition 14.4**

$C \subset X$  is convex  $\Leftrightarrow \forall x, y \in C : \frac{x+y}{2} \in C$

$C \subset X$  is **strictly convex**  $\Leftrightarrow \forall x, y \in C : \frac{x+y}{2} \in C^\circ$

Roughly speaking,  $X$  is uniformly convex if  $\overline{B_1(0)}$  is strictly convex in a quantitative way.

**Theorem 14.8** (Milman-Pettis)

Every uniformly convex Banach space is reflexive.

Recap on reflexivity:

$X$  Banach space.  $X^{**} = (X^*)^*$  is the **bidual space**,  $\mathcal{L}(X^*, \mathbb{R})$

$$\forall x \in X \exists \Lambda_x : X^* \rightarrow \mathbb{R} \text{ defined by } \Lambda_x(L) = Lx \quad \forall L \in X^*$$

We proved that  $\Lambda_x \in X^{**}$ . Thus, we can define the **canonical map**:

$$\begin{aligned} \tau : X &\rightarrow X^{**} \\ x &\mapsto \Lambda_x \end{aligned} \quad (\text{Canonical Map})$$

We stated that  $\tau$  is an isometric isomorphism from  $X$  into  $\tau(X)$ . This is true but for our purpose it's even too much, and it's difficult to prove in details. However, we can prove a slightly weaker result

**Theorem 14.9**

$\tau$  is linear, continuous, and is an isometry

$$\|\tau(x)\|_{X^{**}} = \|x\|_X \quad \forall x \in X$$

Moreover,  $\tau$  is injective. If  $\tau$  is also surjective, it is an isometric isomorphism between  $X$  and  $X^{**}$

**Proof.** There are two parts:

- $\tau$  is linear and continuous: exercise.

$\tau$  is an isometry:  $\|\tau(x)\|_{X^{**}} = \|\Lambda_x\|_{X^{**}} = \|x\|_X$

$\tau$  is injective:  $x \neq y \Rightarrow \tau(x) \neq \tau(y)$ ?

$x \neq y \Rightarrow$  by the second corollary to Hahn-Banach  $\exists L \in X^*$  s.t.  $Lx \neq Ly$ .

$$_{X^{**}} \langle \tau(x), L \rangle_{X^*} = \Lambda_x(L) = Lx \neq Ly = \Lambda_y(L) = _{X^{**}} \langle \tau(y), L \rangle_{X^*}$$

Then,  $\tau(x) \neq \tau(y)$  and  $\tau$  is injective.

- Let now  $\tau$  be surjective. Then  $\tau \in \mathcal{L}(X, X^{**})$  and is bijective  $\Rightarrow$  by a corollary of the open map theorem,  $\tau^{-1} \in \mathcal{L}(X^{**}, X)$

★

### Definition 14.5

$X$  is reflexive if  $\tau$  is surjective. In this case,  $\tau$  is an isometric isomorphism between  $X$  and  $X^{**}$

We formally mentioned that

### Theorem 14.10

If  $(X, \|\cdot\|)$  is uniformly convex  $\Rightarrow (X, \|\cdot\|)$  is reflexive.

### Remarks:

### Proposition 14.2

If  $(X, \|\cdot\|)$  is uniformly convex  $\Rightarrow \overline{B_1(0)}$  is strictly convex.

**Proof.** Is it true that if  $x, y \in \overline{B_1(0)}$ , then  $\frac{x+y}{2} \in B_1(0)$ ? Since  $(X, \|\cdot\|)$  is uniformly convex, we know that  $(\|x - y\| =: \bar{\varepsilon} > 0)$

$$\forall \bar{\varepsilon} > 0 \quad \exists \bar{\delta} > 0 \text{ s.t. } \|x\| \leq 1 \quad \|y\| \leq 1 \quad \|x - y\| > \frac{\bar{\varepsilon}}{2} \Rightarrow \left\| \frac{x+y}{2} \right\| < 1 - \bar{\delta}$$

In particular,

$$\frac{x+y}{2} < 1 - \bar{\delta} < 1 \Rightarrow \frac{x+y}{2} \in B_1(0)$$

★

Consequence:  $(\mathbb{R}^2, \|\cdot\|_1)$  and  $(\mathbb{R}^2, \|\cdot\|_\infty)$  are not uniformly convex.

### Proposition 14.3

$(\mathbb{R}^2, \|\cdot\|_2)$  is uniformly convex

**Proof.** Suppose by contradiction that this is false:  $\exists \bar{\varepsilon} > 0$  and  $\{x_n\}, \{y_n\} \subset \overline{B_1(0)}$  s.t.

$$\|x_n - y_n\| > \bar{\varepsilon}, \text{ but } \left\| \frac{x_n + y_n}{2} \right\| \geq 1 \quad (*)$$

$\overline{B_1(0)}$  is compact (since we are in  $\mathbb{R}^2$ )  $\Rightarrow$  UTS<sup>a</sup>  $x_n \rightarrow \bar{x}, y_n \rightarrow \bar{y}$  as  $n \rightarrow \infty$ . Taking the limit in (\*), we deduce that  $\bar{x}, \bar{y} \in \overline{B_1(0)}$

$$\|\bar{x} - \bar{y}\| \geq \bar{\varepsilon}, \text{ and } \left\| \frac{\bar{x} + \bar{y}}{2} \right\| \geq 1$$

This is not possible, since  $\overline{B_1(0)}$  is strictly convex.

★

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<sup>a</sup>Up To a Subsequence



**Theorem 14.11**

$(X, \mathcal{M}, \mu)$  complete measure space. Then  $L^p(X)$  is reflexive  $\forall p \in (1, \infty)$

**Proof.**  $(L^p(X), \|\cdot\|_p)$  is uniformly convex  $\forall p \in (1, \infty)$  (Clarkson inequalities) ★

$L^1(X)$  and  $L^\infty(X)$  are not uniformly convex, and not reflexive.

**14.5 Riesz representation theorem****Theorem 14.12** (Riesz representation theorem)

$(X, \mathcal{M}, \mu)$  complete measure space,  $p \in (1, \infty)$ . Then

$$\forall L \in (L^p(X))^* \quad \exists! g \in L^q(X)$$

with  $q$  conjugate exponent s.t.  $L = L_g$ , namely

$$Lf = \int_X fg d\mu \quad \forall f \in L^p(X)$$

Moreover  $\|L_g\|_{(L^p)^*} = \|g\|_q$

Thus:  $T : g \in L^q \mapsto L_g \in (L^p)^*$  is an isometric isomorphism.

**Proof.**  $1 < p < \infty$ . Consider  $T : L^q \rightarrow (L^p)^*$  with  $g \mapsto Tg : \langle Tg, f \rangle = \int_X fg d\mu$  (namely  $Tg = L_g$ ). We already know that

$$\|Tg\|_* = \|L_g\|_* = \|g\|_q$$

$T$  is injective: for exercise.

$T$  is surjective. Indeed, let  $F := T(L^q) \subseteq (L^p)^*$  subspace. Since  $T$  is an isometry and  $L^q$  is complete, it can be shown that  $T(L^q)$  is also complete  $\Rightarrow T(L^q)$  is closed.

If by contradiction  $F \neq (L^p)^*$ , then we can apply corollary 3 to Hahn Banach ( $X = (L^p)^*$ ,  $Y = F$ ,  $x_0 = \lambda$ ):

$$\exists h \in (L^p)^{**} \text{ s.t. } \langle h, \lambda \rangle \neq 0 \text{ and } h|_F = 0 : \langle h, Tg \rangle = 0 \quad \forall g \in L^q \quad (1)$$

But  $L^p$  is reflexive ( $1 < p < \infty$ ), then  $h \in L^p \setminus \{0\}$ :

$$\langle h, Tg \rangle = (Tg)h = \int_X hg d\mu$$

Therefore, (1) tells us that

$$\int_X hg d\mu = 0 \quad \forall g \in L^q(X)$$

Take  $g = |h|^{p-2}h$ . Therefore,

$$0 = \int_X hg d\mu = \int_X h|h|^{p-2}h d\mu = \int_X |h|^p d\mu \Rightarrow h = 0 \in L^p$$

which is the desired contradiction.

$T$  is an isomorphism: for exercise. ★

$$(L^p)^* = L^q$$

**Remark 14.3**

$p = 1$ . One can prove the:

**Theorem 14.13**

$(X; \mathcal{M}, \mu)$  complete measure space,  $\sigma$ -finite. Then  $\forall L \in (L^1(X))^* \quad \exists! g \in L^\infty(X)$  s.t.  $L = L_g$ :

$$Lf = \int_X fg \, d\mu \quad \forall f \in L^1(X)$$

Moreover, the map  $L \in (L^1)^* \mapsto g \in L^\infty$  is an isometric isomorphism.

Recall that  $(X, \mathcal{M}, \mu)$  is finite if  $\mu(X) < \infty$ .

**Definition 14.6**

$(X, \mathcal{M}, \mu)$  is  $\sigma$ -finite if either  $\mu(X) < \infty$ , or  $X = \sum_{n=1}^{\infty} X_n$ , where  $\mu(X_n) < \infty$

## 15 Weak Convergence

We know that  $\overline{B_1(0)}$  is never compact in  $\infty$  dimension. This is a problem in proving convergence of sequences. A way to approach this issue consists in weakening the notion of convergence.

### 15.1 Weak convergence in Banach spaces

#### Definition 15.1

$X$  Banach space.  $\{x_n\} \subset X$  sequence,  $x \in X$ . We say that  $x_n$  tends to  $x$  **weakly** (in  $X$ ) as  $n \rightarrow \infty$ ,  $x_n \rightharpoonup x$  in  $X$ , if

$$Lx_n \rightarrow Lx \quad \forall L \in X^*$$

#### Remark 15.1

Assume that  $x_n \rightarrow x$  in  $X$ , namely  $\|x_n - x\|_X \rightarrow 0$ . If  $f : X \rightarrow \mathbb{R}$  is continuous, then

$$\lim_{n \rightarrow \infty} f(x_n) = f(x)$$

In particular, this is true if  $f = L \in X^*$ :

$$\begin{aligned} x_n \rightarrow x \in X &\Rightarrow Lx_n \rightarrow Lx \quad \forall L \in X^* \\ x_n \rightharpoonup x \in X &\Rightarrow x_n \rightarrow x \in X \\ &\Leftrightarrow \end{aligned}$$

#### Remark 15.2

We will be interested in weak convergence in  $L^p$ .

If  $p \in [1, \infty)$ , then

$$f_n \rightharpoonup f \text{ in } L^p(X) \Leftrightarrow \int_X f_n g d\mu \rightarrow \int_X f g d\mu \quad \forall g \in L^q$$

Similarly, in  $\ell^p = L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_c)$

$$x_n \rightharpoonup x \text{ in } \ell^p \Leftrightarrow \sum_{k=1}^{\infty} x_n^{(k)} y^{(k)} \rightarrow \sum_{k=1}^{\infty} x^{(k)} y^{(k)} \quad \forall y \in \ell^q$$

#### Proposition 15.1

The weak limit is unique (if it exists)

**Proof.** By contradiction, suppose that  $\exists \{x_n\} \subset X$  s.t.  $x_n \rightharpoonup x_1$ ,  $x_n \rightharpoonup x_2$  in  $X$ ,  $x_1 \neq x_2$ . Then

$$\begin{aligned} Lx_n &\rightarrow Lx_1 \quad \forall L \in X^* \\ Lx_n &\rightarrow Lx_2 \quad \forall L \in X^* \\ \Rightarrow Lx_1 &= Lx_2 \quad \forall L \in X^* \end{aligned}$$

By Hahn Banach (corollary 2), this implies  $x_1 = x_2$ , a contradiction. ★

#### Proposition 15.2

If  $x_n \rightharpoonup x$  in  $X$ , then  $\{x_n\}$  is bounded, and

$$\|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|$$

weak lower semi continuity of  $\|\cdot\|$

**Proof.** •  $\{x_n\}$  is bounded

$x_n \rightharpoonup x \Rightarrow \{Lx_n\}$  is bounded in  $\mathbb{R}$ ,  $\forall L \in X^*$ . Consider  $\Lambda_n \in X^{**}$  defined by

$$\Lambda_n L = Lx_n \quad \forall L \in X^*$$

$\forall L \in X^* \exists M_L > 0$  s.t.

$$|\Lambda_n L| = |Lx_n| \leq M_L \quad \forall n$$

PB

(pointwise boundedness of  $\{\Lambda_n\} \subset X^{**}$ )

$$\Lambda_n : X^* = \mathcal{L}(X, \mathbb{R}) \rightarrow \mathbb{R}$$

By Banach Steinhaus,  $\{\Lambda_n\}$  is uniformly bounded:

$$\sup_n \|\Lambda_n\|_{\mathcal{L}(X^*, \mathbb{R})} \leq M$$

Moreover, by Hahn Banach,  $\forall n \in \mathbb{N} \exists L_n \in X^*$  s.t.  $\|L_n\|_* = 1$  and  $L_n x_n = \|x_n\|$ . Therefore,

$$\|x_n\| = |L_n x_n| = |\Lambda_n L_n| \leq \|\Lambda_n\|_{\mathcal{L}} \|L_n\|_* \leq M \quad \forall n \in \mathbb{N}$$

•  $x_n \rightharpoonup x$ .

By corollary 1 of Hahn Banach,  $\exists L_x \in X^*$  s.t.  $\|L_x\|_* = 1$  and  $L_x x = \|x\|$ . Then

$$\|x\| = |L_x x| = \lim_n |L_x x_n| = \liminf_n |L_x x_n| \leq \liminf_n \|L_x\|_* \|x_n\|_X = \liminf_n \|x_n\|_X$$

★

### Proposition 15.3

$x_n \rightharpoonup x$  in  $X$ , and  $L_n \rightarrow L$  (strongly) in  $X^*$ . Then

$$L_n x_n \rightarrow Lx \quad \text{in } \mathbb{R}$$

### Proposition 15.4

$X, Y$  Banach,  $T \in \mathcal{L}(X, Y)$

$$x_n \rightharpoonup x \Rightarrow Tx_n \rightharpoonup Tx$$

We introduced the weak convergence.  $X$  Banach space.  $x_n \subset X$  converges weakly to  $x$ ,  $x_n \rightharpoonup x$  in  $X$ , if

$$Lx_n \rightarrow Lx \text{ in } \mathbb{R}, \quad \forall L \in X^* = \mathcal{L}(X, \mathbb{R})$$

Recall that:

- $x_n \rightarrow x$  strongly in  $X$ , namely  $\|x_n - x\|_X \rightarrow 0 \Rightarrow x_n \rightharpoonup x$  and  $\Leftarrow$
- $x_n \rightharpoonup x \Rightarrow \{x_n\}$  is bounded, the weak limit  $x$  is unique, and

$$\|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|$$

### Remark 15.3

In  $\mathbb{R}^n$  (or any finite dimensional Banach space)  $x_n \rightharpoonup x \Leftrightarrow x_n \rightarrow x$  strongly (ex.)

With the same philosophy we introduce:

**Definition 15.2**

$X$  Banach  $\Rightarrow X^*$  is Banach as well.

**15.2 Weak\* convergence****Definition 15.3**

A sequence  $\{L_n\} \subset X^*$  is **weakly\*** convergent to  $L \in X^*$ , namely  $L_n \rightharpoonup^* L$  in  $X^*$ , if

$$L_n x \rightarrow Lx \in \mathbb{R} \quad \forall x \in X$$

**Remark 15.4**

Observe that a sequence  $\{L_n\}$  tends weakly to  $L$  in  $X^*$  if

$$\Lambda L_n \rightarrow \Lambda L \quad \forall \Lambda \in X^{**}$$

We know that  $\exists \tau : X \rightarrow X^{**}$  canonical map s.t.

$$X^{**} \langle \tau(x), L \rangle_{X^*} = Lx \quad \forall L \in X^*$$

Thus  $L_n \rightharpoonup^* L$  in  $X^* \Rightarrow \langle \tau(x), L_n \rangle \rightarrow \langle \tau(x), L \rangle \quad \forall x \in X$  : namely

$$L_n x \rightarrow Lx \quad \forall x \in X$$

namely  $L_n \rightharpoonup^* L$  in  $X^*$ . In general the converse is false. However

**Proposition 15.5**

If  $X$  is reflexive, then  $L_n \rightharpoonup L$  in  $X^* \Leftrightarrow L_n \rightharpoonup^* L$  in  $X^*$

**Proof.** If  $X$  is reflexive, every element  $\Lambda$  of  $X^{**}$  is of type  $\Lambda = \tau(x)$  for some  $x$  ★

**Proposition 15.6**

$X$  Banach space,  $X^*$  dual space,  $L_n \rightharpoonup^* L$  in  $X^*$ . Then

- The weak\* limit is unique
- $\{L_n\}$  is bounded
- $\|L\|_{X^*} \leq \liminf_{n \rightarrow \infty} \|L_n\|_{X^*}$
- If in addition  $x_n \rightarrow x$  strongly in  $X \Rightarrow L_n x_n \rightarrow Lx$

**15.3 Banach-Alaoglu theorem****Theorem 15.1** (Banach Alaoglu)

$X$  separable Banach space. Then every bounded sequence in  $X^*$  has a weakly\* convergent subsequence. (bounded sets in  $X^*$  sequentially compact for the weak\* convergence)

**Proof.**  $\{L_n\}$  bounded sequence in  $X^*$ , namely

$$\sup_n \|L_n\|_{X^*} = M < \infty$$

Since  $X$  is separable,  $\exists \{x_k\}_{k \in \mathbb{N}}$  dense in  $X$ . Now, consider  $\{L_n x_1\}$  : it is bounded in  $\mathbb{R}$ :

$$|L_n x_1| \leq \|L_n\|_{X^*} \|x_1\|_X \leq M \|x_1\|_X < \infty$$

$\Rightarrow \exists \{L_{n_j}\}$  s.t.  $L_{n_j}x_1 \rightarrow l_1$  in  $\mathbb{R}$ . Now, consider  $\{L_{n_j}x_2\}$ : it is bounded,

$$|L_{n_j}x_2| \leq \|L_{n_j}\|_{X^*} \|x_2\|_X \leq M \|x_2\|_X < \infty$$

$\Rightarrow \exists \{L_{n_{ij}}\}$  subsequence of  $\{L_{n_j}\}$  s.t.  $L_{n_{ij}}x_2 \rightarrow l_2$  in  $\mathbb{R}$ . We can iterate the process.  $\forall k \{L_n^k\}$  is a subsequence of  $\{L_n^{k-1}\}$ .  $\Rightarrow \{L_n^k\}$  is a subsequence of  $\{L_n^j\} \forall j < k$ . In particular,

$$L_n^k x_j \rightarrow l_j \quad \forall j \leq k$$

We pick up  $T_n = L_n^n$  (diagonal selection). By construction,  $\forall m \in \mathbb{N}$  fixed,  $\{T_n : n \geq m\}$  is a subsequence of  $\{L_n^m : n \geq m\}$

$$\Rightarrow T_n x_m \rightarrow l_m \quad \text{as } n \rightarrow \infty$$

We want to show now that  $T_n x \rightarrow l_x \forall x \in X$ , and that  $l_x = Tx$  is such that  $T \in X^*$ . Since  $\{x_k\}$  is dense,  $\forall x \in X$  and  $\forall \varepsilon > 0 \exists k \in \mathbb{N}$  s.t.

$$\|x - x_k\|_X < \frac{\varepsilon}{2M}$$

Thus

$$\begin{aligned} |T_n x - T_m x| &\leq |T_n x - T_n x_k| + |T_n x_k - T_m x_k| + |T_m x_k - T_m x| \leq \\ &\leq \|T_n\|_{X^*} \|x - x_k\|_X + |T_n x_k - T_m x_k| + \|T_m\|_{X^*} \|x - x_k\|_X \leq \\ &\leq M \frac{\varepsilon}{2M} + |T_n x_k - T_m x_k| + M \frac{\varepsilon}{2M} < \varepsilon + |T_n x_k - T_m x_k| < 2\varepsilon \end{aligned}$$

$\forall n, m > \bar{n}$ , since  $\{T_n x_k\}$  is convergent and so a Cauchy sequence.

This means that  $\{T_n x\}$  is a Cauchy sequence in  $\mathbb{R}$

$$T_n x \rightarrow l_x \text{ in } \mathbb{R} \quad \forall x \in \mathbb{R}$$

It only remains to show that  $l_x = Tx$  for some  $T \in X^*$ . This is a consequence of a corollary of Banach Steinhaus.

To sum up:  $\{L_n\}$  bounded in  $X^*$

$$\Rightarrow \exists \{T_n\} \text{ subsequence s.t. } T_n x \rightarrow Tx$$

for every  $x \in X$ , namely  $T_n \rightharpoonup^* T$  in  $X^*$

★

**Theorem 15.2** (Variant of BA for reflexive spaces)

$X$  reflexive and Banach. Then every bounded sequence in  $X$  has a weakly convergent subsequence

**Proof.** For simplicity, we assume that  $X$  is separable (not necessary).  $X$  separable and reflexive  $\Rightarrow X^*$  is separable.  $\tau : X \rightarrow X^{**}$  canonical map: it is an isometric isometry.

$\{x_n\}$  bounded sequence in  $X \Leftrightarrow \{\tau(x_n)\}$  is bounded in  $X^{**} = (X^*)^*$

$\Rightarrow$  by Banach Alaoglu,  $\exists \{x_{n_k}\}$  s.t.  $\tau(x_{n_k}) \rightharpoonup^* \Lambda$  in  $X^{**}$ :

$$X^{**} \langle \tau(x_{n_k}), L \rangle_{X^*} \rightarrow X^{**} \langle \Lambda, L \rangle_{X^*} \quad k \rightarrow \infty, \quad \forall L \in X^*$$

Since  $X$  is reflexive,  $\forall \Lambda \in X^{**} \exists ! x \in X$  s.t.  $\Lambda = \tau(x)$ . Therefore,

$$Lx_{n_k} = X^{**} \langle \tau(x_{n_k}), L \rangle_{X^*} \rightarrow X^{**} \langle \Lambda, L \rangle_{X^*} = Lx, \quad \forall L \in X^*$$

We proved that

$$\lim_{k \rightarrow \infty} Lx_{n_k} = Lx \quad \forall L \in X^*$$

namely  $x_{n_k} \rightharpoonup x$  in  $X$



## 16 Compact Operators

$X, Y$  Banach spaces.

### Definition 16.1

A linear operator  $K : X \rightarrow Y$  is said to be compact if  $\forall E \subseteq X$  bounded, the set  $K(E)$  is relatively compact, namely  $\overline{K(E)}$  is compact.

Equivalently,  $K$  is compact if  $\forall \{x_n\} \subset X$  bounded, the sequence  $\{K(x_n)\}$  has a strongly convergent subsequence.

### Proposition 16.1

$K : X \rightarrow Y$  linear and compact. Then  $K \in \mathcal{L}(X, Y)$

**Proof.** Define  $B := \overline{B_1(0)}$  in  $X$ . If  $K$  is compact  $\Rightarrow K(B)$  is relatively compact  $\Rightarrow \overline{K(B)}$  is compact  $\Rightarrow \overline{K(B)}$  is bounded  $\Rightarrow K(B)$  is bounded:  $\exists M > 0$  s.t.

$$\begin{aligned} \|Kx\|_Y &\leq M \quad \forall x \in \overline{B_1(0)} = B \\ &\Rightarrow \underbrace{\sup_{\|x\| \leq 1} \|Kx\|_Y}_{\|K\|_{\mathcal{L}(X, Y)}} \leq M \end{aligned}$$

★

### Definition 16.2

$T \in \mathcal{L}(X, Y)$  has finite rank if

$$\text{the image of } T = \{y \in Y : y = Tx\} \text{ for some } x \in X < \infty_{\dim(T(X))}$$

$T(X) \subset Y$  is a subspace.

### Proposition 16.2

$T \in \mathcal{L}(X, Y)$  has finite rank  $\Rightarrow T$  is compact.

**Proof.**  $A \subset X$  bounded. Since  $T \in \mathcal{L}(X, Y)$ , then  $T(A)$  is bounded.  $T(A) \subset T(X) \approx \mathbb{R}^n$ , since  $T$  has finite rank.

Thus,  $T(A)$  is a bounded set of  $\mathbb{R}^n \Rightarrow T(A)$  is relatively compact.

★

### Definition 16.3

We denote by  $\mathcal{K}(X, Y)$  the class of linear compact operators from  $X$  to  $Y$ . This is a linear subspace.

If  $Y = X$ , we write  $\mathcal{K}(X)$

### Proposition 16.3

$X, Y$  Banach spaces,  $T : X \rightarrow Y$  linear and compact,  $Y$  in  $\infty$  dim. Then  $T$  cannot be surjective.

**Proof.** Recall that  $C$  compact set,  $S \subset C$  closed  $\Rightarrow S$  is compact (in any metric space). Assume by contradiction that  $T$  is surjective. By the OMT,  $T$  is an open map. Take

$$\emptyset \neq A \subset X$$

open and bounded.  $T(A)$  is relatively compact (since  $T$  is compact), and is open (since  $T$  is



an open map) and  $\neq \emptyset$

$$\Rightarrow T(A) \supset B_r(y_0)$$

for some  $y_0 \in Y$  and  $r > 0$ . Thus,

$$\overline{T(A)} \supset \overline{B_r(y_0)} \Rightarrow \overline{B_r(y_0)} \text{ is compact in } Y.$$

This contradicts the Riesz theorem, since in  $\infty$  dimension balls are never compact. ★

#### Proposition 16.4

$X, Y, Z$  Banach spaces.  $T \in \mathcal{L}(X, Y)$ ,  $S \in \mathcal{K}(Y, Z)$  (or  $T \in \mathcal{K}(X, Y)$ ,  $S \in \mathcal{L}(Y, Z)$ ). Then  $S \circ T$  is compact.

#### Theorem 16.1

$\mathcal{K}(X, Y)$  is a closed subspace of  $\mathcal{L}(X, Y)$ .  $\Rightarrow (\mathcal{K}(X, Y), \|\cdot\|_{\mathcal{K}(X, Y)})$  is a Banach space.

Consequence: if we want to check that  $T \in \mathcal{L}(X, Y)$  is compact, we can prove that  $\exists \{T_n\} \subseteq \mathcal{K}(X, Y)$  s.t.

$$\|T_n - T\|_{\mathcal{L}} \rightarrow 0$$

Since  $\mathcal{K}(X, Y)$ , it follows that  $T$  is compact.

## 17 Hilbert Spaces

### Definition 17.1

$H$  vector space on  $\mathbb{R}$ . A function  $p : H \times H \rightarrow \mathbb{R}$  is called **scalar (or inner) product** if it is positive definite, symmetric, and bilinear; namely if

- (1)  $p(x, x) \geq 0 \ \forall x \in H$  and  $p(x, x) = 0 \Rightarrow x = 0$
- (2)  $p(x, y) = p(y, x) \ \forall x, y \in H$
- (3)  $p(\alpha x_1 + \beta x_2, y) = \alpha p(x_1, y) + \beta p(x_2, y) \ \forall \alpha, \beta \in \mathbb{R}, x_1, x_2, y \in H$

Notation:  $p(x, y) = \langle x, y \rangle = (x, y) = x \cdot y$

### Definition 17.2

A vector space  $H$  with a scalar product is called a pre Hilbertian space.

### Proposition 17.1

$(H, \langle \cdot, \cdot \rangle)$  pre Hilbertian space.

- Cauchy Schwarz inequality

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle} \quad \forall x, y \in H$$

- $\sqrt{\langle x, x \rangle} =: \|x\|$  is a norm on  $H$

$(H, \langle \cdot, \cdot \rangle)$  pre Hilbert  $\rightarrow (H, \|\cdot\|)$  normed space  $\rightarrow (H, d)$  metric space where  $d(x, y) = \|x - y\|$

## 17.1 Hilbert spaces

### Definition 17.3

We say that  $(H, \langle \cdot, \cdot \rangle)$  is a Hilbert space if  $(H, \|\cdot\|)$  is a Banach space. (namely, if  $(H, d)$  is a complete metric space)

**Example 17.1** •  $\mathbb{R}^n, \langle x, y \rangle = \sum_{i=1}^n x_i y_i$

- $L^2(X, \mathcal{M}, \mu)$   $(X, \mathcal{M}, \mu)$  complete measure space.

$\langle f, g \rangle = \int_X f g d\mu$ .  $\|f\| = (\int_X f^2 d\mu)^{\frac{1}{2}} = \|f\|_2$ .  $(L^2(X), \|\cdot\|_2)$  is a Banach space  $\Rightarrow (L^2(X), \langle \cdot, \cdot \rangle)$  is a Hilbert space.

- $\ell^2$  is a Hilbert space.  $\langle x, y \rangle = \sum_{k=1}^{\infty} x^{(k)} y^{(k)}$ ,  $x = (x^{(k)})$ ,  $y = (y^{(k)})$

- $(C^0([a, b]), \langle \cdot, \cdot \rangle)$  is a pre Hilbertian space.  $(C^0([a, b]), \|\cdot\|_2)$  is not a Banach space.

### Definition 17.4

$x, y$  are orthogonal if  $\langle x, y \rangle = 0$ . We write  $x \perp y$

## 17.2 Parallelogram identity

### Remark 17.1

Hilbert spaces are particular cases of Banach spaces. The converse is not true. In any Hilbert space, the norm induced by  $\langle \cdot, \cdot \rangle$  must satisfy the parallelogram rule

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in H$$

PR

**Proposition 17.2**

$H$  Banach space with respect to  $\|\cdot\|$ . If  $\|\cdot\|$  satisfies (PR), then  $H$  is a Hilbert space with scalar product

$$\langle x, y \rangle := \frac{1}{2}[\|x + y\|^2 - \|x\|^2 - \|y\|^2], \quad \langle x, x \rangle = \|x\|^2$$

Consequence: we can check that a Banach space is not a Hilbert space by showing that (PR) does not hold. Ex:  $(L^p, \|\cdot\|_p)$  is not a Hilbert space  $\forall p \neq 2$ . The same for  $(C^0([a, b]), \|\cdot\|_\infty)$

**17.3 Orthogonal projections**

Recall:

**Definition 17.5**

$C \subset H$  is convex if  $\forall x, y \in C : \frac{x+y}{2} \in C$

**Definition 17.6**

$S \subset H, f \in H$ .

$$\text{dist}(f, S) = \inf_{g \in S} \|f - g\|$$

**Theorem 17.1** (projection on closed convex sets)

$H$  Hilbert space. Let  $S \subseteq H$  non-empty, closed, convex. Then  $\forall f \in H \quad \exists! h \in S$  s.t.

$$\|f - h\| = \text{dist}(f, S) = \min_{g \in S} \|f - g\| \quad (1)$$

Moreover,  $h$  is characterized by the variational inequality:

$$\langle f - h, g - h \rangle \leq 0 \quad \forall g \in S \quad *$$

namely  $h$  is the projection of  $f$  on  $S$  ( $f$  satisfies (1)  $\Leftrightarrow$  (\*) holds)

**Remark 17.2**

$h$  satisfies 1:  $h$  is the projection of  $f$  on  $S, h = P_S f$

**Proof.** Only of the existence of  $h$ .

$S \subset H, \text{dist}(f, S) > 0$  ( $f \notin S$ ).  $\exists \{v_n\} \subset S$  s.t.

$$\|v_n - f\| \rightarrow d := \text{dist}(f, S)$$

We show that  $\{v_n\}$  is a Cauchy sequence. Let  $m, n$ , then  $\frac{v_m + v_n}{2} \in S$ , since  $S$  is convex. Then

$$\left\| f - \frac{v_m + v_n}{2} \right\| \geq d \Rightarrow \|2f - (v_m + v_n)\| \geq 2d \quad 2$$

By the (PR), with  $x = f - v_n, y = v_m - f$

$$\begin{aligned} \|v_m - v_n\|^2 &= \|v_m \pm f - v_n\|^2 \stackrel{PR}{=} \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \|x - y\|^2 = \\ &= 2\|f - v_n\|^2 + 2\|v_m - f\|^2 - \|2f - (v_m + v_n)\|^2 \stackrel{(2)}{\leq} 2\|f - v_n\|^2 + 2\|v_m - f\|^2 - 4d^2 \leq (*) \end{aligned}$$

Up to now, we only used that  $v_n, v_m \in S$ . Since  $\|v_n - f\|^2 \rightarrow d^2$  as  $n \rightarrow \infty, \forall \varepsilon > 0 \exists \bar{n}$  s.t.  $n, m > \bar{n}$

$$\Rightarrow \|v_n - f\|^2 < (d + \varepsilon)^2 \quad \|v_m - f\|^2 < (d + \varepsilon)^2$$

Coming back to  $(*) < 4((d+\varepsilon)^2 - d^2)$ , provided that  $n, m > \bar{n}$ . Since  $\varepsilon$  was arbitrarily chosen, the right-hand side can be made arbitrary small (it tends to 0 if  $\varepsilon \rightarrow 0$ ). We proved that we can make  $\|v_n - v_m\|^2$  arbitrarily small, provided that  $n, m$  are sufficiently large. Namely,  $\{v_n\}$  is Cauchy. Since  $(H, \|\cdot\|)$  is Banach,  $\exists v \in H$  s.t.  $v_n \rightarrow v$ .  $v \in S$ , since  $S$  is closed. And, by continuity,

$$\|f - v\| = \lim_n \|f - v_n\| = d$$

So  $v$  is the desired  $h$ .

About uniqueness.

Let  $\bar{v}$  and  $v'$  2 elements in  $S$  such that

$$\|f - \bar{v}\| = \|f - v'\| = d$$

Then, exactly as before,

$$\|\bar{v} - v'\|^2 = 2(\|\bar{v} - f\|^2 + \|v' - f\|^2) - \|2f - (\bar{v} + v')\|^2 \leq 2(d^2 + d^2) - 4d^2 = 0$$

$$\Rightarrow \bar{v} = v'$$

★

### Remark 17.3

A particular case:  $S$  closed subspace (it is always convex). In this case, the variational inequality becomes an equality:

$$h = P_S f \Leftrightarrow \langle f - h, g \rangle = 0 \quad \forall g \in S$$

### Definition 17.7

$H$  Hilbert space.  $S \subset H$  subset. We define the **orthogonal complement** of  $S$  as

$$S^\perp = \{x \in H : \langle x, y \rangle = 0 \quad \forall y \in S\}$$

### Example 17.2

$S^\perp$  is always a closed subspace of  $H$

### Example 17.3

If  $S$  is a subspace, then  $S \cap S^\perp = \{0\}$

### Definition 17.8

$V, W$  subspace of  $H$ , orthogonal one to each other:

$$\forall v \in V, \quad w \in W : v \perp w$$

We can define the **orthogonal sum** of  $V$  and  $W$  as

$$V \oplus W = \{v + w : v \in V, w \in W\}$$

### Example 17.4

If  $x \in V \oplus W \Rightarrow \exists! (v, w) \in V \times W$  s.t.  $x = v + w$

### Theorem 17.2

$H$  Hilbert space. Let  $V \subseteq H$  be a closed subspace. Then

$$H = V \oplus V^\perp$$

**Definition 17.9**

From the theorem, given any  $x \in H$  we can define

$$\begin{aligned} P_v : H &\rightarrow V \\ x = v + w &\mapsto v \\ P_{v^\perp} : H &\rightarrow V^\perp \\ x &\mapsto w \end{aligned} \quad \text{orthogonal projections}$$

**Example 17.5**

$P_v$  and  $P_{v^\perp}$  are linear bounded operators, with norms 1.

**17.4 Dual space of a Hilbert space**

Observe that, if  $y \in H$ , then we can define  $\Lambda_y : H \rightarrow \mathbb{R}$  as

$$\Lambda_y x = \langle y, x \rangle$$

It is linear ( $\langle \cdot, \cdot \rangle$  is bilinear), and it is bounded:

$$|\Lambda_y x| = |\langle y, x \rangle| \leq \|y\| \|x\| \quad \forall x, y$$

$\Rightarrow \Lambda_y$  is bounded, with  $\|\Lambda_y\|_* \leq \|y\|$

Moreover,

$$\begin{aligned} \Lambda_y \left( \frac{y}{\|y\|} \right) &= \left\langle y, \frac{y}{\|y\|} \right\rangle = \|y\| \\ \Rightarrow \|\Lambda_y\|_* &= \sup_{\|x\| \leq 1} |\Lambda_y x| \geq \left| \Lambda_y \left( \frac{y}{\|y\|} \right) \right| = \|y\| \end{aligned}$$

Thus  $\|\Lambda_y\|_* = \|y\|$ , and the map

$$\begin{aligned} i : H &\rightarrow H^* \\ y &\mapsto \Lambda_y \end{aligned}$$

is an isometry from  $H$  into  $i(H) \subset H^*$ .

Are there other elements in  $H^*$ ?

**17.5 Riesz's representation theorem****Theorem 17.3** (Riesz Representation Theorem)

$\forall \Lambda \in H^* \exists! y \in H$  s.t.  $\Lambda = \Lambda_y$ , namely

$$\Lambda x = \langle y, x \rangle \quad \forall x \in H$$

Moreover, the map  $i$  is an isometric isomorphism. We can identify  $H^*$  with  $H$

**Corollary 17.1**

Any Hilbert space is reflexive.

**Remark 17.4**

Any Hilbert space is uniformly convex.

- Riesz in  $L^p$ :  $L^p$  is uniformly convex  $\Rightarrow L^p$  is reflexive. We used this fact to prove Riesz in  $L^p$
- Riesz in Hilbert: direct proof of  $H^* = H \Rightarrow H$  is reflexive.

Both strategies can be adopted in both contexts.

**Proof.** • We show that  $\forall \Lambda \in H^* \exists y \in H$  s.t.  $\Lambda = \Lambda_y$

If  $\Lambda = 0 \Rightarrow \Lambda = \Lambda_0$  ( $\Lambda_0 x = \langle 0, x \rangle = 0$ )

Suppose  $\Lambda \neq 0$ .  $\ker(\Lambda) = \Lambda^{-1}(\{0\})$  is a closed (since  $\Lambda$  is continuous) subspace,  $\neq H$ .  
 $\Rightarrow$  we consider  $\ker(\Lambda)^\perp \neq \{0\}$ . Let

$$z \in \ker(\Lambda)^\perp, \quad \|z\| = 1$$

For  $x \in H$ , we have

$$x - \frac{\Lambda x}{\Lambda z} z \in \ker(\Lambda)$$

Indeed,  $\Lambda \left( \frac{\Lambda x}{\Lambda z} z \right) \stackrel{\text{linearity}}{=} \Lambda x - \frac{\Lambda x}{\Lambda z} \Lambda z = 0$ . Then, since  $z$  is orthogonal to any element of  $\ker(\Lambda)$ ,

$$\langle z, x - \frac{\Lambda x}{\Lambda z} z \rangle = 0 \quad \forall x \in H$$

$\langle \cdot, \cdot \rangle$  is bilinear: the left-hand side is

$$\langle z, x \rangle - \frac{\Lambda x}{\Lambda z} \|z\|^2 \Rightarrow \langle z, x \rangle = \frac{\Lambda x}{\Lambda z}$$

$$\Lambda x = \langle (\Lambda z) z, x \rangle \quad \forall x \in H$$

So the thesis is proved for  $y = (\Lambda z)z$ .

- The uniqueness of  $y$  is easy.

$$\langle x, y_1 \rangle = \langle x, y_2 \rangle \quad \forall x \in H$$

Then  $\langle x, y_1 - y_2 \rangle = 0 \quad \forall x \in H$ . We choose  $x = y_1 - y_2$ :

$$\|y_1 - y_2\|^2 = 0 \Rightarrow y_1 = y_2$$

★

## 17.6 Convergences

Consequence:  $H$  Hilbert space.

$$x_n \rightharpoonup x \text{ in } H \Leftrightarrow \langle x_n, y \rangle \rightarrow \langle x, y \rangle \quad \forall y \in H$$

Sometimes weak convergence + something else  $\Rightarrow$  strong convergence. For instance

### Proposition 17.3

$H$  Hilbert. If  $x_n \rightharpoonup x$  in  $H$ , and  $\|x_n\| \rightarrow \|x\| \Rightarrow x_n \rightarrow x$  in  $H$ , namely  $\|x_n - x\| \rightarrow 0$

**Proof.**

$$\|x_n - x\|^2 = \|x_n\|^2 - 2\langle x_n, x \rangle + \|x\|^2 = (*)$$

$\langle x_n, x \rangle \rightarrow \langle x, x \rangle = \|x\|^2$  by weak convergence.

$$(*) = \|x\|^2 - 2\|x\|^2 + \|x\|^2 = 0$$

★

## 17.7 Orthonormal Basis

In  $\mathbb{R}^n$ , we have the canonical basis

$$e_1, \dots, e_n \in \mathbb{R}^n$$

s.t.

$$e_j^{(k)} = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

There elements are  $\perp$ :  $\langle e_i, e_j \rangle = 0 \ \forall i \neq j$ .  $\|e_i\| = 1 \ \forall i$ .

Moreover,  $e_1, \dots, e_n$  are a basis, namely  $\forall v \in \mathbb{R}^n \ \exists!$  expression

$$v = \sum_{i=1}^n v_i e_i = \sum_{i=1}^n \langle v, e_i \rangle e_i$$

In particular,  $v = 0 \Leftrightarrow \langle v, e_i \rangle = 0 \ \forall i$ . Do we have an analogue in Hilbert spaces?

### Definition 17.10

$S \subset H$  is called orthonormal if

- $x \perp y \ \forall x \neq y, x, y \in S$
- $\|x\| = 1 \ \forall x \in S$

### Definition 17.11

An orthonormal set is a Hilbert Basis (or is **complete**) if  $S^\perp = \{0\}$ , namely if

$$\langle u, x \rangle = 0 \ \forall x \in S \Rightarrow u = 0$$

### Theorem 17.4

$H$  Hilbert space,  $H \neq \{0\}$ . Then  $H$  has a Hilbert basis.

Moreover,  $H$  is a separable Hilbert space  $\Leftrightarrow$  it has a finite and countable Hilbert basis.

**Example 17.6** •  $H = l^2$ .  $H$  is separable

An Hilbert basis is  $\{e_n\}_{n \in \mathbb{N}}$  defined By

$$e_n^{(k)} = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$$

- $H = L^2([-\pi, \pi])$

An Hilbert basis is

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin(nx)}{\sqrt{\pi}}, \frac{\cos(nx)}{\sqrt{\pi}} \right\} \quad n \in \mathbb{N}$$

### Remark 17.5

Hamel basis  $\neq$  Hilbert basis.

$X \infty$ -dimensional  $\Rightarrow$  any Hamel basis of  $X$  is uncountable.

$H \infty$ -dimensional and separable  $\Rightarrow$  any Hilbert basis is countable

The usefulness of Hilbert basis stays in the fact that they allow us to reason component by component.

## 17.8 Bessel inequality

**Theorem 17.5** (Bessel inequality)

$H$  separable Hilbert space.  $\{u_n\}_{n \in \mathbb{N}}$  orthonormal set. Then  $\forall x \in H$ :

$$\sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 \leq \|x\|^2$$

**Theorem 17.6** (Generalized Fourier Series)

$H$  separable Hilbert space,  $\{u_n\}$  Hilbert basis. Then any  $x \in H$  can be written in a unique way as

$$x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n \quad \langle x, u_n \rangle \text{ Fourier coefficient of } x$$

Moreover,  $\forall y \in H$  we have

$$\langle x, y \rangle = \sum_{n=1}^{\infty} \langle x, u_n \rangle \langle y, u_n \rangle$$

and

$$\|x\|^2 = \sum_{n=1}^{\infty} (\langle x, u_n \rangle)^2 \quad \text{Parseval identity}$$

## 17.9 Riesz-Fisher theorem

**Theorem 17.7**

$H$  separable Hilbert space. Then  $H$  is isomorphic to  $l^2$  as Hilbert space: namely  $\exists$  an isomorphism  $\varphi : H \rightarrow l^2$  s.t.

$$\langle x, y \rangle_H = \langle \varphi(x), \varphi(y) \rangle_{l^2} \quad \forall x, y \in H$$

**Proof.**  $\exists$  a countable Hilbert basis, and  $\forall x \in H$

$$x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n$$

Then the desired isomorphism is

$$\begin{aligned} \varphi : H &\rightarrow l^2 \\ x &\mapsto \sum_{n=1}^{\infty} \langle x, u_n \rangle e_n \end{aligned}$$

★

**Corollary 17.2**

$H$  separable Hilbert space,  $\dim H = \infty$ .  $\{u_n\}_{n \in \mathbb{N}}$  Hilbert basis. Then  $u_n \rightharpoonup 0$  in  $H$ , but  $u_n \not\rightarrow 0$  in  $H$ .

**Proof.**

$$\|u_n\| = 1 \quad \forall n \Rightarrow \|u_n - 0\| \not\rightarrow 0$$



On the other hand, we know that  $\forall x \in H$

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2 < \infty$$

It is then necessary that

$$\langle x, u_n \rangle \rightarrow 0 \text{ as } n \rightarrow \infty \quad \forall x \in H$$

By Riesz, this means that  $u_n \rightharpoonup 0$  in  $H$

★

### Example 17.7

$H = L^2([-\pi, \pi])$ . Then the previous corollary tells that (Riemann - Lebesgue lemma)

$$\int_{-\pi}^{\pi} f(x) \sin(nx) dx \rightarrow 0 \text{ as } n \rightarrow \infty$$

$\sin(nx) \rightharpoonup 0$  in  $L^2([-\pi, \pi])$  as  $n \rightarrow \infty$ . Note that  $\{\sin(nx)\}_{n \in \mathbb{N}}$  does not converge for a.e.  $x$ . Weak convergence in  $L^2$  and pointwise or a.e. convergence are not related.

$\{\sin(nx)\}$  does not converge a.e. on  $[-\pi, \pi]$ , not even up to subsequences. The same is true in  $L^p$ ,  $p \neq 2$ . Even in this case

$$\sin(nx) \rightharpoonup 0 \text{ in } L^p([a, b]) \quad (p \in [1, \infty),$$

but we don't have a.e. convergence.

## 18 Linear operator on Hilbert spaces

### Proposition 18.1

$p \in [1, \infty)$ . Suppose that  $f_n \rightharpoonup f$  in  $L^p(X)$ , and that  $f_n \rightarrow g$  a.e. in  $X$ . Then  $f = g$  a.e.

### Proposition 18.2

$X$  Banach,  $V$  subspace of  $X^*$ , dense in  $X^*$ . Suppose that  $\{x_n\} \subset X$  is bounded, and that

$$Lx_n \rightarrow Lx \quad \forall L \in V$$

Then

$$Lx_n \rightarrow Lx \quad L \in X^*$$

namely  $x_n \rightarrow x$  in  $X$ .

**Proof.** (ex) ★

Consequence:  $I \subset \mathbb{R}$  interval ( $I = \mathbb{R}$  is fine).  $\{f_n\} \subseteq L^p(I)$ ,  $p \in (1, \infty)$ .  $\{f_n\}$  bounded in  $L^p$ :  $\exists C > 0$  s.t  $\|f_n\|_{L^p} \leq C$ ,  $\forall n$ . Then:

- If

$$\int_I f_n \varphi \rightarrow \int_I f \varphi \quad \forall \varphi \in \mathcal{C}_c(I)$$

$$\Rightarrow f_n \rightharpoonup f \text{ in } L^p(I)$$

- If

$$\int_a^b f_n \rightarrow \int_a^b f \quad \forall (a, b) \subset I$$

$$\Rightarrow f_n \rightharpoonup f \text{ in } L^p(I)$$

Some useful facts on bounded operators in Hilbert Spaces.

$H$  Hilbert space.

### Proposition 18.3

If  $T \in \mathcal{L}(H)$ , then

$$\|T\|_{\mathcal{L}(H)} = \sup_{\|x\|=\|y\|=1} |\langle Tx, y \rangle|$$

## 18.1 Symmetric operators

### Definition 18.1

$T$  is called **symmetric** (or self adjoint) if

$$\langle Tx, y \rangle = \langle x, Ty \rangle \quad \forall x, y \in H$$

### Proposition 18.4

Let  $T \in \mathcal{L}(H)$  be symmetric. Then

$$\|T\|_{\mathcal{L}(H)} = \sup_{\|x\|=1} |\langle Tx, x \rangle|$$

Example:  $K \in L^2([0, 1] \times [0, 1])$ . Let  $T : L^2([0, 1]) \rightarrow L^2([0, 1])$  be defined by

$$(Tf)(t) = \int_0^1 K(s, t) f(s) ds$$

$T \in \mathcal{L}(L^2([0, 1]))$ . It is symmetric  $\Leftrightarrow K(s, t) = K(t, s) \forall s, t$

## 18.2 Spectral Theory

In what follows,  $E$  is a Banach space and  $T \in \mathcal{L}(E)$ .

### Definition 18.2

The **resolvent** of  $T$  is

$$\rho(T) = \{\lambda \in \mathbb{R} : T - \lambda I \text{ is bijective from } E \text{ to } E\}$$

### Definition 18.3

The **spectrum** of  $T$  is

$$\sigma(T) = \mathbb{R} \setminus \rho(T)$$

### Definition 18.4

$\lambda$  is an **eigenvalue** of  $T$ ,  $\lambda \in EV(T)$ , if

$$\ker(T - \lambda I) \neq \emptyset$$

( $T - \lambda I$  is not injective), namely if  $\exists u \in E$  s.t.  $u \neq 0$  and

$$Tu = \lambda u$$

In this case,  $u$  is called eigenvector and  $\ker(T - \lambda I)$  is the eigenspace of  $\lambda$ .

#### Remark 18.1

$$EV(T) \subset \sigma(T)$$

#### Remark 18.2

In finite dimension, linear operators can be represented by matrices.

A  $n \times n$  matrix. We know that  $x \mapsto Ax$  is bijective  $\Leftrightarrow$  it is injective  $\Leftrightarrow \det A \neq 0$ . In particular, in finite dimension  $\sigma(A) = EV(A)$ . This is false in  $\infty$  dimension.

Basic fact:

### Theorem 18.1

$E$  Banach,  $T \in \mathcal{L}(E)$ . Then  $\sigma(T) \subset \mathbb{R}$  is compact, and

$$\sigma(T) \subset [-\|T\|_{\mathcal{L}}, \|T\|_{\mathcal{L}}]$$

In general, we cannot say much more.

#### Example 18.1

In  $l^2$ , consider the left shift:

$$T_l(x^{(0)}, x^{(1)}, x^{(2)}, \dots, x^{(n)}, \dots) = (x^{(1)}, x^{(2)}, x^{(3)}, \dots, x^{(n+1)}, \dots)$$

$T_l \in \mathcal{L}(l^2)$ ,  $\|T_l\| = 1$ .  $EV(T_l) = ?$ . We have to solve

$$T_l x = \lambda x \quad \text{for some } \lambda \in \mathbb{R}, x \in l^2 \setminus \{0\}$$

$\Rightarrow x^{(1)} = \lambda x^{(0)}$ .  $x^{(n+1)} = \lambda x^{(n)} = \lambda^{n+1} x^{(0)}$ .  $\forall \lambda \in \mathbb{R}$ , the sequence

$$x = x^{(0)} (1, \lambda, \lambda^2, \dots, \lambda^n, \dots)$$

is a solution of  $T_l x = \lambda x$ .

$$x \in l^2 \Leftrightarrow \sum_{n=0}^{\infty} (\lambda^n)^2 < \infty \Leftrightarrow \sum_{n=0}^{\infty} (\lambda^2)^n < \infty \Leftrightarrow |\lambda| < 1$$

Any  $\lambda \in (-1, 1)$  is an e.v. of  $T_l$ . Moreover,  $\sigma(T_l)$  is a compact set which is included in  $[-1, 1]$  and contains  $EV(T_l) = (-1, 1)$

$$\Rightarrow \sigma(T) = [-1, 1]$$

We focus in what follows on the following case:  $H$  separable Hilbert space,  $T \in \mathcal{K}(H)$  and symmetric.

### Proposition 18.5

Let  $d = \|T\|_{\mathcal{L}(H)}$ . Then either  $d$  or  $-d$  is an eigenvalue of  $T$

Recall:  $T \in \mathcal{K}(H)$ ,  $u_n \rightharpoonup u \Rightarrow Tu_n \rightarrow Tu$  strongly in  $H$

**Proof.**  $d \neq 0$  (otherwise  $T = 0$ ). We know that

$$d = \sup_{\|u\|=1} |\langle Tu, u \rangle|$$

Take a maximizing sequence for  $d$ :

$$\exists \{u_n\} \subset H \text{ s.t. } \|u_n\| = 1 \quad |\langle Tu_n, u_n \rangle| \rightarrow d$$

$\{u_n\}$  is bounded  $\Rightarrow$  by Banach Alaoglu in reflexive spaces (any Hilbert space is reflexive) we can extract  $\{u_{n_k}\}$  s.t.  $u_{n_k} \rightharpoonup u$  in  $H$ , for some  $u$ .

By weak strong continuity,  $Tu_{n_k} \rightarrow Tu$  strongly in  $H$ . From this, we deduce that

$$|\langle Tu_{n_k}, u_{n_k} \rangle - \langle Tu, u \rangle| \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

We know

$$|\langle Tu_{n_k}, u_{n_k} \rangle| \leq |\langle Tu_{n_k} - Tu, u_{n_k} \rangle| + |\langle Tu, u_{n_k} - u \rangle| \rightarrow 0$$

and also that  $|\langle Tu_{n_k}, u_{n_k} \rangle| \rightarrow d$

$$\Rightarrow |\langle Tu, u \rangle| = d$$

and hence  $u \neq 0$

- Suppose that  $\langle Tu, u \rangle = d$ . Then

$$\|Tu - du\|^2 = \|Tu\|^2 - 2d\langle Tu, u \rangle + d^2\|u\|^2 \leq d^2 - 2d^2 + d^2 = 0$$

$$\Rightarrow \|Tu - du\| = 0 \Rightarrow Tu = du$$

and  $d$  is an eigenvalue.

- $\langle Tu, u \rangle = -d$ . Then one can prove that  $-d$  is an eigenvalue.

★

### Proposition 18.6

$\lambda \neq 0$  is an eigenvalue of a compact operator  $T \in \mathcal{K}(E)$ ,  $E$  Banach. Let  $V_\lambda$  be the eigenspace of  $\lambda$ . Then  $\dim V_\lambda < \infty$

**Proof.** Recall that  $I : F \rightarrow F$ , with  $F$   $\infty$  dimensional. Banach space, cannot be compact. Assume by contradiction that  $V_\lambda$  has  $\infty$  dim. Consider

$$\frac{1}{\lambda} T|_{V_\lambda} : V_\lambda \rightarrow V_\lambda$$

is the identity  $\frac{1}{\lambda} Tu = \frac{1}{\lambda} \cdot \lambda u = u \forall u \in V_\lambda$ . So  $\frac{1}{\lambda} T|_{V_\lambda}$  cannot be compact. On the other hand,

$\frac{1}{\lambda}T|_{V_\lambda}$  is compact by assumption. ★

### Proposition 18.7

$H$  Hilbert,  $T \in \mathcal{L}(H)$  symmetric. Then eigenvectors associated with different eigenvalues are orthogonal.

**Proof.**

$$\begin{aligned} Tu_1 &= \lambda_1 u_1 \quad u_1, u_2 \neq 0 \quad \lambda_1 \neq \lambda_2 \\ Tu_2 &= \lambda_2 u_2 \\ \lambda_1 \langle u_1, u_2 \rangle &= \langle Tu_1, u_2 \rangle = \langle u_1, Tu_2 \rangle = \lambda_2 \langle u_1, u_2 \rangle \\ \Rightarrow (\lambda_1 - \lambda_2) \langle u_1, u_2 \rangle &= 0 \Rightarrow \langle u_1, u_2 \rangle = 0 \end{aligned}$$

★

### Theorem 18.2 (Spectral Theorem)

$H$  separable Hilbert,  $T \in \mathcal{K}(H)$  symmetric. Then

- (1)  $\sigma(T) \setminus \{0\} = EV(T) \setminus \{0\}$
- (2)  $0 \in \sigma(T)$

and the following alternative holds:

- (1) either  $T$  has infinitely many distinct eigenvalues, and in this case  $0 \in EV(T)$  and  $\ker T$  is infinite dimensional
- (2) or  $EV(T) \setminus \{0\}$  is a sequence tending to 0

Moreover, the eigenvectors can be chosen in such a way to form a Hilbert basis of  $H$  (if necessary adding an orthonormal basis of  $\ker T$ )

### Remark 18.3

Given:

- $\forall$  symmetric matrix  $A$ ,  $n \times n$ ,  $\exists$  an orthonormal basis of  $\mathbb{R}^n$  of eigenvectors
- If  $T \in \mathcal{K}(E)$ ,  $E$  Banach, we can still say that  $0 \in \sigma(T)$  (if  $E$  has  $\infty$  dimension), that  $EV(T) \setminus \{0\} = \sigma(T) \setminus \{0\}$  and that either there are finitely many distinct eigenvectors, or  $EV(T) \setminus \{0\}$  is a sequence tending to 0

**Proof.** •  $0 \in \sigma(T)$  is simple:  $T$  is compact,  $H$  has  $\infty$  dimension  $\Rightarrow$  it can't be surjective.

$$T = T - 0I \text{ is not bijective: } 0 \notin \rho(T)$$

- From proposition (18.5),  $\exists$  an eigenvalue  $\lambda$  with  $|\lambda_0| = \|T\|_{\mathcal{L}(H)}$ . Let  $V_0$  be the associated eigenspace. By proposition (18.6),  $\dim V_0 = N_0 < \infty$ .

Let  $\{w_1^0, \dots, w_{N_0}^0\}$  be an orthonormal basis for  $V_0$ . Consider now  $H_1 = V_0^\perp$ , so that  $H = V_0 \oplus H_1$ . We claim that  $T|_{H_1} \in \mathcal{K}(H_1)$  symmetric.

$T|_{H_1}$  is compact and symmetric, by assumption. We have to check that  $T|_{H_1} : H_1 \rightarrow H_1$

$$u \in H_1 \Leftrightarrow \langle u, w \rangle = 0 \quad \forall w \in V_0$$

$$\langle Tu, w \rangle = \langle u, Tw \rangle = \langle u, \lambda_0 w \rangle = \lambda_0 \langle u, w \rangle$$

$\forall w \in V_0$ , namely  $Tu \in H_1, \forall u \in H_1$ .

$$H_1 = V_0^\perp$$

$\Rightarrow$  it is a closed subspace of  $H \Rightarrow H_1$  is a Hilbert space.  $T|_{H_1}$  is a compact symmetric operator on a separable Hilbert space. Therefore, arguing as before, we have an eigenvalue for  $T$  given by  $\lambda_1$  s.t.

$$|\lambda_1| = \sup_{\substack{\|u\|=1 \\ u \in H_1}} |\langle Tu, u \rangle|$$

Clearly,  $|\lambda_1| \leq |\lambda_0| = \sup |\langle Tu, u \rangle|$ . We have an eigenspace  $V_1$  for  $\lambda_1$ , with dimension  $N_1$ , and an orthonormal basis  $\{w_1^1, \dots, w_{N_1}^1\}$  for  $V_1$ . We iterate the process. Either after a finite number of steps we have

$$\lambda_N = \sup_{\substack{\|u\|=1 \\ u \in H_1}} |\langle Tu, u \rangle| = 0$$

Or  $\{\lambda_n\}$  forms a sequence, s.t.  $|\lambda_n|$  is decreasing.

Case 1: We can say that

$$H = V_0 \oplus V_1 \oplus V_2 \oplus \dots \oplus V_{N-1} \oplus \ker T$$

$\ker T$  is a closed subspace of  $H$ , separable  $\Rightarrow$  we have an orthonormal countable basis  $\{z_1, \dots, z_n\}$  of  $\ker T$ . Then

$$\{w_1^0, \dots, w_{N_0}^0, w_1^1, \dots, w_{N_1}^1, \dots, w_1^{N-1}, \dots, w_{N_{N-1}}^{N-1}, z_0, \dots, z_n\}$$

is an orthonormal basis of  $H$ , made of eigenvectors.

Case 2: at first, we show that  $\lambda_n \rightarrow 0$ . If not,  $|\lambda_n| \rightarrow \eta > 0$ . Consider then  $\{\frac{w_n}{\lambda_n}\}$ , where  $w_n$  is an eigenfunction of  $\lambda$  with  $\|w_n\| = 1$ . Then  $\{\frac{w_n}{\lambda_n}\}$  is bounded, and

$$T(\frac{w_n}{\lambda_n}) = \frac{1}{\lambda_n} Tw_n = \frac{1}{\lambda_n} \lambda_n w_n = w_n$$

$\Rightarrow$  by compactness, there exists a subsequence of  $T(\frac{w_n}{\lambda_n} = w_n)$  which is strongly convergent. This is not possible, since

$$\|w_i - w_j\|^2 = 2 \quad \forall i \neq j$$

$$\|w_i\|^2 + \|w_j\|^2 - 2\langle w_i, w_j \rangle \Rightarrow \lambda_n \rightarrow 0.$$

It remains to show that  $x \in V_i^\perp, \forall i$ , then  $x \in \ker T$ . To this end

$$\|Tx\| = \|T|_{H_i} x\| \leq \|T|_{H_i}\|_{\mathcal{H}_i} \|x\| = |\lambda_i| \|x\| \quad \forall i$$

Taking  $i \rightarrow \infty$ , we deduce

$$\|Tx\| \leq \lim_{i \rightarrow \infty} |\lambda_i| \|x\| = 0$$

$\Rightarrow x \in \ker T$ . Even in this case,

$$H = \ker T \oplus V_1 \oplus V_2 \oplus \dots \oplus V_n \oplus \dots$$

and once again we can consider a basis of eigenvectors.



### 18.3 Fredholm's alternative

**Corollary 18.1** (Fredholm's alternative)

$H$  separable Hilbert space,  $T \in \mathcal{K}(H)$  and symmetric. Then:

(1) either  $\forall y \in H$  the equation

$$x - Tx = y$$

has a unique solution

(2) or  $\lambda = 1$  is an eigenvalue of  $T$ , and in this case  $x - Tx = y$  can have no solution or infinitely many solutions, depending on  $y$ .

**Remark 18.4**

Also:

- Rouché Capelli:  $Ax = y$ .  $A$  matrix. Either  $\det A \neq 0$ , and then  $\forall y \in \mathbb{R}^n \quad \exists!$  solution; or  $Ax = y$  can have 0 or  $\infty$  many solutions.
- $T$  symmetric is not necessary, and the corollary also holds in Banach spaces.
- The corollary is very useful to treat integral equations:

$$u(t) - \int_0^1 K(s, t)u(s)ds = g(t)$$

**Proof.** By the Spectral Theorem,  $\forall x \in H$ , we can write

$$x = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_n$$

where  $\{u_n\}$  is a Hilbert basis of eigenvectors of  $T$ . Also, we have

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, u_n \rangle u_n$$

and

$$y = \sum_{n=1}^{\infty} \langle y, u_n \rangle u_n$$

Then, the equation  $x - Tx = y$  becomes

$$\begin{aligned} \sum_{n=1}^{\infty} (1 - \lambda_n) \langle x, u_n \rangle u_n &= \sum_{n=1}^{\infty} \langle y, u_n \rangle u_n \\ \Rightarrow (1 - \lambda_n) \langle x, u_n \rangle &= \langle y, u_n \rangle \quad \forall n \end{aligned}$$

If  $\lambda_n \neq 1 \quad \forall n$ , then we take

$$\begin{aligned} \langle x, u_n \rangle &= \frac{\langle y, u_n \rangle}{1 - \lambda_n} \quad \forall n \\ x &= \sum_{n=1}^{\infty} \frac{\langle y, u_n \rangle}{1 - \lambda_n} u_n \end{aligned}$$

is the solution. If instead  $\lambda_n = 1$  for some  $n$ , then there are no solution if  $y$  is such that  $\langle y, u_n \rangle \neq 0$ :

$$(1 - \lambda_n) \langle x, u_n \rangle = \langle y, u_n \rangle$$

For  $y$  s.t.  $\langle y, u_n \rangle = 0$ , we have  $\infty$  many solutions. ★