Real and Functional Analysis

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Real Analysis

1 Set Theory

1.1 Collections and sequences of sets

Let X be a set. Then

$$\mathcal{P}(X) = \{Y \mid Y \subseteq X\} \tag{Power Set}$$

Let $I \subseteq \mathbb{R}$ be a set of indexes. A family of sets induced by I is

$$\{E_i\}_{i\in I}, \quad E_i \subseteq X \tag{Family/Collection}$$

If $I = \mathbb{N}$ is called a

$$\{E_n\}_{n\in\mathbb{N}} \tag{Sequence}$$

Definition 1.1
$\{E_n\} \subseteq \mathcal{P}(X)$ is monotone increasing (resp. decreasing) if

$$E_n \subseteq E_{n+1}\, \forall n \qquad (\text{resp. } E_n \supseteq E_{n+1}\, \forall n)$$

and is written as

$$\{E_n\} \nearrow \qquad (\text{resp. } \{E_n\} \searrow)$$

Given a family of sets $\{E_i\}_{i\in I} \subseteq \mathcal{P}(X)$, will be often considered

$$\bigcup_{i\in I} E_i = \{x \in X : \exists\, i \in I\, s.t.\, x \in E_i\}$$

$$\bigcap_{i\in I} E_i = \{x \in X : x \in E_i,\, \forall i \in I\}$$

$\{E_i\}$ is said to be **disjoint** if $E_i \cap E_j = \varnothing\ \forall i \neq j$.
Examples:

$$[a,b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$$

$$(a,b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}]$$

1.2 lim inf, lim sup

Definition 1.2
$\{E_n\} \subseteq \mathcal{P}(X)$. We define

$$\limsup_n E_n := \bigcap_{k=1}^{\infty}\left(\bigcup_{n=k}^{\infty} E_n\right) \qquad \liminf_n E_n := \bigcup_{k=1}^{\infty}\left(\bigcap_{n=k}^{\infty} E_n\right)$$

If these two sets are equal, then

$$\lim_n E_n = \limsup_n E_n = \liminf_n E_n$$

which is the limit of the succession.

1.3 lim of sequences of sets

Proposition 1.1
Some limits are:

- $\limsup_n E_n = \{x \in X : x \in E_n \text{ for } \infty - \text{many indexes } n\}$
- $\liminf_n E_n = \{x \in X : x \in E_n \text{ for all but finitely many indexes } n\}$
- $\liminf_n E_n \subseteq \limsup_n E_n$
- $(\liminf_n E_n)^C = \limsup_n E_n^C$

Proof. We can define:

$$\begin{aligned} x \in \limsup_n E_n \quad &\Leftrightarrow \quad x \in \bigcap_{k=1}^{\infty} \left(\bigcup_{n=k}^{\infty} E_n \right) \\ &\Leftrightarrow \quad \forall k \in \mathbb{N} : \bigcup_{n=k}^{\infty} E_n \\ &\Leftrightarrow \quad \forall k \in \mathbb{N}\ \exists n_k \geq k \text{ s.t. } x \in E_{n_k} \end{aligned}$$

So $x \in \limsup_n E_n \Rightarrow$
$$\begin{aligned} &\exists m_1 = n_1 \text{ s.t. } x \in E_{n_1} \\ &\exists m_2 := n_{m_1+1} \geq m_1 + 1 \text{ s.t. } x \in E_{n_2} \\ &\vdots \\ &\exists m_k := n_{m_{k-1}+1} \geq m_{k-1} + 1 \text{ s.t. } x \in E_{n_k} \\ &\vdots \\ &x \in E_{m_1}, \dots, E_{m_k}, \dots \end{aligned}$$
On the other hand, assume that $x \in E_n$ for ∞-many indexes. We claim that $\forall k \in \mathbb{N}, \exists n_k \geq k$ s.t. $x \in E_{n_k} \Leftrightarrow x \in \limsup_n E_n$. If that claim is not true, then $\exists\, \bar{k}$ s.t. $x \notin E_n \quad \forall n > \bar{k} \Rightarrow x$ belongs at most to $E_1, \dots, E_{\bar{k}}$, a contradiction. ★

1.4 Cover and subcover of a set

Definition 1.3
$\{E_i\}_{i \in I}$ is a **covering** of X if
$$X \subseteq \bigcup_{i \in I} E_i$$
A subfamily of E_i that is still a covering is called a **subcovering**

1.5 Characteristic function of a set

Definition 1.4
Let $E \subseteq X$. The function $\chi_E : X \to \mathbb{R}$
$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in X \setminus E \end{cases}$$
is called **characteristic function** of E

Let E_1, E_2 be sets:

$$\chi_{E_1 \cap E_2} = \chi_{E_1} \cdot \chi_{E_2}$$

$$\chi_{E_1 \cup E_2} = \chi_{E_1} + \chi_{E_2} - \chi_{E_1 \cap E_2}$$

$$\{E_n\} \subseteq \mathcal{P}(X), \text{ disjoint}, E = \bigcup_{n=1}^{\infty} E_n \Rightarrow \mathcal{X}_{\mathcal{E}} = \sum_{n=1}^{\infty} \chi_{E_n}$$

$$\{E_n\} \subseteq \mathcal{P}, P = \liminf_n E_n, Q = \limsup_n E_n \Rightarrow \chi_P = \liminf \chi_{E_n}, \chi_Q = \limsup_n \chi_{E_n}$$

Recall that $\limsup_n a_n = \lim_{k\to\infty} \sup_{n\geq k} a_n$ and $\liminf_n a_n = \lim_{k\to\infty} \inf_{n\geq k} a_n$
Let's also check that $\chi_Q = \limsup_n \chi_{E_n}$

$$\begin{aligned} x \in \limsup_n E_n \quad &\Leftrightarrow \quad \chi_Q(x) = 1 \\ &\Leftrightarrow \quad \forall\, k \in \mathbb{N}\, \exists\, n_k \geq k \ s.t. \ x \in E_{n_k} \end{aligned}$$

If we fix k then

$$\sup_{n\geq k} \chi_{E_n}(x) = \chi_{E_{n_k}}(x) = 1$$

$$\lim_{k\to\infty} \sup_{n\geq k} \chi_{E_n}(x) = \limsup_n \chi_{E_n}(x) = 1$$

Let now $x \notin \limsup E_n \Leftrightarrow \chi_Q(x) = 0$. Then x belongs at most to finitely many $E_n \Rightarrow \exists\, \bar{k} \ s.t. \ x \notin E_n, \forall n \geq \bar{k}$
If $k \geq \bar{k}$, then $\sup_{n\geq k} \chi_{E_n}(x) = 0 \Rightarrow \lim_{k\to\infty} \sup_{n\geq k} \chi_{E_n}(x) = \limsup_n \chi_{E_n}(x) = 0$

1.6 Equivalence relations

Given X, Y sets, is called a **relation** of X and Y a subset of $X \times Y$

$$R \subseteq X + Y \quad R = \{(x, y) \,:\, x \in X, y \in Y\}$$

$$(x, y) \in R \Leftrightarrow xRy$$

$$X = \{0, 1, 2, 3\} \quad R = \{(0,1), (1,2), (2,1)\} \text{ is a relation in } X$$

Definition 1.5
A **function** from X to Y is a relation R s.t. for any element x of X $\exists!$ element y of Y s.t. xRy

Definition 1.6
R on X is an **equivalence relation** if

(1) $xRx \ \forall\, x \in X$ (R is **reflexive**)

(2) $xRy \Rightarrow yRx$ (R is **symmetric**)

(3) $xRy, yRz \Rightarrow xRz$ (R is **transitive**)

If R is an equivalence relation, the set $E_X := \{y \in X \,:\, yRx\}$, $x \in X$ is called the **equivalence class** of X

Definition 1.7
$\frac{X}{R} := \{E_X \,:\, x \in X\}$ is the **quotient set**

Ex: $X = \mathbb{Z}$, let's say that nRm if $n - m$ is even. This is an equivalence relation.

$$E_n = \{\ldots, n-4, n-2, n, n+2, n+4, \ldots\}$$

in this case if n is even, $E_n = \{\text{even numbers}\}$ and if n is odd, $E_n = \{\text{odd numbers}\}$

2 Measure Spaces

2.1 σ-algebra

Definition 2.1
A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is called a σ**-algebra** if

(1) $X \in \mathcal{M}$

(2) $E \in \mathcal{M} \Rightarrow E^C = X \setminus E \in \mathcal{M}$

(3) If $E = \bigcup_{n\in\mathbb{N}} E_n$ and $E_n \in \mathcal{M}\ \forall\, n$, then $E \in \mathcal{M}$

2.2 Measurable space and sets

If $\mathcal{M}$ is a σ-algebra, $(X, \mathcal{M})$ is called **measurable space** and the sets in $\mathcal{M}$ are called **measurable**. Ex:

- $(X, \mathcal{P}(X))$ is a measurable space
- Let X be a set, then $\{\varnothing, X\}$ is a σ-algebra

Remark 2.1
σ is often used to denote the closure with respect to countably many operators. If we replace the countable unions with finite unions in the definition of σ-algebra, we obtain an **algebra**.

Some **basic properties** of a measurable space $(X, \mathcal{M})$:

(1) $\varnothing \in \mathcal{M}$: $\varnothing = X^C$ and $X \in \mathcal{M}$

(2) $\mathcal{M}$ is an algebra, and $E_1, \dots, E_n \in \mathcal{M}$

$$E_1 \cup \ldots \cup E_n = E_1 \cup \ldots \cup E_n \cup \underbrace{\varnothing}_{\in\mathcal{M}} \cup \varnothing \ldots \in \mathcal{M}$$

(3) $E_n \in \mathcal{M}$, $\bigcap_{n\in\mathbb{N}} E_n \in \mathcal{M}$

$$\bigcap_{n\in\mathbb{N}} E_n = \underbrace{\left(\bigcup_{n\in\mathbb{N}} \underbrace{E_n^C}_{\in\mathcal{M}}\right)}_{\in\mathcal{M}}^C \qquad (\mathcal{M} \text{ is also closed under finite intersection})$$

- $E, F \in \mathcal{M} \Rightarrow E \setminus F \in \mathcal{M} = E \setminus F = E \cap F^C \in \mathcal{M}$
- If $\Omega \subset X$, then the **restriction** of $\mathcal{M}$ to Ω, written as

  $$\mathcal{M}|_\Omega := \{F \subseteq \Omega : F = E \cap \Omega,\ \text{with } E \in \mathcal{M}\}$$

  is a σ-algebra on Ω

2.3 σ-algebra generated by a set

Theorem 2.1
$\mathcal{S} \subseteq \mathcal{P}(X)$. Then it is well defined the smallest σ-algebra containing $\mathcal{S}$, the σ-algebra generated by $\mathcal{S} := \sigma_0(\mathcal{S})$:

- $\mathcal{S} \subseteq \sigma_0(\mathcal{S})$ and thus is a σ-algebra
- $\forall \sigma(\mathcal{M})$ s.t. $\mathcal{M} \supseteq \mathcal{S}$, we have $\mathcal{M} \supseteq \sigma_0(\mathcal{S})$

Proof idea.

$$\mathcal{V} = \{\mathcal{M} \subseteq \mathcal{P}(X) : \mathcal{M} \text{ is a } \sigma\text{-algebra and } \mathcal{S} \subseteq \mathcal{M}\} \neq \varnothing \text{ since } \mathcal{P}(X) \in \mathcal{V}$$

We define $\sigma_0(\mathcal{S}) = \bigcap \{\mathcal{M} \ : \ \mathcal{M} \in \mathcal{V}\}$, so it can be proved that this is the desired σ-algebra ★

2.4 Borel sets

Given (X, d) metric space, the σ-algebra generated by the open sets is called **Borel** σ-algebra, written as $\mathcal{B}(X)$. The sets in $\mathcal{B}(X)$ are called **Borel sets**. The following sets are Borel sets:

- open sets
- closed sets
- countable intersections of open sets: G_σ sets
- countable unions of closed sets: F_σ sets

2.5 Generation of $\mathcal{B}(\mathbb{R})$

Remark 2.2
$\mathcal{B}(\mathbb{R})$ can be equivalently defined as the σ-algebra generated by

$$\begin{gathered}\{(a,b) : \ a,b \in \mathbb{R}, a<b\} \\ \{(-\infty,b) : \ b \in \mathbb{R}\} \\ \{(a,+\infty) : \ a \in \mathbb{R}\} \\ \{[a,b) : \ a,b \in \mathbb{R}, a<b\} \\ \vdots\end{gathered}$$

Question: What is $\mathcal{B}(\mathbb{R})$? Is $\mathcal{B}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$? No.

2.6 Measure

Definition 2.2
$(X, \mathcal{M})$ measurable space. A function $\mu : \ \mathcal{M} \to [0, +\infty]$ is called a **positive measure** if $\mu(\varnothing) = 0$ and if μ is countably additive, that is

$$\forall \{E_n\} \subseteq \mathcal{M} \quad \text{disjoint}$$

we have that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) \qquad \sigma\text{-additivity}$$

Remark 2.3
A set A is countable if $\exists\, f$ $1-1$ s.t. $f : A \to \mathbb{N}$
Examples: $\mathbb{Z}, \mathbb{Q}$ are countable, while $\mathbb{R}$ is not, also $(0,1)$ is uncountable.

We always assume that $\exists\ E \neq \varnothing, E \in \mathcal{M}$ s.t. $\mu(E) \neq \infty$.

2.7 Measure space

If $(X, \mathcal{M})$ is a measurable space, and μ is a measure on it, then $(X, \mathcal{M}, \mu)$ is a measure space. Then:

(1) μ is **finitely additive**:

$$\forall\, E, F \in \mathcal{M},\ \text{with } E \cap F \neq \varnothing \Rightarrow \mu(E \cup F) = \mu(E) + \mu(F)$$

(2) the **excision property**

$$\forall\, E, F \in \mathcal{M},\ \text{with } E \subset F \text{ and } \mu(E) < +\infty \Rightarrow \mu(F \setminus E) = \mu(F) - \mu(E)$$

(3) **monotonicity**

$$\forall\, E, F \in \mathcal{M},\ \text{with } E \subset F \Rightarrow \mu(E) \leq \mu(F)$$

(4) if $\Omega \in \mathcal{M}$ then $(\Omega, \mathcal{M}|_\Omega, \mu|_{\mathcal{M}|_\Omega})$ is a measure space

Proof. (1) $E_1 = E, E_2 = F, E_3 = \ldots = E_n = \ldots = \varnothing$ This is a disjoint sequence $\Rightarrow$ by σ-additivity.

$$\mu(E \cup F) = \mu\left(\bigcup_n E_n\right) = \sum_n \mu(E_n) = \mu(E) + \mu(F) + \underbrace{\mu(E_k)}_{=\mu(\varnothing)}$$

(2) $E \subset F$, so $F = E \cup (F \setminus E)$ and this is disjoint $\overset{(i)}{\Rightarrow} \mu(F) = \mu(E) + \mu(F \setminus E)$, and since $\mu(E) < \infty$, the property follows.

(3) $E \subset F \Rightarrow \mu(F) = \mu(E) + \underbrace{\mu(F \setminus E)}_{\geq 0} \geq \mu(E)$

(4)

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2.8 Basic properties of a measure

Definition 2.3
$(X, \mathcal{M}, \mu)$ measure space.

- If $\mu(X) < +\infty$, we say that μ is **finite**.
- If $\mu(X) = +\infty$, and $\exists\ \{E_n\} \subset \mathcal{M}$ s.t. $X = \bigcup_n E_n$ and each E_n has finite measure, then we say that μ is σ-finite.
- If $\mu(X) = 1$ we say that μ is a **probability measure**.

Some examples:

- Trivial Measure: $(X, \mathcal{M})$ measurable space. $\mu : \mathcal{M} \to [0, \infty]$ defined by $\mu(E) = 0 \quad \forall\, E \in \mathcal{M}$
- Counting Measure: $(X, \mathcal{P}(X))$ measurable space. We define

$$\mu_C : \mathcal{P}(X) \to [0, \infty], \quad \mu_C(E) = \begin{cases} n & \text{if } E \text{ has } n \text{ elements} \\ \infty & \text{if } E \text{ has } \infty\text{-many elements} \end{cases}$$

- Dirac Measure: $(X, \mathcal{P}(X))$ measurable space, $t \in X$. We define

$$\delta_t : \mathcal{P}(X) \to [0, +\infty], \quad \delta_t(E) = \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{otherwise} \end{cases}$$

2.9 Continuity of the measure along monotone sequences

$(X, \mathcal{M}, \mu)$ measure space

(1) $\{E_i\} \subset \mathcal{M},\ E_i \subseteq E_{i+1}\ \forall i$ and let

$$E = \bigcup_{i=1}^{\infty} E_i = \lim_i E_i$$

Then:

$$\mu(E) = \lim_i \mu(E_i)$$

(2) $\{E_i\} \subset \mathcal{M},\ E_{i+1} \subseteq E_i\ \forall i$ and let $E = \bigcap_{i=1}^{\infty} E_i = \lim_i E_i$.

Proof. (1) if $\exists\ i$ s.t. $\mu(E_i) = +\infty$, then is trivial. Assume then that every E_i has a finite measure, so that $E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=0}^{\infty} (E_{i+1} \setminus E_i)$ with $E_0 = \varnothing$.

So, by σ-additivity

$$\mu(E) = \mu\left(\bigcup_{i=0}^{\infty} (E_{i+1} \setminus E_i)\right) =$$

$$= \sum_{i=0}^{\infty} \mu(E_{i+1} \setminus E_i) \overset{(excision)}{=} \sum_{i=0}^{\infty} (\mu(E_{i+1} - \mu(E_i))) =$$

$$\overset{(telescopic\ series)}{=} \lim_n \mu(E_n) - \underbrace{\mu(E_0)}_{=0} = \lim_n \mu(E_n)$$

(2) For simplicity, suppose $\tau = 1$, and define $F_k = E_i \setminus E_k$

$$\{E_k\} \searrow \Rightarrow \{F_k\} \nearrow$$

$$\mu(E_i) = \mu(E_k) + \mu(F_k) \text{ and } \bigcup_k F_k = E_i \setminus (\bigcap_k E_k)$$

$$\mu(E_i) = \mu(\bigcup_k F_k) + \underbrace{\mu(\bigcap_k E_k)}_{\mu(E)} =$$

$$\overset{(i)}{=} \lim_k \mu(F_k) + \mu(E) = \lim_k (\mu(E_i) - \mu(E_k)) + \mu(E)$$

Since $\mu(E_i) < \infty$ we can subtract it from both sides

$$0 = -\lim_k \mu(E_k) + \mu(E)$$

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Counterexample: given $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_C)$ measure space. Let $E_n = \{n, n+1, n+2, \ldots\}$. In this case $\mu_C(E_n) = +\infty, E_{n+1} \subseteq E_n \forall\ n$, but $\bigcap_n E_n = \varnothing \Rightarrow \mu\left(\bigcap_n E_n\right) = 0$

2.10 σ-subadditivity of the measure

Theorem 2.2 (σ-subadditivity of the measure)
$(X, \mathcal{M}, \mu)$ is a measure space. $\forall \{E_n\} \subseteq \mathcal{M}$ (not necessarily disjoint): $\mu\left(\bigcup_n E_n\right) \leq \sum_n \mu(E_n)$

Proof. $E_1, E_2 \in \mathcal{M}$ and also $E_1 \cup E_2 = E_1 \cup (E_2 \setminus E_1)$ disjoint sets.

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(\underbrace{E_2 \setminus E_1}_{\subseteq E_2}) \overset{(monotonicity)}{\leq} \mu(E_1) + \mu(E_2)$$

that means that we have the subadditivity for finitely many sets.
Given $A = \bigcup_{n=1}^{\infty} E_n, A_k = \bigcup_{n=1}^{k} E_n, \{A_k\} \nearrow,\ A_{k+1} \supseteq A_k,\ \lim_k A_k = A:$

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \overset{(continuity)}{=} \lim_k \mu(A_k) = \lim_k \mu\left(\bigcup_{n=1}^{k} E_n\right) \leq \lim_k \sum_{n=1}^{k} \mu(E_n) = \sum_{n=1}^{\infty} \mu(E_n)$$

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Exercise: $(X, \mathcal{M})$ measurable space. $\mu : \mathcal{M} \to [0, +\infty]$ s.t. μ is finitely additive, σ-subadditive and $\mu(\varnothing) = 0 \Rightarrow \mu$ is σ-additive, and hence is a measure.

2.11 Borel-Cantelli Lemma

The Borel-Cantelli lemma states that, given $(X, \mathcal{M}, \mu)$ and $\{E_n\} \subseteq \mathcal{M}$. Then

$$\sum_{n=0}^{\infty} \mu(E_n) < \infty \Rightarrow \mu(\limsup_n E_n) = 0$$

It can be phrased as:

> If the series of the probability of the events E_n is convergent, then the probability that ∞-many events occur is 0

Proof. The thesis is:

$$\mu(\limsup_n E_n) = \mu\Big(\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k \geq n} E_k}_{A_n := \bigcup_{k \geq n} E_k}\Big)$$

Is it true that $\{A_n\} \searrow$? Yes.

$$A_{n+1} = \bigcup_{k \geq n+1} E_k \subseteq \bigcup_{k \geq n} E_k = A_n$$

Does some A_n have a finite measure?

$$\mu(A_n) = \mu\left(\bigcup_{k \geq n} E_k\right) \leq \sum_{k \geq n} \mu(E_k) < \infty$$

by assumption. Therefore, we can use the continuity along decreasing sequences:

$$\mu(\limsup_n E_n) = \lim_n \mu(A_n) = \lim_n \mu\left(\bigcup_{k \geq n} E_k\right) \overset{\sigma-sub.}{\leq} \lim_n \sum_{k=n}^{\infty} \mu(E_k) = 0$$

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2.12 Sets of 0 measure, negligible sets

$(X, \mathcal{M}, \mu)$ measure space.

- $N \subseteq X$ is a set of 0 measure if $N \in \mathcal{M}$ and $\mu(N) = 0$
- $E \subseteq X$ is called **negligible set** if $\exists\ N \in \mathcal{M}$ with 0 measure s.t. $E \subseteq N$ (E does not necessarily stays in $\mathcal{M}$)

2.13 Complete measure space

Definition 2.4
$(X, \mathcal{M}, \mu)$ measure space s.t. every negligible set is measurable (and hence of 0 measure), then $(X, \mathcal{M}, \mu)$ is said to be a **complete measure space**.
A measure space may not be complete. However, let

$$\overline{\mathcal{M}} := \{E \subseteq X : \exists\ F, G \in \mathcal{M} \text{ with } F \subseteq E \subseteq G \text{ and } \mu(G \setminus F) = 0\}$$

Clearly $\mathcal{M} \subseteq \overline{\mathcal{M}}$. For $E \in \overline{\mathcal{M}}$, take F and G as above and let $\bar{\mu}(E) = \bar{\mu}(F)$ then $\bar{\mu}|_{\mathcal{M}} = \mu$, and moreover:

Theorem 2.3
$(X, \mathcal{M}, \mu)$ is a complete measure space. Let's observe that $\bar{\mu}$ is well defined: let $E \subseteq X$ and F_1, F_2, G_1, G_2 s.t. $F_i \subset E \subset G_i \quad i = 1, 2$. Then $\mu(G_i \setminus F_i) = 0$. Now we have to check that $\mu(F_1) = \mu(F_2)$.
Since

$$F_1 \setminus F_2 \subseteq E \setminus F_2 \subseteq G_2 \setminus F_2$$

and $G_2 \setminus F_2$ has 0 measure $\Rightarrow \mu(F_1 \setminus F_2) = 0$. Then $F_1 = (F_1 \setminus F_2) \cup (F_1 \cap F_2) \Rightarrow \mu(F_1) = \mu(F_1 \cap F_2)$.
In the same way, $\mu(F_2) = \mu(F_1 \cap F_2)$

The elements of $\overline{\mathcal{M}}$ are sets of the type $E \cup N$, with $E \in \mathcal{M}$ and $\bar{\mu}(N) = 0$.

2.14 Outer measure

We wish to define a measure λ "on $\mathbb{R}$" with the following properties:

(1) $\lambda((a, b)) = b - a$

(2) $\lambda(E + t)^{\dagger} = \lambda(E)$ for every measurable set $E \subset \mathbb{R}$ and $t \in \mathbb{R}$

It would be nice to define such a measure on $\mathcal{P}(\mathbb{R})$. In such case, note that $\lambda(\{x\}) = 0, \forall\ x \in \mathbb{R}$
But then
Theorem 2.4 (Ulam)
The only measure on $\mathcal{P}(\mathbb{R})$ s.t. $\lambda(\{x\}) = 0 \quad \forall\ x$ is the trivial measure. Thus, a measure satisfying the two properties of the outer measure cannot be defined on $\mathcal{P}(\mathbb{R})$

We'll learn in what follows how to create a measure space on $\mathbb{R}$, with a σ-algebra including all the Borel sets, and a measure satisfying properties of the outer measure. This is the so called **Lebesgue measure**.
Definition 2.5
Given a set X. An **outer measure** is a function $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ s.t.

- $\mu^*(\varnothing) = 0$
- $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$ (Monotonicity)
- $\mu^*(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$ (σ-subadditivity)

$^{\dagger}\{x \in \mathbb{R} : x = y + t, \text{ with } y \in E\}$

2.15 Generation of an outer measure

The common way to define an outer measure is to start with a family of elementary sets $\mathcal{E}$ on which a notion of measure is defined (e.g. intervals on $\mathbb{R}$, rectangles on $\mathbb{R}^2, \dots$) and then to approximate arbitrary sets from outside by **countable** unions of members of $\mathcal{E}$.

Proposition 2.1
Let $\mathcal{E} \subset \mathcal{P}(\mathbb{R})$ and $\rho : \mathcal{E} \to [0, +\infty]$ be such that $\varnothing \in \mathcal{E}, X \in \mathcal{E}$ and $\rho(\varnothing) = 0$. For any $A \in \mathcal{P}(X)$, let

$$\mu^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \rho(E_n) : E_n \in \mathcal{E} \text{ and } A \subset \bigcup_{n=1}^{\infty} E_n \right\}$$

Then μ^* is an outer measure, the outer measure generated by $(\mathcal{E}, \rho)$.

Proof. $\forall\ A \subset X\ \exists \{E_n\} \subset \mathcal{E}$ s.t. $A \subset \bigcup_n E_n$: take $E_n = X\ \forall\ n$, then μ^* is well defined. Obviously, $\mu^*(\varnothing) = 0$ (with $E_n = \varnothing \quad \forall\ n$), and $\mu^*(A) \leq \mu^*(B)$ for $A \subset B$ (any covering of B with elements of $\mathcal{E}$ is also a covering of A.)
We have to prove the σ-subadditivity.
Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X)$ and $\varepsilon > 0$. For each n, $\exists \left\{E_{n_j}\right\}_{j\in\mathbb{N}} \in \mathcal{E}$ s.t. $A_n \subset \bigcup_{i=1}^{\infty} E_{n_j}$ and $\sum_{j=1}^{\infty} \rho(E_{n_j}) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}$. But then, if $A = \bigcup_{n=1}^{\infty} A_n$, we have that $A \subset \bigcup_{n,j\in\mathbb{N}^2} E_{n_j}$ and

$$\mu^*(A) \leq \sum_{n,j} \rho(E_{n_j}) \leq \sum_{n} \left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_{n} \mu^*(A_n) + \varepsilon$$

Since ε is arbitrary, we are done. ★

Ex:

(1) $X \in \mathbb{R}, \mathcal{E} = \{(a, b) : a \leq b, a, b \in \mathbb{R}\}$ family of open intervals:

$$\rho((a, b)) = b - a$$

(2) $X = \mathbb{R}^n, \mathcal{E} = \{(a_1, b_1) \times \ldots \times (a_n, b_n) : a_i \leq b_i, a_i, b_i \in \mathbb{R}\}$:

$$\rho((a_1, b_1) \times \ldots \times (a_n, b_n)) = (b_1 - a_1) \cdot \ldots \cdot (b_n - a_n)$$

Remark 2.4
$E \in \mathcal{E} \Rightarrow \mu^*(E) = \rho(E)$.
In examples 1 and 2, we have in fact

$$\mu^*((a, b)) = b - a, \mu^*\left((a_1, b_1) \times \ldots \times (a_n, b_n)\right) = \prod_{i=1}^{n} (b_i - a_i)$$

2.16 Caratheodory condition

To pass from the outer measure to a measure there is a condition:
Definition 2.6 (Caratheodory condition)
If μ^* is an outer measure on X, a set $A \subset X$ is called μ^*-**measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall\ E \subset X$$

Remark 2.5
If E is a "nice" set containing A, then the above equality says that the outer measure of A, $\mu^*(E \cap A)$, is equal to $\mu^*(E) - \mu^*(E \cap A^C)$, which can be thought as an "inner measure". So basically we are saying that A is measurable if the outer and inner measure coincide. (Like the definition of Riemann integration with lower and upper sum)

Remark 2.6
μ^* is subadditive by def $\Rightarrow$ $\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall\ E, A \subset X$. So, to prove that a set is μ^*-measurable it is enough to prove the reverse inequality, $\forall\ E \subset X$. In fact, if $\mu^*(E) = +\infty$, then $+\infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^C)$, and hence A is μ^*-measurable iff

$$\mu^*(E) \geq \mu(E \cap A) + \mu^*(E \cap A^C) \quad \forall\ E \subset X \text{ with } \mu^*(E) < +\infty$$

2.17 Measure induced by an outer measure

Their relevance to the notion of μ^*-measurability is clarified by the following
Theorem 2.5 (Caratheodory)
If μ^* is an outer measure on X, the family

$$\mathcal{M} = \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\}$$

is a σ-algebra and $\mu^*|_{\mathcal{M}}$ is a complete measure.

Lemma 2.1
If $A \subset X$ and $\mu^*(A) = 0$, then A is μ^*-measurable.

Proof. Let $E \subset X$ with $\mu^*(E) < +\infty$. Then

$$\mu^*(E) \geq \mu^*(E) + \mu^*(A) \overset{\ddagger}{\geq} \mu^*(E \cap A) + \mu^*(E \cap A^C)$$

This implies that A is μ^*-measurable. ★

To sum up: X set, $(\mathcal{E}, \rho)$ elementary and measurable sets, so μ^* is an outer measure. Then given μ^* and the Caratheodory condition, we have $(X, \mathcal{M}, \mu)$ that is a complete measure space.
Remark 2.7
So far we did not prove that $\mathcal{E} \subseteq \mathcal{M}$. We will do it in a particular case.

2.18 Lebesgue measure on $\mathbb{R}^n$

- $X = \mathbb{R}$, $\mathcal{E}$ family of open intervals, $\rho((a,b)) = b - a = \lambda((a,b))$, the complete measure space is $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ with $\mathcal{L}(\mathbb{R})$ the Lebesgue-measurable sets on $\mathbb{R}$ and λ the Lebesgue measure on $\mathbb{R}$.
- $X = \mathbb{R}^n$, $\mathcal{E} = \{\prod_{k=1}^n (a_k, b_k) : a_k \leq b_k \quad \forall\ k = 1, \dots, n\}$, $\rho\left(\prod_{k=1}^n (a_k, b_k)\right) = \prod_{k=1}^n (b_k - a_k)$ and this is a complete measure space $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$

$\mathcal{E} =$ family of open intervals (a,b), $a, b \in \mathbb{R}^*, a < b$. $\rho =$ length l. $\rho((a,b)) = b - a$.
Notations: open interval I with length $l(I)$ $E \subset \mathbb{R}$. The outer measure of E is

$$\lambda^*(E) = \inf \left\{ \sum_{n=1}^{+\infty} l(I_n) | I_n \text{ is an open interval, } E \subset \bigcup_{n=1}^{\infty} I_n \right\}$$

$A \subset \mathbb{R}$ is λ^*-measurable if

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \cap A^C) \qquad \forall\ E \subset \mathbb{R}$$

$$\{A \subset \mathbb{R} : A \text{ is } \lambda^*\text{-measurable}\} =: \mathcal{L}(\mathbb{R}) \qquad \text{(Lebesgue } \sigma\text{-algebra)}$$

$$\lambda := \lambda^*|_{\mathcal{L}(\mathbb{R})} \qquad \text{(Lebesgue measure on } \mathbb{R}\text{)}$$

Then, $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is a complete measure space. In particular, $\lambda^*(A) = 0 \Rightarrow A \in \mathcal{L}(\mathbb{R})$ and $\lambda(A) = 0$.

‡ $E \cap A^C \subseteq E$ and $E \cap A \subseteq A$ + monotonicity

Remark 2.8 (CC-Criterion for measurability)
To check that A is λ^*-measurable, it is sufficient to check that

$$\lambda^* \geq \lambda^*(E \cap A) + \lambda^*(E \cap A^C)$$

for every $E \subset \mathbb{R}$ with $\lambda^*(E) < +\infty$

2.19 Every countable set is Lebesgue-measurable

Proposition 2.2
Any countable set is measurable, with 0 Lebesgue measure.

Proof. Let $a \in \mathbb{R}$,

$$\{a\} \subseteq (a-\varepsilon, a+\varepsilon), \forall\, \varepsilon > 0 \overset{\text{by def.}}{\Rightarrow} \lambda^*(\{a\}) \leq 2\varepsilon \overset{\lim}{\Rightarrow} \lambda^*(\{a\}) = 0$$

$\{a\}$ is measurable with $\lambda(\{a\}) = 0, \forall\, a \in \mathbb{R}$. Now if a set A is countable, $A = \{a_n\}_{n\in\mathbb{N}} = \bigcup_n \{a_n\}$ (disjoint) $\Rightarrow \lambda(A) \underset{\sigma-add}{=} \sum_n \lambda(\{a_n\}) = 0$ ★

Remark 2.9
$\lambda(\mathbb{Q} = 0)$. $\mathbb{Q}$ is dense on $\mathbb{R}$, $\bar{\mathbb{Q}} = \mathbb{R}$. In general, measure theoretical info and topological info cannot be compared.

2.20 Relation between $\mathcal{B}(\mathbb{R})$ and $\mathcal{L}(\mathbb{R})$

Proposition 2.3
$\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$

Remark 2.10
So far we didn't prove the fact that open intervals are $\mathcal{L}$-measurable.

Proof. We know that $\mathcal{B}(\mathbb{R})$ is generated by $\{(a, +\infty) : a \in \mathbb{R}\}$. Then, we can directly show that $(a, +\infty) \in \mathcal{L}(\mathbb{R}) \quad \forall\, a \in \mathbb{R}$. Let $a \in \mathbb{R}$ be fixed. We use the criterion for measurability and we check that

$$\lambda^*(E) \geq \lambda^* \underbrace{(E \cap (a, +\infty)}_{=:E_1} + \lambda^* \underbrace{(E \cap (-\infty, a])}_{=:E_2} \quad \forall\, E \subset \mathbb{R},\, \lambda^* < +\infty$$

Since $\lambda^*(E) < +\infty$, $\exists$ a countable union $\bigcup_n I_n \supset E$, where I_n is an open interval $\forall\, n$ and

$$\sum_n l(I_n) \leq \lambda^*(E) + \varepsilon$$

Let $I_n^1 := I_n \cap E_1, I_n^2 := I_n \cap (-\infty, a + \frac{\varepsilon}{2^n})$. These are open intervals:

$$E_1 \subset \bigcup_n I_n^1 \qquad E_2 \subset \bigcup_n I_n^2 \qquad \text{countable unions}$$

and moreover

$$l(I_n) \geq l(I_n^1) + l(I_n^2) - \frac{\varepsilon}{2^n}$$

By definition of λ^*, $\lambda^*(E_1) \leq \sum_n l(I_n^1)$ and $\lambda^*(E_2) \leq \sum_n l(I_n^2)$, therefore

$$\lambda^*(E_1) + \lambda^*(E_2) \leq \sum_n l(I_n^1) + \sum_n l(I_n^2) \leq \sum_n \left(l(I_n) + \frac{\varepsilon}{2^n} \right) = \left(\sum_n l(I_n) \right) + 2\varepsilon \leq \lambda^*(E) + 3\varepsilon$$

Since ε was arbitrarily chosen, we have

$$\lambda^*(E) \geq \lambda^*(E_1) + \lambda^*(E_2)$$

which is the thesis. ★

So, the Lebesgue measure measures all the open, closed G_δ, F_δ sets. Clearly

$$\lambda((a,b)) = b - a$$

One can also show that λ is invariant under translation.
Questions: $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R}) \subseteq \mathcal{P}(\mathbb{R})$, is it a strict inclusion or not?

- By Ulam's theorem, if a measure is such that $\lambda(\{a\}) = 0, \forall\, a$ and all the sets in $\mathcal{P}(\mathbb{R})$ are measurable, then $\lambda \equiv 0$. This and the fact that $\lambda((a,b)) \neq 0$ simply that $\mathcal{L}(\mathbb{R}) \subsetneq$ [‡]$\mathcal{P}(\mathbb{R})$: $\exists$ non-measurable sets called Vitali sets. Every measurable set with positive measure contains a Vitali set. (Explanation)
- $\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{L}(\mathbb{R})$. The construction of a $\mathcal{L}$-measurable se which is not a Borel set will be done during exercise classes.

The relation between $\mathcal{B}(\mathbb{R})$ and $\mathcal{L}(\mathbb{R})$ is clarified by

2.21 Regularity of Lebesgue measure

Theorem 2.6 (Regularity of λ)
The following sentences are equivalent:

(1) $E \in \mathcal{L}(\mathbb{R})$

(2) $\forall\, \varepsilon > 0\ \exists\, A \supset E$, A open s.t.
$$\lambda(A \setminus E) < \varepsilon$$

(3) $\exists\, G \supset E$, G of class G_δ, s.t.
$$\lambda(G \setminus E) = 0$$

(4) $\exists\, C \subset E$, C closed, s.t.
$$\lambda(E \setminus C) = 0$$

(5) $\exists\, F \subset E$, F of class F_δ, s.t.
$$\lambda(E \setminus F) = 0$$

Consequence: $E \in \mathcal{L}(\mathbb{R}) \Rightarrow E = F \cup N$, where F is of class F_δ, and $\lambda(N) = 0$.

Partial proof. For simplicity, we will consider only sets with finite measure.

(1) $\Rightarrow$ (2) $E \in \mathcal{L}(\mathbb{R})$. By definition of λ^*, $\forall\, \varepsilon > 0\ \exists \bigcup_n I_n \supset E$ s.t. each I_n is an open interval, and
$$\lambda(E) = \lambda^*(E) \geq \sum_n l(I_n) - \varepsilon$$
We define $A = \bigcup_n I_n$, which is open. Also $A \supset E$ and
$$\lambda(A) = \lambda\left(\bigcup_n I_n\right) \overset{\sigma\text{–sub.}}{\leq} \sum_n l(I_n) \leq \lambda(E) + \varepsilon$$
Then, by excision
$$\lambda(A \setminus E) = \lambda(A) - \lambda(E) \leq \varepsilon$$

[‡]I had no choice

(2) $\Rightarrow$ (3) Define, for every $K \in \mathbb{N}$, an open set A_k s.t. $A_k \supset E$ and $\lambda(A_k \setminus E) < \frac{1}{k}$. Let $A = \bigcap_k A_k$. This is a G_δ set, it contains E (since each A_k contains E) and

$$\lambda(A \setminus E) \underset{(A \subset A_k \ \forall\, k)}{\leq} \lambda(A_k \setminus E) < \frac{1}{k} \Rightarrow \lambda(A \setminus E) = 0 \quad \forall\, k$$

(3) $\Rightarrow$ (1) If $E \subset \mathbb{R}$ and $\exists\, G \supset E$, with G of class G_δ, s.t. $\lambda(G \setminus E) = 0$, then

$$E = G \setminus (G \setminus E) \text{ is measurable}$$

since G is a Borel set and $(G \setminus E)$ has 0 measure, then both are in $\mathcal{L}$

★

Remark 2.11
Any countable set has 0 measure. The inverse is false. An example is given by the **Cantor set**.
Let $T_0 = [0,1]$. Then we define T_{n+1} stating from T_n in the following way: given T_n, finite union of closed disjoint intervals of length $l_n(\frac{1}{3})^n$, T_{n+1} is obtained by removing from each interval of T_n, the open central subinterval of length $\frac{l_n}{3}$.
The Cantor set is $T := \bigcap_{k=0}^{+\infty}$. It can be proved that T is compact, $\lambda(T) = 0$ and T is uncountable.
If, instead of removing intervals of size $\frac{1}{3}, \frac{1}{9}, \ldots, \frac{1}{3^k}$, we remove sets of size $\left(\frac{\varepsilon}{3}\right)^k$, with $\varepsilon \in (0,1)$, we obtain the **generalized Cantor set** (or **fat Cantor set**) T_ε. T_ε is uncountable, compact and has no interior points (it contains no intervals). However, $\lambda(T_\varepsilon) = \frac{3(1-\varepsilon)}{3-2\varepsilon} > 0$

Remark 2.12
We worked on $\mathbb{R}$, but everything can be adapted to $\mathbb{R}^n$

3 Measurable functions

3.1 Definition of measurable functions

Definition 3.1
$f : X \to Y$, then it is well defined the counterimage

$$f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$$

$$E \to f^{-1}(E) = \{x \in X : f(x) \in E\}$$

Definition 3.2
$(X, \mathcal{M}), (Y, \mathcal{N})$ measurable spaces. $f : X \to Y$ is called **measurable** or $(\mathcal{M}, \mathcal{N})$-measurable if

$$f^{-1}(E) \in \mathcal{M} \text{ for every } E \in \mathcal{M}$$

so, the counterimage of measurable sets in Y is a measurable set on X.

To check if a function is measurable or not, it is often used the following proposition
Proposition 3.1
$(X, \mathcal{M}), (Y, \mathcal{N})$ measurable spaces. Let $\mathcal{F} \subseteq \mathcal{P}(Y)$ be s.t. $\mathcal{N} = \sigma_0(\mathcal{F})$. Then

$$f : X \to Y \text{ is } (\mathcal{M}, \mathcal{N}) - \text{measurable} \Leftrightarrow f^{-1}(E) \in \mathcal{M} \text{ for every } E \in \mathcal{F}$$

We will mainly focus on 2 situations:

(1) $(X, \mathcal{M})$ is a measurable space obtained by means of an outer measure.
Ex: $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n))$, (Y, d_y) metric space $\to (Y, \mathcal{B}(Y))$.

If $X \to Y$ is (Lebesgue) measurable $\Leftrightarrow (\mathcal{M}, \mathcal{B}(Y))$ is measurable

(2) $(X, d_X), (Y, d_Y)$ are metric spaces $\longrightarrow (X, \mathcal{B}(X)), (Y, \mathcal{B}(Y))$
$f : X \to Y$ is Borel measurable $\Leftrightarrow (\mathcal{M}(X), \mathcal{B}(Y))$-measurable.

Remark 3.1
f is Lebesgue measurable if the continuity of the Borel set is a Lebesgue-measurable set.

3.2 Continuous functions

Proposition 3.2
There are two parts:

(1) $(X, d_X), (Y, d_Y)$ metric spaces. If $f : X \to Y$ is continuous, then is Borel measurable

(2) (Y, d_Y) metric space. If $f : \mathbb{R}^n \to Y$ is continuous, then it is a Lebesgue measure.

Proof. The proof is divided in:

(1) f is continuous $\Leftrightarrow f^{-1}(A)$ is open $\forall\ A \subset Y$ open $\Rightarrow f^{-1}(A) \in \mathcal{B}(Y)\ \forall\ A \subset Y$ open. Since $\mathcal{B}(Y) = \sigma_0$ (open sets) by proposition (1) this implies that f is Borel measurable

(2) f is continuous $\overset{(1)}{\Rightarrow}$ f is Borel measurable. $f^{-1}(A) \in \mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{L}(\mathbb{R}^n) \forall\ A \in \mathcal{B}(Y)$. Namely f is Lebesgue measurable

★

Proposition 3.3
$(X, \mathcal{M})$ measurable space, $(X, d_Y), (Y, d_Y)$ metric spaces. If $f : X \to Y$ is $(\mathcal{M}, \mathcal{B}(Y))$-measurable and $g : Y \to Z$ is continuous $\Rightarrow g \circ f : x \to Z$ is $(\mathcal{M}, \mathcal{B}(Y))$-measurable

Proposition 3.4
$(X, \mathcal{M})$ measurable space , $u, v : X \to \mathbb{R}$ measurable functions. Let $\Phi : \mathbb{R}^2 \to Y$ be continuous where (Y, d_Y) is a metric space. Then $h : X \to Y$ defined by $h(x) = \Phi(u(x), v(x))$ is $(\mathcal{M}, \mathcal{B}(Y))$-measurable.

3.3 Measurability of composition

Definition 3.3
u, v measurable $\Rightarrow u + v$ is measurable.

Proof. Define $f : X \to \mathbb{R}^2$, $f(x) = u(x), v(x)$. By definition $h = \Phi \circ f$ by proposition (3) if we show that f is $(\mathcal{M}, \mathcal{B}(\mathbb{R}^2))$-measurable, then h is measurable. It can be proved that

$$\mathcal{B}(\mathbb{R}^2) = \sigma_0(\{\underbrace{(a_1, b_1) \times (a_2, b_2)}_{\text{open rectangle}} : a, b \in \mathbb{R}\})$$

Thanks to proposition (1), to check that f is measurable. We can simply check that $f^{-1}(\mathbb{R}) \in \mathcal{M} \quad \forall$ open rectangle in $\mathbb{R}^2$ and $R = I \times J$, with I and J open intervals:

$$\begin{gathered} F^{-1}(\mathbb{R}) = \{x \in X : (u(x), v(x)) \in \mathbb{R}\} \\ \Updownarrow \\ u(x) \in I \text{ and } v(x) \in J \\ = \{x \in X : u(x) \in I\} \cap \{x \in X : v(x) \in J\} \\ = \underbrace{u^{-1}(I)}_{\in \mathcal{M}} \cap \underbrace{v^{-1}(J)}_{\in \mathcal{M}} \in \mathcal{M} \\ \text{since both } u, v \text{ are measurable} \end{gathered}$$

This completes the proof ★

Consequences: by proposition 3 and 4, if u and v are measurable, then also $u + v$, $u \cdot v$. Other measurable functions include $u^+ = \max\{u, 0\}, u^- = -\min\{u, 0\}, |u| = u^+ + u^-, u^2, \ldots$
Recall that $u = u^+ - u^-$

Remark 3.2
u^+ is measurable since $u^+ = g \circ u$, where:

$$g(x) = \begin{cases} x & \text{where } x \geq 0 \\ 0 & \text{where } x < 0 \end{cases}$$

Most of the times we will work with functions $f : X \to \mathbb{R}$ or $f : X \to \underbrace{\overline{\mathbb{R}}}_{\mathbb{R} \cup \{\pm\infty\}}$ $(X, \mathcal{M})$ measurable space, then such a function f is measurable iff

$$f^{-1}((a, +\infty)]^{\dagger}) \in \mathcal{M} \quad \forall a \in \mathbb{R}$$

or equivalently

$$f^{-1}([a, +\infty)]) \in \mathcal{M} \quad \forall a \in \mathbb{R}$$

Let now $\{f_n\}$ be a sequence of measurable functions from X to $\overline{\mathbb{R}}$. Then we define

$$(\inf_n f_n)(x) = \inf_n f_n(x)$$

$$(\sup_n f_n)(x) = \sup_n f_n(x)$$

$$(\liminf_n f_n)(x) = \liminf_n f_n(x)$$

$$(\limsup_n f_n)(x) = \limsup_n f_n(x)$$

$$(\lim_n f_n)(x) = \lim_n f_n(x) \quad \text{if the limit exists}$$

3.4 Measurability of limits for real valued fucntions

Proposition 3.5
$(X, \mathcal{M})$ measurable space, $f_n : X \to \overline{\mathbb{R}}$ measurable, then

$$\sup_n f_n \;\; \inf_n f_n \;\; \liminf_n f_n \;\; \limsup_n f_n$$

are measurable, in particular if $\lim_n f_n$ is well defined, then f is measurable

† We use) if f takes values in $\mathbb{R}$ and] if f takes values in $\overline{\mathbb{R}}$

Proof. $(\sup f_n)^{-1}((a,\infty]) = \{x \in X : \underset{\exists \text{ some indexes } n \text{ s.t. } f_n(x) > a}{\underset{\Updownarrow}{\sup f_n(x) > a}} \}$

$$\bigcup_n \{x \in X : f_n(x) > a\} = \bigcup_n \underbrace{f_n^{-1}((a,+\infty])}_{\in \mathcal{M}}$$

Then $(\sup f_n)^{-1}((a,\infty])$ is measurable, since it is the countable union of measurable sets.
Now we check that the $\limsup_n f_n$ is measurable

$$\limsup_n f_n(x) = \lim_n \underbrace{(\sup_{k>n} f_k(x))}_{\text{is decreasing on } n} = \inf_n (\sup_{k\geq n} f_k(x))$$

If we write $g_n(x) = \sup_{k\geq n} f_k(x)$, then

- g_n is measurable, by what we proved previously
- $\limsup_n f_n = \inf_n g_n$ is measurable

★

3.5 Simple functions and step functions

Definition 3.4
$(X, \mathcal{M})$ measurable space. A measurable function $s : X \to \overline{\mathbb{R}}$ is said to be simple if $s(X)$ is a finite set.

$$s(X) = \{a_1 \ldots, a_n\} \text{ for some } n \in \mathbb{N}, a_i \neq a_j$$

Then

$$s(x) = \sum_{n=1} a_n \chi_{E_n}(x)$$

where E_n is a measurable set, $E_n = \{x \in X : s(X) = a_n\}$, and $E_i \cap E_j = \varnothing$ for $i \neq j$, and $\bigcup_{n=1}^N E_n = X$.

Particular case: if $s : \mathbb{R} \to \overline{\mathbb{R}}$, and each E_n is a finite union of intervals, then s is said to be a **step** function.

3.6 Simple approximation theorem

The idea is to approximate arbitrary measurable functions with simple functions.

Theorem 3.1
$(X, \mathcal{M})$ measurable space, $f : X \to [0,\infty]$ measurable. Then $\exists$ a sequence $\{s_n\}$ of simple functions s.t.

$$0 \leq s_1 \leq \ldots \leq s_n \leq \ldots \leq f \quad \underset{\forall\, x\in X}{\text{(pointwise)}}$$

and $s_n(x) \to f(x) \quad \forall\, x \in X$ as $n \to \infty$.
Moreover if f is bounded then $s_n \to f$ uniformly on X as $n \to \infty$

Proof - For f bounded. Fix $n \in \mathbb{N}$ and divide $[0,n)$ in $n \cdot 2^n$ intervals called $I_j = [a_j, b_j)$ with length $\frac{1}{2^n}$.
Let $E_0 = f^{-1}([n,+\infty)), E_j = f^{-1}([a_j,b_j))$ for $j = 1,\ldots,n\cdot 2^n$. We let

$$\begin{aligned} s_n(x) &= a_j \quad \text{for } x \in E_j \\ s_n(x) &= n \quad \text{for } x \in E_0 \end{aligned}$$

Namely we define the simple function s_n as

$$s_n(x) = n\chi_{E_0}(X) + \sum_{j=1}^{n\cdot 2^n} a_j \chi_{E_j}(x)$$

Then $s_n \leq s_{n+1}$ by contradiction, and, since f is bounded, $E_0 = \varnothing$ for n sufficiently large ($n > \sup f$).
Then any $x \in X$ stays in $f^{-1}([a_j, b_j))$ for some j

$$\begin{aligned}
&\Rightarrow \underset{s_n(x)}{\underset{\shortparallel}{a_j}} \leq f(x) < b_j \\
&\Rightarrow 0 \leq f(x) - s_n(x) < b_j - a_j = \frac{1}{2^n} \\
&\Rightarrow \sup_{x \in X} |f(x) - s_n(x)| < \frac{1}{2^n} \to 0 \text{ as } n \to \infty
\end{aligned}$$

Namely, $s_n \to f$ uniformly on X. ★

Remark 3.3
On the relation between $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ and $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ ($\lambda =$ Lebesgue measure)
$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ is not complete. In fact, $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is the completion of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$.
Note that, $\forall\, E \in \mathcal{L}(\mathbb{R})\ \exists$ a $G_\delta -$ set A and an $F_\delta -$ set B s.t.

$$\begin{gathered}
A \supset E \text{ and } \lambda(A \setminus E) = 0 \\
B \subset E \text{ and } \lambda(E \setminus B) = 0
\end{gathered}$$

$(X, \mathcal{M}, \mu)$ complete measure space.

Definition 3.5
Let $P(x)$ be a proposition depending on $x \in X$. We say that $P(x)$ is true ($\mu-$) almost everywhere if

$$\mu\left(\{x \in X : P(x) \text{ is false}\}\right) = 0$$

$P(x)$ is true $\underset{(\mu-\text{a.e.})}{\text{a.e.}}$ on X.

Ex: $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$, $f(x) = x^2$. Then $f(x) > 0$ a.e. on $\mathbb{R}$ (for a.e. x):

$$\{f(x) \leq 0\} = \{0\}\,, \text{ and } \lambda(\{0\}) = 0$$

$(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mu_C)$ with μ_C counting measure. Then it is not true that $f(x) > 0$ μ_C-a.e.

$$\mu_C\left(\{0\}\right) = 1$$

It will be useful to consider sequences converging a.e.:

$$f_n \to f \qquad \text{a.e. on } X$$

if $\mu\left(\{x \in X : \lim_n f_n(x) \neq f(x), \text{ or does not exist}\}\right) = 0$

Proposition 3.6
$(X, \mathcal{M}, \mu)$ complete measure space.

(1) $f : X \to \mathbb{R}$ is measurable, and $g = f$ a.e. on X, then g is measurable

(2) $f_n \to f$ a.e. on X, $f_n : X \to \mathbb{R}$ measurable for all n, then f is measurable

4 The Lebesgue integral

Notation:

$$\{x \in X : f(x) \geq 0\} = \{f \geq 0\}$$
$$\{x \in X : f(x) > 0\} = \{f > 0\}$$
$$\vdots$$

$(X, \mathcal{M}, \mu)$ complete measure space. We consider measurable functions $f : X \to [0, +\infty]$
Convention: we define

$$a + \infty = +\infty \quad \forall\, a \in \mathbb{R}$$
$$a \cdot (+\infty) = \begin{cases} +\infty & \text{if } a \neq 0, a > 0 \\ 0 & \text{if } a = 0 \end{cases}$$

With this convention, $+$ and $\cdot$ of measurable functions are measurable functions.

4.1 Integral of nonnegative simple functions

Definition 4.1
Let $s : X \to [0, +\infty]$ be a measurable simple function,

$$s(x) = \sum_{n=1}^{m} a_n \chi_{D_n}(\bar{x})$$

where $D_1, \ldots, D_m$ are measurable, disjoint, and $\bigcup_{n=1}^{m} D_n = X$. Let also $E \in \mathcal{M}$. Then we define

$$\int_E s\, d\mu := \sum_{n=1}^{m} a_n \mu(D_n \cap E)$$

Remark 4.1
Given a simple function s:

$$s : [a, b] \to \mathbb{R}, \lambda = \mu \Rightarrow \int_E s\, d\mu \text{ is the area under the curve}$$

Remark 4.2
There are several points:

- In the definition we have already used the convention $\mu(D_n \cap E = +\infty) \quad$ for some n
- $E \in \mathcal{M} \Rightarrow \chi_E$ is a simple function

  $$\chi_E(x) = 1 \cdot \chi_E + 0 \cdot \chi_{X \setminus E}(x)$$

  In this case

  $$\int_X \chi_E\, d\mu = 1 \cdot \mu(E) + 0 \cdot \mu(X \setminus E) = \mu(E)$$

- $s\chi_E = \sum_{n=1}^{m} a_n \chi_{D_n \cap E} \Rightarrow \int_E s\, d\mu = \int_X s\chi_E\, d\mu$

4.2 Integral of nonnegative measurable functions

Definition 4.2
$f : X \to [0, +\infty]$ measurable, $E \in \mathcal{M}$. The **Lebesgue integral** of f on E, with respect to (w.r.t.) μ, is

$$\int_E f \, d\mu = \sup \left\{ \int_E s \, d\mu \;\middle|\; \begin{array}{l} s \text{ is simple} \\ 0 \leq s \leq f \end{array} \right\}$$

(1) If f is simple, the definitions are consistent

(2) Also for f measurable: $\int_E f \, d\mu = \int_X f \chi_E \, d\mu$

(3) $(\mathbb{N}, \mathbb{N}, \mu_C)$. $f : \mathbb{N} \to \mathbb{R}$ is a sequence $\{a_n\}_{n \in \mathbb{N}}$

$$\int_{\mathbb{N}} \{a_n\} \, d\mu_C = \sum_{n=0}^{\infty} a_n$$

4.3 Basic properties of Lebesgue integral

Let $f, g : X \to [0, \infty]$ measurable. $E, F \in \mathcal{M}$, $\alpha \geq 0$. Then:

(1) $\mu(E) = 0 \Rightarrow \int_E f \, d\mu = 0$

(2) $f \leq g$ on $E \Rightarrow \int_E f \, d\mu \leq \int_E g \, d\mu$

(3) $E \subset F \Rightarrow \int_E f \, d\mu \leq \int_F f \, d\mu$

(4) $\alpha \geq 0 \Rightarrow \int_E \alpha f \, d\mu = \alpha \int_E d \, d\mu$

Remark 4.3
$([0,1], \mathcal{L}([0,1]), \lambda)$
Consider $\chi_{\mathbb{Q}}$, it is the Dirichlet function on $[0,1]$. This is not Riemann integrable.
However, $\int_{[0,1]} \chi_{\mathbb{Q}} \, d\lambda = \lambda(\mathbb{Q} \cap [0,1]) = 0$

4.4 Chebychev's inequality

Theorem 4.1 (Chebychev's inequality)
$f : X \to [0, \infty]$ measurable, $c > 0$. Then

$$\mu(\{f \geq c\}) \leq \frac{1}{c} \int \{f \geq c\} f \, d\mu \leq \frac{1}{c} \int_X f \, d\mu$$

Proof.

$$\int_X f \, d\mu \overset{X \supset \{f \geq c\}}{\geq} \int_{\{f \geq c\}} f \, d\mu \geq \int_{\{f \geq c\}} c \, d\mu = c \int_{\{f \geq c\}} d\mu = c\mu(\{f \geq c\})$$

★

4.5 Measure defined by the integral

Theorem 4.2
$s : X \to [0, \infty]$ simple. Define $\varphi : \mathcal{M} \to [0, \infty]$
$\varphi(E) = \int_E s \, d\mu$
$\Rightarrow \varphi$ is a measure.

Proof. $\mu(\varnothing) = 0 \Rightarrow \varphi(\varnothing) = 0$ by definition.
Definition 4.3 (sigma additivity)
$\{E_n \subset \mathcal{M}\}$ disjoint, and let $E = \bigcup_{n=1}^{\infty} E_n \Rightarrow s = \sum_{k=1}^{m} a_k \chi_{D_k} \; D_k \in \mathcal{M}$

Then, by definition and since μ is a measure and $E \cap D_k = \bigcup_n (E_n \cap D_k)$

$$\varphi(E) = \sum_{k=1}^{m} a_k \mu(D_k \cap E) = \sum_{k=1}^{\infty} a_k \sum_{n=1}^{\infty} \mu(E_n \cap D_k) =$$

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{m} a_k \mu(E_n \cap D_k) \right) = \sum_{n=1}^{\infty} \int_{E_n} s \, d\mu = \sum_{n=1}^{\infty} \varphi(E_n)$$

★

4.6 Vanishing Lemma

Theorem 4.3 (Vanishing Lemma)
$f : X \to [0, \infty]$ measurable. $E \subset X$ measurable

$$\int_E f \, d\mu = 0 \Leftrightarrow f = 0 \text{ a.e. on } E$$

Proof. $\Leftarrow$ easy
$\Rightarrow$ Consider $E \cap \{f > 0\} = \bigcup_{n=1}^{\infty} \underbrace{\left(E \cap \{f \geq \frac{1}{n}\} \right)}_{=:E_n}$
Then $\{E_n\}$ is an increasing sequence. By Chebyshev

$$\mu(E_n) \leq \frac{1}{\frac{1}{n}} \int_E f \, d\mu = 0 \quad \forall \, n \Rightarrow \mu(E_n) = 0 \quad \forall \, n$$

$\mu(E \cup \{f > 0\}) \overset{\text{continuity}}{=} \lim_n \mu(E_n) = 0$, namely $f = 0$ a.e. on E ★

The $\int$ does not see sets with 0 measure.
Definition 4.4
If $f : X \to [0, \infty]$ is measurable, and $\int_X f \, d\mu < \infty$ then we say that f is integrable.

4.7 Monotone Convergence Theorem

Theorem 4.4 (Monotone Convergence - Beppo Levi)
$f_n : X \to [0, \infty]$ measurable. Suppose that

- $f_n(x) \leq f_{n+1}(x)$ for a.e. $x \in X$ for every n
- $f_n \to f$ a.e. on X

Then

$$\int_X f\,d\mu = \lim_n \int_X f_n\,d\mu$$

Proof. Part 1.
Assume that the two hypotesis hold everywhere. First, if f is measurable

$$\int_X f_n\,d\mu \nearrow \quad \Rightarrow \exists\, \alpha = \lim_n \int_X f_n\,d\mu$$

Also, $f_n \le f$ everywhere $\Rightarrow \int_X f_n\,d\mu \le \int_X f\,d\mu \quad \forall\, n$

$$\Rightarrow \alpha \le \int_X f\,d\mu$$

We want to show that also $\ge$ is true. Let s be a simple function s.t. $0 \le s \le f$ and $c \in (0,1)$
Let $E_n = \{f_n \ge cs\} \in \mathcal{M}$

- $E_n \subset E_{n+1}\ \forall\, n:$
  if $x \in E_n$, then $f_n(x) \ge cs(x) \Rightarrow f_{n+1}(x) \ge cs(x)$
  $\Rightarrow f_{n+1}(x) \ge f_n(x) \ge cs(x) \Rightarrow x \in E_{n+1}$
- Moreover, $X = \bigcup_{n=1}^{\infty} E_n$. Indeed:
  - if $f(x) = 0$, then $s(x) = 0 \Rightarrow f_1(x) = 0 = cs(x),\ x \in E_1$
  - if $f(x) > 0$, then $cs(x) < f(x) = \lim_n f_n(x)$ since $s \le f$ and $c < 1$
  $\Rightarrow cs(x) < f_n(x)$ for n sufficiently large, namely $x \in E_n$ for n sufficiently large.

Therefore,

$$\alpha \ge \int_X f_n\,d\mu \ge \int_{E_n} f_n\,d\mu \ge c\int_{E_n} s\,d\mu = c\varphi(E_n)$$

$\forall\, n,\ \forall\, 0 \le s \le f, \forall\, c \in [0,1] \quad \varphi(E_n) = \int_{E_n} s\,d\mu$. φ is a measure, and $\{E_n\} \nearrow$
Therefore, taking the lim when $n \to \infty$ by continuity

$$\alpha \ge \lim_n c\int_{E_n} s\,d\mu = c\int_X s\,d\mu \quad \forall c \in [0,1]$$

Take the limit when $c \to 1^-:\ \alpha \ge \int_X s\,d\mu \quad \forall\, 0 \le s \le f$
Take the sup over s: $\alpha \ge \int_X f\,d\mu$. We proved both inequalities, so the thesis holds.
Part 2.
Note that $\{x \in X :$ assumptions of the theorem do not hold$\}$ is a set of zero measure. Take F. $X = E \cup F$ since we have the assumpion on E and $\mu(F) = 0$.
Then, by the Vanishing Lemma, since $(f - f\chi_E) = 0$ a.e. and $(f_n - f_n\chi_E) = 0$ we have that

$$\int_X f\,d\mu = \int_E f\,d\mu = \lim_n \int_E f_n\,d\mu = \lim_n \int_X f_n\,d\mu$$

★

Corollary 4.1 (Monotone convergence for series)
$f_n : X \to [0,+\infty]$ measurable, then

$$\int_X \left(\sum_{n=0}^{\infty} f_n\right) d\mu = \sum_{n=0}^{\infty} \int_X f_n\,d\mu$$

Theorem 4.5 (Approximation with simple functions)
Given $(X, \mathcal{M})$ measure space, $f : X \to [0, +\infty]$ measurable, then $\exists$ a sequence $\{s_n\}$ of simple functions s.t.

$$0 \leq s_1 \leq \ldots \leq s_n \leq \ldots \leq f \qquad \text{pointwise } \forall\, x \in X$$

and

$$s_n(x) \to f(x) \qquad \forall\, x \in X \text{as } n \to \infty$$

Moreover, if f is bounded, then $s_n \to f$ uniformly on X as $n \to \infty$.

Remark 4.4
There is also

$$\int_X f\, d\mu = \sup \left\{ \int_X s\, d\mu \,\middle|\, \begin{array}{l} s \text{ is simple} \\ 0 \leq s \leq f \end{array} \right\}$$

But let $\{s_n\}$ be the sequence given by the simple approximation theorem. By monotone convergence

$$\int_X f\, d\mu = \lim_n \int_X s_n\, d\mu$$

Ex: $f, g : X \to [0, +\infty]$. Then

$$\int_X (f + g)\, d\mu = \int_X f\, d\mu + \int_X g\, d\mu$$

4.8 Fatou's Lemma

Lemma 4.1 (Fatou's Lemma)
Given $f_n : X \to [0, +\infty]$ measurable $\forall\, n$. Then

$$\int_X (\liminf_n f_n)\, d\mu \leq \liminf_n \int_X f_n\, d\mu$$

In particular, if $f_n \to f$ a.e. on X, then

$$\int_X f\, d\mu \leq \liminf_n \int_X f_n d\mu$$

Proof. Given that $(\liminf_n f_n)(x) = \lim_n(\underbrace{\inf_{k \geq n} f_k(x)}_{= g_n(x)})$. Now, for every $x \in X$, $\{g_n(x)\} \nearrow$

$$g_{n+1}(x) = \inf_{k \geq n+1} f_k(x) \geq \inf_{k \geq n} f_k(x) = g_n(x)$$

Also, $g_n \geq 0$ on X. Thus, by monotone convergence

$$\int_X \liminf_n f_n\, d\mu = \int_X \lim_n g_n\, d\mu = \lim_n \int_X g_n\, d\mu = \liminf \int_X g_n\, d\mu$$

Now, note that

$$g_n(x) = \inf_{k \geq n} f_k(x) \leq f_n(x) \leq \liminf_n \int_X f_n\, d\mu$$

★

4.9 Integration of series of nonnegative functions

Theorem 4.6 (σ-additivity of $\int$)
Given $(X, \mathcal{M}, \mu)$ measure space, $\phi : X \to [0, +\infty]$. Define $\nu(E) = \int_E \phi \, d\mu$, with $E \in \mathcal{M}$. $\nu : \mathcal{M} \to [0, +\infty]$ is a measure. Moreover, let $f : X \to [0, +\infty]$ measurable

$$\int_X f \, d\nu = \int_X f \phi \, d\mu \qquad *$$

Proof. $\underline{\nu \text{ is a measure}}$:
$\nu(\varnothing) = 0$, since $\mu(\varnothing) = 0$. Now, let $E = \bigcup_{k=1}^{\infty} E_k$, $\{E_k\}$ disjoint. Then

$$\nu(E) = \int_X \phi \chi_E \, d\mu = \int_X \phi \sum_n \chi_{E_n} \, d\mu \underset{\substack{\text{monot. conv.} \\ \text{for } \sum}}{=} \sum_n \int_X \phi \chi_{E_n} \, d\mu = \sum_n \int_{E_n} \phi \, d\mu = \sum_n \nu(E_n)$$

We have proven σ additivity, so ν is a measure.
$\underline{(\text{*}) \text{ holds}}$: Let $E \in \mathcal{M}$. Then

$$\int_X \chi_E \, d\nu = \int_E 1 \, d\nu = \nu(E) = \int_E \phi \, d\mu = \int_X \phi \chi_E \, d\mu$$

This shows that $(*)$ holds for χ_E, $\forall \, E$. Then it holds for simple functions.
Let now f be any measurable function, positive. Then we can take $\{s_n\}$ given by the simple approximation theorem

$$\int_X f \, d\nu \overset{\text{monot}}{=} \lim_n \int_X s_n \, d\nu = \lim_n \int_X s_n \phi \, d\mu \overset{\text{monot}}{=} \int_X f \phi \, d\mu$$

which is $(*)$ ★

Remark 4.5
$X, \mathcal{M}, \mu$ complete measure space. Then, by the vanishing lemma, it is not difficult to deduce that

$$f = g \text{ a.e. on } X \Leftrightarrow \int_E f \, d\mu = \int_E g \, d\mu \qquad \forall \, E \in \mathcal{M}$$

The $\int$ does not see differences of sets with 0 measure. As a consequence, in all the theorems, it is sufficient to assume that the assumptions are satisfied a.e.

4.10 Integrable functionss

$X, \mathcal{M}, \mu$ complete measure space.
$f : X \to \overline{\mathbb{R}} = [-\infty, \infty]$ measurable. Recall $f = f^+ - f^-$ where $f^+ = \max\{f, 0\}$, $f^- = -\min\{f, 0\}$ and $|f| = f^+ + f^-$. Note that both are positive and measurable.

Definition 4.5
we say that $f : X \to \overline{\mathbb{R}}$ measurable is integrable on X if

$$\int_X |f| \, d\mu < \infty$$

If f is integrable, we define $\int_X f \, d\mu = \int_X f^+ \, d\mu + \int_X f^- \, d\mu$

4.11 The set $\mathcal{L}^1$

The set of integrable functions is denoted by

$$\mathcal{L}^1(X, \mathcal{M}, \mu) := \{f : X \to \overline{\mathbb{R}} \text{ integrable}\}$$

$$\mathcal{L}^1(X, \mathcal{M}, \mu) = \mathcal{L}^1(X) = \mathcal{L}^1$$

If $E \in \mathcal{M}$, we define

$$\int_E f\, d\mu = \int_X f \chi_E\, d\mu$$

Remark 4.6
$f \in \mathcal{L}^1(X) \Rightarrow \int_X f\, d\mu \in \mathbb{R}$. Indeed $0 \le f^{\pm} \le |f|$

$$\Rightarrow 0 \le \int_X f^+\, d\mu,\ \int_X f^-\, d\mu \le \int_X |f|\, d\mu < \infty$$

$$\Rightarrow \int_X f\, d\mu = \int_X f^+\, d\mu - \int_X f^-\, d\mu \in \mathbb{R}$$

4.12 Triangle inequality

Proposition 4.1
$f : X \to \overline{\mathbb{R}}$ measurable. Then

(1) $f \in \mathcal{L}^1 \Leftrightarrow |f| \in \mathcal{L}^1 \Leftrightarrow$ both $f^+,\ f^- \in \mathcal{L}^1$

(2) $f \in \mathcal{L}^1$, then

$$\left|\int_X f\, d\mu\right| \le \int_X |f|\, d\mu \qquad \text{(triangle inequality)}$$

Proof. Of the second part.

$$\left|\int_X f\, d\mu\right| = \left|\int_X f^+\, d\mu + \int_X f^-\, d\mu\right| \le \int_X f^+\, d\mu + \int_X f^-\, d\mu = \int_X |f|\, d\mu$$

★

4.13 $\mathcal{L}^1$ is a vector space

Proposition 4.2
$\mathcal{L}^1(X, \mathcal{M}, \mu)$ is a vector space, and $f,\ g \in \mathcal{L}^1$, $\alpha \in \mathbb{R}$

$$\Rightarrow \int_X (\alpha f + g)\ d\mu = \alpha \int_X f\, d\mu + \int_X g\, d\mu$$

by linearity of the integrals.

Many results can be extended from non negative functions to general functions.

Theorem 4.7
$(X, \mathcal{M}, \mu)$ complete measure space. $f,\ g \in \mathcal{L}^1$. Then

$$f = g \text{ a.e. on } X \Leftrightarrow \int_X |f - g|\, d\mu = 0 \Leftrightarrow \int_E f\, d\mu = \int_E g\, d\mu \qquad \forall\, E \in \mathcal{M}$$

4.14 Dominated convergence theorem

The most relevant theorem for convergence is the following

Theorem 4.8 (Dominated convergence theorem)

$\{f_n\}$ sequence of measurable functions $X \to \overline{\mathbb{R}}$. Suppose that

(1) $f_n \to f$ a.e. on X

(2) $\exists\, g : X \to \overline{\mathbb{R}}$, $g \in \mathcal{L}^1(X)$, such that $|f_n(x)| \leq g(x)$ a.e. on X $\forall\, n \in \mathbb{N}$

Then $f \in \mathcal{L}^1$ and

$$\lim_n \int_X |f_n - f|\, d\mu = 0 \qquad \left(\Rightarrow \int_X f\, d\mu = \lim_n \int_X f_n\, d\mu \right)$$

Proof. Note that $f_n \in \mathcal{L}^1$ $\forall\, n$, since $|f_n| \leq g$ and we have the monotonicity of $\int$ for non negative functions

$$|f_n(x)| \leq g(x) \quad n \to \infty \qquad |f(x)| \leq g(x) \text{ a.e. on } X$$

$$\Rightarrow f \in \mathcal{L}^1(X)$$

Now, consider $\phi_n = 2g - |f_n - f|$. We have

$$|f_n - f| \leq |f_n| + |f| \leq 2g \quad \text{ a.e. on } X \quad \phi_n \geq 0 \quad \text{ a.e. on } X$$

We can use Fatou's lemma:

$$\int_X \underbrace{(\liminf_n \phi_n)}_{=\text{ 2g a.e.}}\, d\mu \leq \liminf_n \int_X \phi_n\, d\mu = \liminf_n \int_X (2g - |f_n - f|)\, d\mu =$$

$$\int_X 2g\, d\mu$$

$$= \int_X 2g\, d\mu + \liminf_n (-\int_X |f_n - f|\, d\mu) = \int_X 2g\, d\mu - \limsup_n \int_X |f_n - f|\, d\mu$$

Subtracting $\int_X 2g\, d\mu$ from both sides

$$0 \leq -\limsup_n \int_X |f_n - f|\, d\mu \Rightarrow 0 \leq \liminf_n \int_X |f_n - f|\, d\mu \leq \limsup_n \int_X |f_n - f|\, d\mu \leq 0$$

★

Remark 4.7

If $\mu(X) < +\infty$, and $\exists\, M > 0$ s.t. $|f_n| \leq M$ a.e. on X, $\forall\, n$, then we can take $g = M$ as dominating function.

4.15 Comparison between Riemann and Lebesgue integrals

Let $f : I \subset \mathbb{R} \to \mathbb{R}$, I interval, be bounded. Assume also that I is closed and bounded.

Theorem 4.9

Let f be Riemann-integrable on I ($f \in R(I)$). Then

$$f \in \mathcal{L}^1(I, \mathcal{L}(I), \lambda)$$

and

$$\int_I f\, d\lambda = \int_I f(x)\, dx$$

Theorem 4.10
$f \in R(I) \Leftrightarrow f$ is continuous on x, for a.e. $x \in I$.

Ex: $\chi_{\mathbb{Q}}$ on $[0,1]$ is not Riemann integrable, because it is discontinuous at any point. Note that, instead, $\chi_{\mathbb{Q}} = 0$ a.e. on $[0,1] \Rightarrow \int_{[0,1]} \chi_{\mathbb{Q}}\, d\lambda = 0$ Let $f \notin R(I)$. Is it true that $\exists\, g \in R(I)$ s.t. $g = f$ a.e. on I? No.
For instance, consider $T_{\mathcal{E}}$, the generalized Cantor set ($\lambda(T_{\mathcal{E}}) = 0$) and then consider $\chi_{T_{\mathcal{E}}}$.
In general, χ_A is discontinuous on δA. But $T_{\mathcal{E}}$ has no interior parts $\Rightarrow T_{\mathcal{E}} = \delta T_{\mathcal{E}} \Rightarrow \chi_{T_{\mathcal{E}}}$ is discontinuous on $T_{\mathcal{E}}$, which has positive measure $\Rightarrow$ by the last theorem, χ_{T_ε} is not $R(I)$
Clearly

$$\int_{[0,1]} \chi_{T_{\mathcal{E}}} d\lambda = \lambda(T_{\mathcal{E}})$$

so $\chi_{T_{\mathcal{E}}} \in \mathcal{L}^1([0,1])$.
If $g = \chi_{T_{\mathcal{E}}}$ a.e., then g is discontinuous at almost every part of $T_{\mathcal{E}} \Rightarrow g$ is discontinuous on a set of positive measure $\Rightarrow g \notin R(I)$. So, the Lebesgue integral is a true extension of the Riemann one.
Regarding generalized integrals we have

Theorem 4.11
$-\infty \leq a < b \leq +\infty, \quad f \in R^g([a,b])$ where

$$R^g([a,b]) = \{\text{Riemann-int functions on } [a,b] \text{ in the generalized sense}\}$$

Then, f is $([a,b], \mathcal{L}([a,b]))$-measurable. Moreover

(1) $f \geq 0$ on $[a,b] \Rightarrow f \in \mathcal{L}^1([a,b])$

(2) $|f| \in R^g([a,b]) \Rightarrow f \in \mathcal{L}^1([a,b])$

and in both cases

$$\int_{[a,b]} f d\lambda = \int_a^b f(x)dx$$

If f is in $R^g([a,b])$, but $|f| \notin R^g([a,b])$, then the two notions of $\int$ are not really related

Ex: $f(x) = \frac{\sin x}{x}, x \in [1, \infty]$

$$\int_1^\infty |f(x)|dx = +\infty \Rightarrow f \notin \mathcal{L}^1([1,+\infty])$$

But on the other hand

$$\int_1^\infty \frac{\sin x}{x} dx = \lim_{\omega \to \infty} \int_1^\omega \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Proposition 4.3
$(X, \mathcal{M}, \mu)$ complete measure space. Let $\{f_n\} \subseteq \mathcal{L}^1(X, \mathcal{M}, \mu)$.
Suppose that $\sum_{n=1}^\infty \int_X |f_n|\, d\mu < \infty$ Then the series $\sum_{n=1}^\infty f_n$ converges a.e. on X, it is in $\mathcal{L}^1(X)$ and

$$\int_X \left(\sum_{n=1}^\infty f_n \right) d\mu = \sum_{n=1}^\infty \int_X f_n\, d\mu$$

4.16 The spaces $\mathcal{L}^1$ and $\mathcal{L}^\infty$

$(X, \mathcal{M}, \mu)$ complete measure space.

$$\mathcal{L}^1 = \left\{ f : X \to \overline{\mathbb{R}} : \text{ f is integrable} \right\}$$

$\mathcal{L}^1$ is a vector space. On $\mathcal{L}^1$ we can introduce $d : \mathcal{L}^1 \times \mathcal{L}^1 \to [0, +\infty)$ defined by

$$d_1(f,g) = \int_X |f - g|$$

It is immediate to check that

$$d_1(f,g) = d_1(g,f) \qquad \text{(symmetry)}$$

$$d_1(f,g) \leq d_1(f,h) + d_1(h,g) \ \ \forall f,g,h \in \mathcal{L}^1(X) \qquad \text{(triangular inequality)}$$

However, d_1 is not a distance on $\mathcal{L}^1(X)$, since

$$d_1(f,g) = 0 \Rightarrow f = g \quad \text{a.e on } X \qquad \text{(pseudo-distance)}$$

To overcome this problem, we introduce an equivalent relation in $\mathcal{L}^1(X)$: we say that

$$f \sim g \Leftrightarrow f = g \quad \text{a.e. on } X$$

If $f \in \mathcal{L}^1(X)$, we can consider the equivalence class

$$[f] = \left\{ g \in \mathcal{L}^1(X) : g = f \text{ a.e on } X \right\}$$

We define

$$L^1(X) = \frac{\mathcal{L}^1(X)}{\sim} = \{[f] : f \in \mathcal{L}^1(X)\}$$

$L^1(X)$ is a vector space, and on $L^1(X)$ the function d_1 is a distance:

$$d_1([f],[g]) = 0 \Leftrightarrow \int_X |[f] - [g]| \, d\mu = 0 \Leftrightarrow [f] = [g] \text{ a.e.} \ \ \Leftrightarrow f = g \text{ a.e.}$$

To simplify the notations, the elements of $L^1(X)$ are called functions, and one writes $f \in L^1(X)$. With this, we means that we choose a representative in $[f]$, and f denotes both the representative and the equivalence class. The representative can be arbitrarily modified on any set with 0 measure.
Another relevant space of measurable functions is the space of **essentially bounded** functions.

Definition 4.6
$f : X \to \overline{\mathbb{R}}$ measurable is called essentially bounded if $\exists\, M > 0$ s.t.

$$\mu(\{x \in X : |f(x)| \geq M\}) = 0$$

Ex:

$$f(x) = \begin{cases} 1 & x > 0 \\ +\infty & x = 0 \\ 0 & x < 0 \end{cases}$$

For $M > 1$, $\lambda(\{x \in \mathbb{R} : |f(x)| > M\}) = \lambda(\{0\}) = 0 \Rightarrow f$ is essentially bounded.
If f is essentially bounded, it is well defined the **essential supremum** of f.

$$\operatorname*{ess\,sup}_X f := \inf\{M > 0 \text{ s.t. } f \leq M \text{ a.e. on } X\} = \inf\{M > 0 \text{ s.t. } \mu(\{f \geq M\}) = 0\}$$

It can also be defined an essential inf.

Remark 4.8
Note that, by def of inf, $\forall\, \varepsilon > 0$ we have

$$f \leq (\operatorname*{ess\,sup}_X f) + \varepsilon \qquad \text{a.e. on } X$$

We define

$$L^\infty(X, \mathcal{M}, \mu) = \frac{\mathcal{L}^\infty(X, \mathcal{M}, \mu)}{\sim}$$

$L^\infty(X)$ is a vector space, and it is also a metric space for $d_\infty(f, g) = \operatorname*{ess\,sup}_X |f - g|$

5 Types of convergence

5.1 Various types of convergence

$\{f_n\}$ sequence of measurable functions $X \to \overline{\mathbb{R}}$

- $f_n \to f$ pointwise (everywhere) on X if $f_n(x) \overset{n\to\infty}{\to} f(x)\ \forall\, x \in X$
- $f_n \to f$ uniformly on X if $\sup_{x\in X} |f_n(x) - f(x)| \overset{n\to\infty}{\to} 0$
- $f_n \to f$ a.e. on X if

$$\mu\left(\left\{x \in X : \lim_n f_n(x) \neq f(x) \text{ or does not exist}\right\}\right) = 0$$
$$\Updownarrow$$
$$f_n(x) \to f(x) \text{ for a.e } x \in X$$

- <u>Convergence in $L^1(X)$</u>: $f_n \to f$ in $L^1(X)$ if

$$\underbrace{\int_X |f_n - f|\, d\mu}_{= d_1(f_n, f)} \overset{n\to\infty}{\to} 0$$

- <u>Convergence in measure/probability</u>: $f_n \to f$ in measure if $\forall\, \alpha > 0$

$$\lim_{n\to\infty} \mu\left(\{|f_n - f| \geq \alpha\}\right) = 0$$

<u>Basic facts</u>: uniformly convergence $\underset{\not\Leftarrow}{\Rightarrow}$ pointwise $\underset{\not\Leftarrow}{\Rightarrow}$ a.e. convergence.
<u>Ex:</u> $f_n(x) = \exp\{-nx\}, x \in [0, 1]$

$$f(x) = 0, \quad g(x) = \begin{cases} 0 & x \in (0, 1] \\ 1 & x = 0 \end{cases}$$

Then $f_n \to g$ pointwise, $g = f$ a.e. $\Rightarrow f_n \to f$ a.e. on $[0, 1]$. But $f(0) \neq g(0) \Rightarrow f_n \to f$ pointwise.

$f_n \nrightarrow g$ uniformly on $[0, 1]$ $\left|\ \begin{array}{l} f_n \in \mathcal{C}([0, 1]) \\ f_n \to g \Rightarrow g \in \mathcal{C}([0, 1]) \end{array}\right.$

a.e. $\not\Rightarrow$ uniform, but not all is lost...

5.2 Egorov's theorem

Theorem 5.1 (Egorov)
Let $\mu(X) < +\infty$, and suppose that $f_n \to f$ a.e. on X. Then, $\forall\, \varepsilon > 0, \exists\, X_\varepsilon \subset X$, measurable, s.t.

$$\mu(X \setminus X_\varepsilon) < \varepsilon$$

and $f_n \to f$ uniformly on X_ε

Ex: in an example $f_n \to 0$ a.e., $f_n \to 0$ uniformly on $[0,1]$, but $f_n \to 0$ uniformly on $[\varepsilon, 1]$.
Regarding a.e. convergence and in measure convergence there is the following theorem

Theorem 5.2
If $\mu(X) < +\infty$ and $f_n \to f$ a.e. on $X \Rightarrow f_n \to f$ in measure on X

Proof. Let $\alpha > 0$. We want to show that $\forall\, \varepsilon > 0\ \exists\, \bar{n} \in \mathbb{N}$ s.t.

$$n > \bar{n} \Rightarrow \mu(\{|f_n - f| \geq \alpha\}) < \varepsilon$$

$f_n \to f$ a.e. on X, $\mu(X) < +\infty \overset{\text{Egorov}}{\Rightarrow} \exists\, X_\varepsilon \subseteq X$ s.t. $\mu(X \setminus X_\varepsilon) < \varepsilon$ and $f_n \to f$ uniformly on $X_\varepsilon \Leftrightarrow \sup_{X_\varepsilon} |f_n - f| \overset{n\to\infty}{\to} 0$.
In particular, this means that $\exists\, \bar{n} \in \mathbb{N}$ s.t. $n > \bar{n} \Rightarrow |f_n - f| < \alpha$ on X_ε.
Therefore

$$\{|f_n - f| \geq \alpha\} \cap X_\varepsilon = \varnothing \Rightarrow \{|f_n - f| \geq \alpha\} \subseteq X \setminus X_\varepsilon \qquad \text{for } n > \bar{n}$$

By monotonicity of μ, we deduce that

$$\mu\left(\{|f_n - f| \geq \alpha\}\right) \leq \mu(X \setminus X_\varepsilon) < \varepsilon \quad \text{for } n > \bar{n}$$

Namely, $f_n \to f$ in measure. ★

Remark 5.1
$\mu(X) < +\infty$ is essential

For example, in $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ consider

$$f_n(x) = \chi_{[n,n+1)}(x)$$

$f_n(x) \to 0$ for every $x \in \mathbb{R}$. However, $\lambda(\left\{|f_n| \geq \frac{1}{2}\right\}) = \lambda([n, n+1)) = 1$ not 0

5.3 The typewriter sequence

Remark 5.2
Convergence in measure $\Rightarrow$ a.e convergence?
No, not even if $\mu(X) < +\infty$.
Consider $\chi_{n,k} = \chi_{[\frac{k-1}{n}, \frac{k}{n}]}$ with $n \in \mathbb{N}, k = 1, \ldots, n$

$$\begin{array}{lll}
\chi_{1,1}(x) = \chi_{[0,1]}(x) & & \\
\chi_{2,1}(x) = \chi_{[0,\frac{1}{2}]}(x) & \chi_{2,2}(x) = \chi_{[\frac{1}{2},1]}(x) & \\
\chi_{3,1}(x) = \chi_{[0,\frac{1}{3}]}(x) & \chi_{3,2}(x) = \chi_{[\frac{1}{3},\frac{2}{3}]}(x) & \chi_{3,3}(x) = \chi_{[\frac{2}{3},1]}(x)
\end{array}$$

For n fixed and k variable, we move the χ from the left to right. When the χ reaches 1, we switch n, and χ reappear from the left, being thinner.

$$\int_{[0,1]} \chi_{n,k}\, d\lambda = \frac{1}{n} \quad \int_{[0,1]} \chi_{n+1,k}\, d\lambda = \frac{1}{n+1}$$

We can order the elements of $\chi_{n,k}$ in a sequence, letting $f_p = \chi_{n,k}$ for $p = 1+2+\ldots+(n-1)+k$. We will prove that $\{f_p\}$ converges in measure, but not a.e.
This is the **typewriter sequence** ($p(n,k)$). For every $x \in [0,1]$ there are ∞ many indexes s.t. $f_p(x) = 1$ and ∞ many indexes s.t. $f_p(x) = 0$, meaning that $\nexists \lim_{p\to\infty} f_p(x)$ $f_p \nrightarrow 0$ a.e. on $[0,1]$.
But we do have convergence in measure to 0: $\alpha \in (0,1)$

$$\lambda\left(\left\{\left|f_{p(n,k)}\right| \geq \alpha\right\}\right) = \lambda\left(\left[\frac{k-1}{n}, \frac{k}{n}\right]\right) = \frac{1}{n} \to 0 \text{ as } \begin{array}{c} n\to\infty \\ \updownarrow \\ p\to\infty \end{array}$$

Remark 5.3
So, $f_p \nrightarrow 0$ a.e. on $[0,1]$. But consider $\{f_{p(n,1)} : n \in \mathbb{N}\}$. This is a subsequence and, by definition

$$f_{p(n,1)}(x) = \chi_{n,1}(x) = \chi_{[0,\frac{0}{n}]}(x)$$

For this subsequence, we have $f_{p(n,1)}(x) \to 0$ as $n \to \infty$ $\forall x \in (0,1]$, then a.e. on $[0,1]$
This is not random!

Proposition 5.1
If $\mu(X) < \infty$ and $f_n \to f$ in measure, then $\exists$ a subsequence $\{f_{n_k}\}$ s.t. $f_{n_k} \to f$ a.e. on X.

Now we analize the relation between convergence in $L^1(X)$ and the other convergences.

Theorem 5.3
$\{f_n\} \subset L^1(X), f \in L^1(X)$. If $f_n \to f$ in $L^1(X)$ then $f_n \to f$ in measure on X

Proof. By contradiction. Suppose that $f_n \nrightarrow f$ in measure on X: $\exists\, \bar{\alpha} > 0$ s.t.

$$\limsup_{n\to\infty} \mu(\{|f_n - f| \geq \bar{\alpha}\}) > 0$$

$\Rightarrow \exists\, \bar{\varepsilon}$ and a subsequence $\{f_{n_k}\}$ s.t.

$$\mu(\{|f_{n_k} - f| \geq \bar{\alpha}\}) > \bar{\varepsilon}$$

Consider then

$$d_1(f_{n_k}, f) = \int_X |f_{n_k} - f|\, d\mu \overset{\text{monot. } \int}{\geq} \int_{\{|f_{n_k}-f|\geq\bar{\alpha}\}} |f_{n_k} - f|\, d\mu \geq$$
$$\geq \int_{\{|f_{n_k}-f|\geq\bar{\alpha}\}} \bar{\alpha}\, d\mu = \bar{\alpha}\ \mu(\{|f_{n_k} - f| \geq \bar{\alpha}\}) > \bar{\alpha}\ \bar{\varepsilon}$$

But, by assumption, $d_1(f_n, f) \to 0$

$$\Rightarrow d_1(f_{n_k}, f) \to 0$$

Contradiction. ★

Remark 5.4
The convergence in measure doesn't imply the convergence in L^1.
For example, consider

$$f_n(x) = n\chi_{[0,\frac{1}{n}]}(x)$$

$\underbrace{\mu\left(\{|f_n| \geq \alpha\}\right)}_{=\frac{1}{n}} \to 0$ for every α
On the other hand

$$\int_{[0,1]} n\chi_{[0,\frac{1}{n}]}\, d\lambda = \int_{[0,\frac{1}{n}]} n\, d\lambda = n\frac{1}{n} = 1$$

$f_n \nrightarrow 0$ in L^1

Convergence a.e. $\not\Rightarrow$ convergence in L^1:
Use the same example above, $f_n \to 0$ a.e. on $[0,1] \not\Rightarrow f_n \to 0$ in L^1
Convergence in $L^1 \not\Rightarrow$ convergence a.e.:
Consider the typewriter sequence: $d_1(f_{p(n,k)}, 0) \to 0$ when $p \to \infty$
But we don't have a.e. convergence.
However, recall the dominated convergence theorem: (DOM)

$$f_n \to f \text{ a.e. } + \exists \text{ of a dominating function } \Rightarrow d(f_n, f) \to 0$$

It is also possible to show a reverse DOM:
If $f_n \to f$ in $L^1(X)$, then $\exists$ a subsequence $\{f_{n_k}\}$ and $w \in L^1(X)$ s.t.

(1) $f_{n_k} \to f$ a.e. on X

(2) $|f_{n_k}(x)| \leq w(x)$ for a.e. $x \in X$

6 Derivative of a measure

6.1 Radon-Nykodym derivative

$(X, \mathcal{M}, \mu)$ measure space, $\phi : X \to [0, \infty]$ measurable.
We learned that $\nu : \mathcal{M} \to [0, \infty]$ by

$$\nu(E) = \int_E \phi \, d\mu \text{ is a measure on } (X, \mathcal{M})$$

If the equation above holds, then we say that ϕ is the **Radon Nykodym derivative** of ν with respect to μ and we write

$$\phi = \frac{d\nu}{d\mu}$$

Definition 6.1
μ, ν measures on $(X, \mathcal{M})$. We say that ν is absolutely continuous with respect to μ, $\nu << \mu$ if

$$\mu(E) = 0 \Rightarrow \nu(E) = 0$$

Lemma 6.1
There is a necessary condition:

$$\exists \, \frac{d\nu}{d\mu} \Rightarrow \nu << \mu$$

Proof.

$$\nu(E) = \int_E \left(\frac{d\nu}{d\mu}\right) d\mu = 0$$

if $\mu(E) = 0$ by basic properties of $\int$ ★

6.2 Radon-Nykodim theorem

Theorem 6.1 (Radon Nykodim Theorem)
$(X, \mathcal{M})$ measurable space, μ, ν measures.
If $\nu << \mu$ and moreover μ is σ-finite, then $\phi : X \to [0, \infty]$ measurable s.t.

$$\phi = \frac{d\nu}{d\mu} \quad \text{namely } \nu(E) = \int_E \phi \, d\mu \quad \forall \, E \in \mathcal{M}$$

Remark 6.1
If μ is not sigma finite the theorem may fail.
In $([0,1], \mathcal{L}([0,1]))$ consider the counting measure $\mu = \mu_C$ and the Lebesgue measure $\nu = \lambda$
$\nu << \mu$ since $\mu(E) = 0 \Leftrightarrow E = \varnothing \Rightarrow \lambda(E) = \nu(E) = 0$
But we can check that $\nexists\, \phi : [0,1] \to [0,\infty]$ measurable s.t. $\lambda(E) = \int_E \phi \, d\mu_C$

Check by contradiction: assume that ϕ does exist, and take $x_0 \in [0,1]$

$$0 = \lambda(\{x_0\}) = \int_{\{x_0\}} \phi \, d\mu_C = \phi(x_0) \overbrace{\mu_C(\{x_0\})}^{=1} = \phi(x_0)$$

$\Rightarrow \phi(x_0) = 0 \; \forall \, x_0 \in [0,1]$.
But then $1 = \lambda([0,1]) = \int_{[0,1]} 0 \, d\mu_C = 0$. Contradiction
Note that $\mu_C([0,1]) = \infty$ and $([0,1], \mathcal{L}([0,1]), \mu_C)$ is not σ-finite ($[0,1]$ is uncountable)

7 Product measures space

7.1 Construction of product measure spaces

$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ measure spaces. The goal is to define a measure space on $X \times Y$
Definition 7.1
We call **measurable rectangle** in $X \times Y$ a set of type $A \times B$ where $A \in \mathcal{M}, B \in \mathcal{N}$

$$R = \{A \times B \subset X \times Y \text{ s.t. } A \in \mathcal{M}, B \in \mathcal{N}\}$$

We define the product σ-algebra $\mathcal{M} \otimes \mathcal{N}$ as $\sigma_0(R)$.
This is a σ-algebra in $X \times Y$

Definition 7.2
Let $E \subset X \times Y$. For $\bar{x} \in X$ and $\bar{y} \in Y$ we define

$$\begin{array}{ll} E_{\bar{x}} = \{y \in Y : (\bar{x}, y) \in E\} \subseteq Y & \qquad \bar{x}\text{-section of } E \\ E_{\bar{y}} = \{x \in X : (x, \bar{y}) \in E\} \subseteq X & \qquad \bar{y}\text{-section of } E \end{array}$$

Proposition 7.1
$(X, \mathcal{M}), (Y, \mathcal{N})$ measurable spaces. $E \in \mathcal{M} \otimes \mathcal{N}$
Then $E_x \in \mathcal{M}$ and $E_y \in \mathcal{N} \Rightarrow$ we can define

$$\begin{array}{rl} \varphi : & X \to [0,\infty] \\ & x \mapsto \nu(E_x) \end{array} \qquad \begin{array}{rl} \psi : & Y \to [0,\infty] \\ & y \mapsto \mu(E_y) \end{array}$$

Theorem 7.1
If $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ finite spaces, then:

(1) φ is $\mathcal{M}$-measurable and ψ is $\mathcal{N}$-measurable

(2) we have that $\int_X \nu(E_x) \, d\mu = \int_Y \mu(E_y) \, d\nu$

Using the fact that μ and ν are measures, and that $\int$ of non negative function is a measure, we deduce the following

Theorem 7.2 (Iterated integrals for characteristic functions)
$\mu \otimes \nu : \mathcal{M} \otimes \mathcal{N} \to \mathbb{R}$ defined by

$$(\mu \otimes \nu)(E) = \int_X \nu(E_x)\, d\mu = \int_Y \mu(E_y)\, d\nu$$

is a measure, the product measure.

Remark 7.1 (On the complection of product measure spaces)
$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ complete measures spaces.
In general it is not true that $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \otimes \nu)$ is complete.

Example: $X = Y = \mathbb{R}$, $\mathcal{M} = \mathcal{N} = \mathcal{L}(\mathbb{R}), \mu = \nu = \lambda$.
Given A non meas. set , $A \subseteq [0,1], B = \{y_0\}, E = A \times B$. If E were measurable, then its sections must be measurable. But $E_{y_0} = A$ which is not measurable.
However, E is negligible:

$$E \subseteq [0,1] \times \{y_0\}, \text{ and } (\lambda \otimes \lambda)([0,1] \times \{y_0\}) = 0$$

Then $(\lambda \otimes \lambda)$ is not complete

$$\Rightarrow (\mathbb{R}^2, \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R}), \lambda \otimes \lambda) \neq (\mathbb{R}^2, \mathcal{L}(\mathbb{R}^2), \lambda_2)$$

Theorem 7.3
Let λ_n be the Lebesgue measure in $\mathbb{R}^n$. If $n = K + m$, then $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$ is the complection of $(\mathbb{R}^k \times \mathbb{R}^m, \mathcal{L}(\mathbb{R}^k) \otimes \mathcal{L}(\mathbb{R}^m), \lambda_k \otimes \lambda_m)$

Integration on product spaces

$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ measure spaces. $f : X \times Y \to \overline{\mathbb{R}}$ measurable.
If $f \geq 0$, then

$$\iint_{X \times Y} f d\mu \otimes d\nu$$

Goal: obtain a formula of iterated integral like the one in Analysis 2.
$\forall\, \bar{x} \in X$ and $\bar{y} \in Y$, we define

$$f_{\bar{x}} : \begin{array}{l} Y \to \overline{\mathbb{R}} \\ y \mapsto f(\bar{x}, y) \end{array} \qquad f_{\bar{y}} : \begin{array}{l} X \to \overline{\mathbb{R}} \\ x \mapsto f(x, \bar{y}) \end{array}$$

Proposition 7.2
If f is measurable $\Rightarrow$ $f_{\bar{x}}$ is $(\mathcal{N}, \mathcal{B}(\mathbb{R}))$-measurable and $f_{\bar{y}}$ is $(\mathcal{M}, \mathcal{B}(\overline{\mathbb{R}}))$-measurable. Then we can consider

$$\varphi : X \to \overline{\mathbb{R}} \quad \varphi(x) = \int_Y f_x d\nu = \int_Y f(x,y)\, \underbrace{d\nu(y)}_{dy}$$

$$\psi : Y \to \overline{\mathbb{R}} \quad \psi(y) = \int_X f_y d\mu = \int_X f(x,y) d\mu(x)$$

7.2 Tonelli's theorem

Questions: what is the solution of $\iint_{X \times Y}$, φ and ψ?

Theorem 7.4 (Tonelli and Fubini's theorem)
$(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ complete measure spaces and σ-finite.
Suppose that f is $(\mathcal{M} \otimes \mathcal{N}, \mathcal{B}(\overline{\mathbb{R}}))$-measurable and that $f > 0$ a.e. on $X \times Y$. Then ψ and φ are measurable and

$$\iint_{X\times Y} f d\mu \otimes d\nu = \int_X \varphi(x)\, d\mu(x) = \int_Y \psi(y)\, d\nu(y) \qquad \text{Integration formula}$$

Equally holds also if one of the integrals is ∞.

$$\begin{aligned}\int_X \varphi(x)\, d\mu(x) &= \int_X \left(\int_Y f(x,y)\, d\nu(y)\right) d\mu(x)\\ \int_Y \psi(y)\, d\nu(y) &= \int_Y \left(\int_X f(x,y)\, d\mu(x)\right) d\nu(y)\end{aligned}$$

Remark 7.2
The double integral can be reduced to single integrals, iterated. Moreover we can always change the order of the integrals. For sign changing functions the situation is more involved.

Theorem 7.5 (Fubini's theorem)
$(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ complete measure spaces and σ-finite. If $f \in L^1(X \times Y)$, then ψ and φ defined above are measurable, the integration formula holds, and all the integrals are finite.

Question: how to check if $f \in L^1(X \times Y)$? Typically, to check that $f \in L^1(X \times Y)$ one uses Tonelli:

$$f \in L^1(X \times Y) \Leftrightarrow \iint_{X\times Y} |f|\, d\mu \otimes d\nu$$

We use Tonelli to check that this is finite. If $\iint_{X\times Y} |f| d\mu \otimes d\nu < \infty$ then we can apply Fubini for $\iint_{X\times Y} f d\mu \otimes d\nu$
Remark 7.3
the proof of Fubini's and Tonelli's theorems is based for the iterated integrals for characteristic functions. Note that

$$(\mu \otimes \nu)(E) = \begin{aligned}\int_X \varphi(x)\, d\mu(x) &= \int_X \left(\int_Y f(x,y)\, d\nu(y)\right) d\mu(x)\\ \int_Y \psi(y)\, d\nu(y) &= \int_Y \left(\int_X f(x,y)\, d\mu(x)\right) d\nu(y)\end{aligned}$$

Remark 7.4
Sometimes double integrals are very useful to compute single integrals.

Ex: $\int_{-\infty}^{+\infty} e^{-x^2} = \sqrt{\pi}$

8 AC and BV functions

8.1 Lebesgue points

Consider $f \in L^1([a,b])$. We can define the **integral function**

$$F(x) = \int_{[a,b]} f d\lambda = \int_a^b f(t) dt, \quad x \in [a,b]$$

If $f \in \mathcal{C}([a,b])$, then F is differentiable on $[a,b]$, and $F'(x) = f(x)$
What happens if $f \in L^1([a,b])$?

Definition 8.1
Given $f \in L^1([a,b])$. We say that $x \in [a,b]$ is a **Lebesgue point** for f if

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\, dt = 0$$

If $x = a$ or $x = b$, this is the left/right lim.

Remark 8.1
A point x is called a Lebesgue point for f if f 'does not oscillate too much' close to x:

- $f\ \mathcal{C}([a,b]) \to$ every $x \in [a,b]$ is a Lebesgue point.
-
$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$
$$\lim_{h\to 0} \frac{1}{h} \int_0^h |f(t) - f(0)|\, dt = \lim_{h\to 0} \frac{1}{|h|} \int_0^h |0 - 1|\, dt = 0$$

Theorem 8.1 (Lebesgue)
If $f \in L^1([a.b])$ then a.e. $x \in [a,b]$ is a Lebesgue point for f

Remark 8.2
In the definition of Lebesgue point, the pointwise values of f are relevant

$$f = g \in L^1 \Leftrightarrow f = g \text{ a.e.}$$

Then the Lebesgue point of f could be different from the one of g. This is not a big problem if $f = g$ a.e. on $[a,b] \Rightarrow f = g \in [a,b] \setminus N$ where $\lambda(N) = 0$; x is a Lebesgue point for f, $\forall\, x \in [a,b] \setminus M,\ \lambda(M) = 0$
$\Rightarrow x$ is a Lebesgue point for $g,\ \ \forall x \in [a,b] \setminus (M \cup N)$
$[a,b] \setminus (M \cup N)$ is a set of full measure of Lebesgue points for f and g.

To speak about Lebesgue points, one has to choose a specific representative $f \in L^1([a,b])$. If you change representative, you obtain the same set of Lebesgue points up to sets with 0-measure.

8.2 First fundamental theorem of calculus

Theorem 8.2 (First fundamental theorem of calculus)
Given $f \in L^1([a,b]),\ F(x) = \int_a^x f(t)\, dt$
Then f is differentiable a.e. on $[a,b]$ and $F'(x) = f(x)$ a.e. in $[a,b]$

Proof. Let $x \in [a,b]$ for any Lebesgue point for f (a.e. $x \in [a,b]$ is fine). Consider

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| = \left| \frac{1}{h} \int_x^{x+h} (f(t) - f(x))\, dt \right| \leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\, dt \to 0$$

Since x is a Lebesgue point. ★

Definition 8.2
Given $f : I \to \mathbb{R}$ is called **absolutely continuous** in I, $f \in AC(I)$, if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t.

$$\bigcup_{k=1}^{n} [a_k, b_k] \subseteq I \text{ with disjoint interiors}$$

$$\lambda(\bigcup_{k=1}^{n}[a_k,b_k]) = \sum_{k=1}^{n}(b_k - a_k) < \delta$$
$$\Rightarrow \sum_{k=1}^{n}|f(b_k) - f(a_k)| < \varepsilon$$

Remark 8.3
f is uniformly continuous on $[a,b]$ if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t.

$$|t - \tau| < \delta \Rightarrow |f(t) - f(\tau)| < \varepsilon$$

An absolutely continuous function is also uniformly continuous.
But the converse is false.

- If f is Lipschitz on $[a,b] \Rightarrow f \in \mathrm{AC}([a,b])$

Recall that $f \in \mathrm{Lip}([a,b])$ if $\exists\, L > 0$ s.t.

$$|f(x) - f(y)| \leq L|x - y| \qquad \forall x, y \in [a,b]$$

Check: For any $\varepsilon > 0$, and consider

$$\sum_{k=1}^{n}|f(b_k) - f(a_k)| \leq \sum_{k=1}^{n}L(b_k - a_k) = L\sum_{k=1}^{n}(b_k - a_k)$$

If we take $\delta = \delta(\varepsilon) = \frac{\varepsilon}{L}$, then

$$\sum_{k=1}^{n}(b_x, a_x) < \delta \Rightarrow \sum_{k=1}^{n}|f(b_k) - f(a_k)| \leq L\sum_{k=1}^{n}(b_k - a_k)$$

★

$$\mathrm{Lip}([a,b]) \subsetneq \mathrm{AC}([a,b]) \subsetneq \mathrm{UC}([a,b])$$

Theorem 8.3 (Regularity of integral functions)
Given $f \in L^1([a,b]), F(x) = \int_a^x f(t)\,dt$, then $F \in \mathrm{AC}([a,b])$

To prove the theorem we need the
Theorem 8.4 (Absolute continuity of the integral)
Given $f \in L^1(X, \mathcal{M}, \mu)$. Then $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t.

$$\begin{matrix} E \in \mathcal{M} \\ \mu(E) < \delta \end{matrix} \Rightarrow \int_E |f|\,d\mu < \varepsilon$$

Proof. We fix $\varepsilon > 0$. Let $F_n := \{|f| < n\}$, $n \in \mathbb{N}$. Also $F_n \in \mathcal{M}\ \forall\, n$, $F_n \subseteq F_{n+1}$ and

$$\bigcup_{n=1}^{\infty} F_n = \{|f| < \infty\} =: F$$

$f \in L^1 \Rightarrow |f|$ is finite a.e.: $\mu(X \setminus F) = 0$. Therefore:

$$\int_X |f|\,d\mu = \int_{X\setminus F}|f|\,d\mu + \int_F |f|\,d\mu = \lim_{n\to\infty}\int_{F_n}|f|\,d\mu$$

$$\lim_{n\to\infty} \int_X |f| \left(\chi_{F_n^C}\right) \, d\mu = 0$$

$\forall\, \varepsilon > 0 \; \exists\, \bar{n} \in \mathbb{N}$ s.t.

$$n > \bar{n} \Rightarrow \left| \int_X |f| \chi_{F_n^C} \, d\mu \right| < \frac{\varepsilon}{2}$$

Now, fix $\varepsilon > 0$, and take $n > \bar{n}(\varepsilon)$. If $E \in \mathcal{M}$, then

$$\int_E |f| \, d\mu = \int_{E \cap F_n} |f| \, d\mu + \int_{E \cap F_n^C} |f| \, d\mu \leq n \int_E 1 \, d\mu + \int_{F_n^C} |f| \, d\mu$$

If we suppose that $\mu(E) < \frac{\varepsilon}{2n} =: \delta(\varepsilon)$, we deduce that

$$n \int_E 1 \, d\mu = n\mu(E) < \frac{\varepsilon}{2}$$

Also, since $n > \bar{n}$

$$\int_{F_n^C} |f| \, d\mu < \frac{\varepsilon}{2}$$

$$\Rightarrow \int_E |f| \, d\mu < \varepsilon$$

★

Proof - Regularity of integral functions. Let $\varepsilon > 0$, and $\delta = \delta(\varepsilon) > 0$ be the value given by the absolute continuity of $\int |f| \, d\mu$. Take

$$E = \bigcup_{k=1}^{n} [a_k, b_k] \qquad E \subseteq [a, b]$$

If $\lambda(E) < \delta$, then

$$\sum_{k=1}^{n} |F(b_k) - F(a_k)| = \sum_{k=1}^{n} \left| \int_{a_k}^{b_k} f(t) \, dt \right| \leq \sum_{k=1}^{n} \int_{a_k}^{b_k} |f(t)| \, dt = \int_E |f| \, d\lambda < \varepsilon$$

by absolute continuity of $\int$ ★

Remark 8.4
$\sqrt{x}$ is AC($[0,1]$), but is not Lip($[0,1]$).

$$\sqrt{x} = \int_0^x \frac{1}{2\sqrt{t}} \, dt$$

$\Rightarrow \sqrt{x}$ is the $\int$ function of a L^1 function
$\Rightarrow \sqrt{x} \in$ AC($[0,1]$)

To sum up: the $\int$ function of a L^1 function is AC, it is differentiable a.e., and

$$F(x) - F(a) = \int_a^x F'(t) \, dt \qquad \text{FC}$$

Suppose G is differentiable a.e. on $[a, b]$ and FC holds for G:

$$G(x) - G(a) = \int_a^x G'(t) \, dt$$

What can we say about G?

Remark 8.5
If $G \in \mathcal{C}^1([a,b]) \Rightarrow$ FC holds.
If FC holds, then $G' \in L^1([a,b])$ (necessary condition). Is the necessary condition also sufficient? In general not. Take $v(x)$, the Vital Cantor function: $v \in \mathcal{C}([0,1]), v(0) = 0, v(1) = 1$. v is differentiable a.e. on $[0,1]$ but the calculus formula doesn't hold!

Remark 8.6
A function which is differentiable a.e. on an interval can behave very badly

Theorem 8.5
$G \in \mathrm{AC}([a,b])$. Then G is differentiable a.e. on $[a,b]$, $G' \in L^1([a,b])$, and FC holds.

Remark 8.7
These theorems say that AC function are precisely the ones for which FC holds:

- $G \in \mathrm{AC} \Rightarrow$ FC holds.
- If FC holds, then $G' \in L^1([a,b])$

$\Rightarrow \int_a^x G'(t)\,dt \in \mathrm{AC}$
$\Rightarrow G(x) - G(a) = \int_a^x G'(t)\,dt \in \mathrm{AC}$

Remark 8.8
$v \in \mathrm{UC}([0,1])$ by continuity and Heine Cantor, but $v \notin \mathrm{AC}([0,1])$ because FC does not hold.

The proof of the second fundamental theorem of calculus is divided into two steps.
Lemma 8.1
The second fundamental theorem hold under the additional assumption that G is monotone.

Second step: to get rid of the monotonicity.
For step 2, is it useful to give the
Definition 8.3
$[a,b] \subset \mathbb{R}$. Let

$$\mathcal{P}_{[a,b]} := \{(x_0, x_1, \ldots, x_n) : n \in \mathbb{N} \text{ and } a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

For $P \in \mathcal{P}_{[a,b]}$ and $f : [a,b] \to \overline{\mathbb{R}}$, define

$$v_a^b(f, P) := \sum_{k=1}^n |f(x_k) - f(x_{k-1})|$$

The total variation of f on $[a,b]$ is

$$V_a^b(f) := \sup_{P \in \mathcal{P}_{[a,b]}} v_a^b(f,P) = \sup\{\sum_{k=1}^n |f(x_k) - f(x_{k-1})| : n \in \mathbb{N}, a = x_0 < x_1 < \cdots < x_n = b\}$$

If $V_a^b(f) < \infty$, we say that f is a function with **bounded variation**, $f \in \mathrm{BV}\ ([a,b])$
Theorem 8.6 (The 2^{nd} fundamental theorem of calculus.)
$G \in \mathrm{AC}([a,b]) \Leftrightarrow G$ is differentiable a.e. on $[a,b]$, $G' \in L^1([a,b])$, and (FC) holds.

Example and comments:

- If f is bounded and monotone $\Rightarrow f \in \mathrm{BV}$
$$V_a^b(f) = |f(b) - f(a)|$$
Note that f may not be continuous
$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \Rightarrow f \in \mathrm{BV}([-1,1])$$

- $f \in \mathrm{BV}([a,b]) \Rightarrow f$ is bounded. Indeed
$$\sup_{x\in[a,b]} |f(x)| \leq |f(x)| + V_a^b(f) \overset{f\in\mathrm{BV}}{<} +\infty$$

- f is continuous on $[a,b]$, or even if f is differentiable everywhere in $[a,b] \not\Rightarrow f \in \mathrm{BV}([a,b])$
$$f(x) = \begin{cases} x^2 \cos \frac{2\pi}{x^2} & x \in (0,1] \\ 0 & x = 0 \end{cases}$$
It is continuous in $[0,1]$, but $f \notin \mathrm{BV}([0,1])$

- $f \in \mathrm{BV}([a,b]) \cap \mathrm{UC}([a,b]) \not\Rightarrow f \in \mathrm{AC}([a,b])$
$$\begin{aligned} &v \text{ a Vitali-Cantor function} \\ &v \text{ is bounded and monotone } \Rightarrow v \in \mathrm{BV}([0,1]) \\ &v \in \mathrm{UC}([0,1]) \end{aligned}$$
But $v \notin \mathrm{AC}([0,1])$

- If $f \in \mathrm{BV}([a,b]) \Rightarrow f$ is differentiable a.e. on $[a,b]$, and $f' \in L^1([a,b])$

We can now come back to the proof of Lemma 1 of the last lesson.
Preliminary result: $A \in \mathbb{R}$ open. Then
$$A = \bigcup_{n=1}^{\infty} (a_n, b_n) \text{ disjoint}$$
any open set of $\mathbb{R}$ is the (at most) countable union of open disjoint intervals.
Preliminary result (equivalent definition for AC): $f \in \mathrm{AC}([a,b]) \Leftrightarrow \forall\, \varepsilon > 0 \exists\, \delta > 0$ depending on ε s.t.
$$\forall \bigcup_{n=1}^{\infty} [a_n, b_n], \quad [a_n, b_n] \text{ have disjoint interiors}$$
$$\sum_{n=1}^{\infty} (b_n - a_n) < \delta \Rightarrow \sum_{n=1}^{\infty} |f(b_n) - f(a_n)| < \varepsilon$$

Proof. We defined λ starting from two properties

- invariance under translations
- $\lambda((x,y)) = y - x \quad \forall\, a \leq y \leq b$

Now, G is monotone, say G increasing (if $G \searrow$, take $-G$). We can repeat the construction of λ in order to obtain a measure μ s.t.

- μ is invariant under translations
- $\mu((x,y)) = \underbrace{G(y) - G(x)}_{\geq 0}\ \forall\, a \leq x < y \leq b$ (for λ, take $G(t) = t$)

It can be proved that we obtain a measure on $(\mathbb{R}, \mathcal{L}(\mathbb{R}))$, complete.
On $(\mathbb{R}, \mathcal{L}(\mathbb{R}))$ we have two measures: λ and μ.
Idea: We take these measures on $([a,b], \mathcal{L}([a,b]))$, and we want to show that $\exists\ \frac{d\mu}{d\Lambda}$ (Radon-Nikodym)
We can check the hypothesis of the Radon-Nikodym theorem:

- λ is σ-finite: $\lambda([a,b]) = b - a < +\infty$
- $\mu << \lambda$: $E \in \mathcal{L}([a,b])$, $\lambda(E) = 0 \Rightarrow \mu(E) = 0$

Assume $\lambda(E) = 0$. G is AC($[a,b]$): then $\forall\, \varepsilon > 0\ \exists\, \delta = \delta(\varepsilon) > 0$ s.t.

$$\forall \bigcup_{n=1}^{\infty}[a_n, b_n], \quad [a_n, b_n] \text{ have disjoint interiors}$$

$$\lambda\left(\bigcup_{n=1}^{\infty}[a_n, b_n]\right) < \delta \Rightarrow \sum_{n=1}^{\infty}|G(b_n) - G(a_n)| < \varepsilon$$

Take this δ. By regularity of λ, $\exists\, A$ open set of $[a,b]$s.t. $A \supset E$ and $\lambda(A) < \delta$

$$A \text{ is open} \Rightarrow A = \left(\bigcup_{n=1}^{\infty} I_n{}^{\dagger}\right), \text{ disjoint}$$

it is a countable union of open intervals (maybe two of them contains a or b)

$$\lambda(A) < \delta \Leftrightarrow \sum_{n=1}^{\infty}(y_n - x_n) < \delta$$

But then, since μ is a measure it is countably additive

$$\mu(E) \leq \mu(A) = \sum_n \mu(I_n) = \sum_n G(y_n) - G(x_n) < \varepsilon$$

by the choice of δ and the fact that $G \in$ AC. We proved that

$$\lambda(E) = 0 \Rightarrow \forall\, \varepsilon > 0 :\ \mu(E) < \varepsilon \Rightarrow \mu(E) = 0$$

So $\mu << \lambda$. We can apply Radon Nikodym $\exists\, \phi : [a,b] \to [0, \infty]$ s.t.

$$G(x) - G(a) = \int_a^x \phi\, d\lambda$$

Since G is bounded, then $\phi \in L^1([a,b])$

$$G(x) = G(a) + \int_a^x \phi(t)\, dt$$

By the first fundamental theorem of calculus, this is differentiable a.e.

$$\Rightarrow G'(x) = \phi(x) \text{ a.e. on } [a,b]$$

$$\Rightarrow G'(x) = G(a) + \int_a^x G'(t)\, dt$$

★

$\dagger$ open intervals $= (x_n, y_n)$

Now we want to get rid of the additional assumption (monotonicity).
Preliminary result: $f \in \mathrm{BV}([a,b])$. Then

$$\varphi(x) = V_a^x(f), \quad \forall\, x \in [a,b]$$

is an increasing function.

Proof. By $a \leq x < y \leq b$. Then

$$V_a^y(f) = V_a^x(f) + \underbrace{V_x^y(f)}_{\geq 0} \geq V_a^x(f)$$

★

Preliminary result: If $G \in \mathrm{AC}([a,b])$, then $G \in \mathrm{BV}([a,b])$, and moreover

$$\varphi(x) = V_a^x(G) \text{ is in } \mathrm{AC}([a,b])$$

Proof of the second fundamental theorem of calculus in the general case. $G \in \mathrm{AC}([a,b])$
We want to write $G = G_1 + G_2$ where $G_1 \nearrow$ and $G_2 \searrow$, both AC.
Then the second fundamental theorem holds for G_1 and G_2 so it holds for G by linearity of the integral.
We pose:

$$G_1(x) = \frac{G(x) + V_a^x(G)}{2}$$
$$G_2(x) = \frac{G(x) - V_a^x(G)}{2}$$

Clearly, $G_1 + G_2 = G$, G_1, G_2 are AC, by the last preliminary result.
$G_1 \nearrow$: Let $a \leq x < y \leq b$

$$|G(y) - G(x)| \leq V_x^y(G)$$

Therefore,

$$G_1(y) - G_1(x) = \frac{1}{2}(\underbrace{G(y) - G(x)}_{\substack{\geq -|G(y)-G(x)| \\ \geq -V_x^y(G)}} + V_a^y(G) + V_a^x(G)) \geq \frac{1}{2}(-V_x^y(G) + V_x^y(G)) = 0$$

So G_1 is decreasing. In an analogue way, we can prove that G_2 is decreasing. ★

Functional Analysis

9 Review on metric spaces

Normed spaces and Banach spaces

Definition 9.1

Given X vector space, a norm on X is a function $\|\cdot\| : X \to [0,\infty)$ s.t.

- $\|x\| = 0 \Leftrightarrow x = 0$
- $\forall\, \alpha \in \mathbb{R}, \forall\, x \in X$:
$$\|\alpha x\| = |\alpha|\|x\| \qquad \text{(positive homogeneity)}$$
- $\forall\, x, y \in X$:
$$\|x + y\| \leq \|x\| + \|y\| \qquad \text{(triangle inequality)}$$

Then, $(X, \|\cdot\|)$ is called a **normed space**

Ex: $|\|x\| - \|y\|| \leq \|x - y\| \;\forall\, x, y \in X$

Proposition 9.1

$(X, \|\cdot\|)$ normed space. Then (X, d) is a metric space for

$$d(x, y) = \|x - y\|$$

Remark 9.1

Normed space $\underset{\not\leftarrow}{\to}$ metric space

Examples:

- $\mathbb{R}^N$
$$\|x\|_p := (\sum_{i=1}^{N} |x_i|^p)^{\frac{1}{p}} \;\; \forall p \in [1, +\infty) \quad \|x\|_\infty := \max_{i=1,\dots,N} |x_i|$$
- $\mathcal{C}^0([a, b])$
$$\|f\|_\infty := \max_{x\in[a,b]} |f(x)|$$
- $L^1(X, \mathcal{M}, \mu)$
$$\|f\|_1 := \int_X |f|\, d\mu$$
This is a norm in L^1, but not on $\mathcal{L}^1$ ($\int_x |f|\, d\mu = 0 \Rightarrow f = 0$ a.e.)
- $L^\infty(X, \mathcal{M}, \mu)$
$$\|f\|_\infty := \operatorname*{ess\,sup}_{[a,b]} |f|$$

$(X, \|\cdot\|)$ normed space $\to$ (X, d) metric space $\to$ convergent sequences on X: $\{x_n\} \subset X$ is convergent in X iff

$$d(x_n, x) \to 0 \Leftrightarrow \|x_n - x\| \to 0 \text{ as } n \to \infty$$

Ex: $x_n \to x$ in X, then $\|x_n\| \to \|x\|$ (the norm is a continuous function on X)

Definition 9.2

$\{x_n\}$ is a **Cauchy sequence** in $(X, \|\cdot\|)$ if $\forall\, \varepsilon > 0\; \exists\, \bar{n} \in \mathbb{N}$ s.t.

$$n, m > \bar{n} \Rightarrow \|x_n - x_m\| < \varepsilon$$

Banach Spaces

Definition 9.3
$(X, \|\cdot\|)$ is called a **Banach space** if (X, d) is complete, namely if any Cauchy sequence in (X, d) is convergent.

If $(X, \|\cdot\|)$ is a normed space, we can speak about series in X. Let $\{x_n\} \subset X$ and $s_n = x_0 + x_1 + \ldots + x_n$, then $\sum_{n=0}^{+\infty} x_n = \{s_n\}$.
Then $\sum x_n$ is convergent if $\{s_n\}$ is convergent. If $\sum x_n$ is convergent, we write

$$s = \sum_{n=0}^{+\infty} x_n \Leftrightarrow s_n \to s$$

For numerical series

$$\sum_{n=1}^{\infty} |a_n| < +\infty \Rightarrow \sum a_n \text{ is convergent}$$

In general, in normed spaces

$$\sum_{n=1}^{\infty} \|x_n\| < +\infty \not\Rightarrow \sum_{n=1}^{\infty} x_n \text{ is convergent}$$

Characterization

$(X, \|\cdot\|)$ is a Banach space $\Leftrightarrow$ every series s.t. $\sum \|x_n\| < +\infty$ is also s.t. $\sum x_n$ is convergent.
$(X, \|\cdot\|) \to (X, d) \to$ open sets, closed sets, bounded sets....
In $\mathbb{R}^n$ we are used to work with $\|\cdot\|_2$, but we could have many different norms.

Definition 9.4
Let $\|\cdot\|$ and $\|\cdot\|_2$ be two norms on the same vector space X. We say that these norms are **equivalent** if $\exists\, m, M > 0$ s.t.

$$m\|x\| \leq \|x\|_2 \leq M\|x\| \quad \forall\, x \in X$$

It can be proved that if two norms are equivalent they lead to different metric spaces, but to the same open sets, closed sets, convergent sequences, compact sets ...

Theorem 9.1
If X is any finite dimension vector space, then all the norms on X are equivalent.

Remark 9.2
This is why in $\mathbb{R}^n$ usually one does not specify the choice of the norm. One choose the Euclidean norm, since it comes from a scalar product. (ref. Hilbert spaces)

Preliminary fact: The set $S_1 = \{s \in \mathbb{R}^n : \|s\|_1 = 1\}$ is compact in $(\mathbb{R}^n, d)$

Proof. We show that any norm is equivalent to $\|\cdot\|_1 = \sum_{i=1}^n |x_i|$

$$x = \sum_{i=1}^n x_i e_i \qquad \{e_i\}_{i=1,\ldots,n} \text{ canonical basis}$$

Let's introduce the norm star

$$\|x\|_* = \left\| \sum_{i=1}^n x_i e_i \right\|_* \leq \sum_{i=1}^n \|x_i e_i\|_* = \sum_{i=1}^n |x_i| \|e_i\|_* \leq \left(\max_{1 \leq i \leq n} \|e_i\|_* \right) \sum_{i=1}^n |x_i| = M\|x\|_1$$

We proved that $\exists\, M > 0$ s.t.

$$\|x\|_* \le M\|x\|_1 \quad \forall\, x \in X \tag{1}$$

Note that this proves that $\varphi(x) = \|x\|_*$ is continuous in (X, d). Indeed

$$x_n \to x \Leftrightarrow d_1(x_n, x) \to 0$$

then

$$|\varphi(x_n) - \varphi(x)| = |\|x_n\|_* - \|x\|| \le \|x_n - x\|_* \overset{(1)}{\le} M\|x_n - x\|_1 \to 0$$

Therefore, by the Weierstrass theorem, $\exists$ a minimum point $x_0 \in S_1$ s.t.

$$\varphi(x) \ge \varphi(x_0) = m \quad \forall\, x \in S_1$$

(recall that S_1 is compact)

$$\|x\|_* \ge m \quad \forall\, x \in S_1$$

We claim that $m > 0$. If $m = 0$ then $\|x_0\|_* = 0 \Rightarrow x_0 = 0$ that is impossible, since $x_0 \in S_1$. Thus $m > 0$. Let now $y \in \mathbb{R}^n, y \neq 0$. Then

$$\frac{y}{\|y\|_1} \in S_1 \Rightarrow \left\| \frac{y}{\|y\|_1} \right\|_* \ge m \Rightarrow \frac{1}{\|y\|_1}\|y\|_* \ge m \Rightarrow \|y\|_* \ge m\|y\|_1 \quad \forall\, y \in \mathbb{R}^n$$

★

If $\dim X = +\infty$, then there are many non-equivalent norms.
Ex: In $\mathcal{C}^0([a, b])$, we can define $\|\cdot\|_\infty$ and $\|f\|_1 = \int_a^b |f(t)|\, dt$.
This is a norm in $\mathcal{C}^0$, but these norms are not equivalent.

Separability

(X, d) metric space.

Definition 9.5
We say that X is separable if $\exists\, A \subset X$ which is dense ($\bar{A} = X$) and countable

In $\mathbb{R}^n$, $\mathbb{Q}^n$ which is dense and countable. In $\infty-\dim$ we can have separable spaces or not. For instance, $(L^\infty, \|\cdot\|_\infty)$ is not separable. Instead $(\mathcal{C}^0([a, b]), \|\cdot\|_\infty)$ is a separable space.

Sketch of the proof. We will use the **Stone-Weierstrass theorem**.
The set of polynomials is dense on $\mathcal{C}^0([a, b])$ and is an uncountable set. However it can be proved that the set of polynomials with coefficients in $\mathbb{Q}$ is dense in the set of all polynomials Moreover this set is countable. Then, by Stone-Weierstrass this is a countable dense set in $\mathcal{C}^0([a, b])$ ★

Remark 9.3
One can show that $\mathcal{C}^0(K)$ is separable whenever K is a compact set of a metric space (X, d)

Compactness

In finite dimension (in $\mathbb{R}^n$), one has that

$$E \subset X \text{ is compact } \Leftrightarrow E \text{ is closed and bounded}$$

If $\dim X = \infty$, then only '$\Rightarrow$' is true. In finite dimension, we know that the closed unit ball is compact

$$\bar{B}_1(0) = \{x \in \mathbb{R}^n : \|x\| \le 1\}$$

What happens now if $(X, \|\cdot\|)$ is on $\infty-\dim$ normed space?

Theorem 9.2 (Riesz's theorem)
X normed space, $\dim X = +\infty \Rightarrow \overline{B_1(0)}$ is not compact

Remark 9.4
It is well known that if $E \in \mathbb{R}^n$ is compact, then $\forall\ \{x_n\} \in E\ \exists\ \{x_{n_k}\}$ subsequence s.t. $x_{n_k} \to x \in E$. This proposition is much harder to prove in $\infty - \dim$.

The proof of the Riesz's theorem is based on the Riesz's **quasi-orthogonality lemma**.
Lemma 9.1 (Riesz Quasi-Orthogonality Lemma)
Let X be a normed space, $E \subsetneq X$ a closed subspace. Then $\forall\, \varepsilon \in (0,1)\ \exists\, x \in X$ s.t.

$$\|x\| = 1 \text{ and } \operatorname{dist}(x,E) = \inf_{y \in E} \|x-y\| \geq 1-\varepsilon$$

Proof. Of the Riesz's theorem. Assume that $\overline{B_1(0)}$ is compact, and X has infinite dimension. $\exists$ a sequence $\{E_n\}$ of finite dimensional subspaces (hence closed) of X s.t.

$$E_{n-1} \subset E_n \text{ and } E_{n-1} \neq E_n$$

E_{n-1} is a proper closed subspace of $E_n\ \forall\, n$
We can apply the Riesz Lemma with $X = E_n$, $E = E_{n-1}$, $\varepsilon = \frac{1}{2}$. Then $\forall\, n\ \exists\, u_n \in E_n$ s.t. $\|u_n\| = 1$ and $\operatorname{dist}(u_n, E_{n-1}) \geq \frac{1}{2}\quad \forall\, n$
Therefore, we have a sequence $\{u_n\}$ with the following properties

$$\|u_n\| = 1 \qquad \forall\, n$$

$$\|u_n - u_m\| \geq \frac{1}{2} \qquad \forall\, n \neq m$$

$\Rightarrow$ this sequence cannot have any convergent subsequence. But then $\overline{B_1(0)} \supseteq \{u_n\}$, this implies that $\overline{B_1(0)}$ is not compact. Contradiction.
(In any $(X, \|\cdot\|)$ normed space, if E is compact, then $\forall\, \{x_n\} \subset E\ \exists\, \{x_{n_k}\}$ s.t. $x_{n_k} \to x \in E$) ★

(X,d) metric space.
Definition 9.6
$E \subset X$ is compact if for any open covering $\{A_i\}_{i \in I}$ has a finite subcover.

Definition 9.7
$E \subset X$ is sequentially compact if $\forall\, \{x_n\} \subset E$ there exists $\{x_{n_k}\}$ subsequence convergent to some limit $x \in E$

Well known fact: if (X,d) is a metric space, then E is compact $\Leftrightarrow E$ is sequentially compact.
Theorem 9.3 (Riesz Theorem)
X normed space, $\dim X = \infty \Leftrightarrow \overline{B_1(0)}$ is not compact.

Lemma 9.2 (Riesz quasi orthogonality Lemma)
X normed space, $E \subsetneq X$ closed subspace. Then $\forall\, \varepsilon \in (0,1)\ \exists\, x \in X$ s.t.

$$\|x\| = 1 \text{ and } \operatorname{dist}(x,E) = \inf_{y \in E} \|x-y\| \geq 1-\varepsilon$$

Remark 9.5
Also:

- $E \in X$ closed. Then $\operatorname{dist}(x,E) = 0 \Leftrightarrow x \in E$

- By definition of infimum, if $d = \operatorname{dist}(x, E)$, then $\forall\, \rho > 0\ \exists\, z \in E$ s.t.

$$\|x - z\| < (1+\rho)d$$

Proof. Let $y \in X \setminus E$, and $d := \operatorname{dist}(y, E) > 0$, since E is closed.
$\forall\, \rho > 0\ \exists\, z \in E$ s.t.

$$\|y - z\| \leq (1+\rho)d = \frac{d}{1-\varepsilon} \tag{1}$$

since we choose ρ s.t. $1+\rho = \frac{1}{1-\varepsilon}$. Now we set $x = \frac{y-z}{\|y-z\|}$.
Clearly $\|x\| = 1$. Moreover, $\forall\, u \in E$, we have that

$$\|x - u\| = \left\| \frac{y-z}{\|y-z\|} - u \right\| = \left\| \frac{y - z - \|y-z\|u}{\|y-z\|} \right\| = \frac{1}{\|y-z\|} \|y - (z + \|y-z\|u)\| =$$

$$= \frac{1}{\|y-z\|}\|y - w\| \geq \frac{1}{\|y-z\|}\operatorname{dist}(y, E) \overset{(1)}{\geq} \frac{1-\varepsilon}{d} d = 1 - \varepsilon$$

Since this is true $\forall\, u \in E$, we deduce that

$$\operatorname{dist}(x, E) \geq 1 - \varepsilon$$

★

10 The space $\mathcal{C}^0$

Definition 10.1
$\{f_n\}$ sequence in $\mathcal{C}^0([a,b])$. We say that $\{f_n\}$ is **uniformly equicontinuous** in $[a,b]$ if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ depending only on ε s.t.

$$|t - \tau| < \delta \Rightarrow \|f_n(t) - f_n(\tau)\| < \varepsilon \qquad \forall\, n$$

Remark 10.1
With respect to the uniform continuity, in this case δ does not depend on f. δ is the same for all the f_n s

Theorem 10.1 (Ascolì Arzelà)
$\{f_n\} \subseteq \mathcal{C}^0([a,b])$. Suppose that:

- $\{f_n\}$ is uniformly equi-continuous
- $\{f_n\}$ is equi-bounded: $\exists\, M > 0$ s.t. $\|f_n\|_\infty < M \qquad \forall\, n$

Then $\exists$ a subsequence $\{f_{n_k}\}$ and $f \in \mathcal{C}^0([a,b])$ s.t. $f_{n_k} \to f$ uniformly.

11 Lebesgue spaces

$(X, \mathcal{M}, \mu)$ measure space, $p \in [1, \infty]$. We defined $L^1(X)$ and $L^\infty(X)$. In a similar way, we define $L^p(X)\ \forall\, p \in [1, \infty]$

$$\mathcal{L}^p(X, \mathcal{M}, \mu) := \{f : X \to \overline{\mathbb{R}} \text{ measurable s.t. } \int_X |f|^p \, d\mu < \infty\}$$

On $\mathcal{L}^p$ we introduce the equivalent relation

$$f \sim g \text{ in } \mathcal{L}^p \Leftrightarrow f = g \text{ a.e. on } X$$

and define

$$L^p(X, \mathcal{M}, \mu) := \frac{\mathcal{L}^p(X, \mathcal{M}, \mu)}{\sim}$$

We want to show that this is a normed space with

$$\|f\|_p := \begin{cases} \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} & p \in [1, \infty) \\ \operatorname*{ess\,sup}_X |f| & p = \infty \end{cases}$$

The fact that L^p is a vector space is easy to prove. The only non trivial part is that $f, g \in L^p \Rightarrow f + g \in L^p$.
This comes directly from the

Lemma 11.1
$p \in [1, \infty),\ a, b \geq 0$. Then

$$(a+b)^p \leq 2^{p-1}\left(a^p + b^p\right)$$

$f, g \in L^p,\ p \in [1, \infty)$

$$\int_X |f+g|^p \, d\mu \leq \int_X (|f| + |g|)^p \, d\mu \leq 2^{p-1} \int_X (|f|^p + |g|^p) \, d\mu$$

$$= 2^{p-1} \int_X |f|^p \, d\mu + 2^{p-1} \int_X |g|^p \, d\mu < \infty$$

L^p is a vector space, $\forall\, p \in [1, \infty)$.
$f, g \in L^\infty$. Then a.e.

$$\Rightarrow |f+g| \leq |f| + |g| \leq \|f\|_\infty + \|g\|_\infty < \infty \Rightarrow f + g \in L^\infty$$

L^∞ is a vector space.

Remark 11.1
$l^p := L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_c)$. l^p is a particular case of L^p

$$\begin{aligned} l^p &= \{x = \left(x^{(k)}\right)_{k \in \mathbb{N}} : \sum_{k=1}^{\infty} |x^{(k)}|^p < \infty\} \quad \|x\|_p = \left(\sum_{k=1}^{\infty} |x^{(k)}|^p\right)^{\frac{1}{p}} \quad p \in [1, \infty) \\ l^\infty &= \{x = \left(x^{(k)}\right)_{k \in \mathbb{N}} : \sup_{k \in \mathbb{N}} |x^{(k)}| < \infty\} \quad \|x\|_\infty = \sup_{k \in \mathbb{N}} |x^{(k)}| \end{aligned}$$

Now we prove that $\|.\|_p$ is actually a norm in L^p. We will concentrate on $p < \infty$ ($p = \infty$ is the easy case)
Properties 1 and 2 of the norm are immediate to check:

(1) $\|f\|_p = 0 \Leftrightarrow \int_X |f|^p \, d\mu = 0 \Leftrightarrow f = 0$ a.e. on $X \Leftrightarrow f = 0 \in L^p$

(2) Obvious, by linearity

(3) About triangle inequality? We need some preliminaries

Theorem 11.1 (Young's Inequality)
Let $p \in (1, \infty)$, $a, b \geq 0$. We say that q is the conjugate exponent of p if

$$\frac{1}{p} + \frac{1}{q} = 1 \Leftrightarrow q = \frac{p}{p-1}$$

Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Remark 11.2
$p \in (1, \infty) \Rightarrow q \in (1, \infty)$. Moreover, we say that 1 and ∞ are conjugate

Proof. $\varphi(x) = e^x$ is convex:

$$\varphi((1-t)x + ty) \leq (1-t)\varphi(x) + t\varphi(y) \qquad \forall x, y \in \mathbb{R} \quad \forall\, t \in [0,1]$$

If $a = 0$ or $b = 0$, then the thesis holds.
If $a, b > 0$

$$ab = e^{\log a} e^{\log b} = e^{\log a^{\frac{p}{p}}} e^{\log b^{\frac{q}{q}}} = e^{\frac{1}{p}\log a^p} e^{\frac{1}{q}\log b^q}$$

Since φ is convex

$$\frac{1}{p} e^{\log a^p} + \frac{1}{q} e^{\log b^q} = \frac{1}{p} a^p + \frac{1}{q} b^q$$

$x = \log a^p$, $y = \log b^q$ $\qquad 1 - t = \frac{1}{p}$, $t = \frac{1}{q}$ ★

Theorem 11.2 (Holder's Inequality)
$(X, \mathcal{M}, \mu)$ measure space. f, g measurable functions. $p, q \in [1, \infty]$ conjugate exponents.
Then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Proof. Case $p, q \in (1, \infty)$. Obvious if $\|f\|_p \|g\|_q = \infty$.
If $\|f\|_p \|g\|_q = 0 \Rightarrow$ either $f = 0$ a.e. on X or $g = 0$ a.e. on X $\Rightarrow fg = 0$ a.e. on $X \Rightarrow \|fg\|_1 = 0$.
Let then $\|f\|_p$, $\|g\|_p \in (0, \infty)$.
For $x \in X$, we set

$$a := \frac{|f(x)|}{\|f\|_p},\ b := \frac{|g(x)|}{\|g\|_q}$$

and use Young:

$$\frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \leq \frac{1}{p} \frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q} \frac{|g(x)|^q}{\|g\|_q^q}$$

$\forall\, x \in X$. By integrating, we obtain

$$\frac{1}{\|f\|_p \|g\|_q} \int_X |fg|\, d\mu \leq \frac{1}{p\|f\|_p^p} \int_X |f|^p\, d\mu + \frac{1}{q\|g\|_q^q} \int_X |g|^q\, d\mu = \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \|fg\| \leq \|f\|_p \|g\|_q$$

Case $p = 1$, $q = \infty$. Exercise ★

Theorem 11.3 (Minkowski Inequality)
$f, g \in L^p(X, \mathcal{M}, \mu)$, $p \in [1, \infty]$. Then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Proof. $p \in (1,\infty)$

$$\|f+g\|_p^p = \int_X |f+g|^p \, d\mu = \int_X |f+g||f+g|^{p-1} \, d\mu$$

$$\leq \int_X (|f|+|g|)\,|f+g|^{p-1} \, d\mu = \int_X |f||f+g|^{p-1} \, d\mu + \int_X |g||f+g|^{p-1} \, d\mu$$

Using Holder with p, $q = \frac{p}{p-1}$

$$\leq \|f\|_p \left(\int_X \left(|f+g|^{p-1}\right)^{\frac{p}{p-1}} \, d\mu \right)^{\frac{p-1}{p}} + \|g\|_p \left(\int_X \left(|f+g|^{p-1}\right)^{\frac{p}{p-1}} \, d\mu \right)^{\frac{p-1}{p}}$$

$$= \|f\|_p \|f+g\|_p^{p-1} + \|g\|_p \|f+g\|_p^{p-1}$$

We divide left hand side and right hand side by $\|f+g\|_p^{p-1}$:

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

★

We introduced $L^p(X, \mathcal{M}, \mu)$, and we proved that this is a normed space with

$$\|f\|_p := \begin{cases} \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} & \text{if } p \in [1,+\infty) \\ \operatorname*{ess\,sup}_X |f| & \text{if } p = +\infty \end{cases}$$

Inclusion of L^p spaces

Theorem 11.4
Suppose that $\mu(X) < +\infty$. Then

$$1 \leq p \leq q \leq \infty \Rightarrow L^q(X) \subseteq L^p(X)$$

Meaning that any $f \in L^q$ is also in L^p. More precisely, $\exists\, C > 0$ depending on $\mu(X), p, q$ s.t.

$$\|f\|_p \leq C \|f\|_q \quad f \in L^q(X)$$

Proof. If $q = +\infty$
$f \in L^\infty(X)$: then $|f(x)| \leq \operatorname*{ess\,sup}_X |f| = \|f\|_\infty$ for a.e. $x \in X$, say $\forall\, x \in X \setminus A$, with $\mu(A) = 0$.
Then

$$\int_X |f|^p \, d\mu = \int_{X\setminus A} |f|^p \, d\mu \leq \|f\|_\infty^p \int_{X \setminus A} 1 \, d\mu = \|f\|_\infty^p \underbrace{\mu(X)}_{=\mu(X\setminus A)}$$

If $q < +\infty$
Then $\frac{q}{p} > 1$, and we can use Hölder$\left(\frac{q}{p}, \left(\frac{q}{p}\right)'\right)$, where $\left(\frac{q}{p}\right)' = \frac{\frac{q}{p}}{\frac{q}{p}-1} = \frac{q}{q-p}$

$$\|f\|_p^p = \int_X |f|^p \, d\mu \overset{\text{Hölder}}{\leq} \left(\int_X \left(|f|^{\cancel{p}}\right)^{\frac{q}{\cancel{p}}} \, d\mu \right)^{\frac{p}{q}} \cdot \left(\int_X 1 \, d\mu \right)^{\frac{q-p}{p}} = \left(\int_X |f|^q \, d\mu \right)^{\frac{p}{q}} \cdot (\mu(X))^{\frac{q-p}{p}}$$

$$\Rightarrow \|f\|_p \leq \mu(X)^{\frac{q-p}{qp}} \|f\|_q$$

★

The assumption $\mu(X) < \infty$ is essential. For example, in $X = [1, \infty]$

$$\frac{1}{x} \in L^2([1,\infty]) \Leftrightarrow \int_1^\infty \frac{dx}{x^2} < \infty$$

$$\frac{1}{x} \notin L^1([1,\infty]) \Leftrightarrow \int_1^\infty \frac{dx}{x} = \infty$$

In particular, the previous theorem is false for l^p-spaces

$$l^p = L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_C)$$

$$1 \le p \le q \le \infty \Rightarrow l^p \subseteq l^q, \text{ and } \exists\, C > 0 \text{ s.t. } \|x\|_q \le C\|x\|_p \quad \forall\, x \in l^p$$

Without assumptions on $\mu(X)$, in general one has the interpolation inequality.

Theorem 11.5

$(X, \mathcal{M}, \mu)$ measure space. Let $1 \le p \le q \le \infty$. If $f \in L^p(X) \cap L^q(X)$, then

$$f \in L^r(X) \quad \forall\, r \in (p, q)$$

and moreover

$$\|f\|_r \le \|f\|_p^\alpha \|f\|_q^{1-\alpha}$$

where α is such that $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$

Proof. For exercise. Use Holder ★

Completeness and Separability

Theorem 11.6

For $1 \le p \le \infty$, $L^p(X, \mathcal{M}, \mu)$ is a Banach space (with reference to $\|\cdot\|_p$)

Proof.

$p < \infty$.

By using the characterization of completeness with the series, we want to show that if $\{f_n\} \subseteq L^p(X)$, and $\sum_{k=1}^\infty \|f_k\|_p < \infty \Rightarrow \sum_{k=1}^\infty f_k$ is convergent in L^p, namely $s_n = \sum_{k=1}^n f_k$ has a limit in L^p: $\|s_n - s\|_p \to 0$ as $n \to \infty$.

Let then $\{f_n\} \subseteq L^p(X)$ s.t.

$$\sum_{k=1}^\infty \|f_k\|_p = M < \infty$$

Define

$$g_n(x) = \sum_{k=1}^n |f_k(x)|$$

By Minkowski, $\|g_n\|_p \le \sum_{k=1}^n \|f_k\|_p \le M < \infty$. Moreover, for every $x \in X$ fixed, $\{g_n(x)\}$ is increasing $\Rightarrow g_n(x) \to g(x)$ as $n \to \infty$, $\forall\, x \in X$

$$\int_X |g|^p \, d\mu \overset{\text{Monot conv}}{=} \lim_n \int_X |g_n|^p \le M^p < \infty \Rightarrow g \in L^p(X)$$

$\Rightarrow |g|^p$ is finite a.e.:

$$\sum_{k=1}^\infty |f_k(x)| < \infty \text{ for a.e. } x \in X$$

$$\Rightarrow \sum_{k=1}^{\infty} f_k(x) \text{ is convergent a.e. to a limit } s(x)$$

Thus, we proved that $s_n(x) = \sum_{k=1}^{n} f_k(x) \to s(x)$ a.e. in X. Namely $|s_n - s|^p \to 0$ a.e. in X. To find a dominating function for $|s_n - s|^p$, we start by observing that

$$|s_n(x)| = \left|\sum_{k=1}^{n} f_k(x)\right| \leq \sum_{k=1}^{n} |f_k(x)| = g_n(x) \leq g(x) \text{ for a.e. } x \in X$$

Therefore

$$|s_n - s|^p \leq 2^{p-1}(|s_n|^p + |s|^p) \leq 2^{p-1}(g^p + g^p) = 2^p g^p \in L^1(X)$$

By the dominated convergence theorem

$$\int_X |s_n - s|^p \, d\mu \to 0 \Leftrightarrow \|s_n - s\|_p \to 0$$

Thus L^p is complete.

$p = \infty$ exercise ★

To speak about separability, we give a

Definition 11.1
$g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. The support of g is

$$\operatorname{supp} g = \overline{\{x \in \Omega : g(x) \neq 0\}}$$

Also

$$\mathcal{C}_C^0 = \left\{f \in \mathcal{C}^0(\Omega) : \operatorname{supp} f \text{ is compact in } \Omega\right\} = \mathcal{C}_O^0(\Omega) = \mathcal{C}_C(\Omega)$$

Theorem 11.7 (Lusin Theorem)
$\Omega \in \mathcal{L}(\mathbb{R}), \lambda(\mathbb{R}) < +\infty$. Let also $f : \mathbb{R} \to \mathbb{R}$ measurable, s.t. $f \equiv 0$ in Ω^C.
Then $\forall\, \varepsilon > 0\ \exists\, g \in \mathcal{C}_C^0(\mathbb{R})$ s.t.

$$\lambda\left(\{x \in \mathbb{R} : g(x) \neq f(x)\}\right) < \varepsilon$$

and

$$\sup_{\mathbb{R}} |g| \leq \sup_{\mathbb{R}} |f|$$

Definition 11.2
Given s simple function $= \sum_{k=1}^{n} a_k \chi_{E_k}$, where $E_1, \ldots, E_n$ are $\mathcal{L}$-measurable sets, $a_1, \ldots, a_n \in \mathbb{R}$.

$$E_1 \cup E_2 \cup \ldots \cup E_n = \mathbb{R}$$

We consider

$$\tilde{\mathcal{S}}(\mathbb{R}) = \{s \text{ simple in } \mathbb{R} \text{ s.t. } \lambda\left(\{s \neq 0\}\right) < +\infty\}$$

What does it mean for a simple function to be in $L^p(\mathbb{R})$?

$$\int_{\mathbb{R}} |s|^p \, d\mu = \sum_{k=1}^{n} a_k^p \lambda(E_k) < +\infty \qquad 1 \leq p \leq +\infty$$

iff $s \equiv 0$ outside a set of finite measure $\Leftrightarrow s \in \tilde{\mathcal{S}}(\mathbb{R})$.
$\tilde{\mathcal{S}}(\mathbb{R})$ is the set of integrable simple functions.

Theorem 11.8
$\tilde{\mathcal{S}}(\mathbb{R})$ is dense in L^p, $\forall\, p \in (1, +\infty)$

Proof. $f \in L^p(\mathbb{R}), f \geq 0$ a.e. in $\mathbb{R}$.
We want to show that $\exists\ \{s_n\} \subseteq \tilde{\mathcal{S}}(\mathbb{R})$ s.t. $\|s_n - f\|_p \to 0$.
By the simple approximation theorem, $\exists\ \{s_n\}$ of simple functions s.t. $\{s_n(x)\}$ is increasing, for every x, and $s_n \to f$ pointwise in $\mathbb{R}$.
Since $|s_n|^p \leq f^p \Rightarrow s_n \in L^p$ for every n $\Rightarrow \{s_n\} \subseteq \tilde{\mathcal{S}}(\mathbb{R})$. Moreover

$$|s_n - f|^p \to 0 \qquad \text{a.e. in } \mathbb{R}$$

$$|s_n - f|^p \leq 2^{p-1}(|s_n|^p + |f|^p) \leq 2^p |f|^p \in L^1$$

$\Rightarrow$ by dominated convergence

$$\int_{\mathbb{R}} |s_n - f|^p \, d\lambda \to 0 \text{ , namely } \|s_n - f\|_p \to 0$$

If f is sign changing, then $f = f^+ - f^-$ and argue as before on f^+ and f^- ★

Theorem 11.9
$\forall p \in [1, \infty)$, the space $L^p(\mathbb{R})$ is separable.

Proof. sketch

- Step 1: $\mathcal{C}^0_C(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, $\forall\ \leq p \leq \infty$.

  Take $s \in \tilde{\mathcal{S}}(\mathbb{R})$. Then, by Lusin theorem, $\exists\ \{f_n\} \subseteq \mathcal{C}^0_C(\mathbb{R})$ s.t. $\|f_n - s\|_p \to 0$. Then, since any $f \in L^p$ can be approximated by simple integrable functions, we have that f can be approximated by functions in $\mathcal{C}^0_C(\mathbb{R})$.

- Step 2:

  By Stone Weierstrass, the set of polinomials $\mathcal{P}(\mathbb{R})$ is dense in $\mathcal{C}^0_C(\mathbb{R})$ with the $\|\cdot\|_\infty$ norm. Since we work with functions with compact support, this implies that $\mathcal{P}(\mathbb{R})$ is dense in $\mathcal{C}^0_C(\mathbb{R})$ also with respect to $\|\cdot\|_p$

  $$\int_{-M}^{M} |f - p_n|^p \, d\lambda \leq \|f - p_n\|^p_\infty 2M \to 0$$

  if $\|f - p_n\|_\infty \to 0$, $\Rightarrow \mathcal{P}(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

  $\tilde{\mathcal{P}}(\mathbb{R}) = \{$polynomials with rational coefficients$\}$. This is countable, and is dense in $(\mathcal{P}(\mathbb{R}), \|\cdot\|_p)$. $\Rightarrow$ is dense in L^p

★

What about $L^\infty(\mathbb{R})$? In this case $\mathcal{C}(\mathbb{R})$ are not dense in $L^\infty(\mathbb{R})$. For example, consider

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

If $g \in L^\infty$ s.t. $\|g - f\|_\infty < \frac{1}{3}$, then g cannot be continuous. Assume by contradiction that $\exists\ g \in \mathcal{C}(\mathbb{R})$ s.t. $\|g - f\|_\infty < \frac{1}{3}$. Then

$$\operatorname{ess\,sup}_{\mathbb{R}} |g(x) - f(x)| < \frac{1}{3}$$

In particular, $g(x) < \frac{1}{3}\ \forall\, x < 0$

$$\Rightarrow \lim_{x \to 0^-} g(x) \leq \frac{1}{3}$$

On the other hand, $g(x) > \frac{2}{3} \quad \forall\, x > 0$

$$\Rightarrow g(0) = \lim_{x \to 0^+} g(x) \geq \frac{2}{3}$$

Quick recap about the 'delirium' on the separability
The thing that you need to know, in $\to L^p(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$, are:

(1) L^p is separable $\forall\, p \in [1, \infty)$

(2) $\tilde{S}(\mathbb{R})$ is dense in $L^p(\mathbb{R})\ \forall\, p \in [1, \infty)$, namely $\forall p \in L^p(\mathbb{R})$ and $\forall\, \varepsilon > 0\ \exists\, s \in \tilde{S}(\mathbb{R})$ s.t.

$$\|f - s\|_p < \varepsilon$$

(3) $\mathcal{C}^0_C(\mathbb{R})$ is dense in L^p, namely $\forall p \in L^p(\mathbb{R})$ and $\forall\, \varepsilon > 0\ \exists\, g \in \mathcal{C}^0_C(\mathbb{R})$ s.t.

$$\|f - g\|_p < \varepsilon$$

Everything remains true if you replace $\mathbb{R}$ with X open or closed, or with $X \in L(\mathbb{R}^n)$, and consider $(X, L(X), \lambda)$.
What happens for $L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$?
$\mathcal{C}(\mathbb{R})$ is not dense in L^∞.
By the simple approximation theorem, we have that simple functions are dense in L^∞.

Theorem 11.10
$L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is not separable.

Proof. $\{\chi_{[-\alpha,\alpha]} : \alpha > 0\} \subseteq L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ $\chi_\alpha = \chi_{[-\alpha,\alpha]}$
This is an uncountable family of functions. $\|\chi_\alpha - \chi_{\alpha'}\|_\infty = 1\ \forall\, \alpha \neq \alpha'$, indeed

$$|\chi_\alpha(x) - \chi_{\alpha'}(x)| = \begin{cases} 0 & \text{if } x \in [-\alpha, \alpha] \cup (\alpha', \infty) \cup (-\infty, -\alpha') \\ 1 & \text{if } x \in (\alpha, \alpha'] \cup [-\alpha', \alpha) \end{cases}$$

In particular, $B_{\frac{1}{2}}(\chi_\alpha) \cap B_{\frac{1}{2}}(\chi_{\alpha'}) = \varnothing\ \forall\, \alpha \neq \alpha'$
Assume by contradiction that $L^\infty(\mathbb{R})$ is separable: $\exists\, Z \subset L^\infty$ which is countable and dense. In particular, $\forall\, f \subset L^\infty\ \exists\, g \in Z$ s.t.

$$\|g - f\|_\infty < \frac{1}{2}$$

Therefore, $\forall\, \alpha$, $\exists\, g_\alpha \in B_{\frac{1}{2}}(\chi_\alpha) \cap Z$. But $B_{\frac{1}{2}}(\chi_\alpha) \cap B_{\frac{1}{2}}(\chi_{\alpha'}) = \varnothing$

$$\Rightarrow \alpha \neq \alpha', \text{ we have } g_\alpha \neq g_{\alpha'}$$

$Z \supseteq \{g_\alpha : \alpha > 0\}$, which is uncountable. This is not possible, since Z is countable. ★

Remark 11.3
The same is true if $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is swapped with $(X, \mathcal{L}(X), \lambda)$, X is open or closed on $\mathbb{R}$ or $\mathbb{R}^n$

12 Linear operators

$(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ normed spaces.

Definition 12.1
$T : D(T) \subseteq X \to Y$ is a **linear operator** (or map) if

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2) \quad \forall\, x_1, x_2, \in D(T) \quad \forall\, \alpha_1, \alpha_2 \in \mathbb{R}$$

$D(T)$ is a linear subspace of X, and is called the domain of T. When $D(T) = X$ and $Y = \mathbb{R}$, T is called linear functional.

Definition 12.2
A linear operator $T : D(T) \subseteq X \to Y$ is bounded if $D(T) = X$ and $\exists\, M > 0$ s.t.

$$\|T_X\|_Y \leq M|x|_X \forall\, x \in X$$

Recall that T is continuous in $x_0 \in X$ iff

$$\forall\ \{x_n\} \subset X, x_n \xrightarrow{X} x_0 \Rightarrow Tx_n \xrightarrow{Y} Tx_0$$

Ex:

- $L : \mathbb{R}^n \to \mathbb{R}$ is a linear functional . Then $\exists\, y \in \mathbb{R}^n$ s.t.

  $$Lx = \langle y, x\rangle = (y, x) = y \cdot x$$

  In particular, then L is continuous on $\mathbb{R}^n$ and bounded:

  $$|L_X| < |\langle y, x\rangle| \overset{\text{Cauchy-Schwarz}}{\leq} \|y\|\|x\| \qquad \forall\, x \in \mathbb{R}^n$$

  So L is bounded with $M = \|y\|$.

- Linear operators in ∞-dim may not be defined everywhere, and many may not be continuous: $(X, \|\cdot\|_X) = (Y, \|\cdot\|_Y) = (\mathcal{C}([0,1]), \|\cdot\|_\infty)$.
  Consider

  $$\begin{aligned} \frac{d}{dx} : \mathcal{C}'([0,1]) \subseteq X &\to Y \\ f &\mapsto f' \end{aligned} \qquad \frac{d}{dx}(\alpha f + \beta g) = \alpha\frac{d}{dx}f + \beta\frac{d}{dx}g$$

  This is not continuous or bounded. For example, take $f_n(x) = \frac{1}{n}\sin 2\pi nx$. $\|f_n\|_\infty \to 0$ but $\|f_n'\|_\infty = 1$
  In this case $f_n \to 0 \not\Rightarrow \frac{d}{dx}f_n \to 0$, then $\frac{d}{dx}$ is not bounded as well.

- Let $(X, \|\cdot\|_X)$ be a normed space. If $\dim X = 0$, is it possible to find linear functionals which are not bounded? Yes.

Definition 12.3
A subset $\{e_i\}_{i \in I}$ is called **Hamel basis** of X if

$$\|e_i\|_X = 1 \quad \forall\, i$$

and if every $x \in X$ can be written in a unique way as

$$x = \sum_{k=1}^{n} x_k e_{i_k}, \quad x_k \in \mathbb{R},\ n \in \mathbb{N}$$

Every x can be written uniquely as a finite linear combination of element of the basis. If $\dim X = \infty$ is not immediate that the Hamel basis exists. This can be proved using the axiom of choice. (Zorn's lemma).
Any normed space has a Hamel basis $\dim X = \infty \Rightarrow \{e_i\}_{i\in I}$ has ∞ many elements.
Let then $(X, \|\cdot\|_X)$ be $\infty -$ dim, with Hamel basis $\{e_i\}_{i\in I}$. I is infinite $\Rightarrow I \supseteq \mathbb{N}$.
We define $L : X \to \mathbb{R}$ in the following way

$$Le_0 = 0 \qquad Le_1 = 1 \quad \dots \quad Le_n = n \quad \dots$$
$$Le_i = 0 \quad \forall\, i \in I \setminus \mathbb{N}$$

Then, for $x \in X$ we set

$$Lx = L\left(\sum_{k=1}^{n} x_k e_{i_k}\right) = \sum_{k=1}^{n} x_k L e_{i_k}$$

L is linear by contradiction, and it is not bounded:

$$|Le_n| = n \to \infty \quad \|e_n\|_X = 1 \;\forall\, n$$
$$\frac{|Le_n|}{\|e_n\|_X} \to \infty \Rightarrow L \text{ is not bounded}$$

Remark 12.1
In practice, Hamel basis are hard to use. They differ from Hilbertian basis.

For linear operators, boundedness and continuity are equivalent.
Theorem 12.1
$T : X \to Y$ linear map. Then the following are equivalent

(1) T is continuous in $0 \in X$

(2) T is continuous everywhere in X

(3) T is bounded

Remark 12.2
T linear $\Rightarrow T0 = 0$. Indeed

$$T0 = T(0x) = 0Tx = 0$$

Proof.
- (2) $\Rightarrow$ (1) obvious.
- (1) $\Rightarrow$ (3) Suppose by contradiction that T is not bounded.

  Then $\exists\, \{x_n\} \subset X$, $x_n \neq 0$, s.t.
  $$\frac{\|Tx_n\|_Y}{\|x_n\|_X} \geq n \quad \forall\, n$$
  Define
  $$z_n := \frac{x_n}{n\|x_n\|_X}$$
  Then $\|z_n\|_X = \frac{1}{n\|x_n\|}\|x_n\|_X \to 0$, namely $z_n \to 0$ in $X \Rightarrow$ (T is continuous in 0) $Tz_n \to T0 = 0$. However,
  $$\|Tz_n\|_Y = \left\|T\left(\frac{x_n}{n\|x_n\|_X}\right)\right\| = \frac{1}{n\|x_n\|_X}\|Tx_n\|_Y \geq 1 \;\forall\, n$$
  Contradiction.

- (3) $\Rightarrow$ (2) We observe that

$$\|Tx_1 - Tx_2\|_Y = \|T(x_1 - x_2)\|_Y \leq M\|x_1 - x_2\|_X \quad \forall\, x_1\, x_2 \in X$$

  Then, let $x \in X$ and let $x_n \to x$ in X: $\|x_n - x\|_X \to 0$. But then

$$\|Tx_n - Tx\|_Y \leq M\|x_n - x\|_X \to 0$$

  namely $Tx_n \to Tx$ in Y. This is the continuity.

★

Definition 12.4
The set of linear operators $T : X \to Y$ which are also bounded (continuous) is denoted by $\mathcal{L}(X, Y)$. If $Y = X$, one simply writes $\mathcal{L}(X)$

This is a vector space. $\forall\, T,\, S \in \mathcal{L}(X, Y)$, $\forall\, \alpha, \beta \in \mathbb{R}$:

$$(\alpha T + \beta S)(x) = \alpha Tx + \beta Sx \qquad \in \mathcal{L}(X, Y)$$

We can also introduce a norm:

$$\|T\|_{\mathcal{L}(X,Y)} = \|T\|_{\mathcal{L}} := \sup_{\|x\|_X \leq 1} \|Tx\|_Y$$

Also,

$$\|T\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X = 1} \|Tx\|_Y = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \inf M > 0 \text{ s.t. } \|Tx\|_Y \leq M\|x\|_X \quad \forall\, x \in X$$

Theorem 12.2
X normed space, Y Banach space. Then $(\mathcal{L}(X, Y), \|\cdot\|_{\mathcal{L}(X,Y)})$ is a Banach space.

Proof. Let $\{T_n\}$ be a Cauchy sequence in $\mathcal{L}(X, Y)$. We want to show that $\exists\, T \in \mathcal{L}(X, Y)$ s.t.

$$\|T_n - T\|_{\mathcal{L}} \to 0$$

$\{T_n\}$ Cauchy: $\forall\, \varepsilon > 0\ \exists\, \bar{n} \in \mathbb{N}$ s.t.

$$n, m > \bar{n} \Rightarrow \|T_n - T_m\|_{\mathcal{L}} < \varepsilon$$

Consider then $\{T_n x\}$, $x \in X$

$$\|T_n x - T_m x\|_Y = \|(T_n - T_m)x\|_Y \leq \|T_n - T_m\|_Y \|x\|_X \leq \varepsilon\|x\|_X \tag{*}$$

This means that $\{T_n x\}$ is a Cauchy sequence in Y, which is complete: then $\forall\, x \in X\ \exists$ a vector $y_x \in Y$ s.t. $T_n x \to y_x$ in Y.
Define

$$T : X \to Y \qquad x \mapsto y_x = Tx$$

T is linear: indeed, $\forall\, x_1,\, x_2 \in X$ and $\alpha_1,\, \alpha_2 \in \mathbb{R}$:

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \lim_{n\to\infty} T_n(\alpha_1 x_1 + \alpha_2 x_2) = \lim_{n\to\infty} (\alpha_1 T_n x_1 + \alpha_2 T_n x_2) = \alpha_1 Tx_1 + \alpha_2 Tx_2$$

So T is linear. It remains to show that T is bounded, and that $\|T_n - T\|_{\mathcal{L}} \to 0$. To show that T is bounded, note that, by (*), $\forall\, \varepsilon > 0\ \exists\, \bar{n}$ s.t.

$$n, m > \bar{n} \Rightarrow \|T_n x - T_m x\|_Y \leq \varepsilon\|x\|_X \quad \forall\, x$$

Take the limit for $m \to \infty$:
$$\|T_n x - Tx\|_Y \le \varepsilon \|x\|_X$$
But then, since T_n is bounded,
$$\|Tx\|_Y = \|Tx \pm T_n x\|_Y \le \|T_n x\|_Y + \|Tx - T_n x\|_Y \le M_n \|x\|_X + \varepsilon \|x\|_X = (M_n + \varepsilon)\|x\|_X$$
and T is bounded. To show that $\|T_n - T\|_{\mathcal{L}} \to 0$, observe that $\forall\, \varepsilon > 0\ \exists\, \bar{n}$ s.t. $n > \bar{n}$
$$\|T_n x - Tx\|_Y \le \varepsilon \|x\|_X \Leftrightarrow \frac{\|(T_n - T)x\|_Y}{\|x\|_X} \le \varepsilon \quad \forall\, x \in X \setminus 0 \overset{\text{take sup over } x \neq 0}{\Rightarrow} \|T_n - T\|_{\mathcal{L}} < \varepsilon$$
namely, $T_n \to T$ in $\mathcal{L}$ ★

Let T be a linear operator from X to Y.

Definition 12.5
The **kernel** of T is the set
$$\ker(T) = \{x \in X : Tx = 0\} \subset X$$
This is a vector subspace of X.
T is injective $\Leftrightarrow \ker(T) = \{0\}$. If T is continuous, $\ker(T)$ is closed
$$\ker(T) = T^{-1}(\{0\})$$

Definition 12.6
X, Y normed spaces. X and Y are isomorphic if $\exists\ T \in \mathcal{L}(X, Y)$ bijective, and such that $T^{-1} \in \mathcal{L}(X, Y)$

Definition 12.7
$T \in \mathcal{L}(X, Y)$ is an isometry if
$$\|Tx\|_Y = \|x\|_X \quad \forall\, x \in X$$

Definition 12.8
If $X \subseteq Y$ is a vector subspace, and $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed space, then we can consider
$$\begin{aligned} J :\ & X \to Y \quad \text{(inclusion map)} \\ & x \mapsto x \end{aligned}$$
If $J \in \mathcal{L}(X, Y)$ (namely, if $\exists\ M > 0$ s.t. $\|x\|_Y \le M\|x\|_X\ \forall\, x \in X$), then we say that J is an embedding of X into Y, and we write $X \hookrightarrow Y$

Ex: $\mu(X) < \infty$, $1 \le p < q \le \infty$
$$L^q(X) \hookrightarrow L^p(X) \qquad \text{(inclusion of } L^p \text{ spaces)}$$

Some fundamentals theorems on linear operators

Definition 12.9
(X, d) metric space. $A \subset X$. $x \in X$ is an **adherence point** of A if $\forall\, r > 0 : B_r(x) \cap A \neq \varnothing$
$$\bar{A} = \{x \in X : x \text{ is an adherence point of } A\} = A \cup \partial A$$

Definition 12.10
$A \subset X$ is **dense** in X if $\bar{A} = X$.

For example, $\mathbb{Q}$ is dense in $\mathbb{R}$, and (a, b) is dense in $[a, b]$.

Definition 12.11

$A \subset X$ is **nowhere dense** if the interior of the closure of A is empty, namely

$$\text{int}(\bar{A}) = \bar{A}^{\text{o}} = \varnothing$$

Ex: $\bar{\{x\}}^{\text{o}} = \{x\}^{\text{o}} = \varnothing$

$\mathbb{Z} \subset \mathbb{R}$: $\bar{\mathbb{Z}}^{\text{o}} = \mathbb{Z}^{\text{o}} = \varnothing$

$\mathbb{Q}$ is not nowhere dense: $(\bar{\mathbb{Q}})^{\text{o}} = (\mathbb{R})^{\text{o}} = \mathbb{R}$

Definition 12.12

$A \subset X$ is called **of first category** (or **meager set**) in X if A is the (at most) countable union of nowhere dense sets.

Ex: $\mathbb{Q}$ is of first category in $\mathbb{R}$: countable union of nowhere dense sets

$$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$$

Definition 12.13

$A \subset X$ is of second category if it is not of first category.

Theorem 12.3 (Baire category theory)

(X, d) complete metric space. Then

- $\{U_n\}_{n=0}^{\infty}$ is a sequence of open and dense sets in $X \Rightarrow \cap_{n=0}^{\infty} U_n$ is dense in X.
- X is of second category in itself: X cannot be the countable union of nowhere dense sets.

<u>Preliminaries</u>:

- $A \subset X$ is dense $\Leftrightarrow \forall\, W \subset X$, W open, $W \neq \varnothing$, we have that $A \cap W \neq \varnothing$
- A is nowhere dense $\Leftrightarrow \left(\bar{A}\right)^C$ is open and dense

Proof. Here's the proof of the two parts of the theorem:

(a) Thanks to the first preliminary, we show that $\forall\, W \subset X$ open and non empty we have $(\cap_n U_n) \cap W \neq \varnothing$

$$U_0 \text{ is open and dense: } \overset{1^{st}\text{prel.}}{\Rightarrow} \underbrace{U_0 \cap W}_{\text{is open}} \neq \varnothing$$

$\Rightarrow$ it contains an open ball

$\Rightarrow (U_0 \cap W) \supset B_{r_0}(x_0)$ for some $x_0 \in X$ and $r_0 > 0$

For $n > 0$, we choose $x_n \in X$ and $r_n > 0$ inductively in the following way: we have

$$U_n \cap B_{r_{n-1}}(x_{n-1}) \neq \varnothing \qquad (1^{st} \text{ prel. } + U_n \text{ is dense})$$

$$\Rightarrow \overline{B_{r_n}(x_n)} \subset (U_n \cap B_{r_{n-1}}(x_{n-1}))$$

all these balls
are included in
$B_{r_0}(x_0)$

with $x_n \in X$ and $0 < r_n < \frac{1}{2^n}$

By the condition on r_n, we see that

$$x_n, x_m \in B_{r_N}(x_N) \quad \forall\, n, m > N$$

$\Rightarrow \{x_n\}$ is a Cauchy sequence in X

$$d(x_n, x_m) \leq \frac{1}{2^N} \quad \forall\, n, m > N$$

X is complete: $x_n \xrightarrow{d} x \in X$ Since

$$\begin{aligned}
&x_n \in B_{r_N}(x_N) && \forall\, n > N \\
&\Rightarrow x = \lim_n x_n \in \overline{B_{r_N}(x_N)} \subset (U_n \cap B_{r_0}(x_0)) \subset (U_N \cap W) && \forall\, n \in \mathbb{N} \\
&\Rightarrow x_n \in \bigcap_n (U_n \cap W) = \left(\bigcap_n U_n\right) \cap W
\end{aligned}$$

This means that $\bigcap_n U_n$ is dense.

(b) It follows from (a):

If $\{E_n\}$ is a sequence of nowhere dense sets in X, then, by the second preliminary $\left\{(E_n)^C\right\}$ is a sequence of open and dense sets. By (a)

$$\bigcap_n (\overline{E_n})^C \neq \varnothing$$

$$\Rightarrow \bigcup_n E_n \subset \bigcup_n \overline{E_n} = X \setminus \left(\overbrace{\bigcap_n (\overline{E_n})^C}^{=\varnothing}\right) \neq X$$

★

<u>Ex</u>: $(X, \|\cdot\|)$ $\infty-\dim$ Banach space. $\{e_i\}_{i\in I}$ Hamel basis.
Then I is uncountable.

Theorem 12.4 (Banach Steinhaus)
X Banach space, Y normed space, $\mathcal{F} \subseteq \mathcal{L}(X,Y)$ family. Suppose that $\mathcal{F}$ is pointwise bounded:

$$\forall\, x \in X \quad \exists\, M_x > 0 \text{ s.t. } \sup_{T\in\mathcal{F}} \|Tx\|_Y \leq M_x \tag{PB}$$

Then $\mathcal{F}$ is uniformly bounded:

$$\exists\, M \geq 0 \text{ s.t. } \sup_{T\in\mathcal{F}} \|T\|_{\mathcal{L}(X,Y)} \leq M \tag{UB}$$

Proof. $\forall\, n \in \mathbb{N}$, let

$$C_n := \{x \in X : \|Tx\|_Y \leq n \quad \forall\, T \in \mathcal{F}\} = \cap_{T\in\mathcal{F}}\{x \in X : \|Tx\|_Y \leq n\}$$

C_n is a closed set $\forall\, n$, since T is continuous. (also $\varphi : X \to \mathbb{R}$ $\varphi(x) = \|Tx\|_Y$ is continuous)
By (PB), every $x \in X$ stays in some C_n: $X = \cup_{n=1}^{\infty} C_n$. <u>Since X</u> is Banach, by the Baire theorem it is necessary that $\exists\, n_0 \in \mathbb{N}$ s.t. ${C_{n_0}}^{\circ} \neq \varnothing \Rightarrow$ a ball $\overline{B_r(x_0)} \subset C_{n_0}$: then

$$\|T(x_0 + rz)\|_Y \leq n_0 \quad \forall\, z \in \overline{B_1(0)}$$

$$\|T(x_0 + rz)\|_Y \overset{\text{linearity}}{=} \|Tx_0 + rTz\|_Y \overset{\text{triangle ineq}}{\leq} r\|Tz\|_Y - \|Tx_0\|_Y \quad \forall\, T \in \mathcal{F}$$

To sum up: $\forall\, T \in \mathcal{F}$, $\forall\, z \in \overline{B_1(0)}$ we have

$$r\|Tz\|_Y - \|Tx_0\|_Y \leq n_0 \Rightarrow \|Tz\|_Y \leq \frac{1}{r}(n_0 + M_{x_0})$$

We take sup over $T \in \mathcal{F}$:

$$\sup_{T\in\mathcal{F}} \|T\|_{\mathcal{L}(X,Y)} \leq \frac{1}{r}\,(n_0 + M_{x_0}) =: M$$

★

Corollary 12.1
X Banach space, Y normed space. $\{T_n\} \subseteq \mathcal{L}(X,Y)$ s.t. $\{T_n x\}$ has a limit, denoted by Tx, $\forall\, x \in X$ (pointwise convergence). Then $T \in \mathcal{L}(X,Y)$

Proof. T is linear:

$$\begin{array}{ccc} T_n(\alpha_1 x_1 + \alpha_2 x_2) & = & \alpha_1 T_n x_1 + \alpha_2 T_n x_2 \\ \downarrow & & \downarrow \\ T(\alpha_1 x_1 + \alpha_2 x_2) & = & \alpha_1 T x_1 + \alpha_2 T x_2 \end{array}$$

Now we observe that we have (PB): if $\{T_n x\}$ is convergent $\Rightarrow$ $\{T_n x\}$ is bounded $\Rightarrow$ by Banach Steinhaus, $\{T_n\}$ is uniformly bounded:

$$\exists M > 0 \text{ s.t. } \sup_n \|T_n\|_{\mathcal{L}(X,Y)} \leq M$$

Therefore, $\forall\, x \in X$:

$$\|Tx\|_Y = \left\|\lim_n (T_n x)\right\|_Y = \lim_n \|T_n x\|_Y \leq \lim_n \|T_n\|_{\mathcal{L}} \|x\|_X \leq \lim_n M\|x\|_X = M\|x\|_X$$

Thus, T is bounded: $T \in \mathcal{L}(X,Y)$ ★

Let X, Y be normed spaces.

Definition 12.14
$T : X \to Y$ is called **open map** if, $\forall\, A \subset X$ open, the set $T(A) \subset Y$ is open.

Remark 12.3
Recall that T is continuous on X if $T^{-1}(O)$ is open on X, $\forall\, O$ open in Y.

<u>Ex</u>: $f(x)$: constant is continuous, but not open. $f((a,b)) = \{\text{const}\}$

Theorem 12.5 (Open map theorem)
X, Y Banach spaces. $T \in \mathcal{L}(X,Y)$ is surjective. Then T is an open map.

Corollary 12.2
X, Y Banach spaces, $T \in \mathcal{L}(X,Y)$ is bijective. Then T is an isomorphism: $T^{-1} \in \mathcal{L}(X,Y)$

Proof.
- $T : Y \to X$ is linear. (Exercise. Hint: Use $T^{-1} \circ T = \text{Id}+$ linearity of T)
- We want now to check that T^{-1} is continuous on Y: $(T^{-1})^{-1}(O)$ is open in Y, $\forall\, O$ open in X. We know that T is an open map thanks to the open map theorem.

$$(T^{-1})^{-1}(O) = \big\{y \in Y, T^{-1}(y) \in O\big\} = \big\{y \in Y, T^{-1}(y) = x, \text{ for some } x \in O\big\} =$$

$$= \{y \in Y, y = Tx, \text{ for some } x \in O\} = T(O) \text{ is open}$$

Since T is an open map, $\forall\, O \subset X$, open.

★

Corollary 12.3
X vector space, $\|\cdot\|, \|\cdot\|_*$ norms on X. Assume $(X, \|\cdot\|), (X, \|\cdot\|_*)$ are Banach spaces. Assume that $\exists\, C_1 > 0$ s.t.

$$\|x\|_* \leq C_1 \|x\| \quad \forall\, x \in X$$

Then $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent, namely $\exists\, C_2 > 0$ s.t.

$$\|x\| \leq C_2 \|x\|_*$$

Proof. Consider

$$\begin{aligned} I: \quad (X, \|\cdot\|) &\to (X, \|\cdot\|_*) \\ x &\mapsto x \end{aligned}$$

By assumption, I is bounded: $\exists\, C_1 > 0$ s.t.

$$\|Ix\|_* = \|x\|_* \leq C_2 \|x\|$$

I is bijective.
Thus, by the corollary before

$$I^{-1} = I \in \mathcal{L}((X, \|\cdot\|_*), (X, \|\cdot\|))$$

namely $\exists\, C_2 > 0$ s.t.

$$\underset{\|x\|}{\underset{\shortparallel}{\|Ix\|}} \leq C_2 \|x\|_*$$

★

Definition 12.15
$T : D(T) \subset X \to Y$ linear operator. We say that T is **closed** if $\forall\ \{x_n\} \subset D(T)$.

$$\left.\begin{array}{ll} x_n \to x & \text{in X} \\ Tx_n \to y & \text{in Y} \end{array}\right\} \Rightarrow x \in D(T) \text{ and } Tx = y$$

<u>Ex</u>: $X = Y = \mathcal{C}^0([0,1])$ with the supremum norm.

$$T = \frac{d}{dx}$$

T is not continuous. But it is closed: it can be proved that if $\{f_n\} \subset \mathcal{C}^1([0,1])$ is s.t.

$$\left.\begin{array}{ll} f_n \to f & \text{uniformly} \\ f_n' \to g & \text{uniformly} \end{array}\right\} \Rightarrow f \text{ is } \mathcal{C}^1([0,1]) \text{ and } f' = g$$

<u>Ex</u>: $T \in \mathcal{L}(X, Y) \Rightarrow T$ is closed
Remark 12.4
T is a closed operator $\Leftrightarrow$ the graph of T is closed.

$$\text{graph}(T) = \{(x, Tx) : x \in X\}$$

Theorem 12.6 (Closed graph theorem)
X, Y Banach spaces.
$T : X \to Y$ linear closed operator ($D(T) = X$).
Then $T \in \mathcal{L}(X, Y)$.

Remark 12.5
In general it is easier to prove that an operator is closed, rather than it it continuous.

Proof. Define on X the graph-norm of T

$$\|x\|_* = \|x\|_X + \|Tx\|_Y$$

Then is a norm on X. If $\{x_n\} \in X$ is a Cauchy sequence for $\|\cdot\|_*$, then $\{x_n\}$ is a Cauchy sequence in $(X, \|\cdot\|_X)$ and $\{Tx_n\}$ is a Cauchy sequence on $(Y, \|\cdot\|_Y)$

$$\Rightarrow \left.\begin{array}{ll} x_n \to x & \text{in X} \\ Tx_n \to y & \text{in Y} \end{array}\right\} \text{since } T \text{ is closed, we deduce that } y = Tx$$

Thus

$$\|x_n - x\|_X + \|Tx_n - Tx\|_Y \to 0$$

This proves that $(X, \|\cdot\|_*)$ is a Banach space. Also, we know that

$$\|x\|_X \leq \|x\|_X + \|Tx\|_Y = \|x\|_*$$

By the last corollary of the open map theorem, $\exists\, C_2$ s.t.

$$\|x\|_* \leq C_2 \|x_X\|$$

$$\|Tx\|_Y \leq \|x\|_* \leq C_2 \|x\|_X \quad \forall\, x \in X$$

This means that T is bounded. ★

13 Duality and reflexivity

X normed space:

$$X^* = \mathcal{L}(X, \mathbb{R}) \text{ is called } \textbf{dual space of } X$$

X normed space, Y Banach space $\Rightarrow \mathcal{L}(X, Y)$ is a Banach space with $\|\cdot\|_{\mathcal{L}}$.
Since $\mathbb{R}$ is a Banach space, the dual space X^* is a Banach space with

$$\|L\|_* = \sup_{\|x\|_X \leq 1} |Lx|$$

<u>Ex</u>:

- In $\mathbb{R}^n$, only linear functional is separated by a scalar product:

  $$L : \mathbb{R}^n \to \mathbb{R} \text{ is linear } \Rightarrow \exists!\, y \in \mathbb{R}^n \text{ s.t. } Lx = \langle y, x \rangle$$

  It can be proved that

  $$L \in (\mathbb{R}^n)^* \mapsto y \in \mathbb{R}^n$$

  is an isometric isomorphism

  $$(\mathbb{R}^n)^* \approxeq \mathbb{R}^n$$

  Then X^* is very complicated.

- Dual of L^p?

  $(X, \mathcal{M}, \mu)$ measure space. $p \in [1, \infty]$, p' conjugate exponent.

$$\frac{1}{p} + \frac{1}{p'} = 1 \Leftrightarrow \begin{cases} p' = \dfrac{p}{p-1} & p \in (1, \infty) \\ p' = \infty & p = 1 \\ p' = 1 & p = \infty \end{cases}$$

  For $g \in L^{p'}(X)$, define $L_g : L^p(X) \to \mathbb{R}$ by

$$L_g f := \int_X fg \, d\mu \qquad \forall\, f \in L^p(\Omega)$$

  This is well defined, by the Holder inequality:

$$\left| \int_X fg \, d\mu \right| \leq \int_X |fg| \, d\mu = \|fg\|_1 \leq \|g\|_{p'} \|f\|_p \tag{*}$$

  If $g \in L^{p'}$, this shows that $L_g f \in \mathbb{R} \quad \forall\, f \in L^p$

Proposition 13.1
If $p \in [1, \infty]$ then $L_g \in (L^p(X)^*)$. Moreover,

- if $p > 1$, then $\|L_g\|_* = \|g\|_{p'}$
- if $p = 1$ then $\|L_g\|_* = \|g\|_\infty$ with more assumptions (they are satisfied in $(X, \mathcal{L}(X), \lambda)$)

Remark 13.1
We are saying that $L^{p'}$ can be identified with a subspace of the dual space $(L^p)^*$ and this identification is an isometry.

Question: are there functional in $(L^p)^*$?

***Proof*.** (of the proposition)

- Case $p = \infty$ ex
- Case $p = 1$ but difficult it's ok if you don't do it
- Case $p \in (1, \infty)$

  L_g is clearly linear, by linearity of $\int$, indeed: $\forall\, \alpha\ \beta \in \mathbb{R}$, $f_1\ f_2 \in L^p(X)$. Then

$$L_g(\alpha f_1 + \beta f_2) = \int_X g(\alpha f_1 + \beta f_2) \, d\mu = \alpha \int_X g f_1 \, d\mu + \beta \int_X g f_2 \, d\mu = \alpha L_g f_1 + \beta L_g f_2$$

  We want to show now that L_g is bounded. We proved in (*) that

$$|L_g f| \leq \|g\|_{p'} \|f\|_p \quad \forall\, f \in L^p(\Omega)$$

  This shows that L_g is bounded, with norm $\|L_g\|_* \leq \|g\|_{p'}$ (remember that $\|T\|_{\mathcal{L}} = \inf\{M > 0 : \|Tx\|_Y \leq M \|x\|_X \quad \forall x \in X\}$)

  We want to show that $\|L_g\|_* = \|g\|_{p'}$. If $\|L_g\|_* < \|g\|_{p'}$, then $\exists\, M < \|g\|_{p'}$ s.t.

$$|L_g f| \leq M \|f\|_p \quad \forall\, f \in L^p$$

We rule out this possibility by choosing an explicit $\tilde{f} \in L^p$ s.t.

$$\left|L_g \tilde{f}\right| = \|g\|_{p'} \left\|\tilde{f}\right\|_p$$

We take

$$\tilde{f} = \frac{|g|^{p'-1}}{\|g\|_{p'}^{p'-1}} \frac{g}{|g|}$$

Now,

$$\left\|\tilde{f}\right\|_p^p = \int_X \left|\tilde{f}\right|^p d\mu = \int_X \frac{|g|^{p(p'-1)}}{\|g\|_p^{p(p'-1)}} d\mu = (*)$$

$(p')' = p \Rightarrow p = \frac{p'}{p'-1} \Rightarrow p(p'-1) = p'$

$$(*) = \frac{1}{\|g\|_{p'}^{p'}} \int_X |g|^{p'} d\mu = \frac{\|g\|_{p'}^{p'}}{\|g\|_{p'}^{p'}} = 1$$

$$\left|L_g \tilde{f}\right| = \left|\int_X \frac{|g|^{p'-1}}{\|g\|_{p'}^{p'-1}} |g| \, d\mu\right| = \left|\int_X \frac{|g|^{p'}}{\|g\|_{p'}^{p'-1}} d\mu\right| = \frac{1}{\|g\|_{p'-1}^{p'}} \|g\|_{p'}^{p'} = \|g\|_{p'} = \|g\|_{p'} \left\|\tilde{f}\right\|_p$$

★

Hahn Banach

Definition 13.1
X vector space. A map $p : X \to \mathbb{R}$ is called **sublinear functional** if

- $p(\alpha x) = \alpha p(x) \qquad \forall\, x \in X,\ \alpha > 0$
- $p(x + y) \le p(x) + p(y) \qquad \forall x, y \in X$

Theorem 13.1 (Hahn Banach)
X real vector space, $p : X \to \mathbb{R}$ sublinear functional. Y subspace of X and suppose that $\exists\, f : Y \to \mathbb{R}$ linear on Y s.t.

$$f(y) \le p(y) \quad \forall\, y \in Y$$

Then $\exists$ a linear functional $F : X \to \mathbb{R}$ s.t.

$$F(y) = f(y) \quad \forall\, y \in Y \qquad\qquad F \text{ is an extension of } f$$

Moreover,

$$F(x) \le p(x) \quad \forall\, x \in X$$

Theorem 13.2 (Hahn-Banach regarding continuous extension)
X (real) normed space. Y subspace of X, $f \in Y^* = \mathcal{L}(Y, \mathbb{R})$
Then $\exists\, F \in X^* = \mathcal{L}(X, \mathbb{R})$ s.t.

$$F(y) = f(y) \qquad \forall\, y \in Y$$
$$\|F\|_{X^*} = \|f\|_{Y^*}$$

Proof. Define $p : X \to \mathbb{R}$, $p(x) = \|f\|_{Y^*}\|x\|_X \ \forall\ x \in X$. Then p is sublinear (from the properties of $\|\cdot\|_X$).
Moreover, $f(y) \leq |f(y)| \leq \|f\|_{Y^*}\|y\|_X = p(y) \ \forall\ y \in Y$. Then, by Hahn-Banach theorem (general version), $\exists\ F : X \to \mathbb{R}$ s.t. F is an extension of f and $F(x) \leq p(x)\ \forall\ x \in X$.
Now, if $F(x) \geq 0$

$$|F(x)| = F(x) \leq p(x) = \|f\|_{Y^*}\|x\|_X$$

If $F(x) < 0$

$$|F(x)| = -F(x) = F(-x) \leq p(-x) = \|f\|_{Y^*}\|-x\|_X = \|f\|_{Y^*}\|x\|_X$$

$\forall\ x \in X$

$$|F(x)| \leq \|f\|_{Y^*}\|x\|_X$$

namely, $F \in X^*$ (it is bounded), and

$$\|F\|_{X^*} \leq \|f\|_{Y^*}$$

Also, $\|F\|_{X^*} \geq \|f\|_{Y^*}$ since F extends f:

$$\|F\|_{X^*} = \sup_{\|x\|_X \leq 1} |F(x)| \geq \sup_{\|y\|_Y \leq 1} |F(y)| = \sup_{\|y\|_X \leq 1, y \in Y} |f(y)| = \|f\|_{Y^*}$$

★

Consequence 1

Theorem 13.3
$(L^\infty(X))^*$ 'strictly contains' $L^1(X)$

Proof. We must show that $\exists\ L \in (L^\infty(X))^*$ s.t. $\nexists\ g \in L^1(X)$ s.t.

$$Lf = \int_X fg\, d\mu \quad \forall f \in L^\infty(X)$$

For simplicity, we consider $(X, \mathcal{M}, \mu) = ([-1,1], \mathcal{L}([-1,1]), \lambda)$. Let Y be the subspace of $L^\infty([-1,1])$ of the bounded continuous functions $\mathcal{C}^0([-1,1])$. On Y we define

$$\Lambda f = f(0) \quad \forall\ f \in Y$$

We can do it since $f \in \mathcal{C}^0([-1,1])$ (for elements in L^∞ we cannot speak about pointwise values!).
Λ is linear:

$$\Lambda(\alpha f + \beta g) = \alpha\Lambda f + \beta\Lambda g$$

Moreover, Λ is in Y^*:

$$|\Lambda f| = |f(0)| < \max_{[-1,1]} |f| = \|f\|_\infty$$

This proves that $\Lambda \in Y^*$, $\|\Lambda\|_{Y^*} \leq 1$. By Hahn-Banach, $\exists\ L \in (L^\infty(X))^*$ which is an extension of Λ, and is s.t.

$$\|L\|_{(L^\infty)^*}$$

Can we have

$$Lf = \int_{-1}^{1} fg\, d\mu \quad \text{for some } g \in L^1(X)?$$

Suppose by contradiction that this is true, take

$$f_n \in \mathcal{C}^0([-1,1])$$

defined in this way:

$$f_n(x) = \varphi(nx)$$

where φ is continuous, $\operatorname{supp}\varphi \subseteq \left[\frac{-1}{2}\frac{1}{2}\right]$

$$\varphi(0) = 1, \varphi(nx) = 0 \quad \forall\, x \text{ s.t. } |nx| > \frac{1}{2} \Leftrightarrow |x| > \frac{1}{2n}$$

By contradiction,

$$\sup f_n \subseteq \left[-\frac{1}{2n}, \frac{1}{2n}\right] \Rightarrow f_n(x) \to 0$$

Therefore, if $g \in L^1([-1,1])$ is s.t.

$$\int_{-1}^{1} f_n g \, d\lambda = L f_n$$

Then, on one side

$$\int_{-1}^{1} f_n g \, d\mu = L f_n = f_n(0) = 1 \quad \forall\, n \tag{1}$$

But on the other side

- $f_n(x)g(x) \to 0$ a.e. in $[-1,1]$
- $|f_n(x)g(x)| \leq g(x) \in L^1([-1,1])$

$$\overset{\text{DOM}}{\Rightarrow} \int_{-1}^{1} f_n g \, d\lambda \to 0 \tag{2}$$

But (1) and (2) are in contradiction. In conclusion, there is no $g \in L^!([-1,1])$ s.t.

$$\int_{-1}^{1} f g \, d\lambda = L f \quad \forall\, f \in L^\infty([-1,1])$$

★

Other consequences of the Hahn-Banach theorem

Corollary 13.1

X (real) normed space, $x_0 \in X \setminus \{0\}$. Then $\exists\, L_{x_0} \in X^*$ s.t.

$$\|L_{x_0}\|_{X^*} = 1 \text{ and } L_{x_0}(x_0) = \|x_0\|_X$$

***Proof*.** Take $Y = \{\lambda x_0 : \lambda \in \mathbb{R}\}$ (1-d vector space generated by x_0)

$$\begin{aligned} L_0 : \quad & Y \to \mathbb{R} \\ & \lambda x_0 \mapsto \lambda \|x_0\|_X \end{aligned}$$

This is linear and continuous on $Y \Rightarrow$ by Hahn-Banach (continuous extension) $\exists\, \tilde{L}_0 \in X^*$ s.t. $\tilde{L}_0$ extends L_0 and

$$\left\|\tilde{L}_0\right\|_{X^*} = \|L_0\|_{Y^*} = \sup_{\substack{\lambda x_0 \in Y \\ \|\lambda x_0\| = 1}} |L_0(\lambda x_0)| = \sup |\lambda \|x_0\|_X| = 1$$

Thus $\tilde{L}_0$ is precisely the desired functional.

$$\tilde{L}_0(x_0) = L_0 = \|x_0\|_X$$

and

$$\left\|\tilde{L}_0\right\|_{X^*} = 1$$

★

Corollary 13.2 (The bounded linear functionals separate points)
If $x, y \in X$ and $Lx = Ly \ \forall\, L \in X^* \Rightarrow x = y$ (if $x \neq y, \exists\, L \in X^*$ s.t. $Lx \neq Ly$)

Proof. Assume $x - y \neq 0$. Then, by the previous corollary, $\exists\, L \in X^*$ s.t.

$$\|L\|_{X^*} \text{ and } L(x-y) = \|x-y\|_X \Rightarrow Lx - Ly = L(x-y) = \|x-y\|_X \neq 0$$

★

Corollary 13.3
X normed space, Y closed subspace of X, $x_0 \in X \setminus Y$.
Then $\exists\, L \in X^*$ s.t. $L|_Y = 0$ and $Lx_0 \neq 0$

Reflexive spaces

X Banach space, X^* dual space.
Notation: $L \in X^* : Lx = L(x) = \langle L, x \rangle = \underset{X^*}{\langle} L, \underset{X}{x \rangle}$
$(X^*)^*$ dual space of X^* is called the **bidual** of X, denoted by X^*

$$X^{**} = \mathcal{L}(X^*, \mathbb{R})$$

We can describe many elements of X^{**} in the following way: for $x \in X$, define

$$\begin{aligned} \Lambda : \quad & X^* \to \mathbb{R} \\ & L \mapsto Lx = \underset{X^*}{\langle} L, \underset{X}{x \rangle} \end{aligned}$$

(Λ_x evaluates functionals in X^* in the point x).
Λ_x is linear:

$$\Lambda_x(\alpha L_1 + \beta L_2) = (\alpha L_1 + \beta L_2)(x) = \alpha L_1 x + \beta L_2 x = \alpha \Lambda_x L_1 + \beta \Lambda_x L_2$$

Moreover, it is bounded

$$|\Lambda_x(L)| = |Lx| \underset{L \in X^*}{\leq} \|L\|_{X^*} \|x\|_X \quad \forall\, L \in X^*$$

Moreover,

$$\|\Lambda_x\|_{\mathcal{L}(X^*, \mathbb{R})} = \sup_{L \neq 0} \frac{|\Lambda_x L|}{\|L\|_X^*}$$

We claim that $\|\Lambda_x\|_{\mathcal{L}} = \|x\|_X$. Indeed, by the first corollary of Hahn-Banach, given any $x \in X \setminus \{0\} \,\exists\, Lx \in X^*$

$$\exists\, L_x \in X^* \text{ s.t } |L_x x| = \|x\|_X, \text{ and } \|L_x\|_{X^*} = 1$$

$$\Rightarrow \sup_{L \neq 0} \frac{|\Lambda_x L|}{\|L\|_{X^*}} = \sup_{L \neq 0} \frac{|Lx|}{\|L\|_{X^*}} \geq \frac{|L_x x|}{\|L_x\|_{X^*}} = \|x\|_X$$
$$\Rightarrow \|\Lambda_x\|_{X^{**}} = \|x\|_X$$

Theorem 13.4
$\exists$ a map

$$\begin{aligned} \tau : \quad & X \to X^{**} \\ & x \mapsto \Lambda_x \end{aligned} \qquad \text{(Canonical Map)}$$

which is linear, continuous and an isometry. Namely, the canonical map is an isometric isomorphism from X into $\tau(X) \subseteq X^{**}$

<u>Question</u>: are there other elements in X^{**}?

Definition 13.2
If the canonical map is surjective, then we say that X is **reflexive**, $X \approxeq X^{**}$. Otherwise, $\tau(X)$ will be a strict close subspace of X.

Remark 13.2
X reflexive $\not\Leftarrow\Rightarrow$ X and X^{**} are isometrically isomorphic.

Theorem 13.5
X reflexive space. Then every closed subspace of X is reflexive.

Theorem 13.6
X Banach.

$$X \text{ reflexive } \Leftrightarrow X^* \text{ reflexive}$$

Theorem 13.7
X Banach.

- If X^* is separable $\Rightarrow$ X is separable
- If X is separable and reflexive $\Rightarrow$ X^* is separable

To show that a space is reflexive, it is convenient to introduce the following notion.

Definition 13.3
X Banach space. X is called **uniformly convex** if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t.

$$\forall\, x, y \in X \text{ with } \|x\| \leq 1, \|y\| \leq 1, \|x - y\| > \varepsilon$$

then we have

$$\left\| \frac{x+y}{2} \right\| < 1 - \delta$$

This is a quantitative version of the strict convexity.

Definition 13.4
$C \subset X$ is convex $\Leftrightarrow \forall\, x, y \in C : \frac{x+y}{2} \in C$
$C \subset X$ is **strictly convex** $\Leftrightarrow \forall\, x, y \in C : \frac{x+y}{2} \in C^{\circ}$

Roughly speaking, X is uniformly convex if $\overline{B_1(0)}$ is strictly convex in a quantitative way.

Theorem 13.8 (Milman-Pettis)
Every uniformly convex Banach space is reflexive.

<u>Recap on reflexivity:</u>
X Banach space. $X^{**} = (X^*)^*$ is the **bidual space**, $\mathcal{L}(X^*, \mathbb{R})$

$$\forall\, x \in X\ \exists\, \Lambda_x : X^* \to \mathbb{R} \text{ defined by } \Lambda_x(L) = Lx \quad \forall\, L \in X^*$$

We proved that $\Lambda_x \in X^{**}$. Thus we can define the **canonical map**:

$$\begin{aligned} \tau: \ X &\to X^{**} \\ x &\mapsto \Lambda_x \end{aligned} \tag{Canonical Map}$$

We stated that τ is an isometric isomorphism from X into $\tau(X)$. This is true but for our purpose it's even too much, and it's difficult to prove in details. However, we can prove a slightly weaker result

Theorem 13.9
τ is linear, continuous, and is an isometry

$$\|\tau(x)\|_{X^{**}} = \|x\|_X \quad \forall\, x \in X$$

Moreover, τ is injective. If τ is also surjective, it is an isometric isomorphism between X and X^{**}

***Proof*.** There are two parts:

- τ is linear and continuous: exercise.

  τ is an isometry: $\|\tau(x)\|_{X^{**}} = \|\Lambda_x\|_{X^{**}} = \|x\|_X$

  τ is injective: $x \neq y \Rightarrow \tau(x) \neq \tau(y)$?

  $x \neq y \Rightarrow$ by the second corollary to Hahn-Banach $\exists\, L \in X^*$ s.t. $Lx \neq Ly$.

  $$\underset{X^{**}}{\langle}\ \tau(x), L \underset{X^*}{\rangle} = \Lambda_x(L) = Lx \neq Ly = \Lambda_y(L) = \underset{X^{**}}{\langle}\ \tau(x), L \underset{X^*}{\rangle}$$

  Then, $\tau(x) \neq \tau(y)$ and τ is injective.

- Let now τ be surjective. Then $\tau \in \mathcal{L}(x, X^{**})$ and is bijective $\Rightarrow$ by a corollary of the open map theorem, $\tau^{-1} \in \mathcal{L}(X^{**}, X)$

★

Definition 13.5
X is reflexive if τ is surjective. In this case, τ is an isometric isomorphism between X and X^{**}

We formally mentioned that
Theorem 13.10
If $(X, \|\cdot\|)$ is uniformly convex $\Rightarrow (X, \|\cdot\|)$ is reflexive.

<u>**Remarks**</u>**:**
Proposition 13.2
If $(X, \|\cdot\|)$ is uniformly convex $\Rightarrow \overline{B_1(0)}$ is strictly convex.

***Proof*.** Is it true that if $x, y \in \overline{B_1(0)}$, then $\frac{x+y}{2} \in B_1(0)$? Since $(X, \|\cdot\|)$ is uniformly convex, we know that $(\|x-y\| =: \overline{\varepsilon} > 0)$

$$\forall\, \overline{\varepsilon} > 0 \quad \exists\, \overline{\delta} > 0 \text{ s.t. } \|x\| \leq 1\ \|y\| \leq\ \|x-y\| > \frac{\overline{\varepsilon}}{2} \Rightarrow \left\|\frac{x+y}{2}\right\| < 1 - \overline{delta}$$

In particular,
$$\frac{x+y}{2} < 1 - \overline{\delta} < 1 \Rightarrow \frac{x+y}{2} \in B_1(0)$$

★

Consequence: $(\mathbb{R}^2, \|\cdot\|_1)$ and $(\mathbb{R}^2, \|\cdot\|_\infty)$ are not uniformly convex.
Proposition 13.3
$(\mathbb{R}^2, \|\cdot\|_2)$ is uniformly convex

Proof. Suppose by contradiction that this is false: $\exists\, \overline{\varepsilon} > 0$ and $\{x_n\}, \{y_n\} \subset \overline{B_1(0)}$ s.t.

$$\|x_n - y_n\| > \overline{\varepsilon}, \text{ but } \left\|\frac{x_n + y_n}{2}\right\| \geq 1 \tag{*}$$

$\overline{B_1(0)}$ is compact (since we are in $\mathbb{R}^2$) $\Rightarrow$ UTS $x_n \to \overline{x}$, $y_n \to \overline{y}$ as $n \to \infty$. Taking the limit in (*), we deduce that $\overline{x}, \overline{y} \in \overline{B_1(0)}$

$$\|\overline{x} - \overline{y}\| \geq \overline{\varepsilon} \text{ , and } \left\|\frac{\overline{x} + \overline{y}}{2}\right\| \geq 1$$

This is not possible, since $\overline{B_1(0)}$ is strictly convex. ★

Theorem 13.11
$(X, \mathcal{M}, \mu)$ complete measure space. Then $L^p(X)$ is reflexive $\forall\, p \in (1, \infty)$

Proof. $(L^p(X), \|\cdot\|_p)$ is uniformly convex $\forall\, p \in (1, \infty)$ (Clarkson inequalities) ★

$L^1(X)$ and $L^\infty(X)$ are not uniformly convex, and not reflexive.

Dual space of L^p

Theorem 13.12 (Riesz representation theorem)
$(X, \mathcal{M}, \mu)$ complete measure space, $p \in (1, \infty)$. Then

$$\forall\, L \in (L^p(X))^* \quad \exists!\, g \in L^{p'}(X)$$

with p' conjugate exponent s.t. $L = L_g$, namely

$$Lf = \int_X fg\, d\mu \qquad \forall\, f \in L^p(X)$$

Moreover $\|L_g\|_{(L^p)^*} = \|g\|_{p'}$
Thus: $T : g \in L^{p'} \mapsto L_g \in (L^p)^*$ is an isometric isomorphism.

Proof. $1 < p < \infty$. Consider $T : L^{p'} \to (L^p)^*$ with $g \mapsto Tg : \langle Tg, f\rangle = \int_X fg\, d\mu$ (namely $Tg = L_g$). We already know that

$$\|Tg\|_* = \|L_g\|_* = \|g\|_{p'}$$

T is injective: for exercise.
T is surjective. Indeed, let $F := T(L^{p'}) \subseteq (L^p)^*$ subspace. Since T is an isometry and $L^{p'}$ is complete, it can be shown that $T(L^{p'})$ is also complete $\Rightarrow T(L^{p'})$ is closed.
If by contradiction $F \neq (L^p)^*$, then we can apply corollary 3 to Hahn Banach $(X = (L^p)^*), Y = F, x_0 = \lambda$:

$$\exists h \in (L^p)^{**} \text{ s.t. } \langle h, \lambda\rangle \neq 0 \text{ and } h|_F = 0 : \langle h, Tg\rangle = 0 \quad \forall g \in L^{p'} \tag{1}$$

But L^p is reflexive ($1 < p < \infty$), then $h \in L^p \setminus \{0\}$:

$$\langle h, Tg\rangle = (Tg)h = \int_X hg\, d\mu$$

Therefore, (1) tells us that

$$\int_X hg\, d\mu = 0 \quad \forall\, g \in L^{p'}(X)$$

Take $g = |h|^{p-2}h$. Therefore

$$0 = \int_X hg\, d\mu = \int_X h|h|^{p-2}h\, d\mu = \int_X |h|^p\, d\mu \Rightarrow h = 0 \in L^p$$

which is the desired contradiction.
T is an isomorphism: for exercise. ★

$$(L^p)^* = L^{p'}$$

Remark 13.3
$p = 1$. One can prove the:

Theorem 13.13
$(X; \mathcal{M}, \mu)$ complete measure space, σ-finite. Then $\forall\, L \in (L^1(X))^* \quad \exists!\, g \in L^\infty(X)$ s.t. $L = L_g$:

$$Lf = \int_X fg\, d\mu \quad \forall f \in L^1(X)$$

Moreover, the map $L \in (L^1)^* \mapsto g \in L^\infty$ is an isometric isomorphism.

Recall that $(X, \mathcal{M}, \mu)$ is finite if $\mu(X) < \infty$.

Definition 13.6
$(X, \mathcal{M}, \mu)$ is σ-finite if either $\mu(X) < \infty$, or $X = \sum_{n=1}^{\infty} X_n$, where $\mu(X_n) < \infty$

14 Weak Convergence

We know that $\overline{B_1(0)}$ is never compact in ∞ dimension. This is a problem is in proving convergence of sequences. A way to approach this issue consists in weakening the notion of convergence.

Definition 14.1
X Banach space. $\{x_n\} \subset X$ sequence, $x \in X$. We say that x_n tends to x **weakly** (in X) as $n \to \infty$, $x_n \rightharpoonup x$ in X, if

$$Lx_n \to Lx \qquad \forall\, L \in X^*$$

Remark 14.1
Assume that $x_n \to x$ in X, namely $\|x_n - x\|_X \to 0$. If $f : X \to \mathbb{R}$ is continuous, then

$$\lim_{n\to\infty} f(x_n) = f(x)$$

In particular, this is true if $f = L \in X^*$:

$$\begin{aligned}
x_n \to x \in X \Rightarrow\ & Lx_n \to Lx \qquad\qquad \forall\, L \in X^* \\
& x_n \rightharpoonup x \in X \\
x_n \to x \in X \Rightarrow\ & x_n \rightharpoonup x \text{ weakly } \in X \\
\not\Leftarrow\ &
\end{aligned}$$

Remark 14.2
We will be interested in weak convergence in L^p.
If $p \in [1, \infty)$, then

$$f_n \rightharpoonup f \text{ weakly in } L^p(X) \Leftrightarrow \int_X f_n g\, d\mu \to \int_X fg\, d\mu \quad \forall\, g \in L^{p'}$$

Similarily, in $l^p = L^P(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_c)$

$$x_n \rightharpoonup x \text{ weakly in } l^p \Leftrightarrow \sum_{k=1}^{\infty} x_n^{(k)} y^{(k)} \to \sum_{k=1}^{\infty} x^{(k)} y^{(k)} \quad \forall\, y \in l^{p'}$$

Proposition 14.1
The weak limit is unique (if it exists)

Proof. By contradiction, suppose that $\exists\{x_n\} \subset X$ s.t. $x_n \rightharpoonup x_1$, $x_n \rightharpoonup x_2$ weakly in X, $x_1 \neq x_2$. Then

$$\begin{aligned} &Lx_n \to Lx_1 && \forall\, L \in X^* \\ &Lx_n \to Lx_2 && \forall\, L \in X^* \\ &\Rightarrow Lx_1 = Lx_2 && \forall\, L \in X^* \end{aligned}$$

By Hahn Banach (corollary 2), this implies $x_1 = x_2$, a contradiction. ★

Proposition 14.2
If $x_n \rightharpoonup x$ weakly in X, then $\{x_n\}$ is bounded, and

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\| \qquad \text{weak lower semi continuity of } \|\cdot\|$$

Proof.

- $\{x_n\}$ is bounded

  $x_n \rightharpoonup x$ weakly $\Rightarrow \{Lx_n\}$ is bounded in $\mathbb{R}$, $\forall\, L \in X^*$. Consider $\Lambda_n \in X^{**}$ def by

  $$\Lambda_n L = Lx_n \quad \forall\, L \in X^*$$

  $\forall\, L \in X^* \exists\, M_L > 0$ s.t.

  $$|\Lambda_n L| = |Lx_n| \leq M_L \qquad \forall\, n \qquad \text{PB}$$

  (pointwise boundedness of $\{\Lambda_n\} \subset X^{**}$)

  $$\Lambda_n : X^* = \mathcal{L}(X, \mathbb{R}) \to \mathbb{R}$$

  By Banach Steinhaus, $\{\Lambda_n\}$ is uniformly bounded:

  $$\sup_n \|\Lambda_n\|_{\mathcal{L}(X^*,\mathbb{R})} \leq M$$

  Moreover, by Hahn Banach, $\forall\, n \in \mathbb{N}$ $\exists L_n \in X^*$ s.t. $\|L_n\|_* = 1$ and $L_n x_n = \|x_n\|$. Therefore

  $$\|x_n\| = |L_n x_n| = |\Lambda_n L_n| \leq \|\Lambda_n\|_{\mathcal{L}} \|L_n\|_* \leq M \quad \forall n \in \mathbb{N}$$

- $x_n \rightharpoonup x$ weakly.

  By corollary 1 of Hahn Banach, $\exists L_x \in x^*$ s.t. $\|L_x\|_* = 1$ and $L_x x = \|x\|$. Then

  $$\|x\| = |L_x x| = \lim_n |L_x x_n| = \liminf_n |L_x x_n| \leq \liminf_n \|L_x\|_* \|x_n\|_X = \liminf_n \|x_n\|_X$$

★

Proposition 14.3
$x_n \rightharpoonup x$ in X weakly, and $L_n \to L$ (strongly) in X^*. Then

$$L_n x_n \to Lx \quad \text{in } \mathbb{R}$$

Proposition 14.4
X, Y Banach, $T \in \mathcal{L}(X, Y)$

$$x_n \rightharpoonup x \text{ weakly } \Rightarrow Tx_n \rightharpoonup Tx \text{ weakly}$$

We introduced the weak convergence. X banach space. $x_n \subset X$ converges weakly to x, $x_n \rightharpoonup x$ weakly in X, if

$$Lx_n \to Lx \text{ in } \mathbb{R}, \quad \forall\, L \in X^* = \mathcal{L}(X, \mathbb{R})$$

Recall that:

- $x_n \to x$ strongly in X, namely $\|x_n - x\|_X \to 0 \Rightarrow x_n \rightharpoonup x$ and $\not\Leftarrow$
- $x_n \rightharpoonup x \Rightarrow \{x_n\}$ is bounded, the weak limit x is unique, and

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|$$

Remark 14.3
In $\mathbb{R}^n$ (or any finite dimensional Banach space) $x_n \rightharpoonup x$ weakly $\Leftrightarrow x_n \to x$ strongly (ex.)

With the same philosophy we introduce:
Definition 14.2
X Banach $\Rightarrow X^*$ is Banach as well.

Definition 14.3
A sequence $\{L_n\} \subset X^*$ is **weakly*** convergent to $L \in X^*$, namely $L_n \rightharpoonup^* L$ in X^*, if

$$L_n x \to Lx \in \mathbb{R} \quad \forall\, x \in X$$

Remark 14.4
Observe that a sequence $\{L_n\}$ tends weakly to L in X^* if

$$\Lambda L_n \to \Lambda L \quad \forall \Lambda \in X^{**}$$

We know that $\exists\, \tau : X \to X^{**}$ canonical map s.t.

$$\underset{X^{**}}{\langle}\, \tau(x), L \underset{X^*}{\rangle} = Lx \quad \forall\, L \in X^*$$

Thus $L_n \rightharpoonup L$ weakly in $X^* \to \langle \tau(x), L_n \rangle \to \langle \tau(x), L \rangle\ \forall x \in X$: namely

$$L_n x \to Lx \qquad \forall\, x \in X$$

namely $L_n \rightharpoonup^*$ weakly* in X^*. In general the converse is false. However
Proposition 14.5
If X is reflexive, then $L_n \rightharpoonup L$ weakly in $X^* \Leftrightarrow L_n \rightharpoonup^* L$ weakly* in X^*

Proof. If X is reflexive, every element Λ of X^{**} is of type $\Lambda = \tau(x)$ for some x ★

Proposition 14.6
X banach space, X^* dual space, $L_n \rightharpoonup^* L$ in X^*. Then

- The weak * limit is unique
- $\{L_n\}$ is bounded
- $\|L\|_{X^*} \leq \liminf_{n\to\infty} \|L_n\|_{X^*}$
- If in addition $x_n \to x$ strongly in $X \Rightarrow L_n x_n \to Lx$

Theorem 14.1 (Banach Alaoglu)
X separable Banach space. Then every bounded sequence in X^* has a weakly* convergent subsequence. (bounded sets in X^* sequentially compact for the weak* convergence)

Proof. $\{L_n\}$ bounded sequence in X^*, namely

$$\sup_n \|L_n\|_{X^*} = M < \infty$$

Since X is separable, $\exists\ \{x_k\}_{k\in\mathbb{N}}$ dense in X. Now, consider $\{L_n x_1\}$: it is bounded in $\mathbb{R}$:

$$|L_n x_1| \le \|L_n\|_{X^*}\|x_1\|_X \le M\|x_1\|_X < \infty$$

$\Rightarrow \exists\ \{L_{n_j}\}$ s.t. $L_{n_j}x_1 \to l_j$ in $\mathbb{R}$. Now, consider $\{L_{n_j}x_2\}$: it is bounded,

$$\left|L_{n_j}x_2\right| \le \left\|L_{n_j}\right\|_{X^*}\|x_2\|_X \le M\|x_2\|_X < \infty$$

$\Rightarrow \exists\ \{L_{n_{ij}}\}$ subsequence of $\{L_{n_j}\}$ s.t. $L_{n_{ij}}x_2 \to l_2$ in $\mathbb{R}$ We can iterate the process. $\forall\ k \quad \{L_n^k\}$ is a subsequence of $\{L_n^{k-1}\}$. $\Rightarrow \{L_n^k\}$ is a subsequence of $\{L_n^j\}$ $\forall i < k$. In particular,

$$L_n^k x_j \to l_j \quad \forall\ j \le k$$

We pick up $T_n = L_n^n$ (diagonal selection). By construction, $\forall m \in \mathbb{N}$ fixed, $\{T_n : n \ge m\}$ is a subsequence of $\{L_n^m : n \ge m\}$

$$\Rightarrow T_n x_m \to l_m \qquad \text{as } n \to \infty$$

We want to show now that $T_n x \to l_x$ $\forall x \in X$, and that $l_x = Tx$ is such that $T \in X^*$. Since $\{x_k\}$ is dense, $\forall\ x \in X$ and $\forall\ \varepsilon > 0\ \exists\ k \in \mathbb{N}$ s.t.

$$\|x - x_k\|_X < \frac{\varepsilon}{2M}$$

Thus

$$\begin{aligned}
|T_n x - T_m x| &\le |T_n x - T_n x_k| + |T_n x_k - T_m x_k| + |T_m x_k - T_m x| \le \\
&\le \|T_n\|_{X^*}\|x - x_k\|_X + |T_n x_k - T_m x_k| + \|T_m\|_{X^*}\|x - x_k\|_X \le \\
&\le M\frac{\varepsilon}{2M} + |T_n x_k - T_m x_k| + M\frac{\varepsilon}{2M} < \varepsilon + |T_n x_k - T_m x_k| < 2\varepsilon
\end{aligned}$$

$\forall\ n, m > \overline{n}$, since $\{T_n x_k\}$ is convergent and so a Cauchy sequence.
This means that $\{T_n x\}$ is a Cauchy sequence in $\mathbb{R}$

$$T_n x \to l_x \text{ in } \mathbb{R} \quad \forall x \in \mathbb{R}$$

It only remains to show that $l_x = Tx$ for some $T \in X^*$. This is a consequence of a corollary of Banach Steinhaus.
To sum up: $\{L_n\}$ bounded in X^*

$$\Rightarrow \exists\ \{T_n\} \text{ subsequence s.t. } T_n x \to Tx$$

for every $x \in X$, namely $T_n \rightharpoonup^* T$ in X^* ★

Theorem 14.2 (Variant of BA for reflexive spaces)
X reflexive and Banach. Then every bounded sequence in X has a weakly convergent subsequence

Proof. For simplicity, we assume that X is separable (not necessary). X separable and reflexive $\Rightarrow X^*$ is separable. $\tau : X \to X^{**}$ canonical map: it is an isometric isometry.
$\{x_n\}$ bounded sequence in $X \Leftrightarrow \{\tau(x_n)\}$ is bounded in $X^{**} = (X^*)^*$
$\Rightarrow$ by Banach Alaoglu, $\exists \, \{x_{n_k}\}$ s.t. $\tau(x_{n_k}) \rightharpoonup^* \Lambda$ in X^{**}:

$$\underset{X^{**}}{\langle} \tau(x_{n_k}), L \underset{X^*}{\rangle} \to \underset{X^{**}}{\langle} \Lambda, L \underset{X^*}{\rangle} \qquad K \to \infty$$

$\forall \, L \in X^*$. Since X is reflexive, $\forall \, \Lambda \in X^{**} \, \exists! \, x \in X$ s.t. $\Lambda = \tau(x)$. Therefore,

$$Lx_{n_k} = \underset{X^{**}}{\langle} \tau(x_{n_k}), L \underset{X^*}{\rangle} \to \underset{X^{**}}{\langle} \Lambda, L \underset{X^*}{\rangle} = Lx$$

$\forall \, L \in X^*$. We proved that

$$\lim_{k \to \infty} Lx_{n_k} = Lx \qquad \forall \, L \in X^*$$

namely $x_{n_k} \rightharpoonup x$ in X ★

15 Compact Operators

X, Y Banach spaces.

Definition 15.1
A linear operator $K : X \to Y$ is said to be compact if $\forall \, E \subseteq X$ bounded, the set $K(E)$ is relatively compact, namely $\overline{K(E)}$ is compact.
Equivalently, K is compact if $\forall \, \{x_n\} \subset X$ bounded, the sequence $\{K(x_n)\}$ has a strongly convergent subsequence.

Proposition 15.1
$K : X \to Y$ linear and compact. Then $K \in \mathcal{L}(X, Y)$

Proof. Define $B := \overline{B_1(0)}$ in X. If K is compact $\Rightarrow K(B)$ is relatively compact $\Rightarrow \overline{K(B)}$ is compact $\Rightarrow \overline{K(B)}$ is bounded $\Rightarrow K(B)$ is bounded: $\exists \, M > 0$ s.t.

$$\|Kx\|_Y \le M \qquad \forall \, x \in \overline{B_1(0)} = B$$

$$\Rightarrow \underbrace{\sup_{\|x\| \le 1} \|Kx\|_Y}_{\substack{\shortparallel \\ \|K\|_{\mathcal{L}(X,Y)}}} \le M$$

★

Definition 15.2
$T \in \mathcal{L}(X, Y)$ has finite rank if

$$\text{the image of } T = \underset{\dim(T(X))}{\{y \in Y : y = Tx\}} \text{ for some } x \in X < \infty$$

$T(X) \subset Y$ is a subspace.

Proposition 15.2
$T \in \mathcal{L}(X, Y)$ has finite rank $\Rightarrow T$ is compact.

Proof. $A \subset X$ bounded. Since $T \in \mathcal{L}(X, Y)$, then $T(A)$ is bounded. $T(A) \subset T(X) \approx \mathbb{R}^n$, since T has finite rank.
Thus $T(A)$ is a bounded set of $\mathbb{R}^n \Rightarrow T(A)$ is relatively compact. ★

Definition 15.3
We denote by $\mathcal{K}(X,Y)$ the class of linear compact operators from X to Y. This is a linear subspace.
If $Y = X$, we write $\mathcal{K}(X)$

Proposition 15.3
X, Y Banach spaces, $T : X \to Y$ linear and compact, Y in ∞ dim. Then T cannot be surjective.

Proof. Recall that C compact set, $S \subset C$ closed $\Rightarrow S$ is compact (in any metric space)
Assume by contradiction that K is surjective. By the OMT, T is an open map. Take

$$\varnothing \neq A \subset X$$

open and bounded. $T(A)$ is relatively compact (since T is compact), and is open (since T is an open map) and $\neq \varnothing$

$$\Rightarrow T(A) \supset B_r(y_0)$$

for some $y_0 \in Y$ and $r > 0$. Thus

$$\overline{T(A)} \supset \overline{B_r(y_0)} \Rightarrow \overline{B_R(y_0)}$$

is compact in Y. This contradicts the Riesz theorem, since in ∞ dimension balls are never compact. ★

Proposition 15.4
X, Y, Z Banach spaces. $T \in \mathcal{L}(X,Y)$, $S \in \mathcal{K}(Y,Z)$ (or $T \in \mathcal{K}(X,Y)$, $S \in \mathcal{L}(Y,Z)$). Then $S \circ T$ is compact.

Theorem 15.1
$\mathcal{K}(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$. $\Rightarrow (\mathcal{K}(X,Y), \|\cdot\|_{\mathcal{K}(X,Y)})$ is a Banach space.

Consequence: if we want to check that $T \in \mathcal{L}(X,Y)$ is compact, we can prove that $\exists \{T_n\} \subseteq \mathcal{K}(X,Y)$ s.t.

$$\|T_n - T\|_{\mathcal{L}} \to 0$$

Since $\mathcal{K}(X,Y)$, it follows that T is compact.

16 Hilbert Spaces

Definition 16.1
H vector space on $\mathbb{R}$. A function $p : H \times H \to \mathbb{R}$ is called **scalar (or inner) product** if it is positive definite, symmetric, and bilinear; namely if

(1) $p(x,x) \geq 0 \ \forall \, x \in H$ and $p(x,x) = 0 \Rightarrow x = 0$

(2) $p(x,y) = p(y,x) \ \forall \, x,y \in H$

(3) $p(\alpha x_1 + \beta x_2, y) = \alpha p(x_1,y) + \beta p(x_2,y) \ \forall \, \alpha, \beta \in \mathbb{R}$, $x_1, x_2, y \in H$

Notation: $p(x,y) = \langle x,y \rangle = (x,y) = x \cdot y$

Definition 16.2
A vector space H with a scalar product is called a pre Hilbertian space.

Proposition 16.1
$(H, \langle \cdot, \cdot \rangle)$ pre Hilbertian space.

- Cauchy Schwarz inequality

$$|\langle x,y\rangle| \leq \sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle} \quad \forall\, x,y \in H$$

- $\sqrt{\langle x,x\rangle} =: \|x\|$ is a norm on H

$(H, \langle\cdot,\cdot\rangle$ pre Hilbert $\to (H, \|\cdot\|)$ normed space $\to (H,d)$ metric space where $d(x,y) = \|x-y\|$

Definition 16.3
We say that $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space if $(H, \|\cdot\|)$ is a Banach space. (namely, if (H,d) is a complete metric space)

Examples:

- $\mathbb{R}^n$, $\langle x,y\rangle = \sum_{i=1}^n x_i y_i$
- $L^2(X,\mathcal{M},\mu)$ $(X,\mathcal{M},\mu)$ complete measure space.
  $\langle f,g\rangle = \int_X fg\,d\mu$. $\|f\| = (\int_X f^2\,d\mu)^{\frac{1}{2}} = \|f\|_2$. $(L^2(X), \|\cdot\|_2)$ is a Banach space $\Rightarrow$ $(L^2(X), \langle\cdot,\cdot\rangle)$ is a Hilbert space.
- l^2 is a Hilbert space. $\langle x,y\rangle = \sum_{k=1}^\infty x^{(k)}y^{(k)}$, $x = (x^{(k)})$, $y = (y^{(k)})$
- $(\mathcal{C}^0([a,b]), \langle\cdot,\cdot\rangle)$ is a pre Hilbertian space. $(\mathcal{C}^0([a,b]), \|\cdot\|_2)$ is not a Banach space.

Definition 16.4
x,y are orthogonal if $\langle x,y\rangle = 0$. We write $x \perp y$

Remark 16.1
Hilbert spaces are particular cases of Banach spaces. The converse is not true. In any Hilbert space, the norm induced by $\langle\cdot,\cdot\rangle$ must satisfy the parallelogram rule

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall\, x,y \in H \qquad \text{PR}$$

Proposition 16.2
H Banach space with respect to $\|\cdot\|$. If $\|\cdot\|$ satisfies (PR), then H is a Hilbert space with scalar product

$$\langle x,y\rangle := \frac{1}{2}[\|x+y\|^2 - \|x\|^2 - \|y\|^2], \quad \langle x,x\rangle = \|x\|^2$$

Consequence: we can check that a Banach space is not a Hilbert space by showing that (PR) does not hold. Ex: $(L^p, \|\cdot\|_p)$ is not a Hilbert space $\forall\, p \neq 2$. The same for $(\mathcal{C}^0([a,b]), \|\cdot\|_\infty)$

Orthogonal projection

Recall:

Definition 16.5
$C \subset H$ is convex if $\forall\, x,y \in C$: $\frac{x+y}{2} \in C$

Definition 16.6
$S \subset H$, $f \in H$.

$$dist(f,S) = \inf_{g\in S} \|f-g\|$$

Theorem 16.1 (projection on closed convex sets)
H Hilbert space. Let $S \subseteq H$ non empty, closed, convex. Then $\forall\, f \in H \quad \exists!\, h \in S$ s.t.

$$\|f-h\| = dist(f,S) = \min_{g\in S} \|f-g\| \tag{1}$$

Moreover, h is characterized by the variational inequality:

$$\langle f-h, g-h\rangle \leq 0 \quad \forall\, g \in S \tag{*}$$

namely h is the projection of f on S (f satisfies (1)) $\Leftrightarrow$ $(*)$ holds

Remark 16.2
h satisfies 1: h is the projection of f on S, $h = P_S f$

Proof. Only of the existence of h.
$S \subset H$. $dist(f,S) > 0$ ($f \notin S$). $\exists\, \{v_n\} \subset S$ s.t.

$$\|v_n - f\| \to d := dist(f,S)$$

We show that $\{v_n\}$ is a Cauchy sequence. Let m,n, then $\frac{v_m+v_n}{2} \in S$, since S is convex. Then

$$\left\| f - \frac{v_m+v_n}{2} \right\| \geq d \Rightarrow \|2f-(v_m+v_n)\| \geq 2d \tag{2}$$

By the (PR), with with $x = f - v_n$, $x = v_m - f$

$$\|v_m - v_n\|^2 = \|v_m \pm f - v_n\|^2 \overset{PR}{=} \|x+y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \|x-y\|^2 =$$

$$= 2\|f-v_n\|^2 + 2\|v_m - f\|^2 - \|2f-(v_m+v_n)\|^2 \overset{(2)}{\leq} 2\|f-v_n\|^2 + 2\|v_m - f\|^2 - 4d^2 \leq (*)$$

Up to now, we only used that $v_n, v_m \in S$. Since $\|v_n - f\|^2 \to d^2$ as $n \to \infty$, $\forall\, \varepsilon > 0\ \exists\, \overline{n}$ s.t. $n,m > \overline{n}$

$$\Rightarrow \|v_n - f\|^2 < (d+\varepsilon)^2 \quad \|v_m - f\|^2 < (d+\varepsilon)^2$$

Coming back to $(*) < 4((d+\varepsilon)^2 - d^2)$, provided that $n,m > \overline{n}$. Since ε was arbitrarily chosen, the right hand side can be made arbitrary small (it tends to 0 if $\varepsilon \to 0$). We proved that we can make $\|v_n - v_m\|^2$ arbitrarily small, provided that n,m are sufficiently large.
Namely $\{v_n\}$ is Cauchy. Since $(H, \|\cdot\|)$ is Banach, $\exists\, v \in H$ s.t. $v_n \to v$. $v \in S$, since S is closed. And, by continuity,

$$\|f - v\| = \lim_n \|f - v_n\| = d$$

So v is the desired h.
About uniqueness.
Let $\overline{v}$ and v' 2 elements in S such that

$$\|f - \overline{v}\| = \|f - v'\| = d$$

Then, exactly as before,

$$\|\overline{v} - v'\|^2 = 2(\|\overline{v} - f\|^2 + \|v' - f\|^2) - \|2f - (\overline{v} + v')\|^2 \leq 2(d^2+d^2) - 4d^2 = 0$$

$\Rightarrow \overline{v} = v'$ ★

Remark 16.3
A particular case: S closed subspace (it is always convex). In this case, the variational inequality becomes an equality:

$$h = P_S f \Leftrightarrow \langle f-h, g\rangle = 0 \quad \forall\, g \in S$$

H Hilbert space.
Definition 16.7
$S \subset H$ subset. We define the **orthogonal complement** of S as

$$S^\perp = \{x \in H : \langle x, y \rangle = 0 \quad \forall\, y \in S\}$$

Ex: $S^\perp$ is always a closed subspace of H Ex: if S is a subspace, then $S \cap S^\perp = \{0\}$
Definition 16.8
V, W subspace of H, orthogonal one to each other:

$$\forall\, v \in V, \quad w \in W : v \perp w$$

We can define the **orthogonal sum** of V and W as

$$V \oplus W = \{v + w : v \in V, w \in W\}$$

Ex: if $x \in V \oplus W \Rightarrow \exists!\ (v, w) \in V \times W$ s.t. $x = v + w$
Theorem 16.2
H Hilbert space. Let $V \subseteq H$ be a closed subspace. Then

$$H = V \oplus V^\perp$$

Definition 16.9
From the theorem, given any $x \in H$ we can define

$$\begin{aligned} P_v :\ & H \to V \\ & x = v + w \mapsto v \\ P_{v^\perp} :\ & H \to V^\perp \\ & x \mapsto w \end{aligned} \qquad \text{orthogonal projections}$$

Ex: P_v and $P_{v^\perp}$ are linear bounded operators, with norms 1.

Dual space of a Hilbert space

Observe that, if $y \in H$, then we can define $\Lambda_y : H \to \mathbb{R}$ as

$$\Lambda_y x = \langle y, x \rangle$$

It is linear ($\langle \cdot, \cdot \rangle$ is bilinear), and it is bounded:

$$|\Lambda_y x| = |\langle y, x \rangle| \leq \|y\| \|x\| \quad \forall\, x, y$$

$\Rightarrow \Lambda_y$ is bounded, with $\|\Lambda_y\|_* \leq \|y\|$
Moreover,

$$\Lambda_y \left(\frac{y}{\|y\|} \right) = \langle y, \frac{y}{\|y\|} \rangle = \|y\|$$

$$\Rightarrow \|\Lambda_y\|_* = \sup_{\|x\| \leq 1} |\Lambda_y x| \geq \left| \Lambda_y (\frac{y}{\|y\|}) \right| = \|y\|$$

Thus $\|\Lambda_y\|_* = \|y\|$, and the map

$$\begin{aligned} i :\ & H \to H^* \\ & y \mapsto \Lambda_y \end{aligned}$$

is an isometry from H into $i(H) \subset H^*$.
Are there other elements in H^*?

Theorem 16.3 (Riesz Representation Theorem)
$\forall\, \Lambda \in H^* \; \exists!\; y \in H$ s.t. $\Lambda = \Lambda_y$, namely

$$\Lambda x = \langle y, x\rangle \qquad \forall x \in H$$

Moreover, the map i is an isometric isomorphism. We can identify H^* with H

Corollary 16.1
Any Hilbert space is reflexive.

Remark 16.4
Any Hilbert space is uniformly convex.

- Riesz in L^p: L^p is uniformly convex $\Rightarrow$ L^p is reflexive. We used this fact to prove Riesz in L^p
- Riesz in Hilbert: direct proof of $H^* = H \Rightarrow H$ is reflexive.

Both strategies can be adopted in both contexts.

Proof.

- We show that $\forall\, \Lambda \in H^* \; \exists\, y \in H$ s.t. $\Lambda = \Lambda_y$

  If $\Lambda = 0 \Rightarrow \Lambda = \Lambda_0$ ($\Lambda_0 x = \langle 0, x\rangle = 0$)

  Suppose $\Lambda \neq 0$. $\ker(\Lambda) = \Lambda^{-1}(\{0\})$ is a closed (since Λ is continuous) subspace, $\neq H$. $\Rightarrow$ we consider $\ker(\Lambda)^{\perp} \neq \{0\}$. Let

  $$z \in \ker(\Lambda)^{\perp}, \quad \|z\| = 1$$

  For $x \in H$, we have

  $$x - \frac{\Lambda x}{\Lambda z} z \in \ker(\Lambda)$$

  Indeed, $\Lambda\left(\frac{\Lambda x}{\Lambda z} z\right) \overset{linearity}{=} \Lambda x - \frac{\Lambda x}{\Lambda z}\Lambda z = 0$. Then, since z is orthogonal to any element of $\ker(\Lambda)$,

  $$\langle z, x - \frac{\Lambda x}{\Lambda z} z\rangle = 0 \qquad \forall\, x \in H$$

  $\langle\cdot,\cdot\rangle$ is bilinear: the left hand side is

  $$\langle z, x\rangle - \frac{\Lambda x}{\Lambda z}\|z\|^2 \Rightarrow \langle z, x\rangle = \frac{\Lambda x}{\Lambda z}$$

  $$\Lambda x = \langle (\Lambda z) z, x\rangle \qquad \forall\, x \in H$$

  So the thesis is proved for $y = (\Lambda z) z$.

- The uniqueness of y is easy.

  $$\langle x, y_1\rangle = \langle x, y_2\rangle \quad \forall\, x \in H$$

  Then $\langle x, y_1 - y_2\rangle = 0 \;\forall\, x \in H$. We choose $x = y_1 - y_2$:

  $$\|y_1 - y_2\|^2 = 0 \Rightarrow y_1 = y_2$$

★

Consequence: H Hilbert space.

$$x_n \rightharpoonup x \text{ weakly in } H \Leftrightarrow \langle x_n, y\rangle \to \langle x, y\rangle \quad \forall\, y \in H$$

Sometimes weak convergence + something else $\Rightarrow$ strong convergence. For instance

Proposition 16.3

H Hilbert. If $x_n \rightharpoonup x$ weakly in H, and $\|x_n\| \to \|x\| \Rightarrow x_n \to x$ in H, namely $\|x_n - x\| \to 0$

***Proof*.**

$$\|x_n - x\|^2 = \|x_n\|^2 - 2\langle x_n, x\rangle + \|x\|^2 = (*)$$

$\langle x_n, x\rangle \to \langle x, x\rangle = \|x\|^2$ by weak convergence.

$$(*) = \|x\|^2 - 2\|x\|^2 + \|x\|^2 = 0$$

★

Orthonormal Basis

In $\mathbb{R}^n$, we have the canonical basis

$$e_1, ..., e_n \in \mathbb{R}^n$$

s.t.

$$e_j^{(k)} = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

There elements are $\perp$: $\langle e_i, e_j\rangle = 0\ \forall\, i \neq j$. $\|e_i\| = 1\ \forall\, i$.
Moreover, $e_1, ..., e_n$ are a basis, namely $\forall\, v \in \mathbb{R}^n\ \exists!$ expression

$$v = \sum_{i=1}^{n} v_i e_i = \sum_{i=1}^{n} \langle v, e_i\rangle e_i$$

In particular, $v = 0 \Leftrightarrow \langle v, e_i\rangle = 0\ \forall\, i$. Do we have an analogue in Hilbert spaces?

Definition 16.10

$S \subset H$ is called orthonormal if

- $x \perp y\ \forall\, x \neq y,\ x, y \in S$
- $\|x\| = 1\ \forall x \in S$

Definition 16.11

An orthonormal set is an Hilbert Basis (or is **complete**) if $S^\perp = \{0\}$, namely if

$$\langle u, x\rangle = 0 \quad \forall x \in S \Rightarrow u = 0$$

Theorem 16.4

H Hilbert space, $H \neq \{0\}$. Then H has an Hilbert basis.
Moreover, H is a separable Hilbert space $\Leftrightarrow$ it has a finite and countable Hilbert basis.

Example:

- $H = l^2$. H is separable

  An Hilbert basis is $\{e_n\}_{n \in \mathbb{N}}$ defined By

$$e_n^{(k)} = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$$

- $H = L^2([-\pi, \pi])$

  An Hilbert basis is
$$\{\frac{1}{\sqrt{2\pi}}, \frac{\sin(nx)}{\sqrt{\pi}}, \frac{cos(nx)}{\sqrt{\pi}}\} \quad n \in \mathbb{N}$$

Remark 16.5
Hamel basis $\neq$ Hilbert basis.
X $\infty-$ dimensional $\Rightarrow$ any Hamel basis of X is uncountable.
H $\infty-$ dimensional and separable $\Rightarrow$ any Hilbert basis is countable

The usefulness of Hilbert basis stays in the fact that they allow us to reason component by component.

Theorem 16.5 (Bessel inequality)
H separable Hilbert space. $\{u_n\}_{n\in\mathbb{N}}$ orthonormal set. Then $\forall\, x \in H$:
$$\sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 \leq \|x\|^2$$

Theorem 16.6 (Generalized Fourier Series)
H separable Hilbert space, $\{u_n\}$ Hilbert basis. Then any $x \in H$ can be written in a unique way as
$$x = \sum_{n=1}^{\infty} \langle x, u_n\rangle u_n \qquad \langle x, u_n\rangle \text{ Fourier coefficient of } x$$
Moreover, $\forall\, y \in H$ we have
$$\langle x, y\rangle = \sum_{n=1}^{\infty} \langle x, u_n\rangle\langle y, u_n\rangle$$
and
$$\|x\|^2 = \sum_{n=1}^{\infty} (\langle x, u_n\rangle)^2 \qquad \text{Parseval identity}$$

Theorem 16.7
H separable Hilbert space. Then H is isomorphic to l^2 as Hilbert space: namely $\exists$ an isomorphism $\varphi : H \to l^2$ s.t.
$$\langle x, y\rangle_H = \langle \varphi(x), \varphi(y)\rangle_{l^2} \qquad \forall x, y \in H$$

Proof. $\exists$ a countable Hilbert basis, and $\forall\, x \in H$
$$x = \sum_{n=1}^{\infty} \langle x, u_n\rangle u_n$$
Then the desired isomorphism is
$$\begin{aligned} \varphi :\ & H \to l^2 \\ & x \to \sum_{n=1}^{\infty} \langle x, u_n\rangle e_n \end{aligned}$$

★

Corollary 16.2
H separable Hilbert space, $\dim H = \infty$. $\{u_n\}_{n\in\mathbb{N}}$ Hilbert basis. Then $u_n \rightharpoonup 0$ weakly in H, but $u_n \nrightarrow 0$ in H.

Proof.
$$\|u_n\| = 1 \quad \forall\, n \Rightarrow \|u_n - 0\| \nrightarrow 0$$
On the other hand, we know that $\forall\, x \in H$
$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2 < \infty$$
It is then necessary that
$$\langle x, u_n\rangle \to 0 \text{ as } n \to \infty \qquad \forall\, x \in H$$
By Riesz, this means that $u_n \rightharpoonup 0$ in H ★

Ex: $H = L^2([-\pi, \pi])$. Then the previous corollary tells that (Riemann - Lebesgue lemma)
$$\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \to 0 \text{ as } n \to \infty$$
$\sin(nx) \rightharpoonup 0$ in $L^2([-\pi,\pi])$ as $n \to \infty$. Note that $\{sin(nx)\}_{n\in\mathbb{N}}$ does not converge for a.e. x. Weak convergence in L^2 and pointwise or a.e. convergence are not related.
$\{sin(nx)\}$ does not converge a.e. on $[-\pi,\pi]$, not even up to subsequences. The same is true in L^p, $p \neq 2$. Even in this case
$$\sin(nx) \rightharpoonup 0 \text{ weakly in } L^p([a,b]) \quad (p \in [1,\infty)$$
but we don't have a.e. convergence.

17 Linear operator on Hilbert spaces

Proposition 17.1
$p \in [1,\infty)$. Suppose that $f_n \rightharpoonup f$ in $L^p(X)$, and that $f_n \to g$ a.e. in X. Then $f = g$ a.e.

Proposition 17.2
X Banach, V subspace of X^*, dense in X^*. Suppose that $\{x_n\} \subset X$ is bounded, and that
$$Lx_n \to Lx \qquad \forall\, L \in V$$
Then
$$Lx_n \to Lx \qquad L \in X^*$$
namely $x_n \rightharpoonup x$ weakly in X.

Proof. (ex) ★

Consequence: $I \subset \mathbb{R}$ interval ($I = \mathbb{R}$ is fine). $\{f_n\} \subseteq L^p(I)$, $p \in (1,\infty)$. $\{f_n\}$ bounded in L^p: $\exists\, C > 0$ s.t $\|f_n\|_{L^p} \le C$, $\forall\, n$. Then:

- If
$$\int_I f_n\varphi \to \int_I f\varphi \qquad \forall\, \varphi \in \mathcal{C}_C(I)$$
$\Rightarrow f_n \rightharpoonup f$ weakly in $L^p(I)$
- If
$$\int_a^b f_n \to \int_a^b f \quad \forall (a,b) \subset I$$
$\Rightarrow f_n \rightharpoonup f$ weakly in $L^p(I)$

Some useful facts on bounded operators in Hilbert Spaces.
H Hilbert space.
Proposition 17.3
If $T \in \mathcal{L}(H)$, then

$$\|T\|_{\mathcal{L}(H)} = \sup_{\|x\|=\|y\|=1} |\langle Tx, y\rangle|$$

Definition 17.1
T is called **symmetric** (or self adjoint) if

$$\langle Tx, y\rangle = \langle x, Ty\rangle \qquad \forall x, y \in H$$

Proposition 17.4
Let $T \in \mathcal{L}(H)$ be symmetric. Then

$$\|T\|_{\mathcal{L}(H)} = \sup_{\|x\|=1} |\langle Tx, x\rangle|$$

Example: $K \in L^2([0,1]x[0,1])$. Let $T : L^2([0,1]) \to L^2([0,1])$ be defined by

$$(Tf)(t) = \int_0^1 K(s,t) f(s)\, ds$$

$T \in \mathcal{L}(L^2([0,1]))$. It is symmetric $\Leftrightarrow K(s,t) = K(t,s)\ \forall\ s,t$

Spectral Theory

In what follows, E is a Banach space and $T \in \mathcal{L}(E)$.
Definition 17.2
The **resolvent** of T is

$$\rho(T) = \{\lambda \in \mathbb{R} : T - \lambda I \text{ is bijective from } E \text{ to } E\}$$

Definition 17.3
The **spectrum** of T is

$$\sigma(T) = \mathbb{R} \setminus \rho(T)$$

Definition 17.4
λ is an **eigenvalue** of T, $\lambda \in EV(T)$, if

$$\ker(T - \lambda I) \neq \varnothing$$

($T - \lambda I$ is not injective), namely if $\exists\ u \in E$ s.t. $u \neq 0$ and

$$Tu = \lambda u$$

In this case, u is called eigenvector and $\ker(T - \lambda I)$ is the eigenspace of λ.

Remark 17.1
$EV(T) \subset \sigma(T)$

Remark 17.2
In finite dimension, linear operators can be represented by matrices.
A $n \times n$ matrix. We know that $x \mapsto Ax$ is bijective $\Leftrightarrow$ it is injective $\Leftrightarrow$ $\det A \neq 0$. In particular, in finite dimension $\sigma(A) = EV(A)$. This is false in ∞ dimension.

Basic fact:

Theorem 17.1
E Banach, $T \in \mathcal{L}(E)$. Then $\sigma(T) \subset \mathbb{R}$ is compact, and

$$\sigma(T) \subset [-\|T\|_{\mathcal{L}}, \|T\|_{\mathcal{L}}]$$

In general we cannot say much more.
Ex: in l^2, consider the left shift:

$$T_l(x^{(0)}, x^{(1)}, x^{(2)}, ...x^{(n)}, ...) = (x^{(1)}, x^{(2)}, x^{(3)}, ...x^{(n+1)}, ...)$$

$T_l \in \mathcal{L}(l^2)$, $\|T_l\| = 1$. $EV(T_l) =?$. We have to solve

$$T_l x = \lambda x \qquad \text{for some } \lambda \in \mathbb{R}, x \in l^2 \setminus \{0\}$$

$\Rightarrow x^{(1)} = \lambda x^{(0)}$. $x^{(n+1)} = \lambda x^{(n)} = \lambda^{n+1} x^{(0)}$. $\forall\ \lambda \in \mathbb{R}$, the sequence

$$x = x^{(0)} \left(1, \lambda, \lambda^2, ..., \lambda^n, ...\right)$$

is a solution of $T_l x = \lambda x$.

$$x \in l^2 \Leftrightarrow \sum_{n=0}^{\infty} (\lambda^n)^2 < \infty \Leftrightarrow \sum_{n=0}^{\infty} (\lambda^2)^n < \infty \Leftrightarrow |\lambda| < 1$$

Any $\lambda \in (-1, 1)$ is an e.v. of T_l. Moreover, $\sigma(T_l)$ is a compact set which is included in $[-1, 1]$ and contains $EV(T_l) = (-1, 1)$

$$\Rightarrow \sigma(T) = [-1, 1]$$

We focus in what follows on the following case: H separable Hilbert space, $T \in \mathcal{K}(H)$ and symmetric.

Proposition 17.5
Let $d = \|T\|_{\mathcal{L}(H)}$. Then either d or $-d$ is an eigenvalue of T

Recall: $T \in \mathcal{K}(H)$, $u_n \rightharpoonup u$ weakly $\Rightarrow Tu_n \to Tu$ strongly in H

Proof. $d \neq 0$ (otherwise $T = 0$). We know that

$$d = \sup_{\|u\|=1} |\langle Tu, u\rangle|$$

Take a maximizing sequence for d:

$$\exists\ \{u_n\} \subset H \text{ s.t. } \|u_n\| = 1 \qquad |\langle Tu_n, u_n\rangle| \to d$$

$\{u_n\}$ is bounded $\Rightarrow$ by Banach Alaoglu in reflexive spaces (any Hilbert space is reflexive) we can extract $\{u_{n_k}\}$ s.t. $u_{n_k} \rightharpoonup u$ weakly in H, for some u.
By weak strong continuity, $Tu_{n_k} \to Tu$ strongly in H. From this, we deduce that

$$|\langle Tu_{n_k}, u_{n_k}\rangle - \langle Tu, u\rangle \to 0| \qquad \text{as } K \to \infty$$

We know

$$|\langle Tu_{n_k}, u_{n_k}\rangle| \leq |\langle Tu_{n_k} - Tu, u_{n_k}\rangle| + |\langle Tu, u_{n_k} - u\rangle| \to 0$$

and also that $|\langle Tu_{n_k}, u_{n_k}\rangle| \to d$

$$\Rightarrow |\langle Tu, u\rangle| = d$$

and hence $u \neq 0$

- Suppose that $\langle Tu, u\rangle = d$. Then

$$\|Tu - du\|^2 = \|Tu\|^2 - 2d\langle Tu, u\rangle + d^2\|u\|^2 \leq d^2 - 2d^2 + d^2 = 0$$
$$\Rightarrow \|Tu - du\| = 0 \Rightarrow Tu = du$$

  and d is an eigenvalue.

- $\langle Tu, u\rangle = -d$. Then one can prove that $-d$ is an eigenvalue.

★

Proposition 17.6
$\lambda \neq 0$ is an eigenvalue of a compact operator $T \in \mathcal{K}(E)$, E Banach. Let V_λ be the eigenspace of λ. Then $\dim V_\lambda < \infty$

Proof. Recall that $I : F \to F$, with F ∞ dimensional. Banach space, cannot be compact. Assume by contradiction that V_λ has ∞ dim. Consider

$$\frac{1}{\lambda}T|_{V_\lambda} : V_\lambda \to V_\lambda$$

is the identity $\frac{1}{\lambda}Tu = \frac{1}{\lambda}\cdot\lambda u = u \ \forall\ u \in V_\lambda$. So $\frac{1}{\lambda}T|_{V_\lambda}$ cannot be compact. On the other hand, $\frac{1}{\lambda}T|_{V_\lambda}$ is compact by assumption. ★

Proposition 17.7
H Hilbert, $T \in \mathcal{L}(H)$ symmetric. Then eigenvectors associated with different eigenvalues are orthogonal.

Proof.

$$\begin{aligned} &Tu_1 = \lambda_1 u_1 \quad u_1, u_2 \neq 0 \quad \lambda_1 \neq \lambda_2 \\ &Tu_2 = \lambda_2 u_2 \end{aligned}$$

$$\lambda_1\langle u_1, u_2\rangle = \langle Tu_1, u_2\rangle = \langle u_1, Tu_2\rangle = \lambda_2\langle u_1, u_2\rangle$$
$$\Rightarrow (\lambda_1 - \lambda_2)\langle u_1, u_2\rangle = 0 \Rightarrow \langle u_1, u_2\rangle = 0$$

★

Theorem 17.2 (Spectral Theorem)
H separable Hilbert, $T \in \mathcal{K}(H)$ symmetric. Then

(1) $\sigma(T) \setminus \{0\} = EV(T) \setminus \{0\}$

(2) $0 \in \sigma(T)$

and the following alternative holds:

(1) either T has infinitely many distinct eigenvalues, and in this case $0 \in EV(T)$ and $\ker T$ is infinite dimensional

(2) or $EV(T) \setminus \{0\}$ is a sequence tending to 0

Moreover, the eigenvectors can be chosen in such a way to form a Hilbert basis of H (if necessary adding an orthonormal basis of $\ker T$)

Remark 17.3 • $\forall$ symmetric matrix A, nxn, $\exists$ an orthonormal basis of $\mathbb{R}^n$ of eigenvectors

- If $T \in \mathcal{K}(E)$, E Banach, we can still say that $0 \in \sigma(T)$ (if E has ∞ dimension), that $EV(T) \setminus \{0\} = \sigma(T) \setminus \{0\}$ and that either there are finitely many distinct eigenvectors, or $EV(T) \setminus \{0\}$ is a sequence tending to 0

Proof.

- $0 \in \sigma(T)$ is simple: T is compact, H has ∞ dimension $\Rightarrow$ it can't be surjective.

$$T = T - 0I \text{ is not bijective: } 0 \notin \rho(T)$$

- From proposition 1, $\exists$ an eigenvalue λ with $|\lambda_0| = \|T\|_{\mathcal{L}(H)}$. Let V_0 be the associated eigenspace. By proposition 2, $\dim V_0 = N_0 \langle \infty$.

  Let $\{w_1^0, ..., w_{N_0}^0\}$ be an orthonormal basis for V_0. Consider now $H_1 = V_0^\perp$, so that $H = V_0 \oplus H_1$. We claim that $T|_{H_1} \in \mathcal{K}(H_1)$ symmetric.

  $T|_{H_1}$ is compact and symmetric, by assumption. We have to check that $T|_{H_1} : H_1 \to H_1$

$$u \in H_1 \Leftrightarrow \langle u, w \rangle = 0 \quad \forall\, w \in V_0$$

$$\langle Tu, w \rangle = \langle u, Tw \rangle = \langle u, \lambda_0 w \rangle = \lambda_0 \langle u, w \rangle$$

  $\forall\, w \in V_0$, namely $Tu \in H_1$, $\forall\, u \in H_1$.

$$H_1 = V_0^\perp$$

  $\Rightarrow$ it is a closed subspace of H $\Rightarrow$ H_1 is a Hilbert space. $T|_{H_1}$ is a compact symmetric operator on a separable Hilbert space. Therefore, arguing as before, we have an eigenvalue for T given by λ_1 s.t.

$$|\lambda_1| = \sup_{\substack{\|u\| = 1 \\ u \in H_1}} |\langle Tu, u \rangle|$$

  Clearly, $|\lambda_1| \le |\lambda_0| = \sup |\langle Tu, u \rangle|$. We have an eigenspace V_1 for λ_1, with dimension N_1, and an orthonormal basis $\{w_1^1, ..., w_{N_1}^1\}$ for V_1. We iterate the process. Either after a finite number of steps we have

$$\lambda_N = \sup \begin{matrix} \|u\| = 1 \\ u \in H_1 \end{matrix} |\langle Tu, u \rangle| = 0$$

  Or $\{\lambda_n\}$ forms a sequence, s.t. $|\lambda_n|$ is decreasing.

  Case 1: We can say that

$$H = V_0 \oplus V_1 \oplus V_2 \oplus ... V_{N-1} \oplus \ker T$$

  $\ker T$ is a closed subspace of H, separable $\Rightarrow$ we have an orthonormal countable basis $\{z_1, ..., z_n\}$ of $\ker T$. Then

$$\{w_1^0, ..., w_{N_0}^0, w_1^1, ..., w_{N_1}^1, ..., w_1^{N-1}, ..., w_{N_{N_1}} N - 1, z_0, ..., z_n\}$$

  is an orthonormal basis of H, made of eigenvectors.

  Case 2: at first, we show that $\lambda_n \to 0$. If not, $|\lambda_n| \to \eta > 0$. Consider then $\{\frac{w_n}{\lambda_n}\}$, where w_n is an eigenfunction of λ with $\|w_n\| = 1$. Then $\{\frac{w_n}{\lambda_n}\}$ is bounded, and

$$T(\frac{w_n}{\lambda_n}) = \frac{1}{\lambda_n} T w_n = \frac{1}{\lambda_n} \lambda_n w_n = w_n$$

$\Rightarrow$ by compactness, there exists a subsequence of $T(\frac{w_n}{\lambda_n} = w_n)$ which is strongly convergent. This is not possible, since

$$\|w_i - w_j\|^2 = 2 \quad \forall\, i \neq j$$

$\|w_i\|^2 + \|w_j\|^2 - 2\langle w_i, w_j\rangle \Rightarrow \lambda_n \to 0.$

It remains to show that $x \in V_i^{\perp}$, $\forall\, i$, then $x \in \ker T$. To this end

$$\|Tx\| = \|T|_{H_i} x\| \leq \|T|_{H_i}\|_{\mathcal{H}_\rangle} \|x\| = |\lambda_i| \|x\| \quad \forall\, i$$

Taking $i \to \infty$, we deduce

$$\|Tx\| \leq \lim_{i\to\infty} |\lambda_i| \|x\| = 0$$

$\Rightarrow x \in \ker T$. Even in this case,

$$H = \ker T \oplus V_1 \oplus V_2 \oplus ... \oplus V_n \oplus ...$$

and once again we can consider a basis of eigenvectors.

★

Corollary 17.1 (Fredholm Alternative)
H separable Hilbert space, $T \in \mathcal{K}(H)$ and symmetric. Then:

(1) either $\forall\, y \in H$ the equation

$$x - Tx = y$$

has a unique solution

(2) or $\lambda = 1$ is an eigenvalue of T, and in this case $x - Tx = y$ can have no solution or infinitely many solutions, depending on y.

Remark 17.4

- Rouché Capelli: $Ax = y$. A matrix. Either $\det A \neq 0$, and then $\forall\, y \in \mathbb{R}^n \quad \exists!$ solution; or $Ax = y$ can have 0 or ∞ many solution.
- T symmetric is not necessary, and the corollary also holds in Banach spaces.
- The corollary is very useful to treat integral equations:

$$u(t) - \int_0^1 K(s,t) u(s) ds = g(t)$$

Proof. By the Spectral Theorem, $\forall\, x \in H$, we can write

$$x = \sum_{n=1}^{\infty} \langle x, u_n\rangle u_n$$

where $\{u_n\}$ is a Hilbert basis of eigenvectors of T. Also, we have

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, u_n\rangle u_n$$

and

$$y = \sum_{n=1}^{\infty} \langle y, u_n\rangle u_n$$

Then, the equation $x - Tx = y$ becomes

$$\sum_{n=1}^{\infty}(1-\lambda_n)\langle x, u_n\rangle u_n = \sum_{n=1}^{\infty}\langle y, u_n\rangle u_n$$

$$\Rightarrow (1-\lambda_n)\langle x, u_n\rangle = \langle y, u_n\rangle \quad \forall\, n$$

If $\lambda_n \neq 1 \;\forall\, n$, then we take

$$\langle x, u_n\rangle = \frac{\langle y, u_n\rangle}{1-\lambda_n} \quad \forall\, n$$

$$x = \sum_{n=1}^{\infty}\frac{\langle y, u_n\rangle}{1-\lambda_n}u_n$$

is the solution. If instead $\lambda_n = 1$ for some n, then there are no solution if y is such that $\langle y, u_n\rangle \neq 0$:

$$(1-\lambda_n)\langle x, u_n\rangle = \langle y, u_n\rangle$$

For y s.t. $\langle y, u_n\rangle = 0$, we have ∞ many solutions. ★