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Type-2 Fuzzy Sets—A Tribal Parody

In the beginning, Zadeh came down from the mountain of system theory with fuzzy sets, or some might say that he 'fell from grace'. He was shunned and ridiculed by former colleagues so he wandered into the fuzzy wilderness. Everywhere he went, some people listened to his new message, liked what they heard, and many became converted. Soon, what seemed like fuzzy math was given sound mathematical meaning and approval, at least within the band of converts. Due to Zadeh's perseverance, patience, persistence and kind personality, this group grew considerably into a rather large tribe know as the *Fuzzyites*.

Theory begat more theory, but the practical-minded of the tribe asked, "What can you do with these fuzzy sets?" Alas, Zadeh was also troubled by this question because he was an engineer after all, even though he had joined the system gods on the top of the mountain of system theory. But he had come down from that mountain, and in a flash it came to him—*linguistic variables and rule-based systems that use fuzzy sets as models for such variables*! Disciples Mamdani of England, and Takagi, Sugeno and Kang of Japan, saw the light and showed how fuzzy sets could be used to design rule-based controllers—fuzzy logic controllers. The main tribe of fuzzyites now became two tribes, the second being the *Fuzzy-Logic Controlites*. Things looked good, products emerged and money was made. In later years, many more application-minded tribes appeared.

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For more than a decade, Zadeh's mind had been swimming with new ideas about fuzzy sets, and so, in 1975, he collected his ideas in a monumental three-part paper, sort of a[1] *Guide for the Perplexed* about fuzzy sets. In the first part of this paper, Zadeh introduced *fuzzy sets of Type-2,* later shortened by others to *Type-2 Fuzzy Sets* (T2FS) and *fuzzy sets with Interval-Valued Membership Functions*, later shortened by others to *Interval-Valued Fuzzy Sets* (IVFS)[2].

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In time, the *Fuzzyite* tribe became even larger and spun off more new tribes, two of which were the *Type-2 Fuzzy Set Tribe* and the *Interval-Valued Fuzzy Set Tribe*. As it frequently happens, when two brothers compete for the love of their father, they travel down very different paths and so these two tribes—the *T2 FS-ites* and the *IVFS-ites,* left the main tribe to set out on their own.

The *T2 FS-ites*, a rather small tribe in the late '70s and early 80s, developed some interesting theories In 1976, Mizumoto and Tanaka, both from Japan, studied the set theoretic operations of general T2 FSs (henceforth called T2 FSs) and properties of membership grades of such sets. In 1981, they also examined T2 FSs under the operations of algebraic product and algebraic sum. In 1977, Nieminen of Finland provided more detail about the algebraic structure of T2 FSs. From 1978 to 1980, Dubois and Prade, both from France, discussed fuzzy valued logic and gave a formula for the composition of T2 relations as an extension of the T1 sup-star composition, for the minimum t-norm.

Unfortunately, the timing was not good for the *T2 FS-ites*, for the world was paying attention mainly to the *Fuzzy Logic Controlites*, as it was making money and T2 FSs seemed to be way ahead of their time, because computations for such sets using 1970's computers were slow and quite limited. They had to be done *one vertical slice*[3] *at a time* and there were many such slices. In time, this tribe almost vanished, leaving only a handful of records buried in the fuzzy sands of time, and seemingly put to sleep like Rip van Winkle.

[1] The original *Guide for the Perplexed* was written in the 12th century by Rabbi Moshe ben Maimon, better known as Maimonides, as a three volume letter, just as Zadeh's paper was written in three parts.

[2] A *type-2 fuzzy* set can be thought of as a type-1 fuzzy set on steroids. Its membership function no longer has a single value at each value of the primary variable, but instead is a blurred version of that function, i.e., at each value of the primary variable the membership is itself a function, called a *secondary membership function*. When the secondary membership function is a constant equal to 1, the type-2 fuzzy *set* is called an *interval* type-2 fuzzy set or an *interval-valued fuzzy set*; otherwise, it is called a *general type-2 fuzzy set*.

[3] The membership function of a T2 FS is three-dimensional, with *x-axis* called the *primary variable*, *y-axis* called the *secondary variable* (or *primary membership*) and *z-axis* called the *MF value* (or *secondary MF value*). A vertical slice is a plane that is parallel to the MF-value axis.

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Meanwhile, the *IVFS-ites* prospered and grew into a larger tribe, but they also stood in the shadow of the *Fuzzy Logic Controlites*. In 1975, Sambuc of Marseilles France, did very early work on IVFSs, but alas, this was only published in his medical-school dissertation and was written in French. Hence, it was not, and is still not available in English to the general scientific community.

The *IVFS-ites* were much luckier than the *T2 FS-ites* because computations for IVFSs were easy. Most of them used a very well-established interval arithmetic, which had been developed by a totally different and non-fuzzy tribe—the *Intervalites,* one of whose leaders was Kreinovich of El Paso Texas. Kreinovich quickly saw connections between the interval arithmetic of the *Intervalites* and *IVFS-ites*. Calculations were also done on vertical slices but were relatively easy to perform.

And so, unknown to the two now-separated tribes—the *T2 FS-ites* and the *IVFS-ites*—both tribes had become *Verticalites*.

Returning to the journey of the *IVFS-ites*, in the 1980s they seemed to splinter into two sub-tribes, one led by Turksen of Toronto, Canada (now also of Ankara Turkey) and the other by Gorzcalczany of Poland. Turksen used IVFSs to model the uncertainties of the connector words AND and OR. Not many have gone down this path with him, but he continues to travel it with a small number of other *Connectorites*. Gorzcalczany showed how IVFSs could be used in rule-based systems, but the world paid little attention to this work because it had yet to be applied to real systems. But as the tribe had reached more than critical mass, it continued and even now, continues to thrive and produce many interesting results, mostly extending the results of the main fuzzy set tribe to interval-valued fuzzy sets, including the very important works of Bustince, of Spain, who now is a potentate of the *IVFS-ites*.

The *IVFS-ites* later branched into yet another tribe called the *Intuitionites*, headed by Atanassov of Bulgaria, whose theme was captured by Shakespeare's Hamlet when he said, "To be or not to be, that is the question," or whoever said "every coin has two sides."

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Then, in the mid-1990's, the *T2 FS-ites* awoke from their long sleep and found themselves separated, as though the winds of time had carried the sleeping Type-2 spores across the Channel and across the Atlantic. Now, there were two *T2 FS-ite tribes,* each not knowing about the other, one headed by John of Leicester England and the other by Mendel of Los Angeles, California. As fortune would have it John and Mendel met and liked each other so they re-merged the two tribes into the *General T2 FS-ites*.

Time had been kind to the two *T2 FS-ite* tribes. By the mid 90s, computers were powerful enough so the T2 FS computations could be performed. In the beginning, it was easy to fall back to the old ways of the *Verticalites*; but the *footprint of uncertainty*[4] (FOU) for a T2 FS set emerged, and it was realized that the FOU could not only be covered by vertical slices, the mantra of the *Verticalites*, but it could also be covered by wavy slices[5], the mantra of the newly-reunited *General T2 FS-ites*, now nicknamed the *Wavyites*.

There were now *Verticalites* and *Wavyites*. *Verticalites* followed the old and trusted ways while *Wavyites* explored new ways because their path was untraveled. *Wavyites* extended rule-based fuzzy systems from T1 to T2 FSs. They developed a *Representation Theorem* that decomposed a T2 FS into the union of wavy slices, and *Type-Reduction* that let a T2 FS be projected onto an IVFS (on the way to its defuzzification into a crisp number). The *Representation Theorem* became almost god-like in that it led to many new results, including the centroid of a T2 FS, the widely used Karnik-Mendel (KM) Algorithms, and many other computations.

But all was not manna in the camp of the *Wavyites*. Computational complexity was killing them. The third-dimension of a T2 FS became a burden to many members of this tribe.

Sometimes things happen without even a plan—destiny calls. So it came to be that all applications of rule-based T2 fuzzy systems used the simplest of all T2 FSs, one in which the third dimension became irrelevant[6]. These so-called *Interval Type-2 Fuzzy Sets* (IT2 FSs) are described completely by their FOU, and all computations are pretty simple because they only involve interval sets. With this, money began to be made. This tribe that focused on IT2 FSs could be called *Interval Wavyites*.

So, the fuzzy dominion now had two interval tribes, the *Verticalites* and the *Interval Wavyites*, each wandering in distant lands, seemingly unaware of each other, when in fact they were twin tribes but with different languages and traditions. Not only can the FOU be represented as the union of all of its wavy slices, it can also be represented as the union of all of its vertical slices. There is not one *Representation Theorem (RT)*, but two—the wavy slice RT and the vertical slice RT. Whereas wavy slices lead to the structure of a problem's solution very fast, vertical slices are sometimes, but not always, used to implement that solution.

[4] The *FOU* of a T2 FS lies on the *x-y plane* (i.e., the primary and secondary variable plane) and includes all points on that plane for which the MF value is non-zero. It is the 2D-domain on which sit the secondary membership values.

[5] A *wavy slice* is actually a foil, and is also called an *embedded T2 FS*. It sits on a T1 FS that is contained within the FOU (that is also called an *embedded T1 FS*) and has a non-zero MF value that sits atop that T1 FS. So, it is a very simple T2 FS.

[6] Because an *IT2 FS* is a T2 FS all of whose secondary MF values equal one, there is no information contained in those secondary MF values, and so an IT2 FS is characterized just by its FOU.

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Computing the union, intersection and complement of IT2 FSs is done using vertical slices; however, computing the centroid, cardinality, fuzziness, variance and skewness of an IT2 FS is done using wavy slices. And computing the similarity between two IT2 FSs can be done using wavy slices or vertical slices. These two twin tribes had much to learn from each other. In time, they acknowledged each other, and are still getting to know each other, knowing at last that to be a *Verticalite* is to also be an *Interval Wavyite* and vice-versa.

The *Interval Wavyites* rejoiced because their dominion had increased. But all was not well within the original *Wavyites*, for not all agreed to abandon the third dimension and focus only on IT2 FSs. Hence, the *Wavyites* split into two tribes- the *Interval Wavyites* under Mendel and the *General Wavyites* under John.

The *Interval Wavyites* multiplied and prospered. Hagras of Essex England, Castillo and Melin of Tijuana Mexico, Tan of Singapore, Rhee and Hwang of Seoul Korea, Liang of Arlington Texas, Zeng and Liu of Hong Kong, Melgarejo of Columbia, Niewiadomski of Lodz Poland, as well as many others, produced notable results. Hongwei Wu, now of Savannah Georgia in the USA, developed minimax uncertainty bounds that let type reduction be bypassed. And so, there was much rejoicing by a branch of the *Fuzzy Logic Controlites*, known as the *IT2 FL Controlites*, for now IT2 FSs could be used in real time. Applications abounded and these ranged from control to signal processing to pattern recognition to geography to image processing to many others. Publications proliferated and the word spread.

Alas, there was still some unrest in the tents of the *Interval Wavyites*. Some of the tribe, especially the *IT2 FL Controlites*, did not like type reduction (TR) because it has no closed-form solution. They also did not like the uncertainty bounds because they were difficult to use in their analyses. Much research occurred and is still occurring on ways to bypass TR so that IT2 FL controllers can be used in real time. Lest one forgets (the Oracle now appears and steps up onto his soap box), the type-reduced set provides a measure of the uncertainty that has propagated through all calculations. Heed the type-2 oracle—*beware of throwing out the baby with the bath water*—or, *would a probabilist ignore standard deviation*? (The Oracle now steps down—or is pulled down—from his soapbox.)

The *General Wavyites* were also busy. They were not content to live just on a plateau, as were the *Interval Wavyites*. They wanted to climb the highest (normal, and for Starczewski of Czestochowa Poland, triangular) mountain. *Computational geometry* became their mantra, leading to a sub-tribe known as the *CG-ites*. Because the membership function of a general T2 FS is three-dimensional, it resembles a mountain that sits upon its FOU, one whose amplitudes over its FOU are the union of its secondary membership functions.

The *CG-ites*, under Coupland, also of Leicester England, and John, approximate the mountain by using regular geometric shapes, most notably triangles. Operations such as union and intersection can then be carried out using methods from computational geometry. Type-reduction is by-passed by directly using one coordinate of the geometric centroid for the defuzzified value of the T2 FS. (Oh-Oh! The Oracle is back—*don't throw out the champagne with the cork*.)

And it came to pass that in 1996, Zadeh cried out from the Mecca of fuzziness in Berkeley California, a far cry from the wilderness, the words of a new paradigm—*Compute With Words*! Heads turned, some with belief and others with disdain. Had Zadeh sipped one too many frequent-flier martinis? But there was more to this message. It was *Fuzzy Logic=Computing With Words (CWW)*.

Once again two tribes emerged, the *Type-1 CWW-ites*, and the *IT2 CWW-ites*, each trying to figure out what Zadeh meant and how to implement this new paradigm. The mantra of the former tribe is *to compute with prototypical words that can be modeled using T1 FSs*, so the Type-1 CWW-ites could be called the *Prototypicalites*, whereas the mantra of the latter tribe is *words mean different things to different people and so must be modeled using at least IT2 FSs*. The *IT2 CWW-ites* argued, using the scientific philosopher Sir Karl Popper's theory of Falsificationism, that if words mean different things to different people, then it is a contradiction to model a word that is uncertain by a T1 FS that is completely certain once its membership function parameters are fixed. In this way, they falsified using T1 FSs as a model for words. It seems that the *IT2 CWW-ites* could now be called *Popperites*. Each tribe is presently busy developing CWW theories and computing methodologies. It is too soon to know where all of this will lead to.

There arose, again in the kingdom of Los Angeles under Mendel and Dongrui Wu, one instantiation of CWW, the *Perceptual Computer* that has led to concrete implementations of CWW, but only for the specific application of assisting people in making subjective judgments. This sub-tribe of the *IT2 CWW-ites* could be called the *Perceptualites*, or possibly the *Judgmentalites*.

The building blocks of the Perceptual Computer are threefold: Encoder that maps each word in an application's vocabulary into an FOU by using data collected from a group of subjects about each word; CWW Engine that maps IT2 FSs into other IT2 FSs; and, Decoder that maps those CWW output IT2 FSs into a recommendation such as

a word, rank or class. All *T2 fuzzy-ites* can bask in the knowledge that word uncertainties flow from the Encoder to the Decoder using IT2 FSs; that all words have been modeled in such a way that their uncertainties have been captured by an FOU, using an *Interval Approach* developed by Liu (formerly of Los Angeles but now of Richmond, California) and Mendel; and, that many things that were developed for IT2 FLSs were needed to implement the Perceptual Computer (Oracle appears again, nearly falling off of his soapbox, but helped back onto it by *IT2 Fuzzy-ites*, all chanting: *Let those who have shunned T2, and have kept it from their lands, take note (using whatever notation they choose)!*)

Discontented to represent knowledge with simple words[7], a splinter group led by Rickard of Larkspur, Colorado and Aisbett & Gibbon of Newcastle, Australia entered the darkness of cognitive science and proceeded to shine the T2 light on conceptual space knowledge representations. This theory had been developed in a castle in Lund by Gardenförs of the Kingdom of Sweden, to answer ancient questions of the relationship between properties and concepts that have been asked before the time of Aristotle. Under the banner of *Conceptualites*, the T2 crusaders attacked the over-confident notions of concepts and properties long held by the cognitive science community, fuzzifying them to a degree to which Zadeh would heartily approve. They then brought forth subsethood in higher levels of fuzziness as a fundamental means of inference within such realms. CWW invaded lands occupied by other disciplines that formerly considered themselves safe within their cloistered-walls.

Hold on to your hats! In addition to vertical and wavy slices for a T2 FS, 2008 produced horizontal slices[8] of such

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sets, called *alpha-planes* or *z-slices*, the former developed by Liu of Richmond, California, and the latter by Wagner and Hagras, both of Essex in England. There are now *Verticalites*, *Wavyites* and *Horizontalites*. Who knows, maybe these sets will be sliced and diced next (the latter being called *Cubites*—okay, okay, by now you get it!). As a result, it now seems possible to use general T2 FSs, because they can be analyzed and computed using horizontal slices. Amazingly, each horizontal slice is like an IT2 FS/IVFS whose secondary grades are all equal to alpha (zed) instead of one, which is the case for the more usual IT2 FS/IVFS. Welcome home *Interval Wavyites* and *Verticalites*. Your journey down these seemingly different paths, from the path of the *General Wavyites*, has not been for naught. Everything that you have learned is now needed by the *Horizontalites*. (Oracle, somewhat (technically) intoxicated, reappears: *It seems that this is an excellent example of not putting all of your eggs—uhh type-2 fuzzy sets—into one basket.*)

And so the Type-2 saga continues. There is no doubt in the mind of this author, and the Oracle, that more tribes will emerge, testifying to the diversity of the original *Fuzzyites* and their immediate descendents the *General Type-2 FS-ites*.

This tour de Force, besides being quite educational and a lot of fun for the author to write, also reveals the vitality of the *Fuzzyites*, who begat a multitude of T2 tribes, a part of whose family tree, which will probably be outdated by the time of the publication of this article, is summarized in Fig. 1.

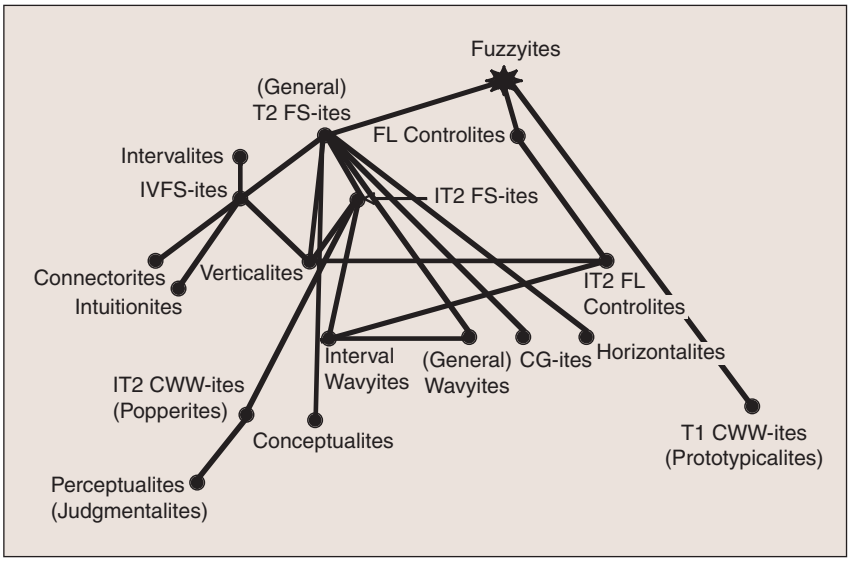

FIGURE 1 A part of the fuzzy family tree.

Acknowledgments

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[7] This paragraph was provided to the author by Dr. John T. Rickard.

[8] A horizontal-slice of a T2 FS is a plane that is parallel to the *x-y plane*.