Analysis H Project:
3D Graphics Generation with Matrices

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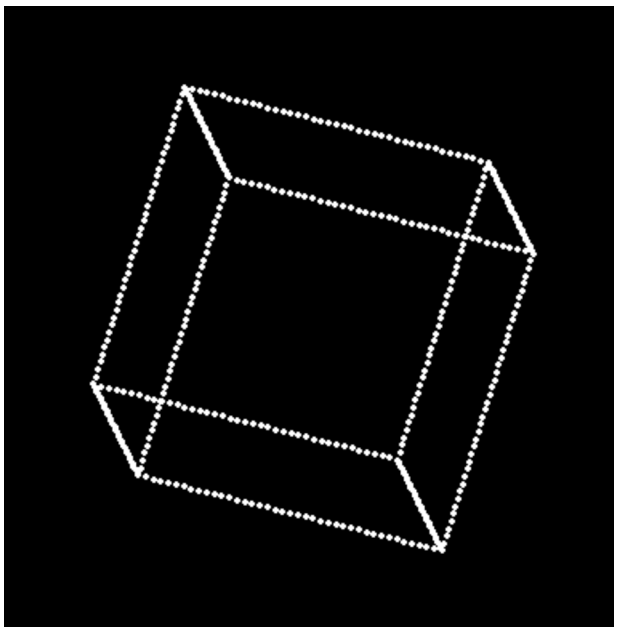


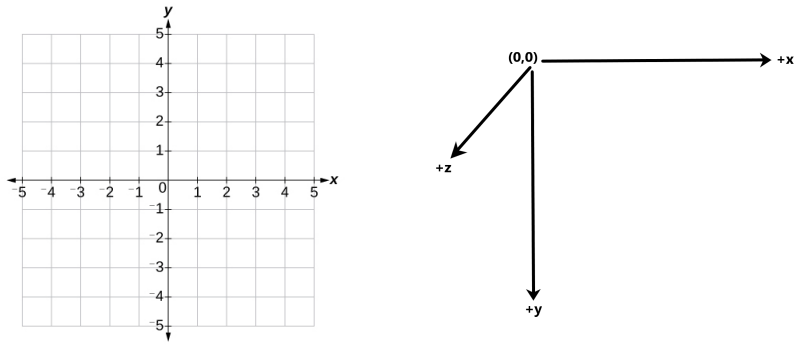
Overview

For our Analysis H project, Steve and I decided to dive into the realm of computer graphics and matrices' applications in this field. Specifically, we wrote code that generates 3D points, projects these 3D points onto a 2D surface(the computer screen), and transforms these points to display an animation(using matrices). We both were fascinated by the numerous applications of matrices and thought this will be a fun project to work on. After doing further research, we learned that Renderman, Pixar's animation software, is heavily based on matrices and linear algebra; so this seemed like an interesting field which we could dip our toes in. This document serves to explain how we came up with the math of this code, challenges we faced, and the ultimate solutions to those challenges. The next section is a brief introduction of how we approached the code and following that is the actual math behind it. Enjoy!

The Math

There are many aspects of the code that we had to come up with in order to display an animated 3D object on a 2D screen. Some of these aspects rely more on math than others and this document will primarily focus on those. The complete code can be found here. Let's begin!

Coordinate Setup & Matrices



(a) (b)

Figure 1: Coordinates

In the figures above, we see figure (a): our regular cartesian coordinate system where (0,0) is at the center of the screen and we have 4 quadrants. However, in figure (b), we see a generic computer coordinate system – only the 4th quadrant is included, making the origin the top left of the screen. Though it may seem more complex to display animations at a point other than the origin, computer screen layouts actually make this problem easier(with a little bit more math of course) since we no longer have to deal with negative values. So why is this important? Well, if we wanted to display an object at the center of the screen, or in fact, any point other than the origin, we'd have to come up with some sort of way to cleverly use matrices and transformations. For now, we'll start of with the generic 2 x 2 rotation matrix.

$$\begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A couple things that are important to note before we proceed are that this matrix rotates points in 2 dimensions and on the xy plane – meaning around the z axis. Since we need to go into 3D, we need to change this matrix to work for 3D points and we can do that by using the concept of an identity matrix. When we multiply the rotation matrix on the left by the 2 x 2 identity matrix on the right, we obviously will still obtain the same rotation matrix. Similarly, a 3 x 3 identity matrix would do the same thing. We can use this concept to come up with a rotation matrix for 3 dimensions. These new matrices are shown below.

$$\begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Just like we can add on another row and column to the identity matrix(to get the matrix for the new

dimension), we can also do this for the rotation matrix. To prove this works, let's take any point and verify its coordinates using both matrices. In the example below, we use the point $(1, 0, 0)$.

$$\begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} cos(\theta) \\ sin(\theta) \\ 0 \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} cos(\frac{\pi}{2}) & -sin(\frac{\pi}{2}) & 0 \\ sin(\frac{\pi}{2}) & cos(\frac{\pi}{2}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} cos(\frac{\pi}{2}) \\ sin(\frac{\pi}{2}) \\ 0 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{3}$$

As we can see above, we get the correct answer of (0, 1, 0) after rotating the point (1, 0, 0) by $\frac{\pi}{2}$ radians counter clockwise. Similarly, we can do this to the rotation matrices of not only the z axis but also the y and x axes. Once we do this, we get the 3 rotation matrices for the x, y, and z axes.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}$$

Rotation Matrix X

$$\begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix}$$

Rotation Matrix Y

$$\begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation Matrix Z

(NOTE: Derivations of these matrices have been left out to keep paper length minimal). Now that we have our 3 main matrices, we can go back to the problem of a computer's coordinate system. If we were to use just these matrices themselves, our points would be rotating about the origin and the center of the cube would not stay fixed. So how might we solve this problem? One way to solve this problem is using translation matrices. First, we translate the entire cube to the origin, rotate it at the origin, and then translate the entire cube back by using the inverse of the first translation matrix. To create these translation matrices, we need to refer back to Figure 1(b). Since, once again, (0,0) is the top left of the screen, the center would be the point $(\frac{w}{2}, \frac{h}{2}, \frac{d}{2})$ where w, h, and d are the width, height and depth of the screen, respectively. Now that we know how much to translate by, we need to figure how to write these matrices.

$$\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -x \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix}$$

Above, we have a translation matrix(left) that translates by x units in the x direction and y units in the y direction, followed by its inverse(right) that translates by $-x$ units in the x direction and $-y$ units in the y direction. However, recall that to translate a 2D point, we need to go into 3D. These matrices above only translate in 2 dimensions – the x direction and the y direction – but that's not what we want. We want to translate a 3 dimensional point and for that, we must go into 4D. These new translational matrices are shown below.

$$\begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation Matrix 1

$$\begin{bmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Translation Matrix 2

We can show that these matrices do their job by applying both of them onto the point $(\frac{w}{2}, \frac{h}{2}, \frac{d}{2})$ to see if we get the same point back. Since we want to translate to the origin first, we'll perform the transformation $T_2 \circ T_1$.

$$\begin{aligned} \left(\begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -z \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} \frac{w}{2} \\ \frac{h}{2} \\ \frac{d}{2} \end{bmatrix} &= \begin{bmatrix} \frac{w}{2} \\ \frac{h}{2} \\ \frac{d}{2} \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{w}{2} \\ \frac{h}{2} \\ \frac{d}{2} \end{bmatrix} &= \begin{bmatrix} \frac{w}{2} \\ \frac{h}{2} \\ \frac{d}{2} \end{bmatrix} \end{aligned} \tag{4}$$

But hold on. Our translation matrices are in 4 dimensions but our rotation matrices are only in 3 dimensions. How might we multiply them together to perform a transformation if they aren't the same dimensions? Since the translation matrices have to be in 4D, our only option is to somehow make the rotation matrices also 4D. We can do that by adding one more identity row and column like shown below.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) & 0 \\ 0 & sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation Matrix X

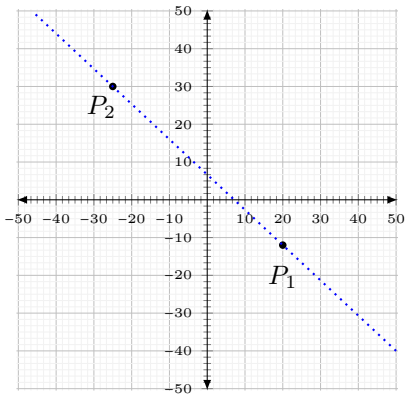
$$\begin{bmatrix} cos(\theta) & 0 & sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\theta) & 0 & cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation Matrix Y

$$\begin{bmatrix} cos(\theta) & -sin(\theta) & 0 & 0 \\ sin(\theta) & cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation Matrix Z

Points Between Function



Above, we have a graph of 2 points: P_1 and P_2. Given these two points, the pointsBetween function aims to return a list of n number of points that lay on the line formed by P_1 and P_2(as shown in blue). To do this, we'll define a few variables: Let n = the number of points we want(given), $\vec{v} = \overleftrightarrow{P_1P_2}$ and $\vec{w} = \frac{\vec{v}}{n}$. Like stated, $\vec{w}$ is a scaled down version of $\vec{v}$ which will mainly be used to generate these points, but more on that in a second. For now, we'll calculate some of the values for these variables and then put them together to solve this problem. In the above diagram, $P_1 = (20, -12)$ and $P_2 = (-25, 30)$.

$$n = 5$$
$$\vec{v} =< x_{P2} - x_{P1}, y_{P2} - y_{P1} >$$
$$\vec{w} = \frac{\vec{v}}{5}$$

$$n = 5$$
$$\vec{v} =< -45, 42 >$$
$$\vec{w} =< \frac{-45}{5}, \frac{42}{5} >$$

Now that we have $\vec{w}$, we can repeatedly add it to P_1 and take note of all the points it takes us to. For example, if $P_1 = (0, 0)$, $P_2 = (0, 10)$, and $n = 10$, the value of w would be $< \frac{0}{10}, \frac{10}{10} >$ or $< 0, 1 >$. Now, we would start off at P_1 and add w giving us the point $(0, 1)$. Adding it again and again, we'd get $(0, 2)$, $(0, 3)$, $(0, 4)$,..., and $(0, 10)$ which are our generated points! To tell our code when to stop adding $\vec{w}$, we can compare the newly generated point to P_2(our destination). Once a generated point $g + \vec{w} = P_2$, we know to stop. This idea is visually represented in Figure 2(b) through vectors.

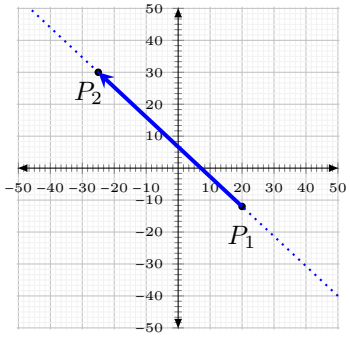


Figure 2(a)

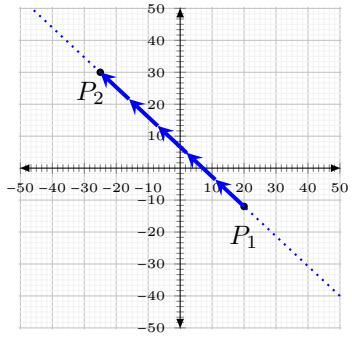


Figure 2(b)

But hmm, this is in 2D and not 3D. So how would we generate points just like we did but in a 3 dimensional space? Well, it turns out that the process is exactly the same. The only thing that would differ is that our points and vectors would have one more parameter: z. For example, $\vec{v}$ might look like $< x, y, z >$ instead of just $< x, y >$ and $\vec{w}$ might look like $< \frac{x}{n}, \frac{y}{n}, \frac{z}{n} >$ instead of just $< \frac{x}{n}, \frac{y}{n} >$. However, we can most definitely still use the concept of scaled vectors to solve this problem.

Conclusion

With 3 rotation matrices, 2 translation matrices, and while loops in code, we can easily build our spinning cube. If each point on our cube were to rotate a small amount of radians, in each iteration of a loop, the cube will look like it is spinning. This is similar to the idea of stop motion animation: where a series of pictures are taken with small adjustments and then played at a fast speed to make it look smooth. In conclusions, I learned a lot about how the math we're learning in Analysis can be applied in various different ways. Walking into this unit, I was not sure when I'd actually use the math we're learning but this project has changed my perspective on that question from *when* to *how*. Overall, this project was super fun to write code for and although everything isn't explained in this document, the

majority of what we went through is. At the end of the day, I had fun while learning a lot, so what more could I ask for.