UNIT-1 ⇒ COMPLEX VARIABLE

$C = \{(a+ib) / a, b \in R\}$

All complex number are generally represented by z.

eg., $z_1 = 3+4i$, $z_2 = 3-6i$ and many more.

We know that, $i = \sqrt{-1}$

$i^2 = -1$

$i^3 \Rightarrow i^2 \cdot i = -1 \times i = -i$

$i^4 \Rightarrow i^2 \cdot i^2 = -1 \times -1 = 1$

$i^5 \Rightarrow i^4 \cdot i = 1 \times i = i$

$i^{100} \Rightarrow (i^2)^{50} = (-1)^{50} = 1$

Let, $z = x + i \cdot y$

Real $(z) = x$ and img $(z) = y$

example, $z = 2+3i$

Here, real $(z) = 2$ and img $(z) = 3$

★ Conjugate of complex number-

Suppose, $z = x + i \cdot y$

then, conjugate of z $(\bar{z}) = x - i \cdot y$

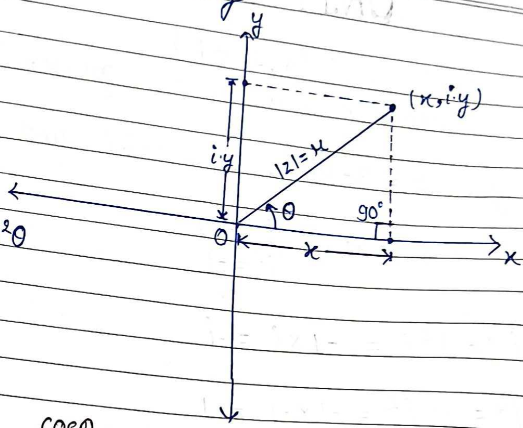
example, $z = -5 + 21i$

$\bar{z} = -5 - 21i$

★ Complex plane - Suppose, $z = x + iy$

$r \Rightarrow |z| = \sqrt{x^2+y^2}$

$\arg(z) \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$



$x^2+y^2 = r^2\cdot\cos^2\theta + r^2\cdot\sin^2\theta$

$x^2+y^2 = r^2$

$\Rightarrow \quad r = \sqrt{x^2+y^2}$

$\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$

$y = r\cdot\sin\theta$ and $x = r\cdot\cos\theta$

• Polar function -

$z = x + iy$

$z = r\cos\theta + ir\cdot\sin\theta$

$z = r\cdot(\cos\theta + i\cdot\sin\theta)$

or

$z = e^{i\cdot\theta}$

★ To find sum or addition, subtraction, product and divide of two complex number -

(1) Addition of two complex number -

Let $z_1 = x_1 + i\cdot y_1$ and $z_2 = x_2 + i\cdot y_2$

$z_1 + z_2 = (x_1 + iy_1) + (x_2 + i\cdot y_2)$

$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)\cdot i$

(2) Subtraction of two complex numbers-

$$z_1 - z_2 = (x_1 + i \cdot y_1) - (x_2 + i \cdot y_2)$$
$$z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2) \cdot i$$

(3) Product of two complex numbers-

$$z_1 \cdot z_2 = (x_1 + i \cdot y_1) \cdot (x_2 + i \cdot y_2)$$
$$z_1 \cdot z_2 = x_1 \cdot x_2 + x_1 \cdot y_2 \cdot i + i \cdot x_2 \cdot y_1 + i^2 \cdot y_1 \cdot y_2$$
$$z_1 \cdot z_2 = x_1 \cdot x_2 + i \cdot (x_1 \cdot y_2 + x_2 \cdot y_1) - y_1 \cdot y_2$$

$$z_1 \cdot z_2 = (x_1 \cdot x_2 - y_1 \cdot y_2) + i \cdot (x_1 \cdot y_2 + y_1 \cdot x_2)$$

(4) Division of two complex numbers-

$$\frac{z_1}{z_2} = \frac{x_1 + i \cdot y_1}{x_2 + i \cdot y_2}$$

$$\frac{z_1}{z_2} = \frac{(x_1 + i \cdot y_1) \cdot (x_2 - i \cdot y_2)}{(x_2 + i \cdot y_2) \cdot (x_2 - i \cdot y_2)}$$

$$\frac{z_1}{z_2} = \frac{x_1 \cdot x_2 - i \cdot x_1 \cdot y_2 + i \cdot y_1 \cdot x_2 - i^2 \cdot y_1 \cdot y_2}{x_2^2 - i \cdot x_2 \cdot y_2 + i \cdot x_2 \cdot y_2 - i^2 y_2^2}$$

$$\frac{z_1}{z_2} = \frac{(x_1 \cdot x_2 + y_1 \cdot y_2) + i \cdot (y_1 x_2 - x_1 \cdot y_2)}{x_2^2 + y_2^2}$$

* Modulus of a complex number-

Let a complex number $z = x + i \cdot y$

$$|z| = \sqrt{x^2 + y^2}$$

• Addition of a complex number and its conjugate-

Let , $z = x + iy$
$\bar{z} = x - iy$

$z + \bar{z} = x + iy + x - iy$
$z + \bar{z} = 2x$
$2x \in R$

Note - $z + \bar{z} = Re(z)$

• Subtraction of a complex number and its conjugate-

$z - \bar{z} = x + iy - x + iy$
$z - \bar{z} = 2i \cdot y$

Note - $z - \bar{z} = Im(z)$

Exercise

(1) If $|z| = 1$, then prove that $\frac{z-1}{z+1}$ is pure imaginary number.

Solution - Given - $|z| = 1$
Let $z = x + iy$

$|z| = \sqrt{x^2 + y^2}$
$1 = \sqrt{x^2 + y^2} \Rightarrow x^2 + y^2 = 1$ ---- (i)

We have $\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} \times \frac{(x+iy-1)}{(x+iy-1)}$

Que. argument of $\frac{z-1}{z+1}$ is example.

$\frac{z-1}{z+1}$ Let us consider, $z = x + iy$

$= \frac{x+iy-1}{x+iy+1}$

$= \frac{x-1+iy}{x+1+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$

$= \frac{((x-1)+iy).((x+1)-iy)}{(x+1)^2-(iy)^2}$

$= \frac{(x-1).(x+1) + iy(x+1) - (x-1)iy - (iy)^2}{x^2+1-2x - i^2y^2}$

$= \frac{x^2-1 + xyi + iy - xyi + iy - i^2y^2}{x^2+y^2+2x+1}$

$= \frac{x^2-1 + 2y.i + y^2}{x^2+y^2+2x+1}$

$= \frac{x^2+y^2 + 2i.y - 1}{x^2+y^2+2x+1}$

$= \frac{x^2+y^2-1 + i.2y}{x^2+y^2+2x+1}$

$= \frac{x^2+y^2-1}{x^2+y^2+2x+1} + i.\frac{2y}{x^2+y^2+2x+1}$

then $\arg\left(\frac{z-1}{z+1}\right) = \tan^{-1}\left[\frac{\frac{2y}{x^2+y^2+2x+1}}{\frac{x^2+y^2-1}{x^2+y^2+2x+1}}\right]$

$\arg\left(\frac{z-1}{z+1}\right) = \tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right)$ ans replacing $i = -i$ in above eq^n

$$x - iy = \sqrt{\frac{a-ib}{c+i\cdot d}} \quad \text{------ (ii)}$$

Multiplying eq^n (i) and eq^n (ii)

$$(x+iy)\cdot(x-iy) = \sqrt{\frac{a+ib}{c-id}} \times \sqrt{\frac{a-ib}{c+id}}$$

$$x^2+y^2 = \sqrt{\frac{(a+ib)\cdot(a-ib)}{(c-id)\cdot(c+id)}}$$

$$x^2+y^2 = \sqrt{\frac{a^2 - i^2 b^2}{c^2 - i^2\cdot d^2}}$$

$$\left[x^2+y^2 = \sqrt{\frac{a^2+b^2}{c^2+d^2}} \right]$$

Hence, Proved

(3) If $A + i\cdot B = \frac{5+7i}{2-3i}$, then find the value of A and B.

Solution :

$$A + iB = \frac{5+7i}{2-3i} \times \frac{2+3i}{2+3i}$$

$$A + iB = \frac{(5+7i)\cdot(2+3i)}{4+9}$$

$$A + iB = \frac{10 + 15i + 14i - 21}{13}$$

$$A + iB = \frac{-11 + 29i}{13}$$

$$A + iB = -\frac{11}{13} + i\cdot\frac{29}{13}$$

On comparing ; we get

$$A = -\frac{11}{13} \quad \text{and} \quad B = \frac{29}{13} \quad \text{Ans}$$

$$\frac{z-1}{z+1} = \frac{[(x+iy)-1]^2}{(x+iy)^2-(1)^2}$$

$$= (x+iy)^2 + (1)^2 - 2(x+iy)$$

$$\frac{z-1}{z+1} \Rightarrow \frac{x+iy-1}{x+iy+1} = \frac{x-1+iy}{x+1+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$$

$$= \frac{[(x-1)+iy] \times [(x+1)-iy]}{(x+1)^2-(iy)^2}$$

$$= \frac{(x-1)\cdot(x+1) - (x-1)\cdot iy + (x+1)\,iy - i^2y^2}{(x+1)^2 - i^2y^2}$$

$$= \frac{x^2-1-i\cdot xy+iy+i\cdot xy+iy+y^2}{(x+1)^2+y^2}$$

$$= \frac{x^2+y^2-1+2iy}{x^2+1+2x+y^2}$$

$$\Rightarrow \frac{1-1+2iy}{1+1+2x} = \frac{2iy}{2+2x}$$

$$\frac{z-1}{z+1} \Rightarrow \frac{2(iy)}{2(1+x)} = i\cdot\left(\frac{y}{1+x}\right)$$

Hence, proved that $\frac{z-1}{z+1}$ for $|z|=1$ is pure imaginary number.

(2) If $x+iy = \sqrt{\frac{a+ib}{c-id}}$, then show that $(x^2+y^2) = \frac{a^2+b^2}{c^2+d^2}$.

Solution: $x+iy = \sqrt{\frac{a+ib}{c-id}}$ ------ (i)

$$\sqrt{(x-1)^2+y^2} = 2\sqrt{(x+1)^2+y^2}$$

$$(x-1)^2+y^2 = 4[(x+1)^2+y^2]$$
$$x^2+1-2x+y^2 = 4(x^2+1+2x+y^2)$$
$$x^2+1-2x+y^2 = 4x^2+4+8x+4y^2$$

$$x^2+1-2x+y^2-4x^2-4-8x-4y^2 = 0$$

$$-3x^2-3y^2-10x-3 = 0$$
$$3x^2+3y^2+10x+3 = 0$$
$$x^2+y^2+\frac{10x}{3}+1 = 0$$

above eqn is the equation of circle.

Que- Find the imaginary part of $z\cdot\bar{z}$.

Que- Find locus of $|z-0_1| \geq 1$ if, $x+iy = \frac{2-3i}{4+7i}$ and find x and y

Solution- $z\cdot\bar{z} = (x+iy)\cdot(x-iy)$
$$= x^2-ixy+ixy-i^2y^2$$
$$z\cdot\bar{z} = x^2+y^2$$

Here imaginary part of $z\cdot\bar{z} = 0$

Que. If $z = \frac{a+ib}{c+id}$ the find $Re(z)$

Qu. If $z_1 = 2+i$ and $z_2 = 3+2i$ find $z_1\cdot z_2$.

Que Find Polar form of $z = -\sqrt{3}\cdot i$.

Que. Find the locus of $|z| = 1$.

Que. If $z = -1+i\sqrt{3}$. Find $|z|$.

Que. $|z_1+z_2|^2 = |z_1|^2+|z_2|^2$ iff

(A) $z_1\bar{z}_2$ is purely imaginary

(B) $z_1\bar{z}_2$ is zero.

(C) $z_1\bar{z}_2$ is purely real.

Que- $\left|\frac{z-1}{z+1}\right| = 2$

We have. $\frac{z-1}{z+1} \Rightarrow \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy}$

$$\frac{z-1}{z+1} = \frac{x^2+y^2-1}{x^2+y^2+2x+1} + \frac{i\cdot 2y}{x^2+y^2+2x+1}$$

$$\left|\frac{z-1}{z+1}\right| = \sqrt{\left(\frac{x^2+y^2-1}{x^2+y^2+2x+1}\right)^2 + \left[\frac{2y}{(x^2+y^2+2x+1)^2}\right]^2}$$

$$(2)^2 = \frac{(x^2+y^2-1)^2}{(x^2+y^2+2x+1)^2} + \frac{(2y)^2}{(x^2+y^2+2x+1)^2}$$

$$4 = \frac{(x^2+y^2-1)^2+(2y)^2}{(x^2+y^2+2x+1)^2}$$

$$4(x^2+y^2+2x+1)^2 = (x^2+y^2-1)^2+(2y)^2$$

$$\cancel{4\left[x^4+y^4+2x^2y^2+4x^2+1+4x+2(x^2+y^2)\cdot(2x+1)\right]}$$

$$\cancel{= x^4+y^2+2x^2+y^2+1-2(x^2+y^2)+4y^2}$$

$$4(x^2+y^2+2x+1)^2 - (x^2+y^2-1)^2 = (2y)^2$$

Solution: $\left|\frac{z-1}{z+1}\right| = 2$

$$\frac{|z-1|}{|z+1|} = 2$$

$$|z-1| = 2|z+1|$$

$z = x+iy$ $\quad |x+iy-1| = 2|x+iy+1|$

Que- find locus of $|z|>1$. then
$x^2+y^2>1$

Que If $z=3+4i$ then modulas of z is-
$|z| = \sqrt{(3)^2+(4)^2}$
$|z| = \sqrt{9+16} = \sqrt{25} = 5$

Que. Conjugate of complex no. $z=-3+i$ is $\bar{z}=-3-i$

Que. What is the polar form of $z=1+i$

polar form $r = \sqrt{(1)^2+(1)^2} = \sqrt{1+1} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(1)$

$\theta = \frac{\pi}{4}$

$z = r\cos\theta + ir\sin\theta$
$z = r\cdot\cos\frac{\pi}{4} + i\cdot r\sin\frac{\pi}{4} \Rightarrow \sqrt{2}\cos\frac{\pi}{4} + i\sqrt{2}\cdot\sin\frac{\pi}{4}$

$$\left[\begin{array}{l} z = \sqrt{2}\times\frac{1}{\sqrt{2}} + i\cdot\sqrt{2}\times\frac{1}{\sqrt{2}} \\ z = 1+i. \end{array}\right]$$ Prove

Que. Find the amplitude of $z = \frac{5}{12}i$.

$\arg.(z) = \tan^{-1}\left(\frac{y}{x}\right)$

$\arg(z) = \tan^{-1}\left(\frac{5}{12\times0}\right)$

$\arg(z) \Rightarrow \tan^{-1}\left(\frac{5}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

Polar form of $z = \sqrt{r}\cdot\cos\theta + ir\cdot\sin\theta$

$= \sqrt{3}\cdot\cos\frac{\pi}{2} + \sqrt{3}\cdot i\cdot\sin\frac{\pi}{2}$

polar form of $z = \cancel{\sqrt{3}\cos\pi + \sqrt{3}\cdot i\cdot\sin\pi}$

$= \sqrt{3}\left(\cos\frac{\pi}{2} + i\sin\pi/2\right)$

$e^{i\theta} = (\cos\theta + i\sin\theta)$

$= \sqrt{2}\left(e^{i\pi/2}\right)$

(4) Solution- $Z = \frac{(a+ib) \times (c-id)}{(c+id) \times (c-id)}$

$$Z = \frac{(a+ib)\times(c-id)}{c^2 - i^2 d^2}$$

$$= \frac{a.c - i a.d + i.bc - i^2 bd}{c^2+d^2}$$

$$= \frac{a.c - i.(ad-bc) + bd}{c^2+d^2}$$

$$= \frac{ac+bd - i(ad-bc)}{c^2+d^2}$$

$$Z = \frac{ac+bd}{c^2+d^2} - i.\frac{(ad-bc)}{c^2+d^2}$$

then $Re(z) = \frac{ac+bd}{c^2+d^2}$

(5) Solution- $Z_1 = 2+i$

$Z_2 = 3+2i$

$$Z_1 \cdot Z_2 = (2+i)\cdot(3+2i)$$

$$= 6+4i+3i+2i^2$$

$$= 6+7i-2$$

$$Z_1 \cdot Z_2 = 4+7i$$

(6) Solution- $Z = -\sqrt{3}\cdot i = 0 - \sqrt{3}i$

$$r = \sqrt{(0)^2 + (-\sqrt{3})^2}$$

$$r = \sqrt{0+3} = -\sqrt{3}$$

and $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-\sqrt{3}}{0}\right) \Rightarrow \pi/(\sqrt{3})$

$\theta \Rightarrow \tan^{-1}(-\infty) = \frac{\pi}{2}$

Que. If $|z-3i|=5$, then find centre of a circle.

Solution

(2) Solution- $|z-1|\geq 1$

Let $z=x+iy$

$|x+iy-1|\geq 1$

$\sqrt{x-1+iy|}\geq 1 \Rightarrow \sqrt{(x-1)^2+(y)^2}\geq 1$

$(x-1)^2+y^2\geq 1$

$x^2+1-2x+y^2\geq 1$

$x^2+y^2-2x+1\geq 1$

$x^2+y^2-2x\geq 0$

(3) Solution- $x+iy = \frac{2-3i}{4+7i}$

$$x+iy = \frac{2-3i}{4+7i} \times \frac{(4-7i)}{(4-7i)}$$

$$= \frac{8-14i-12i+21i^2}{16+(49)}$$

$$= \frac{8-26i-21}{16+49}$$

$$= \frac{-13-26i}{65}$$

$$= \frac{-13}{65} - \frac{26i}{65}$$

$$x+iy = \frac{-1}{5} - i\cdot\frac{2}{5}$$

$\Rightarrow x = \frac{-1}{5}$ and $y = \frac{-2}{5}$

Prove. $\cos(iy) = \cosh y$

Proof. $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

replacing x by iy

$$\cos(iy) = \frac{e^{i \cdot iy} + e^{-i \cdot iy}}{2}$$

$$\cos(iy) = \frac{e^{-y} + e^{y}}{2}$$

$$\cos(iy) = \cosh x$$

(2) Prove that $\sin(iy) = i \sinh y$

Proof - $\sin x = \frac{e^{ix} - e^{-ix}}{2}$

replacing $x \to iy$

$$\sin(iy) = \frac{e^{-y} - e^{y}}{2i} = \frac{-1}{i} \cdot \left(\frac{e^{y} - e^{-y}}{2}\right)$$

$$\sin(iy) = \frac{-i}{i^2} \cdot \sinh y$$

$$[\sin(iy) = i \cdot \sinh y]$$

(3) Prove that $\tan(iy) = i \cdot \tanh y$

Proof - $\tan x = \frac{\sin x}{\cos x}$

$$\tan(iy) = \frac{\sin(iy)}{\cos(iy)}$$

$$\tan(iy) = \frac{i \cdot \sin(hy)}{\cos(hy)}$$

$$[\tan(iy) = i \cdot \tan(hy)]$$

Proved

$$e^{ix} + e^{-ix} = 2\cos x$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Now subtracting e^{ix} from e^{ix}

$$e^{ix} - e^{-ix} = \left[1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} - - - - - -\right]$$
$$- \left[1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} + - - - - -\right]$$
$$= \left[i \cdot 2x - \frac{i \cdot 2x^3}{3!} + \frac{i\,2x^5}{5!} + - - -\right]$$
$$= i \cdot 2 \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + - - - -\right]$$
$$e^{ix} - e^{-ix} = i \cdot 2 \sin x$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{e^{ix} - e^{-ix}}{i \cdot (e^{ix} + e^{-ix})}$$

$$\cot x = \frac{\cos x}{\sin x} = \frac{i\,(e^{ix} + e^{-ix})}{(e^{ix} - e^{ix})}$$

Hyperbolic function-

$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$ and $$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

Prove that $\cos x = \frac{e^{ix} + e^{-ix}}{2}$.

Proof - As we know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \text{--------}$$

$[\cos\theta + i\sin\theta = e^{i\theta}]$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \text{-------}$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \text{-------}$$

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \text{----}\right) + i\left(x - \frac{x^3}{3!} \text{-----}\right)$$

$[e^{ix} = \cos x + i\cdot\sin x]$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \text{-----------}$$

$$e^{-ix} = 1 - ix + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \text{----------}$$

$$e^{-ix} = 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} + \text{----------}$$

$$e^{ix} + e^{-ix} = \left(1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \text{----}\right) + \left(1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} + \text{------}\right)$$

$$e^{ix} + e^{-ix} = 2 - \frac{2x^2}{2!} + \frac{2x^4}{4!} + \text{------}$$

$$e^{ix} + e^{-ix} = 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \text{------}\right)$$

example, Find $arg(z)$ if $z_1 = 1+i$, $z_2 = 1-i$, $z_3 = -1-i$
$z_4 = -1+i$.

Solution- (i) $z_1 = 1+i$

$$arg(z_1) \Rightarrow \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1)$$

$$arg(z_1) = \frac{\pi}{4}$$ Ans

(ii) $z_2 = 1-i$.

$$arg(z_2) = \tan^{-1}\left(\frac{-1}{1}\right) = \tan^{-1}(-1)$$

$$arg(z_2) = -\frac{\pi}{4}$$ Ans

(iii) $z_3 = -1-i$

$$arg(z_3) = \tan^{-1}\left(\frac{-1}{-1}\right)$$

$$arg(z_3) \Rightarrow \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$$

(iv) $z_4 = -1+i$

$$arg(z_4) = \tan^{-1}\left(\frac{-1}{1}\right)$$

$$arg(z_4) \Rightarrow \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

★ Polar form of a complex number-

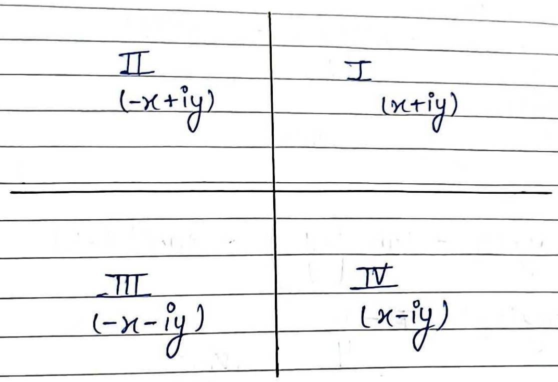
Let $z = x + i \cdot y$, then polar form

$$r(\cos\theta + i\cdot\sin\theta) = r\cos\theta + ir\cdot\sin\theta$$

where, $r = \sqrt{x^2+y^2}$

$\theta = arg(z)$

Argument of a complex number and its polar form-



Argument of complex number in I- Quadrants-
Let z be a complex number
$arg(z) = \theta$

Argument of complex number in II- quadrants-
$arg(z) = \pi - \theta$

Argument of complex number in III- quadrants-
$arg(z) = \theta - \pi$

Argument of complex number in IV- quadrant-

$arg(z) = -\theta$

(4) Prove that $\cot(iy) = \frac{1}{i} \coth y$

$$\cot(iy) = \frac{\cos(iy)}{\sin(iy)}$$

$$= \frac{\cosh y}{i \cdot \sinh y}$$

$$\boxed{\cot(iy) = \frac{1}{i} \cdot \coth y}$$ Ans

(5) Prove that $\operatorname{cosec}(iy) = \frac{1}{i} \cdot \operatorname{cosech} y$

$$\operatorname{Cosec}(iy) = \frac{1}{\sin(iy)}$$

$$\operatorname{cosec}(iy) = \frac{1}{i \cdot \sinh y}$$

$$\boxed{\operatorname{cosec}(iy) = \frac{1}{i} \cdot \operatorname{cosech} y}$$ Ans

(6) Prove that $\sec(iy) = \operatorname{sech} y$

$$\sec(iy) = \frac{\cancel{co}\ 1}{\cos(iy)}$$

$$\sec(iy) = \frac{1}{\cosh y}$$

$$\boxed{\sec(iy) = \operatorname{sech} y}$$ Ans

(1) Complex number - $z = 1 + i \cdot 0$

$r = \sqrt{(1)}$

$r = 1$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{0}{1}\right)$

$\theta = \tan^{-1}(0)$

$\theta = 0$

then $\arg(z) = 0$

(2) If $z_1 = 4 - 3i$ and $z_2 = -1 + 2i$. then find $\frac{z_1}{z_2}$.

Solution - $z_1 = 4 - 3i$ and $z_2 = -1 + 2i$

$\bar{z}_2 = -1 - 2i$

$$\frac{z_1}{z_2} = \frac{4-3i}{-1+2i} \times \frac{(-1-2i)}{(-1-2i)}$$

$$= \frac{-4 - 8i + 3i + 6i^2}{(-1)^2 + (2)^2}$$

$$= \frac{-4 - 11i - 6}{1 + 4}$$

$$\Rightarrow \frac{-10 - 11i}{5} \Rightarrow \frac{-10}{5} - i \cdot \frac{11}{5} = -2 - i \cdot \frac{11}{5}$$

and $$\frac{z_1}{\bar{z}_2} = \frac{4-3i}{-1-2i} \times \frac{(-1+2i)}{(-1+2i)}$$

$$= \frac{-4 + 8i + 3i - 6i^2}{(-1)^2 + (2)^2}$$

$$= \frac{-4 + 11i + 6}{1 + 4}$$

$$\frac{z_1}{\bar{z}_2} \Rightarrow \frac{2 + i \cdot 11}{5} = \frac{2}{5} + \frac{i \cdot 11}{5}$$ Ans

Q) Find the polar form of the $z = -\sqrt{3} + i$

$x = -\sqrt{3}$ and $y = 1$

$r \Rightarrow \sqrt{(-\sqrt{3})^2 + (1)^2}$

$r = \sqrt{3+1}$

$r = \sqrt{4} \qquad \Rightarrow \qquad r = 2$

$\theta = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-1}{\sqrt{3}}\right| = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\theta = \frac{\pi}{6}$

$\arg(z) \Rightarrow \pi - \theta = \pi - \frac{\pi}{6}$

$\arg(z) = \frac{5\pi}{6}$

Polar form is $2\left(\cos\frac{5\pi}{6} + i \cdot \sin\frac{5\pi}{6}\right)$

Some results -

(a) $|z_1 + z_2| \leq |z_1| + |z_2|$

(b) $(|z_1 + z_2|)^2 + (|z_1 - z_2|)^2 = 2(|z_1|^2 + |z_2|^2)$

(c) $|z_1 - z_2| \geq |z_1| - |z_2|$

(d) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

(e) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(f) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

example (i) Write polar form $-1+i\sqrt{3}$

Solution- $x=-1$ and $y=\sqrt{3}$

$r \Rightarrow \sqrt{(-1)^2+(\sqrt{3})^2} = \sqrt{1+9} = \sqrt{6}$

$r = 2$

$\theta \Rightarrow \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{\sqrt{3}}{-1}\right|$

$\arg(z) = \pi - \theta$

$\Rightarrow \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

Polar form- $2\left(\cos\frac{2\pi}{3} + i\cdot\sin\frac{2\pi}{3}\right)$

Que- $z = -1-i$ ~~Here~~, Here, $x=-1$ and $y=-1$

$r \Rightarrow \sqrt{(x)^2+(y)^2} = \sqrt{(-1)^2+(-1)^2}$

$r \Rightarrow \sqrt{1+1} = \sqrt{2}$

$\theta \Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-1}{-1}\right)$

$\theta \Rightarrow \tan^{-1}(1) = \frac{\pi}{4}$

$\arg(z) \Rightarrow \theta - \pi = \frac{\pi}{4} - \pi = \frac{-3\pi}{4}$

Polar form of z is $\sqrt{2}\left[\cos\left(\frac{-3\pi}{4}\right) + i\cdot\sin\left(\frac{-3\pi}{4}\right)\right]$

$= \sqrt{2}\left(\cos\frac{3\pi}{4} - i\cdot\sin\frac{3\pi}{4}\right)$ Ans of

(6) If $(x+iy)^{1/3} = u + v\cdot i$
then

Solution- $(x+iy)^{1/3} = u + v\cdot i$

$$x+iy = (u+v\cdot i)^3$$
$$x+iy = u^3 + i^3\cdot v^3 + 3u\cdot vi\,(u+v\cdot i)$$
$$= u^3 - v^3\cdot i + 3u\cdot vi\,(u+v\cdot i)$$
$$= u^3 - v^3\cdot i + 3u^2\cdot vi + 3uv^2 i^2$$
$$= u^3 - v^3 i + 3u^2 vi - 3u\cdot v^2$$
$$x+iy = (u^3 - 3uv^2) + (3u^2 v - v^3)\,i$$

$\Rightarrow$ $x = u^3 - 3uv^2$ and $y = 3u^2 v - v^3$

$\frac{x}{u} = u^2 - 3v^2$ and $\frac{y}{v} = 3u^2 - v^2$

$$\frac{x}{u} + \frac{y}{v} = u^2 - 3v^2 + 3u^2 - v^2$$

$$\frac{x}{u} + \frac{y}{v} \Rightarrow 4u^2 - 4v^2 = 4(u^2 - v^2) \quad \text{Ans}$$

(7) $z + \bar{z} \neq 0$ iff (a) $Re(z) \neq 0$, (b) $Im(z) \neq 0$
(c) $z \neq 0$ (d) $|z| = 0$

Solution : $z = a + ib$
$\bar{z} = a - ib$

$$z + \bar{z} = a + ib + a - ib$$
$$z + \bar{z} = 2a$$

$\Rightarrow$ $2a \neq 0$

$Re(z) \neq 0$ Ans

(3) If $z_1 = 1+i$ and $z_2 = 3-2i$, find the value of $|5z_1 - 4z_2|$.

Solution - $z_1 = 1+i$ and $z_2 = 3-2i$

$5z_1 = 5+5i$ and $4z_2 = 12-8i$

$5z_1 - 4z_2 = 5+5i-12+8i$

$5z_1 - 4z_2 = -7+13i$

$|5z_1 - 4z_2| = \sqrt{(-7)^2+(13)^2} = \sqrt{49+169}$

$|5z_1 - 4z_2| = \sqrt{218}$

(4) Find $(a,b)\cdot(c,d)$.

Solution - $(a+ib)\cdot(c+id)$

$= ac + i\cdot ad + i\cdot bc + i^2\cdot bd$

$= a\cdot c + i\cdot ad + i\cdot bc - b\cdot d$

$= (a\cdot c - bd) + i\cdot(bc+ad)$

$(a,b)\cdot(c,d) = [(a\cdot c - bd), (bc+ad)]$

(5) If z is any complex number, then $\frac{z-\bar{z}}{2i}$.

Solution - $z = a+ib$

$\bar{z} = a-ib$

$\frac{z-\bar{z}}{2i} = \frac{a+ib-a+ib}{2i}$

$= \frac{i\cdot b}{i\cdot 2}$

$= \frac{b}{2}$ (Purely real)

$$\tan(\alpha - i\beta) = \frac{2\sin(\alpha - i\beta) \times \cos(\alpha + i\beta)}{2\cos(\alpha - i\beta) \times \cos(\alpha + i\beta)}$$

$$= \frac{\sin(\alpha - i\beta + \alpha + i\beta) + \sin(\alpha - i\beta - \alpha - i\beta)}{\cos(\alpha - i\beta + \alpha + i\beta) + \cos(\alpha - i\beta - \alpha - i\beta)}$$

$$= \frac{\sin(2\alpha) + \sin(-2i\beta)}{\cos(2\alpha) + \cos(-2i\beta)}$$

$$\Rightarrow \frac{\sin 2\alpha - \sin 2i\beta}{\cos 2\alpha + \cos 2i\beta} = \frac{\sin 2\alpha - i\cdot\sinh 2\beta}{\cos 2\alpha + \cosh 2\beta}$$

$$\tan(\alpha - i\beta) = \frac{\sin 2\alpha \cancel{- i\cdot\sinh 2\beta}}{\cos 2\alpha + \cosh 2\beta} - \frac{i\cdot\sinh 2\beta}{\cos 2\alpha + \cosh 2\beta}$$

(4) $$\tan(\alpha + i\beta) = \frac{2\sin(\alpha + i\beta)}{2\cos(\alpha + i\beta)} \times \frac{\cos(\alpha - i\beta)}{\cos(\alpha - i\beta)}$$

$$= \frac{\sin(\alpha + i\beta + \alpha - i\beta) + \sin(\alpha + i\beta - \alpha + i\beta)}{\cos(\alpha + i\beta + \alpha - i\beta) + \cos(\alpha + i\beta - \alpha + i\beta)}$$

$$= \frac{\sin(2\alpha) + \sin(2i\beta)}{\cos(2\alpha) + \cos(2i\beta)}$$

$$= \frac{\sin(2\alpha) + i\cdot\sinh 2\beta}{\cos(2\alpha) + \cosh 2\beta}$$

$$\tan(\alpha + i\beta) = \frac{\sin 2\alpha}{\cos 2\alpha + \cosh 2\beta} + \frac{i\cdot\sinh 2\beta}{\cos 2\alpha + \cosh 2\beta}$$

$$\operatorname{Re}(\tan(\alpha + i\beta)) = \frac{\sin 2\alpha}{\cos 2\alpha + \cosh 2\beta}$$

$$\operatorname{Im}(\tan(\alpha + i\beta)) = \frac{\sinh 2\beta}{\cos 2\alpha + \cosh 2\beta}$$

- Separate angle of trigonometric function into real and imaginary part -

(1) $\sin(\alpha + i\beta) = \sin\alpha \cdot \cos i\beta + i \cos\alpha \cdot \sin i\beta$

$= \sin\alpha \cosh\beta + i \cdot \cos\alpha \sinh\beta$

$Re(\sin(\alpha + i\beta)) = \sin\alpha \cdot \cosh\beta$

$Im(\sin(\alpha + i\beta)) = \cos\alpha \cdot \sinh\beta$

(2) $\cos(\alpha + i\beta) = \cos\alpha \cdot \cos i\beta - \sin\alpha \cdot \sin i\beta$

$= \cos\alpha \cdot \cosh\beta - \sin\alpha \cdot i \sinh\beta$

$= \cos\alpha \cdot \cosh\beta - i \cdot \sin\alpha \cdot \sinh\beta$

$Re(\cos(\alpha + i\beta)) = \cos\alpha \cdot \cosh\beta$

$Im(\cos(\alpha + i\beta)) = -\sin\alpha \cdot \sinh\beta$

(3) $\tan(\alpha + i\beta) = \cancel{\cos}\ \cancel{\dfrac{\sin(\alpha + i\beta)}{\cos}}$

$\tan(\alpha + i\beta) = \dfrac{\tan\alpha + \tan i\beta}{1 - \tan\alpha \cdot \tan i\beta}$

$= \dfrac{\tan\alpha + i \cdot \tanh\beta}{1 - \tan\alpha \cdot i \cdot \tanh\beta}$

$= \dfrac{\tan\alpha + i \cdot \tanh\beta}{1 - i \cdot (\tan\alpha \cdot \tanh\beta)}$ $\times$.

$= (\tan\alpha + i \cdot \tanh\beta) \times (1$

(3) $\tan(\alpha - i\beta) = \dfrac{\sin(\alpha - i\beta)}{\cos(\alpha - i\beta)}$

$= \dfrac{2}{2} \times \dfrac{\sin(\alpha - i\beta)}{\cos(\alpha - i\beta)} \times \dfrac{\cos(\alpha + i\beta)}{\cos(\alpha + i\beta)}$

(8) Find the polar of complex number - $Z = 3+3i$

$$r = \sqrt{x^2+y^2} = \sqrt{(3)^2+(3)^2}$$
$$r = \sqrt{9+9} = \sqrt{18}$$

$$\theta = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{3}{3}\right| = \tan^{-1}(1)$$

$$\arg(z) = \frac{\pi}{4}$$

Polar form is $\sqrt{18}\left(\cos\left(\frac{\pi}{4}\right) + i\cdot\sin\left(\frac{\pi}{4}\right)\right)$ ans

(9) What is the principle value $\sqrt{3}-i$ find the amplitude.

Solution - $Z = \sqrt{3}-i$

$x = \sqrt{3}$ and $y = (-1)$

$$\theta = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{-1}{\sqrt{3}}\right|$$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$\arg(z) = \frac{-\pi}{6} \quad \text{(amplitude)}$$

$$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d}$$

* ~~Graphic method (maxima and Minima)~~ –

Que – If $\cos(\theta + i\phi) = \rho(\cos\alpha + i\cdot\sin\alpha)$ show that $\phi = \frac{1}{2}\log\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$.

Solution:

Given – $\cos(\theta+i\phi) = \rho(\cos\alpha + i\sin\alpha)$

$$\cos(\theta+i\phi) = \rho\cdot\cos\alpha + \rho i\cdot\sin\alpha$$

$$\cos\theta\cdot\cos i\phi - \sin\theta\cdot\sin i\phi = \rho\cos\alpha + \rho i\cdot\sin\alpha$$

$$\cos\theta\cdot\cosh\phi - i\sin\theta\sinh\phi = \rho\cos\alpha + \rho i\cdot\sin\alpha$$

Comparing the real and imaginary part of both the complex number –

$$\cos\theta\cdot\cosh\phi = \rho\cos\alpha \quad ---(i)$$

$$-\sin\theta\cdot\sinh\phi = \rho\sin\alpha \quad ---(ii)$$

Dividing eqn (ii) by eqn (i)

$$\cancel{\frac{\rho\cos\alpha}{\rho\sin\alpha} = \frac{\cos\theta\cosh\phi}{\sin\theta\sinh\phi}} \qquad \frac{\rho\sin\alpha}{\rho\cos\alpha} = \frac{-\sin\theta\cdot\sinh\phi}{\cos\theta\cdot\cosh\phi}$$

$$\Rightarrow \frac{\sinh\phi}{\cosh\phi} = \frac{-\sin\alpha\cdot\cos\theta}{\cos\alpha\cdot\sin\theta} \Rightarrow \tanh\phi = \frac{-\sin\alpha\cdot\cos\theta}{\cos\alpha\cdot\sin\theta}$$

Applying componendo and dividendo

$$\frac{e^{\phi}+e^{-\phi}+e^{\phi}-e^{-\phi}}{e^{\phi}-e^{-\phi}-e^{\phi}-e^{-\phi}} = \frac{-\sin\alpha\cdot\cos\theta + \cos\alpha\cdot\sin\theta}{-\sin\alpha\cdot\cos\theta - \cos\alpha\cdot\sin\theta}$$

$$\frac{2e^{\phi}}{-2e^{-\phi}} = \frac{-\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$$

(iii) Proof-

$$\frac{r\sin\alpha}{r\cos\alpha} = \frac{\cos\theta \cdot \sinh\phi}{\sin\theta \cdot \cosh\phi}$$

$$\tan\alpha = \cot\theta \cdot \tanh\phi$$

$$[\tan\alpha = \tanh\phi \cdot \cot\theta]$$ Proved

(5) If $\sin(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$
show that (i) $r^2 = \frac{1}{2}(\cosh 2\phi - \cos 2\theta)$

and (ii) $\tan\alpha = \tan\phi \cdot \cot\theta$

Solution - $\sin(\theta + i\phi) = r(\cos\alpha + i\sin\alpha)$

$\sin\theta \cdot \cos i\phi + \cos\theta \cdot \sin i\phi = r(\cos\alpha + i\sin\alpha)$

$\sin\theta \cdot \cosh\phi + i \cdot \cos\theta \cdot \sinh\phi = r\cos\alpha + ri \cdot \sin\alpha$

$r\cos\alpha = \sin\theta \cdot \cosh\phi$ and $r\sin\alpha = \cos\theta \cdot \sinh\phi$

$r^2\cos^2\alpha = \sin^2\theta \cdot \cosh^2\phi$ --- (i) and $r^2\sin^2\alpha = \cos^2\theta \cdot \sinh^2\phi$ --- (ii)

Add (i) and (ii)

$r^2\cos^2\alpha + r^2\sin^2\alpha = \sin^2\theta \cdot \cosh^2\phi + \cos^2\theta \cdot \sinh^2\phi$

$r^2(\cos^2\alpha + \sin^2\alpha) = \sin^2\theta \cdot \cosh^2\phi + \cos^2\theta \cdot \sinh^2\phi$

$$r^2 = \left(\frac{1 - \cos 2\theta}{2}\right) \cdot \left(\frac{1 + \cosh 2\phi}{2}\right) + \left(\frac{1 + \cos 2\theta}{2}\right) \cdot \left(\frac{-1 + \cosh 2\phi}{2}\right)$$

$$r^2 = \frac{1}{4}\left[(1 - \cos 2\theta) \cdot (1 + \cosh 2\phi) + (1 + \cos 2\theta) \cdot (1 - \cosh 2\phi)\right]$$

$$r^2 = \frac{1}{4}\left[1 + \cosh 2\phi - \cos 2\theta - \cos 2\theta \cdot \cosh 2\phi - 1 + \cosh 2\phi - \cos 2\theta + \cos 2\theta \cdot \cosh 2\phi\right]$$

$$r^2 = \frac{1}{4}\left[\cosh 2\phi - \cos 2\theta + \cosh 2\phi - \cos 2\theta\right]$$

$$r^2 = \frac{1}{4}\left[2\cosh 2\phi - 2\cos 2\theta\right]$$

$$\boxed{r^2 = \frac{1}{2}\left[\cosh 2\phi - \cos 2\theta\right]}$$

Proved

(3) If $\sin(\theta + i\phi) = r(\cos\alpha + i\cdot\sin\alpha)$
then prove that $\tan\alpha = \tanh\phi\cdot\cot\theta$

Solution :
$$\sin(\theta+i\phi) = r(\cos\alpha + i\cdot\sin\alpha)$$
$$\sin\theta\cdot\cos i\phi + \cos\theta\cdot\sin i\phi = r\cos\alpha + ri\cdot\sin\alpha$$
$$\sin\theta\cdot\cosh\phi + i\cdot\cos\theta\cdot\sinh\phi = r\cos\alpha + i\cdot r\sin\alpha$$

$$\sin\theta\cdot\cosh\phi = r\cos\alpha \quad ----(i)$$
and
$$\cos\theta\cdot\sinh\phi = r\sin\alpha \quad -----(ii)$$

$$\frac{r\sin\alpha}{r\cos\alpha} = \frac{\cos\theta\cdot\sinh\phi}{\sin\theta\cdot\cosh\phi}$$
$$\cancel{\frac{\sin\alpha \cdot \sin\theta}{\cos\alpha\cdot\cos\theta}} \qquad \tan\alpha = \cot\theta\cdot\tanh\phi$$
$$\left[\tan\alpha = \tanh\phi\cdot\cot\theta\right]$$ Proved

(4) If $\cos(x+iy) = \cos\alpha + i\sin\alpha$, prove that
$\cos 2y + \cos 2x = 2$

Que. If $\sin(A+i\cdot B) = x+i\cdot y$ then prove that
$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1 \quad \text{and} \quad \frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$$

Solution -
$$\sin(A+i\cdot B) = x+i\cdot y$$
$$\sin(A-iB) = x-i\cdot y$$

$$\sin A\cdot\cos iB + \cos A\cdot\sin iB = x+iy$$
$$\sin A\cdot\cosh B + i\cos A\cdot\sinh B = x+iy$$

$$\sin A\cdot\cosh B = x$$
and $\cos A\cdot\sinh B = y$

(b) $x^2+y^2-2y\coth 2B=-1$

$\tan 2iB = i\tan h 2B$

$\cancel{i\tanh 2B} =$

$$\tan 2iB = \tan(iB+iB)$$
$$= \tan(iB+iB+A-A)$$
$$= \tan[(A+iB)-(A-iB)]$$
$$\tan 2iB = \frac{\tan(A+iB)-\tan(A-iB)}{1+\tan(A+iB)\tan(A-iB)}$$
$$= \frac{x+iy-x+iy}{1+(x+iy)(x-iy)}$$

$$\tan 2iB = \frac{2iy}{1+x^2+y^2}$$
$$i\cdot\tan h 2B = \frac{2iy}{1+x^2+y^2}$$
$$\tanh 2B = \frac{2y}{1+x^2+y^2}$$
$$\frac{1}{\coth 2B} = \frac{2y}{1+x^2+y^2}$$
$$1+x^2+y^2 = 2y\coth 2B$$

$\Rightarrow x^2+y^2-2y\coth 2B=-1$ Proved

$$\frac{e^{\phi}}{e^{-\phi}} = \frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$$

$$e^{2\phi} = \frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$$

$$2\phi = \log \frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$$

$$\left[\phi = \frac{1}{2}\log\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right]$$ Proved

(2) If $\tan(A+iB) = x+iy$ (a) $x^2+y^2+2x\cot 2A = 1$
(b) $x^2+y^2-2y\coth 2B = -1$

Solution - (a) $x^2+y^2+2x\cot 2A = 1$

We have $\tan(A+iB) = x+iy$
$\Rightarrow \tan(A-iB) = x-iy$

Now $\tan 2A = \tan(A+A)$

$\tan 2A = \tan(A+A-iB+iB)$

$\tan 2A = \tan[(A+iB)+(A-iB)]$

$$= \frac{\tan(A+iB)+\tan(A-iB)}{1-\tan(A+iB)\cdot\tan(A-iB)}$$

$$= \frac{x+iy+x-iy}{1-(x+iy)(x-iy)}$$

$$\tan 2A = \frac{2x}{1-(x^2+y^2)}$$

$$\frac{1}{\cot 2A} = \frac{2x}{1-(x^2+y^2)}$$

$$1-(x^2+y^2) = 2x\cot 2A$$

$\Rightarrow$ $[x^2+y^2+2x\cot 2A = 1]$ Proved

$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n$

→ Cube roots of a complex number-
Formula (n^{th} root)-

$$(z)^{1/n} = (r)^{1/n} \cdot [\cos(\theta + 2\pi K) + i\sin(\theta + 2\pi K)]^{1/n}$$

where,

$r = \sqrt{x^2 + y^2}$
$\theta = \arg(z)$
$K \in Z$

$$= (r)^{1/n} \left[\cos\frac{(\theta + 2\pi K)}{n} + i\sin\frac{(\theta + 2\pi K)}{n} \right]$$

or

$$= (r)^{1/n} e^{\frac{i(\theta + 2\pi K)}{n}}$$

Note- $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Que- Find cube root of a complex number (i) $1-i$, (ii) $1+i$, (iii) i, and (iv) $-i$

(i) $1-i \Rightarrow$

$\tan\theta = \tan\alpha$

Que- If $\tan(\theta + i\phi) = \cos\alpha + i\cdot\sin\alpha$

$$\theta = \frac{n\pi}{2} + \frac{\pi}{4}$$

Solution- $\tan(\theta + i\phi) = \cos\alpha + i\cdot\sin\alpha$

$\tan(\theta - i\phi) = \cos\alpha - i\cdot\sin\alpha$

$$\tan 2\theta = \tan(\theta+\theta)$$

$$= \tan(\theta + \theta - i\phi + i\phi)$$

$$\tan 2\theta = \tan[(\theta+i\phi) + (\theta - i\phi)]$$

$$\tan 2\theta = \frac{\tan(\theta+i\phi) + \tan(\theta-i\phi)}{1 - \tan(\theta+i\phi)\cdot\tan(\theta+i\phi)}$$

$$= \frac{\cos\alpha + i\cdot\sin\alpha + \cos\alpha - i\cdot\sin\alpha}{1 - (\cos\alpha + i\cdot\sin\alpha)\cdot(\cos\alpha - i\cdot\sin\alpha)}$$

$$= \frac{2\cos\alpha}{1 - (\cos^2\alpha + \sin^2\alpha)}$$

$$\tan 2\theta = \frac{2\cos\alpha}{1-1}$$

$$\tan 2\theta = \infty$$

$$\tan 2\theta = \tan\frac{\pi}{2}$$

$\tan\theta = \tan\alpha$.

$$2\theta = n\pi + \frac{\pi}{2}$$

$\theta = n\pi \pm \alpha$

$$\theta = \frac{n\pi}{2} + \frac{\pi}{4}$$

$\sin^2 A \cdot \cosh^2 B = x^2$ ----- (i)

and $\cos^2 A \cdot \sinh^2 B = y^2$ ---- (ii)

~~Adding eq^n (i) and eq^n (ii)~~

$\cancel{x^2 + y^2 = \sin^2 A \cdot \cosh^2 B + \cos^2 A \cdot \sinh^2 B}$

$\sin^2 A = \dfrac{x^2}{\cosh^2 B}$ -- (iii) and $\cos^2 A = \dfrac{y^2}{\sinh^2 B}$ --- (iv)

Adding eq^n (iii) and eq^n (iv)

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = \sin^2 A + \cos^2 A$$

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$$

Proved

and $\cosh^2 B = \dfrac{x^2}{\sin^2 A}$ and $\sinh^2 B = \dfrac{y^2}{\cos^2 A}$

So

$$\Rightarrow \frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B$$

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$$

Formula- $\cosh^2 B - \sinh^2 B = 1$

$\frac{5\pi}{12} = \frac{5 \times \cancel{180}^{60}}{\cancel{12}_{6}}$

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(90-40)

Que. If $z = -1 - i$, then find the $z^{1/3}$ and $z^{2/3}$.

Solution- $z \Rightarrow -1-i = -$

$z = -1 + (-i)$

$r \Rightarrow \sqrt{(-1)^2 + (-1)^2} = \sqrt{1+1}$

$r = \sqrt{2}$

$\theta = \arg(z)$

$\arg(z) \Rightarrow \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1)$

$\arg(z) \Rightarrow \frac{\pi}{4} - \pi = \frac{-3\pi}{4}$

put $K=0$

$$Z_0 = (1)^{1/3}\left[\cos\left(\frac{-\pi}{2\times3}\right) + i\cdot\sin\left(\frac{-\pi}{2\times3}\right)\right]$$

$$Z_0 = (1)^{1/3}\left[\cos\left(\frac{-\pi}{6}\right) + i\cdot\sin\left(\frac{-\pi}{6}\right)\right]$$

$$Z_0 = (1)^{1/3}\, e^{i(-\pi/6)}$$

(iv) $-i=$

Let $z \Rightarrow -i = 0-i$

$r \Rightarrow \sqrt{0+(-1)^2} = \sqrt{(-1)^2} = \sqrt{1}$

$\boxed{r=1}$

and Here, $\theta = \arg(z)$

and $\arg(z) \Rightarrow \$-\alpha = \tan\left(\frac{-1}{0}\right)$

$$\arg(z) = -\frac{\pi}{2}$$

$$\boxed{\theta = -\frac{\pi}{2}}$$

$$(z)^{1/n} = (r)^{1/n}\left[\cos\frac{(-\pi/2 + 2\pi K)}{n} + i\cdot\sin\frac{(-\pi/2 + 2\pi K)}{3}\right]$$

(iv) $-i = 0 - i = 0 + (-1)i$

$r = \sqrt{(0) + (-1)^2} = \sqrt{1} = 1$

and, $\theta = \arg(z) = \tan\left(\frac{-1}{0}\right)$ $= \tan(\infty) = \frac{\pi}{2}$

$\arg(z) = -\theta = -\frac{\pi}{2} - \theta$

$n = 3$

$(z)^{1/3} = (1)^{1/3}\left[\cos\frac{(-\pi/2 + 2\pi k)}{3} + i\sin\frac{(-\pi/2 + 2\pi k)}{3}\right]$

$\cancel{= (1)^{1/3}}$

put $K = 1$

$$(z^{2/3})_1 = (2)^{1/3}\left[\cos\left(\frac{\pi/2 + 2\pi}{3}\right) + i\cdot\sin\left(\frac{\pi/2 + 2\pi}{3}\right)\right]$$

$$(z^{2/3})_1 = (2)^{1/3}\cdot\left[\cos\left(\frac{5\pi}{6}\right) + i\cdot\sin\left(\frac{5\pi}{6}\right)\right]$$

$$(z^{2/3})_1 = (2)^{1/3}\cdot e^{i\cdot 5\pi/6} \quad \text{Ans}$$

put $K = 2$

$$(z^{2/3})_2 = (2)^{1/3}\left[\cos\left(\frac{\pi/2 + 4\pi}{3}\right) + i\cdot\sin\left(\frac{\pi/2 + 4\pi}{3}\right)\right]$$

$$= (2)^{1/3}\left[\cos\left(\frac{9\pi}{6}\right) + i\cdot\sin\left(\frac{9\pi}{6}\right)\right]$$

$$= (2)^{1/3}\left[\cos\left(\frac{3\pi}{2}\right) + i\cdot\sin\left(\frac{3\pi}{2}\right)\right]$$

$$(z^{2/3})_2 = (2)^{1/3}\cdot e^{i\cdot\frac{3\pi}{2}} \quad \text{Ans}$$

II- method. $(z)^{2/3} = (z^2)^{1/3}$

$$= (2i)^{1/3}$$

$$= (2)^{1/3}(i)^{1/3}$$

cube roots of $z^{1/3}$ are $z_0 = (\sqrt{2})^{1/3} \cdot e^{-i\pi/4}$

$z_1 = (\sqrt{2})^{1/3} \cdot e^{i \cdot 5\pi/4}$

$z_3 = (\sqrt{2})^{1/3} \cdot e^{i \cdot 13\pi/12}$

and now, $z = -1 - i$

$z^2 \Rightarrow z \cdot z = (-1-i)\cdot(-1-i)$

$z^2 = (1 + i + i + i^2)$

$z^2 = (1 + 2i - 1)$

$z^2 = 0 + 2i$

$r \Rightarrow \sqrt{(0)^2 + (2)^2} = \sqrt{(2)^2}$

$r = 2$

$\Rightarrow \quad \theta = \arg(z^2)$

$\arg(z^2) \Rightarrow \tan^{-1}\left(\frac{2}{0}\right) = \tan^{-1}(\infty)$

$\arg(z^2) = \frac{\pi}{2}$

$\theta = \frac{\pi}{2}$

$(z^2)^{1/3}$

$$z^{1/n} = (r)^{1/n}\left[\cos\frac{(\theta + 2\pi k)}{n} + i \cdot \sin\frac{(\theta + 2\pi k)}{n}\right]$$

$$z^{2/3} \Rightarrow (z^2)^{1/3} = (2)^{1/3}\left[\cos\frac{(\pi/2 + 2\pi k)}{3} + i \cdot \sin\frac{(\pi/2 + 2\pi k)}{3}\right]$$

$z^{2/3} = (2)^{1/3}$

put $k = 0$

$$(z^{2/3})_0 = (2)^{1/3}\left[\cos\left(\frac{\pi}{6}\right) + i \cdot \sin\left(\frac{\pi}{6}\right)\right]$$

$$(z^{2/3})_0 = (2)^{1/3} e^{i\pi/6}$$

Ans

$(2/3)$

$z^{\frac{2}{3}} = (z^2)^{1/3}$

$\theta = \frac{-3\pi}{4}$

$z^{1/n} = (r)^{1/n}\left[\cos\frac{(\theta + 2\pi k)}{n} + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$

$z^{1/3} = (\sqrt{2})^{1/3}\left[\cos\frac{(-3\pi/4 + 2\pi k)}{3} + i\sin\frac{(-3\pi/4 + 2\pi k)}{3}\right]$

put $k = 0$

$z_0^{1/3} = (\sqrt{2})^{1/3}\left[\cos\frac{(-3\pi/4)}{3} + i\cdot\sin\frac{(-3\pi/4)}{3}\right]$

$z_0^{1/3} = (\sqrt{2})^{1/3}\left[\cos\left(\frac{-\pi}{4}\right) + i\cdot\sin\left(\frac{-\pi}{4}\right)\right]$

$z_0^{1/3} = (\sqrt{2})^{1/3} e^{-i\pi/4} = (\sqrt{2})^{1/3}\left(\frac{1}{\sqrt{2}} - i\cdot\frac{1}{\sqrt{2}}\right) = (\sqrt{2})^{-2/3}(1-i)$

and $k = 1$

$z_1 = (\sqrt{2})^{1/3}\left[\cos\frac{(-3\pi/4 + 2\pi)}{3} + i\cdot\sin\left(\frac{-3\pi/4 + 2\pi}{3}\right)\right]$

$= (\sqrt{2})^{1/3}\left[\cos\cdot\left(\frac{5\pi/4}{3}\right) + i\cdot\sin\left(\frac{5\pi/4}{3}\right)\right]$

$= (\sqrt{2})^{1/3}\left[\cos\left(\frac{5\pi}{12}\right) + i\cdot\sin\left(\frac{5\pi}{12}\right)\right]$

$z_1 = (\sqrt{2})^{1/3} e^{i\cdot\frac{5\pi}{4}} = (\sqrt{2})^{1/3}$

and put $k = 2$

$z_2 = (\sqrt{2})^{1/3}\left[\cos\frac{(-3\pi/4 + 4\pi)}{3} + i\cdot\sin\left(\frac{-3\pi/4 + 4\pi}{3}\right)\right]$

$z_2 = (\sqrt{2})^{1/3}\left[\cos\left(\frac{13\pi}{12}\right) + i\cdot\sin\left(\frac{13\pi}{12}\right)\right]$

$z_2 = (\sqrt{2})^{1/3} e^{i\cdot\frac{13\pi}{12}}$

(1) Real and Imaginary part of $\log(x+iy)$

Solution – $x+iy = re^{i\theta}$

where $r = \sqrt{x^2+y^2}$

$\theta \Rightarrow \arg(z) = \tan^{-1}\left|\frac{y}{x}\right|$

$$\log(x+iy) = \log(re^{i\theta})$$
$$= \log r + \log e^{i\theta}$$
$$= \log r + i\theta \cdot \log e$$
$$= \log r + i\theta$$

$$\log(x+iy) = \log(x^2+y^2)^{1/2} + i \cdot \tan^{-1}\left(\frac{y}{x}\right)$$

$$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i \cdot \tan^{-1}\left(\frac{y}{x}\right)$$

$$\text{Real}(\log(x+iy)) = \frac{1}{2}\log(x^2+y^2)$$

$$\text{Im}(\log(x+iy)) = \tan^{-1}\left(\frac{y}{x}\right)$$

Que-(1) Find real and imaginary part of $\log(1+i)$.

Solution – ~~Let~~, $z = \log(1+i)$ Here, $x = 1$ and $y = 1$

$$\text{Re}(\log(1+i)) = \frac{1}{2}\log(x^2+y^2)$$
$$= \frac{1}{2}\log(1+1)$$
$$= \frac{1}{2}\log(2)$$
$$= \log(2)^{1/2}$$

$$\text{Re}(\log(1+i)) = \log\sqrt{2}$$

$$\omega^2 \Rightarrow \frac{1}{4}(-1+i\sqrt{3})^2 = \frac{1}{4}(1+i^2 \cdot 3 - 2i\sqrt{3})$$

$$\omega^2 = \frac{1}{4}(1-3-2\sqrt{3}\cdot i)$$

$$\omega^2 \Rightarrow \frac{1}{4}(-2-2\sqrt{3}i) = \frac{1}{4} \times 2(-1-i\sqrt{3})$$

$$\omega^2 \Rightarrow \frac{1}{2}(-1-i\sqrt{3}) = \frac{-1-i\sqrt{3}}{2}$$

Adding all three cube roots of unity–

$$1+\omega+\omega^2 = 1+\frac{(-1+i\sqrt{3})}{2}+\frac{(-1-i\sqrt{3})}{2}$$

$$1+\omega+\omega^2 = 1+\frac{-1+i\sqrt{3}-1-i\sqrt{3}}{2}$$

$$1+\omega+\omega^2 \Rightarrow 1+\frac{(-2)}{2} = 1-1$$

$$\boxed{1+\omega+\omega^2 = 0}$$

$$\omega^3 = \omega^2 \cdot \omega = \left(\frac{-1-i\sqrt{3}}{2}\right)\cdot\left(\frac{-1+i\sqrt{3}}{2}\right)$$

$$\omega^3 = \frac{(-1-i\sqrt{3})\cdot(-1+i\sqrt{3})}{4}$$

$$\omega^3 = \frac{1 \cancel{+} -i\sqrt{3}+i\sqrt{3}-i^2 \cdot 3}{4}$$

$$\omega^3 = \frac{1+3}{4}$$

$$\omega^3 = \frac{4}{4}$$

$$\boxed{\omega^3 = 1}$$

→ Cube root of unity –

Let $\sqrt[3]{1} = x$

$1 = x^3$

$x^3 - 1 = 0$

$x^3 - 1^3 = 0 \Rightarrow (x-1)\cdot(x^2+1+x) = 0$

$(x-1) = 0$ and $x^2+1+x = 0$

$x = 1$ and $x^2+x+1 = 0$

Finding root by quadratic formula of eqⁿ $x^2+x+1=0$

$x \Rightarrow \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{-1 \pm \sqrt{1^2 - 4\times1\times1}}{2\times1}$

$x = \dfrac{-1 \pm \sqrt{-3}}{2}$

taking '+' sign and taking '–' sign

$x = \dfrac{-1+\sqrt{-3}}{2}$ and $x = \dfrac{-1-\sqrt{-3}}{2}$

$x = \dfrac{-1+i\sqrt{3}}{2}$ and $x = \dfrac{-1-i\sqrt{3}}{2}$

So, cube root of unity are 1, $\dfrac{-1+i\sqrt{3}}{2}$ and $\dfrac{-1-i\sqrt{3}}{2}$

A specific relation $(1+\omega+\omega^2 = 0)$

Let $\omega = \dfrac{-1+i\sqrt{3}}{2}$

On squaring both sides.

$\omega^2 = \left(\dfrac{-1+i\cdot\sqrt{3}}{2}\right)^2$

and $$\frac{3+6i}{2-3i} = \frac{3+6i}{2-3i} \times \frac{2+3i}{2+3i}$$

$$= \frac{(3+6i)\cdot(2+3i)}{(2-3i)\cdot(2+3i)}$$

$$= \frac{6+9i+12i-18}{4+9}$$

$$= \frac{-12+21i}{13}$$

$$= \frac{-12}{13} + \frac{21}{13}i. \quad \text{Ans}$$

$$i \cdot 2\beta = \tan^{-1}\left(\frac{x+iy-x+iy}{1+x^2+y^2}\right)$$

$$= \tan^{-1}\left(\frac{2iy}{1+x^2+y^2}\right)$$

$$= \tan^{-1} i\left(\frac{2y}{1+x^2+y^2}\right)$$

$$i \cdot 2\beta = i \tan^{-1}h\left(\frac{2y}{1+x^2+y^2}\right)$$

$\operatorname{Im}(\tan^{-1}(x+iy))$ $$\beta = \frac{1}{2}\tan^{-1}h\left[\frac{2y}{1+x^2+y^2}\right]$$

Que. If $z = (x-1) + i \cdot y$. Find the radius and centre of circle.

Solution- express the $z = 3+6i$ in polar form and also divide it by $2-3i$. $\tan^{-1}(2) = 63.43^\circ$

Solution- $Z = 3+6i$

$$r = \sqrt{(3)^2+(6)^2} = \sqrt{9+36} = \sqrt{45}$$

$$\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) = \theta = \tan^{-1}\left(\frac{6}{3}\right) = \tan^{-1}(2) = 63.43^\circ$$

polar form $$Z = r\left(\cos\theta + i \cdot \sin\theta\right)$$

$$Z = \sqrt{45}\left(\cos(63.43^\circ) + i \cdot \sin(63.43^\circ)\right)$$

$$Im(\log(1+i)) = \tan^{-1}\left|\frac{y}{x}\right|$$

$$= \tan^{-1}\left|\frac{1}{1}\right|$$

$$= \tan^{-1}(1)$$

$$Im(\log(1+i)) = \frac{\pi}{4}$$

(2) Real and imaginary part of $\tan^{-1}(x+iy)$.

Solution - Let $\alpha + i\beta = \tan^{-1}(x+iy)$

$\alpha - i\beta = \tan^{-1}(x-iy)$

Taking $2\alpha = \alpha + \alpha$

$2\alpha = \alpha + \alpha + i\beta - i\beta$

$2\alpha = (\alpha + i\beta) + (\alpha - i\beta)$

$2\alpha = \tan^{-1}(x+iy) + \tan^{-1}(x-iy)$

$$2\alpha = \tan^{-1}\frac{(x+iy) + x - iy}{1-(x+iy)(x-iy)}$$

$$2\alpha = \tan^{-1}\left[\frac{2x}{1-x^2-y^2}\right]$$

$$Re(\tan^{-1}(x+iy)) = \alpha = \frac{1}{2}\tan^{-1}\left[\frac{2x}{1-x^2-y^2}\right]$$

Imaginary part of $\tan^{-1}(x+iy)$

Taking $2i\beta = i\beta + i\beta$

$2i\beta = i\beta + i\beta - \alpha + \alpha$

$i2\beta = (\alpha + i\beta) - (\alpha - i\beta)$

$i \cdot 2\beta = \tan^{-1}(x+iy) - \tan^{-1}(x-iy)$

$x^{1/2}$

$\frac{1}{2x^{1/2}} \quad \frac{1}{2\sqrt{x}}$

$$\nabla f = 2x\hat{i} + 2y\hat{j} + \left(-2z + \frac{1}{z}\right)\cdot\hat{k}$$

$$(\nabla f)_{(1,1,1)} = (2\times1)\hat{i} + (2\times1)\hat{j} + \left(-2 + \frac{1}{1}\right)\hat{k}$$

$$(\nabla f)_{(1,1,1)} = 2\hat{i} + 2\hat{j} - \hat{k} \qquad \underline{\text{Ans}}$$

(3) $f(x,y,z) = \sqrt{x^2+y^2+z^2} + \log(xyz)$ at $(-1,2,-2)$

Solution- $\nabla f = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot\left(\sqrt{x^2+y^2+z^2} + \log(xyz)\right)$

$$\nabla f = \hat{i}\cdot\frac{\partial}{\partial x}\left[\sqrt{x^2+y^2+z^2} + \log(xyz)\right] + \hat{j}\cdot\frac{\partial}{\partial y}\left[\sqrt{x^2+y^2+z^2} + \log(xyz)\right] + \hat{k}\cdot\frac{\partial}{\partial z}\left[\sqrt{x^2+y^2+z^2} + \log(xyz)\right]$$

$$\nabla f = \hat{i}\cdot\left[\frac{1}{2\sqrt{x^2+y^2+z^2}}\times 2x + \frac{1}{(xyz)}\times(yz)\right] + \hat{j}\left[\frac{1\times 2y}{2\sqrt{x^2+y^2+z^2}} + \frac{1}{xyz}\times xz\right] + \hat{k}\cdot\left[\frac{1}{2\sqrt{x^2+y^2+z^2}}\times 2z + \frac{1}{(xyz)}\times(xy)\right]$$

$$\nabla f = \hat{i}\cdot\left[\frac{x}{\sqrt{x^2+y^2+z^2}} + \frac{1}{x}\right] + \hat{j}\cdot\left[\frac{y}{\sqrt{x^2+y^2+z^2}} + \frac{1}{y}\right] + \hat{k}\left[\frac{z}{\sqrt{x^2+y^2+z^2}} + \frac{1}{z}\right]$$

$$\nabla f(-1,2,-2) = \hat{i}\cdot\left[\frac{(-1)}{\sqrt{1+4+4}} + \frac{1}{(-1)}\right] + \hat{j}\left[\frac{2}{\sqrt{1+4+4}} + \frac{1}{2}\right] + \hat{k}\left[\frac{(-2)}{\sqrt{1+4+4}} + \frac{1}{(-2)}\right]$$

$$= \hat{i}\cdot\left[\frac{-1}{3} - 1\right] + \hat{j}\left[\frac{2}{3} + \frac{1}{2}\right] + \hat{k}\cdot\left[\frac{-2}{3} - \frac{1}{2}\right]$$

$$\nabla f_{(-1,2,-2)} = \hat{i}\cdot\left(\frac{-4}{3}\right) + \hat{j}\cdot\left(\frac{7}{6}\right) + \hat{k}\cdot\left(\frac{-7}{6}\right)$$

$\frac{-4-3}{6}$

Exercise

Que- $f(x,y,z) = 3x^2y - y^3z^2$ at $(1,-2,-1)$

$$\nabla f = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot(3x^2y - y^3z^2)$$

$$= \hat{i}\cdot\frac{\partial}{\partial x}(3x^2y - y^3z^2) + \hat{j}\cdot\frac{\partial}{\partial y}(3x^2y - y^3z^2) + \hat{k}\cdot\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= \hat{i}\cdot(6xy - 0) + \hat{j}\cdot(3x^2 - 3y^2z^2) + \hat{k}\cdot(0 - 2z\cdot y^3)$$

$$= \hat{i}\cdot(6xy) + \hat{j}(3x^2 - 3y^2z^2) + \hat{k}(-2y^3z)$$

$$(\nabla f) = (6xy)\hat{i} + (3x^2 - 3y^2z^2)\cdot\hat{j} + (-2y^3z)\hat{k}$$

$$(\nabla f)_{(1,-2,-1)} = 6\times(1)\times(-2)\hat{i} + [3\cdot(1)^2 - 3(-2)^2\cdot(-1)^2]\hat{j} - 2\cdot(-2)^3\cdot(-1)\hat{k}$$

$$(\nabla f)_{(1,-2,-1)} = -12\hat{i} + (3 - 12)\hat{j} - 16\hat{k})$$

$$(\nabla f)_{(1,-2,-1)} = -12\hat{i} - 9\hat{j} - 16\hat{k} \quad \text{Ans}$$

Que- $f(x,y,z) = x^2 + y^2 - z^2 + \log z$ at $(1,1,1)$

$$\nabla f = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot(x^2 + y^2 - z^2 + \log z)$$

$$= \hat{i}\cdot\frac{\partial}{\partial x}(x^2 + y^2 - z^2 + \log z) + \hat{j}\cdot\frac{\partial}{\partial y}(x^2 + y^2 - z^2 + \log z)$$
$$+ \hat{k}\cdot\frac{\partial}{\partial z}(x^2 + y^2 - z^2 + \log z)$$

$$\nabla f = \hat{i}\cdot(2x) + \hat{j}(2y) + \hat{k}\cdot(-2z + \frac{1}{2})$$

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UNIT-2 ⇒ VECTOR CALCULUS

(1) $f(x,y,z) = f(\vec{r}) = x^2y + xy^2 + xyz$ (Scalar value)

(2) $f(x,y,z) = f(\vec{r}) = x^2y\hat{i} + y\hat{j} + yz\hat{k}$ (Vector value)

- Scalar point of function & scalar function
- Vector point of functions & vector function
- Scalar field
- Vector field
- Vector differentiation

eg, (i) $f(t) = (t^2-1)\cdot\hat{i} + (2t)\hat{j} + 3t^3\cdot\hat{k}$

$$\frac{d\vec{f}}{dt} = \frac{d}{dt}\left[(t^2-1)\hat{i} + 2t\hat{j} + 3t^3\hat{k}\right]$$

$$\frac{d\vec{f}}{dt} = 2t\hat{i} + 2\hat{j} + 9t^2\cdot\hat{k}$$ Ans

★ Gradient $(\nabla f)_P$ vector – It gives vector value.

where $\nabla = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)$

$$\nabla f = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot f$$

$$\nabla f = \left(\hat{i}\cdot\frac{\partial f}{\partial x} + \hat{j}\cdot\frac{\partial f}{\partial y} + \hat{k}\cdot\frac{\partial f}{\partial z}\right)$$

$$(\nabla f)_P = \left(\hat{i}\cdot\frac{\partial f}{\partial x} + \hat{j}\cdot\frac{\partial f}{\partial y} + \hat{k}\cdot\frac{\partial f}{\partial z}\right)_P$$

$$\nabla\left(\frac{1}{r}\right) = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot\left[(x^2+y^2+z^2)^{-1/2}\right]$$

$$= \hat{i}\cdot\frac{\partial}{\partial x}\left[(x^2+y^2+z^2)^{-1/2}\right] + \hat{j}\cdot\frac{\partial}{\partial y}\left[(x^2+y^2+z^2)^{-1/2}\right] + \hat{k}\cdot\frac{\partial}{\partial z}\left[(x^2+y^2+z^2)^{-1/2}\right]$$

$$= \hat{i}\cdot\left[\frac{-1}{2}(x^2+y^2+z^2)^{-3/2}\times 2x\right] + \hat{j}\left[\frac{-1}{2}\cdot(x^2+y^2+z^2)^{-3/2}\times 2y\right] + \hat{k}\cdot\left[\frac{-1}{2}(x^2+y^2+z^2)^{-3/2}\cdot\times 2z\right]$$

$$= \hat{i}\left[\frac{-x}{(x^2+y^2+z^2)^{3/2}}\right] + \hat{j}\cdot\left[\frac{-y}{(x^2+y^2+z^2)^{3/2}}\right] + \hat{k}\cdot\left[\frac{-z}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \frac{-x\cdot\hat{i}}{\sqrt{x^2+y^2+z^2}\cdot(x^2+y^2+z^2)} - \frac{y\cdot\hat{j}}{\sqrt{x^2+y^2+z^2}\cdot(x^2+y^2+z^2)} - \frac{z}{\sqrt{x^2+y^2+z^2}\cdot(x^2+y^2+z^2)}$$

$$= \frac{\cancel{r}\cdot\hat{r}}{\cancel{r}} - \frac{(x\hat{i}+y\hat{j}+z\hat{k})}{\sqrt{x^2+y^2+z^2}\cdot(x^2+y^2+z^2)}$$

$$= -\frac{\hat{r}}{r}$$

$$\Rightarrow \boxed{\nabla\left(\frac{1}{r}\right) = \frac{-\hat{r}}{r^2}} \quad \text{Proved}$$

(ii) $\bar{r} = x\hat{i} + y\hat{j} + z\cdot\hat{k}$ prove that $\nabla r = \hat{r}$

Proof - $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$

$|\vec{r}| = \sqrt{x^2+y^2+z^2} = r$

$$\hat{r} = \frac{x\cdot\hat{i} + y\hat{j} + z\cdot\hat{k}}{\sqrt{x^2+y^2+z^2}} \quad \text{---- R.H.S}$$

L.H.S

$$\nabla r = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot \cancel{(x\hat{i} + y\hat{j} + z\cdot\hat{k})} \left(\sqrt{x^2+y^2+z^2}\right)$$

$$\cancel{= \hat{i}\cdot\frac{\partial}{\partial x}(x\hat{i}+y\hat{j}+z\cdot\hat{k}) + \hat{j}\cdot\frac{\partial}{\partial y}(x\hat{i}+y\hat{j}+z\hat{k}) + \hat{k}\cdot\frac{\partial}{\partial z}(x\hat{i}+y\hat{j}+z\hat{k})}$$

$$= \hat{i}\cdot\frac{\partial}{\partial x}\cdot\left(\sqrt{x^2+y^2+z^2}\right) + \hat{j}\cdot\frac{\partial}{\partial y}\left(\sqrt{x^2+y^2+z^2}\right) + \hat{k}\cdot\frac{\partial}{\partial z}\left(\sqrt{x^2+y^2+z^2}\right)$$

$$= \hat{i}\cdot\frac{2x}{2\sqrt{x^2+y^2+z^2}} + \hat{j}\cdot\frac{2y}{2\sqrt{x^2+y^2+z^2}} + \hat{k}\cdot\frac{2z}{2\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x\hat{i}}{\sqrt{x^2+y^2+z^2}} + \frac{y\hat{j}}{\sqrt{x^2+y^2+z^2}} + \frac{z\cdot\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$\nabla r = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}} \quad \text{L.H.S}$$

L.H.S = R.H.S

Proved

(iii) $\vec{r} = x\hat{i} + y\hat{j} + z\cdot\hat{k}$ prove that $\nabla\left(\frac{1}{r}\right) = \frac{-\hat{r}}{r^2}$

Solution - $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{r}| \Rightarrow r = \sqrt{x^2+y^2+z^2}$

and $r^2 = (x^2+y^2+z^2)$

$\hat{r} \Rightarrow \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$

$\nabla f_{(-1,2,-2)} = \frac{-4}{3}\hat{i} + \frac{7}{6}\hat{j} + \frac{-7}{6}\hat{k}$ Ans

(4) (i) $\vec{r} = x\hat{i} + y\hat{j} + z\cdot\hat{k}$ prove that $\nabla r^n = n \cdot r^{n-2}\,\vec{r}$

Solution- When $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{r}| = \sqrt{x^2+y^2+z^2} = r$

L.H.S $\nabla r^n = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot\left(\sqrt{x^2+y^2+z^2}\right)^n$

$= \hat{i}\cdot\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right)^n + \hat{j}\cdot\frac{\partial}{\partial y}\left(\sqrt{x^2+y^2+z^2}\right)^n + \hat{k}\cdot\frac{\partial}{\partial z}\left(\sqrt{x^2+y^2+z^2}\right)^n$

$= \hat{i}\cdot\left(\frac{n\cdot\left(\sqrt{x^2+y^2+z^2}\right)^{n-1}\times 2x}{2\sqrt{x^2+y^2+z^2}}\right) + \hat{j}\cdot\left(\frac{n\cdot\left(\sqrt{x^2+y^2+z^2}\right)^{n-1}\cdot 2y}{2\sqrt{x^2+y^2+z^2}}\right) + \hat{k}\cdot\left(\frac{n\cdot\left(\sqrt{x^2+y^2+z^2}\right)^{n-1}\cdot 2z}{2\sqrt{x^2+y^2+z^2}}\right)$

$= \hat{i}\cdot\left(\frac{nx\cdot\left(\sqrt{x^2+y^2+z^2}\right)^{n-1}}{\sqrt{x^2+y^2+z^2}}\right) + \hat{j}\cdot\left(\frac{ny\cdot\left(\sqrt{x^2+y^2+z^2}\right)^{n-1}}{\sqrt{x^2+y^2+z^2}}\right) + \hat{k}\cdot\left(\frac{nz\cdot\left(\sqrt{x^2+y^2+z^2}\right)^{n-1}}{\sqrt{x^2+y^2+z^2}}\right)$

$= \hat{i}\cdot\frac{nx\,r^{n-1}}{r} + \hat{j}\cdot\frac{ny\,r^{n-1}}{r} + \hat{k}\cdot\frac{nz\,r^{n-1}}{r}$

$= n r^{n-2}\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$

$\nabla r^n = n r^{n-2}\cdot\vec{r}$

L.H.S = R.H.S

Proved

$$\nabla f = \frac{n\cdot(\sqrt{x^2+y^2+z^2})^n}{(x^2+y^2+z^2)} \times \vec{r}$$

$$\nabla f = \frac{n\cdot(\sqrt{x^2+y^2+z^2})^n}{r^2}\cdot\vec{r}$$

$$(\nabla f)\cdot\hat{r} = \frac{n\cdot(\sqrt{x^2+y^2+z^2})^n}{r^2}\cdot\vec{r}\cdot\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{n\cdot r^n\cdot\vec{r}\cdot\vec{r}}{r^2\times r}$$

$$= \frac{n\cdot r^n\cdot(x\hat{i}+y\hat{j}+z\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})}{r^3}$$

$$= \frac{nr^n\times(x^2+y^2+z^2)}{r^3}$$

$$(\nabla f)\cdot\hat{r} \Rightarrow \frac{nr^n\times r^2}{r^3} = \frac{nr^n}{r}$$

$$\left[(\nabla f)\cdot\hat{r} = nr^{n-1}\right]$$ Proved

* Unit Normal Vector -

Formula - (i) $\hat{n} = \frac{(\nabla f)_p}{|(\nabla f)_p|}$

(ii) $\cos\theta = \hat{n}_1\cdot\hat{n}_2$

(iii) $\cos\theta = \frac{(\nabla f_1)_p\cdot(\nabla f_2)_p}{|(\nabla f_1)_p|\cdot|(\nabla f_2)_p|}$

When two ~~poin~~ functions are given with a point

(iv) $\cos\theta = \frac{(\nabla f)_{p_1}\cdot(\nabla f)_{p_2}}{|(\nabla f)_{p_1}|\cdot|(\nabla f)_{p_2}|}$

When one function is given with two different points.

$D_{af} \Rightarrow \frac{16+1+20}{3} = \frac{37}{3}$ Ans

Que – $f(r) = r^n$ in the direction of $\vec{r}$ when $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Solution – $f(r) \Rightarrow r^n = \left(\sqrt{x^2+y^2+z^2}\right)^n$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r \Rightarrow |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$ and $\hat{r} \Rightarrow \frac{\vec{r}}{\sqrt{x^2+y^2+z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot f$$

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\sqrt{x^2+y^2+z^2}\right)^n$$

$$\nabla f = \hat{i}\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right)^n + \hat{j}\frac{\partial}{\partial y}\left(\sqrt{x^2+y^2+z^2}\right)^n + \hat{k}\frac{\partial}{\partial z}\left(\sqrt{x^2+y^2+z^2}\right)^n$$

$$\nabla f = \hat{i}\,\frac{n\left(\sqrt{x^2+y^2+z^2}\right)^{n-1}}{2\sqrt{x^2+y^2+z^2}} \times 2x + \hat{j}\,\frac{n\left(\sqrt{x^2+y^2+z^2}\right)^{n-1}}{2\sqrt{x^2+y^2+z^2}} \times 2y + \hat{k}\,\frac{n\left(\sqrt{x^2+y^2+z^2}\right)^{n-1}}{2\sqrt{x^2+y^2+z^2}} \times 2z$$

$$\nabla f = \hat{i}\,\frac{nx\left(\sqrt{x^2+y^2+z^2}\right)^{n-1}}{\sqrt{x^2+y^2+z^2}} + \hat{j}\,\frac{ny\left(\sqrt{x^2+y^2+z^2}\right)^{n-1}}{\sqrt{x^2+y^2+z^2}} + \hat{k}\,\frac{nz\left(\sqrt{x^2+y^2+z^2}\right)^{n-1}}{\sqrt{x^2+y^2+z^2}}$$

$$\nabla f = \frac{n\left(\sqrt{x^2+y^2+z^2}\right)^{n-1}}{\sqrt{x^2+y^2+z^2}}\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$\nabla f = \frac{n\left(\sqrt{x^2+y^2+z^2}\right)^{n}}{\sqrt{x^2+y^2+z^2}\times\sqrt{x^2+y^2+z^2}} \times \vec{r}$$

* Directional Derivative –

$$Daf = (\nabla f)_p \cdot \hat{a}$$

where, $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

(1) Find the directional derivative of $f(x,y,z) = x^2yz + 4xz^2$ at the point $(1,-2,-1)$ into direction of the vector $2\hat{i}-\hat{j}-2\hat{k}$.

Solution – $daf = (\nabla f)_p \cdot \hat{a}$

Let, $a = 2\hat{i}-\hat{j}-2\hat{k}$, $\hat{a} = \dfrac{2\hat{i}-\hat{j}-2\hat{k}}{\sqrt{4+1+4}} = \dfrac{2\hat{i}-\hat{j}-2\hat{k}}{3}$

$$(\nabla f) = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot(x^2yz + 4xz^2)$$

$$(\nabla f) = \hat{i}\cdot\frac{\partial}{\partial x}(x^2yz + 4xz^2) + \hat{j}\cdot\frac{\partial}{\partial y}(x^2yz + 4xz^2) + \hat{k}\cdot\frac{\partial}{\partial z}(x^2yz + 4xz^2)$$

$$(\nabla f) = \hat{i}\cdot(2xyz + 4z^2) + \hat{j}(x^2z + 0) + \hat{k}\cdot(x^2y + 8xz)$$

$$(\nabla f)_{(1,-2,-1)} = \hat{i}\cdot(2\times1\times(-2)\cdot(-1) + 4\cdot(-1)^2) + \hat{j}\cdot[(1)^2\cdot(-1)] + \hat{k}\cdot[(1)^2\cdot(-2) + 8\times1\cdot(-1)]$$

$$= \hat{i}\cdot(4+4) + \hat{j}(-1) + \hat{k}[-2-8]$$

$$(\nabla f)_{(1,-2,-1)} = 8\hat{i} - \hat{j} - 10\hat{k}$$

$$Daf = (\nabla f)_p \cdot \hat{a}$$

$$Daf = (8\hat{i}-\hat{j}-10\hat{k})\cdot\frac{(2\hat{i}-\hat{j}-2\hat{k})}{3}$$

$$\left[\cos\theta = \frac{14}{3\sqrt{33}}\right]$$ Answer

(2) Find the constant a and b so that surface $ax^2 - byz = (a+2)x$ will be orthogonal / perpendicular to surface $4x^2y + z^3 = 4$ at $(1, -1, 2)$

Solution - $ax^2 - byz - (a+2)x = 0$

$4x^2y + z^3 - 4 = 0$

$$(\nabla f_1) = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot\left(ax^2 - byz - (a+2)x\right)$$

$$(\nabla f_1) = \hat{i}\cdot\frac{\partial}{\partial x}\left[ax^2 - byz - (a+2)x\right] + \hat{j}\cdot\frac{\partial}{\partial y}\left[ax^2 - byz - (a+2)x\right]$$
$$+ \hat{k}\cdot\frac{\partial}{\partial z}\left[ax^2 - byz - (a+2)x\right]$$

$$(\nabla f_1) = \hat{i}\cdot\left[2ax - by - (a+2)\right] + \hat{j}\cdot\left[-bz\right] + \hat{k}\,(-by)$$

(1) Find angle between, the surface $x^2+y^2+z^2=9$ & $x^2+y^2-3=z$ at $(2,-1,2)$

Solution - $x^2+y^2+z^2=9$ at point $(2,-1,2)$

$$(\nabla f_1) = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot(x^2+y^2+z^2-9)$$

$$= \hat{i}\cdot\frac{\partial}{\partial x}(x^2+y^2+z^2-9) + \hat{j}\cdot\frac{\partial}{\partial y}(x^2+y^2+z^2-9) + \hat{k}\cdot\frac{\partial}{\partial z}(x^2+y^2+z^2-9)$$

$$= \hat{i}\cdot(2x) + \hat{j}\,(2y) + \hat{k}\,(2z)$$

$$(\nabla f_1) = 2x\cdot\hat{i} + 2y\,\hat{j} + 2z\cdot\hat{k}$$

$$(\nabla f_1)_P = 2\times2\hat{i} + 2\times-1\hat{j} + 2\times2\hat{k}$$

$$(\nabla f_1)_P = 4\hat{i} - 2\hat{j} + 4\hat{k}$$

$\cancel{\hat{n}}$ ~~∇f_1~~ $|(\nabla f_1)_P| = \sqrt{16+4+16} = \sqrt{36} = 6$

and $x^2+y^2-3=z$

$x^2+y^2-z-3=0$

$$(\nabla f_2) = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)(x^2+y^2-z-3)$$

$$= \hat{i}\cdot(2x) + \hat{j}\cdot(2y) + \hat{k}\cdot(1)$$

$$(\nabla f_2) = 2x\hat{i} + 2y\hat{j} + \hat{k}$$

$$(\nabla f_2)_P = 2\times2\hat{i} + 2\times-2\hat{j} + \hat{k} = 4\hat{i} - 4\hat{j} + \hat{k}$$

$$\cos\theta = \frac{(\nabla f_1)_P\cdot(\nabla f_2)_P}{|(\nabla f_1)_P|\cdot|(\nabla f_1)_P|}$$

$$\cos\theta = \frac{(4\hat{i}-2\hat{j}+4\hat{k})\cdot(4\hat{i}-4\hat{j}+\hat{k})}{6\times\sqrt{33}}$$

$$\cos\theta \Rightarrow \frac{16+8+4}{6\cdot\sqrt{33}} = \frac{\cancel{28}\,14}{{}_3\cancel{6}\sqrt{33}}$$

(1) Find the unit normal vector to the surface $x^2y + 2xz - 4 = 0$ at $(2,-2,3)$

Solution - $(\nabla f) = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)(x^2y + 2xz - 4)$

$$(\nabla f) = \hat{i}\cdot\frac{\partial}{\partial x}(x^2y + 2xz - 4) + \hat{j}\cdot\frac{\partial}{\partial y}(x^2y + 2xz - 4) + \hat{k}\,\frac{\partial}{\partial z}(x^2y + 2xz - 4)$$

$$= \hat{i}\cdot(2xy + 2z - 0) + \hat{j}\cdot(x^2 + 0 - 0) + \hat{k}\cdot(0 + 2x - 0)$$

$$(\nabla f) = (2xy + 2z)\hat{i} + (x^2)\hat{j} + (2x)\hat{k}$$

$$(\nabla f)_{(2,-2,3)} = (2\times 2\times -2 + 2\times 3)\hat{i} + (2^2)\cdot\hat{j} + (2\times 2)\hat{k}$$

$$= (-8+6)\hat{i} + 4\hat{j} + 4\hat{k}$$

$$(\nabla f)_{(2,-2,3)} = -2\hat{i} + 4\hat{j} + 4\hat{k}$$

$$|(\nabla f)_p| \Rightarrow \sqrt{(-2)^2 + (4)^2 + (4)^2} = \sqrt{4+16+16}$$

$$|(\nabla f)_p| = \sqrt{36} = 6$$

$$\hat{n} \Rightarrow \frac{(\nabla f)_p}{|(\nabla f)_p|} = \frac{-2\hat{i} + 4\hat{j} + 4\hat{k}}{6}$$

$$\hat{n} = \frac{-\hat{i} + 2\hat{j} + 2\hat{k}}{3}$$

Answer

Formula -

$$\nabla f_{(0,\frac{\pi}{4},1)} = \hat{i}\cdot\left[\sin\left(\frac{\pi}{4}\right) + \frac{\pi}{4}\cos 0 + \frac{\pi}{4}\cdot\cos 0\right]$$
$$+ \hat{j}\cdot\left[0 + \sin 0 + 0\right] + \hat{k}\left[0 + 0 + \sin 0\right]$$

$$(\nabla f)_{(0,\pi/4,1)} = \hat{i}\left[\frac{1}{\sqrt{2}} + \frac{\pi}{4}\times\frac{\pi}{2} + \frac{\pi}{4}\times\frac{\pi}{2}\right]$$
$$= \hat{i}\cdot\left[\frac{1}{\sqrt{2}} + \frac{\pi^2}{8} + \frac{\pi^2}{8}\right]$$
$$= \hat{i}\left[\frac{1}{\sqrt{2}} + \frac{\pi^2}{4}\right]$$

(iii) $\phi = e^{xy}(x+y+z)$ at $(2,1,1)$

$\cancel{\nabla}\phi = e^{xy}(x+y+z)$

$$\nabla\phi = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\left[e^{xy}(x+y+z)\right]$$

$$\nabla\phi = \hat{i}\,\frac{\partial}{\partial x}\left[e^{xy}(x+y+z)\right] + \hat{j}\cdot\frac{\partial}{\partial y}\left[e^{xy}(x+y+z)\right] + \hat{k}\,\frac{\partial}{\partial z}\left[e^{xy}(x+y+z)\right]$$

$$\cancel{\nabla\phi = \hat{i}\left(e^{xy} + (x+y+z)\cdot e^{xy}\cdot x\right) + \hat{j}\left(e^{xy} + (x+y+z)e^{xy}\cdot y\right] + \hat{k}\left[e^{xy}\right.}$$

$$\nabla\phi = \hat{i}\left[e^{xy} + (x+y+z)e^{xy}\cdot y\right] + \hat{j}\left[e^{xy} + (x+y+z)\cdot e^{xy}\cdot x\right] + \hat{k}\left[e^{xy} + 0\right]$$

$$\nabla\phi = \hat{i}\left[e^{xy} + ye^{xy}(x+y+z)\right] + \hat{j}\left[e^{xy} + xe^{xy}(x+y+z)\right] + \hat{k}\left(e^{xy}\right)$$

(ii). $\phi = 2z^3 - 3(x^2+y^2)z + \tan^{-1}(x)\cdot z$ at $(1,1,1)$

$$\nabla\phi = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot\left[2z^3 - 3x^2z - 3y^2z + \tan^{-1}(x)\,z\right]$$

$$\nabla\phi = \hat{i}\cdot\left(0 - 6xz - 0 + \frac{z}{1+x^2}\right) + \hat{j}\cdot\left(0 - 0 - 6yz + 0\right) + \hat{k}\cdot\left(6z^2 - 3x^2 - 3y^2 + \tan^{-1}(x)\right)$$

$$\nabla\phi = \hat{i}\cdot\left(-6xz + \frac{z}{1+x^2}\right) + \hat{j}\,(-6yz) + \hat{k}\,(6z^2 - 3x^2 - 3y^2 + \tan^{-1}(x))$$

$$(\nabla\phi)_{(1,1,1)} = \hat{i}\cdot\left(-6 + \frac{1}{1+1}\right) + \hat{j}\cdot(-6) + \hat{k}\,(6 - 3 - 3 +$$

(2) (iii) $\phi = x\sin(yz) + y\sin(2x) + z\cdot\sin(xy)$ at $\left(0, \frac{\pi}{4}, 1\right)$

Solution - $\phi = x\sin(yz) + y\sin(2x) + z\cdot\sin(xy)$

$$\nabla f = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\left[(x\sin(yz) + y\sin(2x) + z\sin(xy)\right]$$

$$\nabla f = \hat{i}\cdot\frac{\partial}{\partial x}\left[x\sin(yz) + y\sin(2x) + z(\sin(xy)\right]$$

$$+ \hat{j}\cdot\frac{\partial}{\partial y}\left[x\sin(yz) + y\sin(2x) + z\sin(xy)\right] + \hat{k}\cdot\frac{\partial}{\partial z}\left[x\sin(yz) + y\sin(2x) + z\sin(xy)\right]$$

$$\nabla f = \hat{i}\left[\sin(yz) + y\cdot\cos(2x)\cdot(2) + z\cos(xy)\cdot y\right] + \hat{j}\left[x\cos(yz)\cdot z + \sin(2x) + z\cos(xy)\,x\right] + \hat{k}\left[x\cos(yz)\cdot y + y\cos(2x)\cdot x + \sin(xy)\right]$$

$$\nabla f = \hat{i}\left[\sin(yz) + \cancel{yz}\,2y(\cos(2x) + yz\cos(xy)\right] + \hat{j}\cdot\left[xz\cos(yz) + \sin(2x) + xz\cos(xy)\right] + \hat{k}\left[xy\cos(yz) + xy\cos(z\cdot x) + \sin(xy)\right]$$

Date-

Exercise

(1) Solution- (i) Given- $\phi = 2z^3 - 3(x^2+y^2)z + \tan^{-1}xz$

points $(1,1,1)$

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\phi$$

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[2z^3 - 3(x^2+y^2)z + \tan^{-1}(xz)\right]$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(2z^3 - 3x^2z - 3y^2z + \tan^{-1}(xz)\right)$$

$$= \hat{i}\frac{\partial}{\partial x}\cdot\left[2z^3 - 3x^2z - 3y^2z + \tan^{-1}(xz)\right] + \hat{j}\frac{\partial}{\partial y}\left[2z^3 - 3x^2z - 3y^2z + \tan^{-1}(xz)\right]$$

$$+ \hat{k}\frac{\partial}{\partial z}\left[2z^3 - 3x^2z - 3y^2z + \tan^{-1}(xz)\right]$$

$$\nabla f = \hat{i}\left(0 - 6xz - 0 + \frac{1}{1+x^2z^2}\times z\right) + \hat{j}\cdot\left(0 - 0 - 6yz + 0\right)$$

$$+ \hat{k}\cdot\left(6z^2 - 3x^2 - 3y^2 + \frac{1}{1+x^2z^2}\times x\right)$$

$$\nabla f = \hat{i}\cdot\left[-6xz + \frac{z}{1+x^2z^2}\right] + \hat{j}\cdot(-6yz) + \hat{k}\cdot\left[-3x^2 - 3y^2 + \frac{x}{1+x^2z^2}\right]$$

$$\nabla f = \left(-6xz + \frac{z}{1+x^2z^2}\right)\cdot\hat{i} - (6yz)\hat{j} + \left(6z^2 - 3x^2 - 3y^2 + \frac{x}{1+x^2z^2}\right)\cdot\hat{k}$$

$$(\nabla f)_{(1,1,1)} = \left(-6\times1\times1 + \frac{1}{1+1}\right)\cdot\hat{i} - 6\times1\times1\hat{j} + \left(6 - 3 - 3 + \frac{1}{1+1}\right)\cdot\hat{k}$$

$$(\nabla f)_{(1,1,1)} = \left(-6 + \frac{1}{2}\right)\cdot\hat{i} - 6\hat{j} + \left(\frac{1}{2}\right)\hat{k}$$

$$(\nabla f)_{(1,1,1)} = \frac{-11}{2}\hat{i} - 6\hat{j} + \frac{1}{2}\hat{k}$$

Solenoidal : $\boxed{\nabla \cdot f = 0}$

greatest rate of increase of f function $|\nabla f|$

$$= \frac{\partial}{\partial x}(x r^{-1}) + \frac{\partial}{\partial y}(y r^{-1}) + \frac{\partial}{\partial z}(z r^{-1})$$

$$= r^{-1}(1) + x\cdot(-1)\cdot r^{-2}\cdot\frac{2x}{2\sqrt{x^2+y^2+z^2}} + r^{-1}(1) + \frac{y^2(-1)\cdot r^{-2}}{\sqrt{x^2+y^2+z^2}}$$

$$+ r^{-1}(1) + \frac{z^2\cdot(-1)\cdot r^{-2}}{\sqrt{x^2+y^2+z^2}}$$

$$= 3r^{-1} - \frac{r^{-2}}{\sqrt{x^2+y^2+z^2}}\cdot\left(x^2+y^2+z^2\right)$$

$$= \frac{3}{r} - \frac{r^{-2}\times r^2}{r}$$

$$= \frac{3}{r} - \frac{1}{r}$$

$$= \frac{2}{r}$$ Proved.

Exercise-2

(1) Solution- $V = x^2y\,\hat{i} - 2xy\,\hat{j} + 2yz\,\hat{k}$ at $(1,-1,1)$

~~$V = -\hat{i} + 2\hat{j} - 2\hat{k}$~~

$$\text{div}\cdot V = \nabla V$$

$$= \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot \cancel{(-\hat{i}+2\hat{j}-2\hat{k})}\,(x^2y\,\hat{i} - 2xy\,\hat{j} + 2yz\,\hat{k})$$

$$= \cancel{\frac{\partial}{\partial x}(-1) + \hat{j}\cdot(2}\ \left\{\frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(2yz)\right.$$

$$= 2xy + (-2x) + (2y)$$

$$\text{div}\cdot V = 2xy - 2x + 2y$$

$$(\text{div}\cdot V)_{(1,-1,1)} \Rightarrow -2-2-2 = \underline{-6}$$ Ans

$$\operatorname{div}(r^n \vec{r}) = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot(r^n x\hat{i} + r^n y\hat{j} + r^n z\hat{k})$$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= r^n + x\cdot n\cdot r^{n-1}\cdot\frac{x}{\sqrt{x^2+y^2+z^2}} + r^n + y\cdot n r^{n-1}\cdot\frac{y}{\sqrt{x^2+y^2+z^2}} + r^n + z\cdot n r^{n-1}\cdot\frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$= 3r^n + x^2 n r^{n-2} + y^2\cdot n r^{n-2} + z^2\cdot n r^{n-2}$$

$$= 3r^n + n r^{n-2}(x^2+y^2+z^2)$$

$$= 3r^n + n r^{n-2}\cdot r^2$$

$$= 3r^n + n r^n$$

$$\operatorname{div}(r^n\vec{r}) = r^n(3+n) \quad \text{R.H.S}$$

Proved

(ii) $\operatorname{div}\vec{r} = 3$

$$\operatorname{div}\vec{r} = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1+1+1$$

$$\operatorname{div}\vec{r} = \underline{\underline{3}} \quad \text{R.H.S.}$$

Proved

(iii) $\operatorname{div}\hat{r} = \frac{2}{r}$

L.H.S $\operatorname{div}\hat{r} = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot\left[\frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}\right]$

$$= \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial y}\cdot\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) + \frac{\partial}{\partial z}\cdot\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$(\nabla\phi)_{(2,1,1)} = \hat{i}\,[e^2 + 1\times e^2\,(2+1+1)] + \hat{j}\,[e^2 + 2e^2(2+1+1)] + \hat{k}\,(e^2)$$

$$= \hat{i}\cdot[e^2 + e^2\cdot 4] + \hat{j}\cdot[e^2 + 2e^2\times 4] + \hat{k}\cdot e^2$$

$$(\nabla\phi)_{(2,1,1)} = 5e^2\hat{i} + 9e^2\hat{j} + e^2\hat{k}$$

$$(\nabla\phi)_{(2,1,1)} = e^2\,(5\hat{i} + 9\hat{j} + \hat{k})$$

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→ Divergence –

$$\nabla\bar{f} = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot(x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$$

$$\nabla\bar{f} = \frac{\partial}{\partial x}(x_1) + \frac{\partial}{\partial y}(y_1) + \frac{\partial}{\partial z}(z_1)$$

example, $f(x,y,z) = 2x^2z\hat{i} - xyz\hat{j} + 3yx\hat{k}$

$$div.\,F = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot(2x^2z\hat{i} - xyz\hat{j} + 3yx\cdot\hat{k})$$

$$= \frac{\partial}{\partial x}(2x^2z) + \frac{\partial}{\partial y}(-x^2yz) + \frac{\partial}{\partial z}(3y^2x)$$

$$div.\,\bar{f} = 4xz - 2xyz \qquad \underline{Ans}$$

(1) Solution – $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r = |\vec{r}|^2 = \sqrt{x^2+y^2+z^2}$$

(i) $div.\, r^n\cdot\bar{r} = (n+3)\cdot r^n$

L.H.S $\quad div.\, r^n\bar{r} = \nabla\cdot r^n\cdot\bar{r}$

$$div.\, r^n\bar{r} = \left(\partial\,\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)(r^n\bar{r})$$

(3) Solution – (i) $V = (x+3y)\hat{i} + (y-2z)\hat{j} + (a+az)\hat{k}$

According to question,

$$\text{div. } V = 0$$

$$\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(a+az) = 0$$

$$1 + 1 + a = 0$$

$$2 + a = 0$$

$$\boxed{a = -2} \quad \text{Ans}$$

(ii) $V = a(x+y)\hat{i} + 4y\hat{j} + 3\hat{k}$

According to question,

$$\text{div. } V = 0$$

$$\frac{\partial}{\partial x}(a(x+y)) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(3) = 0$$

$$a + 4 + 0 = 0$$

$$\boxed{a = -4} \quad \text{Ans}$$

(4) Solution – $\phi = 2x^3y^3z^4$

$$xy^2z = 3x + z^2$$

$$xy^2z - 3x - z^2 = 0$$

$$\nabla\phi = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)(2x^3y^3z^4)$$

$$= \hat{i}\cdot\frac{\partial}{\partial x}(2x^3y^3z^4) + \hat{j}\cdot\frac{\partial}{\partial y}(2x^3y^3z^4) + \hat{k}\cdot\frac{\partial}{\partial z}(2x^3y^3z^4)$$

$$\nabla\phi = \hat{i}\cdot(6x^2y^3z^4) + \hat{j}\cdot(6x^3y^2z^4) + \hat{k}\cdot(8x^3y^3z^3)$$

(2) $v = 3x^2\hat{i} + 5xy^2\hat{j} + xyz^3\hat{k}$ at $(1,2,3)$

$$\text{div}.v = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot(3x^2\hat{i} + 5xy^2\hat{j} + xyz^3\hat{k})$$

$$= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(5xy^2) + \frac{\partial}{\partial z}(xyz^3)$$

$$= 6x + 5x \times 2y + xy(3z^2)$$

$$= 6x + 10xy + 3xyz^2$$

$$= 6\times1 + 10\times2\times2 + 3\times1\times2\times9$$

$$= 6 + 20 + 54$$

$$(\text{div}.v)_{(1,2,3)} = 80$$

(2) (i) $u = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$

$$\text{div}.u = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-3z) + \frac{\partial}{\partial z}(x-2z)$$

$$= 1 + 1 - 2$$

$$[\text{div}.u = 0]$$

Above calculated div.u = 0

⇒ Given vector 'u' is solenoidal.

(ii) $u = yz\hat{i} + zx\hat{j} + xy\hat{k}$

$$\text{div}.u = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy)$$

$$\text{div}.u = 0 + 0 + 0$$

$$[\text{div}.u = 0]$$

⇒ Given vector $u = yz\hat{i} + zx\hat{j} + xy\hat{k}$ is solenoidal.

Solenoidal : $\boxed{\nabla \cdot f = 0}$

greatest rate of increase, of f function $|\nabla f|$

$$= \frac{\partial}{\partial x}(x r^{-1}) + \frac{\partial}{\partial y}(y r^{-1}) + \frac{\partial}{\partial z}(z r^{-1})$$

$$= r^{-1}(1) + x\cdot(-1)\cdot r^{-2}\cdot\frac{2x}{2\sqrt{x^2+y^2+z^2}} + r^{-1}(1) + \frac{y^2(-1)\cdot r^{-2}}{\sqrt{x^2+y^2+z^2}}$$

$$+ r^{-1}(1) + \frac{z^2\cdot(-1)\cdot r^{-2}}{\sqrt{x^2+y^2+z^2}}$$

$$= 3r^{-1} - \frac{r^{-2}}{\sqrt{x^2+y^2+z^2}}\cdot\left(x^2+y^2+z^2\right)$$

$$= \frac{3}{r} - \frac{r^{-2}\times r^2}{r}$$

$$= \frac{3}{r} - \frac{1}{r}$$

$$= \frac{2}{r} \quad \text{Proved.}$$

Exercise-2

(1) Solution- $V = x^2y\hat{i} - 2xy\hat{j} + 2yz\hat{k}$ at $(1,-1,1)$

~~$V = -\hat{i} + 2\hat{j} - 2\hat{k}$~~

$$\text{div}\cdot V = \nabla V$$

$$= \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot \cancel{(-\hat{i}+2\hat{j}-2\hat{k})}\,(x^2y\hat{i} - 2xy\hat{j} + 2yz\hat{k})$$

$$= \cancel{\frac{\partial}{\partial x}(-1) + \hat{j}\cdot\frac{\partial}{\partial y}(2}\ \left\{\frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(2yz)\right.$$

$$= 2xy + (-2x) + (2y)$$

$$\text{div}\cdot V = 2xy - 2x + 2y$$

$$(\text{div}\cdot V)_{(1,-1,1)} \Rightarrow -2-2-2 = -6 \quad \text{Ans}$$

Important Points of Vector Calculus -

- $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$
- $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
- $|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$
- $r^2 = x^2+y^2+z^2$
- $r^n = (\sqrt{x^2+y^2+z^2})^n$
- $\frac{\partial}{\partial x}(r^n) = \frac{n \cdot r^{n-1}}{r} \times x$

(1) • Gradient of f (∇f)

where f is a scalar func.

- Directional Derivatives (Daf) $= (\nabla f)_p \cdot \hat{a}$

where, $\hat{a}$ is a unit vector

- Unit Normal vector $(\hat{n}) = \frac{(\nabla f)_p}{|(\nabla f)_p|}$

- Angle between two surfaces

$$\swarrow \qquad \searrow$$

Two func + one point

$\hat{n}_1 = \frac{(\nabla f_1)_p}{|(\nabla f_1)_p|}$

$\hat{n}_2 = \frac{(\nabla f_2)_p}{|(\nabla f_2)_p|}$

$\cos\theta = \hat{n}_1 \cdot \hat{n}_2$

one point + two func

$\hat{n}_1 = \frac{(\nabla f)_{p_1}}{|(\nabla f)_{p_1}|}$

$\hat{n}_2 = \frac{(\nabla f)_{p_2}}{|(\nabla f)_{p_2}|}$

$\cos\theta = \hat{n}_1 \cdot \hat{n}_2$

- Greatest Integer func $(\nabla \phi)$,

where ϕ scalar potential

- For orthogonal vector

$$\nabla \cdot f = 0$$

→ **Curl** – Let $\vec{f} = f_1 \cdot \hat{i} + f_2 \hat{j} + f_3 \cdot \hat{k}$

$$\text{Curl } f \Rightarrow \nabla \times f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Que – $\vec{f} = x\cos z\,\hat{i} + y\log x\,\hat{j} \cancel{+} - z^2\hat{k}$

$$\text{Curl } \vec{f} \Rightarrow \nabla \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x\cos z & y\log x & -z^2 \end{vmatrix}$$

$$\text{Curl } \vec{f} = \hat{i}\left(\frac{\partial}{\partial x}(-z^2) - \frac{\partial}{\partial z} y\log x\right) - \hat{j}\left(\frac{\partial}{\partial x}(-z^2) - \frac{\partial}{\partial y}(x\cos z)\right) + \hat{k}\left(\frac{\partial}{\partial x}(y\log x) - \frac{\partial}{\partial y}(x\cos z)\right)$$

$$\left[\text{curl } \vec{f} = -x\sin x\,\hat{j} + \frac{y}{x}\hat{k}\right] \quad \text{Ans}$$

daf. $(\nabla\phi) = \nabla(\nabla\phi)_{(1,-2,1)}$.

$$\nabla(\nabla\phi) = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)[(6x^2y^3z^4)\hat{i} + (6x^3y^2z^4)\hat{j} + (8x^3y^3z^3)\hat{k}]$$

$$= \frac{\partial}{\partial x}(6x^2y^3z^4) + \frac{\partial}{\partial y}(6x^3y^2z^4) + \frac{\partial}{\partial z}(8x^3y^3z^3)$$

$$g = \nabla(\nabla\phi) = 12xy^3z^4 + 12x^3yz^4 + 24x^3y^3z^2$$

$$\nabla(\nabla\phi)_{(1,-2,1)} = 12\times1\times(-2)^3\times1 + 12\times1\times(-2)\times1 + 24\times1\times(-2)^3\times(1)$$

$$= 12\times8 + (-24) + 24\times(-8)$$

$$= 72 - 24 - 192$$

$$\nabla g \Rightarrow \nabla(\nabla\cdot\nabla\phi) = \left(\hat{i}\cdot\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot(12xy^3z^4 + 12x^3yz^4 + 24x^3y^3z^2)$$

$$\nabla g = \hat{i}\cdot\frac{\partial}{\partial x}($$

(2) Div. of f $(\nabla \cdot \vec{f})$, where $\vec{f}$ is a vector func.

$\nabla \cdot \vec{f} = 0$

Vector to be solenoidal

(3) Curl of $\vec{f}$

$$\nabla \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

if $\nabla \times \vec{f} = 0$

Given vector is irrotational

Que. curl $r^n \cdot \vec{r} = 0$

Solution - curl $r^n \cdot \vec{r} = 0$

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r = \sqrt{x^2+y^2+z^2}$

$r^n = (\sqrt{x^2+y^2+z^2})^n$

$r^n \cdot \vec{r} = (\sqrt{x^2+y^2+z^2})^n \cdot (x\hat{i} + y\hat{j} + z\hat{k})$

$r^n \cdot \vec{r} = (\sqrt{x^2+y^2+z^2})^n \cdot x\hat{i} + (\sqrt{x^2+y^2+z^2})^n \cdot y\hat{j} + (\sqrt{x^2+y^2+z^2})^n \cdot z\hat{k}$

curl $r^n \cdot \vec{r} = \nabla \times r^n \cdot \vec{r}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\sqrt{x^2+y^2+z^2})^n x & (\sqrt{x^2+y^2+z^2})^n \cdot y & (\sqrt{x^2+y^2+z^2})^n \cdot z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\left[(\sqrt{x^2+y^2+z^2})^n \cdot z\right] - \frac{\partial}{\partial z} \cdot (\sqrt{x^2+y^2+z^2})^n \cdot y \right)\hat{i}$$

$$- \hat{j}\left(\frac{\partial}{\partial x}\left[(\sqrt{x^2+y^2+z^2})^n \cdot z\right] - \frac{\partial}{\partial z}\left[(\sqrt{x^2+y^2+z^2})^n \cdot x\right] \right) + \hat{k}\left[\frac{\partial}{\partial x}\left((\sqrt{x^2+y^2+z^2})^n \cdot y\right) - \frac{\partial}{\partial y}\left[(\sqrt{x^2+y^2+z^2})^n \cdot x\right] \right]$$

$$\nabla f(r) = \frac{f'(r)}{\sqrt{x^2+y^2+z^2}} (x\hat{i} + y\hat{j} + z\hat{k})$$

$$\nabla f(r) = \frac{f'(r) \cdot \vec{r}}{r}$$

$$\nabla f(r) = \frac{f'(r) \cdot \nabla r}{\cancel{r}}$$

$$\left[\because \nabla r = \frac{\vec{r}}{r} \right]$$

(2) If $\bar{f} = x\hat{i} + y\hat{j} + z\hat{k}$ then Curl $\cdot \bar{f}$

(a) 1 (b) 0 (c) $\vec{r}$ (d) $\hat{i} + \hat{j} + \hat{k}$

$$\text{Curl} \cdot \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i} \cdot (0-0) - \hat{j} (0-0) + \hat{k} (0-0)$$

Curl $\cdot \bar{f} = 0$ answer

(3) A vector field $\bar{f}$ is $\nabla \times \bar{f} = 0$

(4) $\bar{f} = 2xy\hat{i} + x^2\hat{j} + 2yz\hat{k}$

$$\text{div} \cdot \bar{f} = \nabla \cdot \bar{f} = \left(\hat{i} \cdot \frac{\partial}{\partial x} + \hat{j} \cdot \frac{\partial}{\partial y} + \hat{k} \cdot \frac{\partial}{\partial z} \right) \cdot (2xy\hat{i} + x^2\hat{j} + 2yz\hat{k})$$

$$= \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial z}(2yz)$$

$$= 2y + 0 + 2y$$

$$\text{div} \cdot \bar{f} \Rightarrow 2y + 2y = 4y$$

$$\text{Curl}\cdot(\vec{a}\times\vec{r}) = \hat{i}\left[\frac{\partial}{\partial y}(a_1y - a_2x) + \frac{\partial}{\partial z}(a_1z - a_3x)\right]$$
$$-\hat{j}\left[\frac{\partial}{\partial x}(a_1y - a_2x) - \frac{\partial}{\partial z}(a_2z - a_3y)\right] + \hat{k}\left[\frac{\partial}{\partial x}(-(a_1z - a_3x)) - \frac{\partial}{\partial y}(a_2z - a_3y)\right]$$

$$\text{curl}\cdot(\vec{a}\times\vec{r}) = \hat{i}(a_1+a_1) - \hat{j}(-a_2 - a_2) + \hat{k}(a_3+a_3)$$

$$\text{curl}(\vec{a}\times\vec{r}) = 2a_1\cdot\hat{i} + 2a_2\hat{j} + 2a_3\hat{k}$$ Ans

Objective Question

(1) Gradient of function $f(r)$ is-

(a) $f'(r)$ (b) $f'(r)\cdot\frac{\partial r}{\partial x}$

(c) $f''(r)$ (d) $f'(r)\cdot\nabla r$.

$$\nabla\cdot f(r) = \nabla\cdot\left(\sqrt{x^2+y^2+x^2}\right)$$
$$= \hat{i}\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+x^2}\right) + \hat{j}\frac{\partial}{\partial y}\left(\sqrt{x^2+y^2+z^2}\right) + \hat{k}\frac{\partial}{\partial z}\left(\sqrt{x^2+y^2+z^2}\right)$$
$$= \frac{x\hat{i}}{\sqrt{x^2+y^2+x^2}} + \frac{y\hat{j}}{\sqrt{x^2+y^2+x^2}} + \frac{z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$
$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+x^2}}$$
$$\nabla\cdot f(r) = \frac{\vec{r}}{\sqrt{x^2+y^2+x^2}}$$

$$\nabla\cdot f(r) = \nabla\cdot f(r) = \hat{i}\frac{\partial}{\partial x}[f(r)] + \hat{j}\frac{\partial}{\partial y}[f(r)] + \hat{k}\frac{\partial}{\partial z}[f(r)]$$
$$= f'(r)\cdot\frac{2x}{2\sqrt{x^2+y^2+z^2}}\cdot\hat{i} + f'(r)\cdot\frac{2y}{2\sqrt{x^2+y^2+z^2}} + f'(r)\frac{2z}{2\sqrt{x^2+y^2+z^2}}$$

$$\text{Curl}\cdot r^n\cdot\vec{r} = \hat{i}\cdot\left[\frac{z\cdot n\cdot(\sqrt{x^2+y^2+z^2})^{n-1}}{2\sqrt{x^2+y^2+z^2}}\times 2y - \frac{y\cdot n(\sqrt{x^2+y^2+z^2})^{n-1}}{2\sqrt{x^2+y^2+z^2}}\times 2z\right]$$

$$-\hat{j}\left[\frac{z\cdot n(\sqrt{x^2+y^2+z^2})^{n-1}}{2\sqrt{x^2+y^2+z^2}}\times 2x - \frac{xn(\sqrt{x^2+y^2+z^2})^{n-1}}{2\sqrt{x^2+y^2+z^2}}\times 2z\right]$$

$$+\hat{k}\left[\frac{yn(\sqrt{x^2+y^2+z^2})^{n-1}}{2\sqrt{x^2+y^2+z^2}}\times 2x - \frac{xn(\sqrt{x^2+y^2+z^2})^{n-1}}{2\sqrt{x^2+y^2+z^2}}\times 2y\right]$$

$$\text{Curl}\cdot r^n\cdot\vec{r} = \hat{i}\left[\frac{yz\cdot n\cdot(\sqrt{x^2+y^2+z^2})^{n-1}}{\sqrt{x^2+y^2+z^2}} - \frac{zy\cdot n(\sqrt{x^2+y^2+z^2})^{n-1}}{\sqrt{x^2+y^2+z^2}}\right]$$

$$-\hat{j}\left[\frac{xzn(\sqrt{x^2+y^2+z^2})^{n-1}}{\sqrt{x^2+y^2+z^2}} - \frac{xzn(\sqrt{x^2+y^2+z^2})^{n-1}}{\sqrt{x^2+y^2+z^2}}\right]$$

$$+\hat{k}\left[\frac{xyn(\sqrt{x^2+y^2+z^2})^{n-1}}{\sqrt{x^2+y^2+z^2}} - \frac{xy\cdot n\cdot(\sqrt{x^2+y^2+z^2})^{n-1}}{\sqrt{x^2+y^2+z^2}}\right]$$

$$\text{Curl}\cdot r\cdot\vec{r} = \hat{i}(0) + \hat{j}(0) + \hat{k}(0)$$

$$\text{Curl}\cdot r\cdot\vec{r} = 0$$

Proved

e. curl $(\vec{a}\times\vec{r})$ find where $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

ution - $\vec{a}\times\vec{r} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).(x\hat{i} + y\hat{j} + z\hat{k})$

$$\vec{a}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$\vec{a}\times\vec{r} = \hat{i}(a_2z - a_3y) - \hat{j}(a_1z - a_3x) + \hat{k}(a_1y - a_2x)$$

$$\times\vec{r}) = \nabla\times(\vec{a}\times\vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (a_2z-a_3y) & -(a_1z-a_3x) & (a_1y-a_2x) \end{vmatrix}$$

$$\text{Curl}\cdot\hat{r} = \hat{i}\left[\frac{-z\times 2y}{(x^2+y^2+z^2)^{3/2}} + \frac{y\times 2z}{(x^2+y^2+z^2)^{3/2}}\right] - \hat{j}\left(\frac{-z\times 2x}{(x^2+y^2+z^2)^{3/2}} + \frac{x\times 2z}{(x^2+y^2+z^2)^{3/2}}\right) + \hat{k}\left[\frac{-y\times 2x}{(x^2+y^2+z^2)^{3/2}} + \frac{x\times 2y}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$\text{Curl}\cdot\hat{r} = \hat{i}\,(0) - \hat{j}\,(0) + \hat{k}\,(0)$$

$\text{Curl}\cdot\hat{r} = 0$ Answer

(10) If $\phi(x,y,z) = x+z$, then directional derivatives in the direction of $\vec{a} = \hat{i}+\hat{j}$ is :

Solution - $\phi(x,y,z) = x+z$

del $\cdot\,\phi$

Q (11) A vector field is solenoidal if $\text{div}\cdot\bar{f} = 0$

$$\vec{a}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2z-a_3y)\hat{i} - (a_1z-a_3x)\hat{j} + (a_1y-a_2x)\hat{k}$$

$$\text{div}\cdot(\vec{a}\times\vec{r}) = \nabla\cdot(\vec{a}\times\vec{r})$$

$$= \frac{\partial}{\partial x}(a_2z-a_3y) - \frac{\partial}{\partial y}(a_1z-a_3x) + \frac{\partial}{\partial z}(a_1y-a_2x)$$

$$= 0-0+0$$

$\text{div}\cdot(\vec{a}\times\vec{r}) = 0$ 'Answer

(9) What is curl of $\hat{r}$?

Solution- $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\cdot\hat{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}$$

$$\text{Curl}\cdot\hat{r} = \nabla\times\hat{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix}$$

$$\text{curl}\cdot\hat{r} = \hat{i}\left[\frac{\partial}{\partial y}\cdot\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\right]$$

$$-\hat{j}\cdot\left[\frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\right]$$

$$+\hat{k}\left(\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right) - \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\right]$$

(5) A vector field $\vec{f}$ is irrotational if for scalar potential ϕ is

Ans $\bar{F} = \nabla \phi$

(6) $f(x,y,z) = x^2yz + yxz^2$ at $(1,2,-1)$, then ∇f

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot(x^2yz + yxz^2)$$

$$\nabla f = \hat{i}\cdot\frac{\partial}{\partial x}(x^2yz + yxz^2) + \hat{j}\cdot\frac{\partial}{\partial y}(x^2yz + yxz^2) + \hat{k}\cdot\frac{\partial}{\partial z}(x^2yz + yxz^2)$$

$$\nabla f = \hat{i}\cdot(2xyz + yz^2) + \hat{j}\,(x^2z + xz^2) + \hat{k}\,(x^2y + 2xyz)$$

$$(\nabla f)_{(1,2,-1)} = \hat{i}\,(2\times1\times2\times(-1) + 2\times(-1)^2) + \hat{j}\,((1)^2\times(-1) + 1\cdot(-1)^2) + \hat{k}\,(1\times2 + 2\times(1)\times(2)\times(-1))$$

$$= \hat{i}(-2) + \hat{j}(0) + \hat{k}(-2) \qquad = -2\hat{i} - 2\hat{k}$$ Answer

(7) If $\vec{f} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is solenoidal.

$$\nabla\cdot\vec{f} = 0$$

$$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\cdot\frac{\partial}{\partial y} + \hat{k}\cdot\frac{\partial}{\partial z}\right)\cdot\vec{f} = 0$$

$$\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0$$

$$1 + 1 + a = 0$$

$$2 + a = 0$$

$$[a = -2]$$

(8) Divergence of $\vec{a}\times\vec{H}$

$$\text{div}\ \vec{a}\times\vec{H} = \nabla\cdot(\vec{a}\times\vec{H})$$

$$= \frac{\partial}{\partial x}$$

⇝ Fourier Series ⇜

Rule's formula for fourier series–

(α to $\alpha+2\pi$) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where $a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cdot dx$

$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cdot \cos nx \cdot dx$

$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cdot \sin nx dx$

where $\alpha = 0$ then $(0, 2\pi)$

$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$

$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \cos nx dx$

$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \sin nx \cdot dx$

Where $\alpha = -\pi$ then $(-\pi, \pi)$

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos nx dx$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin x dx$

General formula

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$\nabla \cdot (\vec{a} \cdot \vec{r}) = \left(\hat{i} \cdot \frac{\partial}{\partial x} + \hat{j} \cdot \frac{\partial}{\partial y} + \hat{k} \cdot \frac{\partial}{\partial z} \right) \cdot (a_1 x + a_2 y + a_3 z)$$

$$\nabla \cdot (\vec{a} \cdot \vec{r}) = \hat{i} \cdot \frac{\partial}{\partial x} (a_1 x + a_2 y + a_3 z) + \hat{j} \cdot \frac{\partial}{\partial y} (a_1 x + a_2 y + a_3 z) + \hat{k} \cdot \frac{\partial}{\partial z} (a_1 x + a_2 y + a_3 z)$$

$$\nabla \cdot (\vec{a} \cdot \vec{r}) = \hat{i} (a_1) + \hat{j} (a_2) + \hat{k} (a_3)$$

$$\nabla \cdot (\vec{a} \cdot \vec{r}) = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$$

$$\nabla \cdot (\vec{a} \cdot \vec{r}) = \vec{a}$$ Answer

(14) If $\phi = x^2 y + y^2 x + z^2$ then $\nabla \phi$ at $(1, 1, 1)$ is

Solution - $$\nabla \cdot \phi = \hat{i} \cdot \frac{\partial}{\partial x} (x^2 y + y^2 x + z^2) + \hat{j} \cdot \frac{\partial}{\partial y} (x^2 y + y^2 x + z^2) + \hat{k} \cdot \frac{\partial}{\partial z} (x^2 y + y^2 x + z^2)$$

$$\nabla \cdot \phi = \hat{i} (2xy + y^2) + \hat{j} (x^2 + 2xy) + \hat{k} \cdot (2z)$$

$$\nabla \cdot \phi = \hat{i} (2+1) + \hat{j} \cdot (1+2) + \hat{k} (2)$$

$$\nabla \cdot \phi = 3\hat{i} + 3\hat{j} + 2\hat{k}$$ Answer

(12) If $f = \sin r$ and $r = \sqrt{x^2+y^2+z^2}$, then grad. (r) is.

Solution – $f = \sin(r)$

$$\text{grad}(f) = \nabla \cdot f$$

$$= \left(\hat{i} \cdot \frac{\partial}{\partial x} + \hat{j} \cdot \frac{\partial}{\partial y} + \hat{k} \cdot \frac{\partial}{\partial z}\right) \cdot (\sin(r))$$

$$= \hat{i} \cdot \frac{\partial}{\partial x}(\sin r) + \hat{j} \cdot \frac{\partial}{\partial y}(\sin r) + \hat{k} \frac{\partial}{\partial z}(\sin r)$$

$$= \hat{i} \cdot \cos r \times \frac{2x}{2\sqrt{x^2+y^2+x^2}} + \hat{j} \cdot \cos r \cdot x \frac{2y}{2\sqrt{x^2+y^2+z^2}}$$

$$+ \hat{k}\left(\cos r \cdot \frac{2z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{\hat{i} \cdot \cos r \cdot x}{\sqrt{x^2+y^2+z^2}} + \frac{\hat{j} \cdot \cos r \cdot y}{\sqrt{x^2+y^2+z^2}} + \frac{\hat{k} \cdot \cos r \cdot z}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{\cos r}{\sqrt{x^2+y^2+z^2}} \cdot (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\cos r}{r} \times \vec{r}$$

$$\text{grad}\cdot(f) = \cos r \cdot \nabla \cdot r$$ Answer

(13) If $\vec{a}$ is constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then $\nabla \cdot (\vec{a} \cdot \vec{r})$ is –

Solution – $\vec{a} \cdot \vec{r} = \vec{a} \cdot (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$

$$\vec{a} \cdot \vec{r} = a_1 x + a_2 y + a_3 \cdot z$$

(2) Find fourier-series of $f(x) = x \quad , \quad 0 \le x \le 2\pi$

Solution - $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ ---(a)

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\,dx$$

$$a_0 \Rightarrow \frac{1}{\pi} \int_0^{2\pi} (x)\cdot dx \quad = \frac{1}{\pi} \left[\frac{x^2}{2}\right]_0^{2\pi}$$

$$a_0 \Rightarrow \frac{1}{2\pi} [x^2]_0^{2\pi} \quad = \frac{1}{2\pi} [4\pi^2 - 0]$$

$$a_0 \Rightarrow \frac{4\pi^2}{2\pi} \quad = 2\pi \quad \text{-----} (i)$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \cdot x \cdot \cos nx\, dx$$

$$a_n = \frac{1}{\pi} \left\{ \left[\frac{x \cdot \sin nx}{n}\right]_0^{2\pi} - \frac{1}{n} \int_0^{2\pi} \sin nx \right\}$$

$$a_n = \frac{1}{\pi} \left\{ \frac{2\pi \sin 2\pi n}{n} - 0 + \frac{1}{n^2} [\cos nx]_0^{2\pi} \right\}$$

$$a_n = \frac{1}{\pi} \left\{ 0 - 0 + \frac{1}{n^2} [\cos 2\pi \cdot n - \cos 0] \right\}$$

$$a_n = \frac{1}{\pi} \left\{ \frac{1}{n^2} ((-1)^{2n} - 1) \right\}$$

$$a_n \Rightarrow \frac{1}{n^2\pi} [(-1^2)^n - 1] \quad = \frac{1}{n^2\pi} [1 - 1]$$

$$[a_n = 0] \quad \text{-----} (ii)$$

Further, $\quad b_n = \frac{1}{\pi} \int_0^{2\pi} x \cdot \sin nx \cdot dx$

$$b_n = \frac{1}{\pi} \left\{ \left[\frac{-x \cos nx}{n}\right]_0^{2\pi} + \int_0^{2\pi} \frac{\cos nx}{n} \right\}$$

$$a_n = \frac{1}{\pi}\left\{0 - 0 + \frac{1}{n^2}[\cos n\pi - \cos n\pi]\right.$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n^2}(0-0)\right]$$

$$\boxed{a_n = 0} \quad \text{-----} \; (ii)$$

and,
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \sin nx\,dx$$

$$b_n = \frac{1}{\pi}\cdot\int_{-\pi}^{\pi} x\cdot \sin nx\,dx$$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{-x\cdot\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos nx\cdot dx}{n}\right\}$$

$$b_n = \frac{1}{\pi}\left\{\frac{-\pi\cdot\cos n\pi}{n} - \frac{\pi\cdot\cos n\pi}{n} + \frac{1}{n^2}[\sin nx]_{-\pi}^{\pi}\right\}$$

$$b_n = \frac{1}{\pi}\left\{\frac{-2\pi\cos n\pi}{n} + \frac{1}{n^2}(0)\right\}$$

$$b_n = \frac{1}{\pi}\frac{(-2\pi)\cdot(-1)^n}{n}$$

$$b_n = \frac{-2(-1)^n}{n} = \frac{(-1)^1\cdot 2\cdot(-1)^n}{n}$$

$$\boxed{b_n = \frac{2(-1)^{n+1}}{n}} \quad \text{-----} \; (iii)$$

Now, putting values of eq^n (i), (ii) and (iii) in eq^n (a)

$$x = \frac{0}{2} + \sum_{n=1}^{\infty} 0\cdot\cos nx + \frac{2(-1)^{n+1}}{n}\cdot\sin nx$$

$$x = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\cdot\sin nx$$

$$x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\sin nx \Rightarrow 2\cdot\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right)$$

where $a_0 = \frac{2}{b-a} \int_a^b f(x)dx$

$a_n = \frac{2}{b-a} \int_a^b f(x) \cdot \cos nx dx$

$b_n = \frac{2}{b-a} \int_a^b f(x) \cdot \sin nx dx$

Note - $\sin n\pi = 0$, $n \in Z$

$\cos n\pi = (-1)^n$, $n \in Z$

Exercise

(1) Find fourier series of $f(x) = x$, $-\pi \le x \le \pi$

Solution - $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ ----(a)

~~where~~ $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot dx$

$a_0 \Rightarrow \frac{1}{\pi} \int_{-\pi}^{\pi} (x) \cdot dx = \frac{1}{\pi} \left[\frac{x^2}{2} \right]_{-\pi}^{\pi}$

$a_0 \Rightarrow \frac{1}{2\pi} \cdot [x^2]_{-\pi}^{\pi} = \frac{1}{2\pi} [\pi^2 - \pi^2]$

$\boxed{a_0 = 0}$ ----(i)

Further, $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos nx dx$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cdot \cos nx dx$

$a_n = \frac{1}{\pi} \left\{ \left[\frac{x \cdot \sin nx}{n} \right]_{-\pi}^{\pi} - \int_{\pi}^{\pi} \frac{\sin x}{n} \right\}$

$a_n = \frac{1}{\pi} \left\{ \frac{\pi \sin n\pi}{n} + (-\pi) \cdot \frac{\sin n\pi}{n} + \frac{1}{n^2} [\cos nx]_{\pi}^{\pi} \right\}$

(4) Find the fourier series of $f(x) = x\sin x$, $-\pi \le x \le \pi$

Solution- $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ ----- (a)

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \sin x \, dx$

$= \frac{1}{\pi}\left\{ \left[\frac{-x\cos x}{\cancel{\pi}}\right]_{\pi}^{\pi} + \int_{-\pi}^{\pi} \cancel{x\sin x} \cos x \cdot dx \right\}$

$= \frac{1}{\pi}\left\{ -\pi\cos\pi + (-\pi)\cdot\cos\pi + \left[\sin x\right]_{-\pi}^{\pi} \right\}$

$= \frac{1}{\pi}\left\{ -\pi\cos\pi - \pi\cos\pi + \sin\pi + \sin\pi \right\}$

$= \frac{1}{\pi}\left\{ -2\pi\cos\pi + 2\sin\pi \right\}$

$a_0 = \cancel{1} - 2\cos\pi + 2\sin\pi \quad = -2\cdot(-1) + 2\times 0$

$\boxed{a_0 = 2}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \cos nx \, dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x \cdot \cos nx \, dx$

$$a_0 = \frac{1}{\pi}\left\{ -2\pi \cos 2\pi + 0 + 1 \left[\int_0^{2\pi} \sin x \right] \right\}$$

$$a_0 = \frac{1}{\pi}\left\{ -2\pi \cdot \cos 2\pi + 1 \left[\sin 2\pi - \sin 0 \right] \right\}$$

$$a_0 = \frac{1}{\pi}\left(-2\pi \cdot \cos 2\pi \right)$$

$$a_0 = \frac{1}{\pi}(-2\pi) = (-2) \quad \text{----(i)}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \cos nx \cdot dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cdot \cos nx \cdot dx$$

$$\cancel{a_n = \frac{1}{2\pi}\int_0^{2\pi} x \cdot \sin 2x \cdot dx} \quad a_n = \frac{1}{2\pi}\int_0^{2\pi} x\left(\sin(n+1)x - \sin(n-1)x \right) dx$$

$$\cancel{a_n = \frac{1}{2\pi}\left\{ \left[-x\cos 2x \right. \right.}$$

$$a_n \Rightarrow \frac{\cancel{2\pi}(-1)^{(n-1)}}{n-1} - \frac{\cancel{2\pi}(-1)^{n+1}}{n+1} = \frac{(-1)^n \cdot (-1)^{-1}}{(n-1)} - \frac{(-1)^n(-1)}{(n+1)}$$

$$a_n = \frac{(-1)^n}{(n+1)} - \frac{(-1)^n}{(n-1)} = (-1)^n\left(\frac{n-1-n-1}{n^2-1} \right)$$

$$a_n = \frac{-(-1)^n \cdot 2}{n^2-1}$$

$$b_n \Rightarrow \frac{1}{\pi}\int_0^{2\pi} f(x) \cdot \sin nx \cdot dx = \frac{1}{\pi}\int_0^{2\pi} x \cdot \sin x \cdot \sin nx \cdot dx$$

$$b_n \Rightarrow \frac{1}{2\pi}\int_0^{2\pi} x\left(\cos(x - nx) - \cos(x + nx) \right) \cdot dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x - \cos(n+1)x \right] \cdot dx$$

$$b_n = \frac{1}{\pi}\left\{ \frac{-2\pi \cos 2\pi n}{n} + 0 + \frac{1}{n^2}\left[\sin nx\right]_0^{2\pi} \right\}$$

$$b_n = \frac{1}{\pi}\left\{ \frac{-2\pi (-1)^{2\pi}}{n} + \frac{1}{n^2}\left[\sin 2\pi n - 0\right] \right\}$$

$$b_n = \frac{1}{\pi}\left\{ \frac{-2\pi}{n} + \frac{1}{n^2}\left[0-0\right] \right\}$$

$$b_n = \frac{1}{\pi} \times \frac{(-2\pi)}{n}$$

$$\left[b_n = \frac{-2}{n} \right] \quad \text{-----} \ (iii)$$

Putting values of a_0, a_n, b_n in eq^n (a)

$$f(x) = \frac{2\pi}{2} + \sum_{n=1}^{\infty} 0 \cdot \cos nx + \frac{(-2)}{n} \sin nx$$

$$f(x) = \pi + \sum_{n=1}^{\infty} 0 + \frac{(-2)}{n} \sin nx$$

$$f(x) = \pi + (-2\sin x) - \sin 2x - \frac{2}{3} \sin 3x + \text{-------}$$

$$f(x) = \pi - \left[2\sin x + \sin 2x + \frac{2}{3} \sin 3x + \text{---} \right]$$

(3) Find fourier- series of $f(x) = x\sin x$, $0 \le x \le 2\pi$

Solution - $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} x \sin x\, dx$$

$$a_0 = \frac{1}{\pi} \left\{ \left[\frac{-x \cos x}{1} \right]_0^{2\pi} + \int_0^{2\pi} \frac{\cos x}{1}\, dx \right\}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \sin nx\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x-x^2)\sin nx \cdot dx$$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{-(x-x^2)\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} (1-2x)\cdot \frac{\cos nx}{n}\cdot dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{-(x-x^2)\cdot\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{(1-2x)\cdot\cos nx\cdot dx}{n}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-2\pi\cos n\pi}{n} + \frac{1}{n}\left\{\left[\frac{(1-2x)\cdot\sin nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(-2)\cdot\frac{\sin nx}{n}\,dx\right\}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-2\pi(-1)^n}{n} + \frac{1}{n}\left\{0 \cancel{+} - \frac{2}{n}\left[\frac{\cos nx}{n}\right]_{-\pi}^{\pi}\right\}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-2\pi(-1)^n}{n} + \frac{1}{n}\left\{-\frac{2}{n^2}\left[\cos n\pi - \cos n\pi\right]\right\}\right\}$$

$$b_n = \frac{-2(-1)^n}{n}$$

Fourier series.

$$x - x^2 = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n^2}\cdot\cos nx - \frac{2(-1)^n}{n}\sin nx$$

put $x = \pi$

$$\pi - \pi^2 = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n}\cos n\pi$$

$$\cancel{\pi - \pi^2 = \frac{-\pi^3}{3}}$$

$$\pi - \pi^2 = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left(-\frac{4}{n^2}\right) \quad \text{---- (i)}$$

put $x = -\pi$

$$-\pi - \pi^2 = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n}\cos n\pi$$

$$-\pi - \pi^2 = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left(-\frac{4}{n^2}\right) \quad \text{---- (ii)}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\,dx$$

$$\Rightarrow \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^3}{3} - \frac{\pi^2}{2} - \frac{\pi^3}{3}\right]$$

$$a_0 = -\frac{2\pi^2}{3} \quad \text{- - - - (i)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\cdot\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\left[(x - x^2)\cdot\frac{\sin nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} (1-2x)\frac{\sin nx}{n}\right\}$$

$$= \frac{1}{\pi}\left\{0 - \frac{1}{n}\left\{\left[\frac{-(1-2x)\cos nx}{n}\right]_{-\pi}^{\pi} + 2\int_{-\pi}^{\pi}\frac{(-\cos nx)}{n}\right\}\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}\left[\frac{-(1-2\pi)\cos n\pi}{n} + \frac{(1+2\pi)\cdot\cos n\pi}{n} - \frac{2}{n^2}\left[\sin nx\right]_{-\pi}^{\pi}\right]\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}\left[\frac{-\cos n\pi}{n} + \frac{2\pi\cos n\pi}{n} + \frac{\cos n\pi}{n} + \frac{2\pi\cos n\pi}{n} - \frac{2}{n^2}\left[\sin n\pi + \sin n\pi\right]\right]\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}\left[\frac{4\pi(-1)^n}{n} - 0\right]\right\}$$

$$= -\frac{1}{\pi}\times\frac{1}{n^2}\times 4\pi(-1)^n$$

$$a_n = \frac{-4(-1)^n}{n^2} \quad \text{- - - - - - (ii)}$$

Que. $f(x) = x - x^2$, $-\pi < x < 2\pi$

To prove that $\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - - - - -$

Solution - $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$

$a_0 = \frac{1}{\pi} \int_{-\pi}^{2\pi} (x - x^2) \cdot dx$

$= \frac{1}{\pi} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{-\pi}^{2\pi}$

$a_0 = \frac{1}{\pi} \left[\frac{4\pi^2}{2} - \frac{8\pi^3}{3} - \frac{\pi^2}{2} - \frac{\pi^3}{3} \right]$

$a_0 = \frac{1}{\pi} \left[\frac{3\pi^2}{2} - \frac{9\pi^3}{3} \right]$

$a_0 = \frac{1}{\pi} \left[\frac{9\pi^2 - 12\pi^3}{6} \right]$

$a_0 = \frac{3\pi}{6\pi} \left[3\pi - 4\pi^2 \right]$

$a_0 = \frac{3\pi - 4\pi^2}{2}$

$a_n = \frac{1}{\pi} \int_{-\pi}^{2\pi} f(x) \cdot \cos nx dx$

$a_n = \frac{1}{\pi} \int_{-\pi}^{2\pi} (x - x^2) \cdot \cos nx \cdot dx$

$= \frac{1}{\pi} \left\{ \left[(x - x^2) \cdot \frac{\sin nx}{n} \right]_{-\pi}^{2\pi} - \int_{-\pi}^{2\pi} (1 - 2x) \cdot \frac{\sin nx}{n} \cdot dx \right\}$

$= \frac{1}{\pi} \left\{ 0 - \frac{1}{n} \left[\int (1 - 2x) \cdot \right. \right.$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \sin nx\,dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi - x}{2}\right) \cdot \sin nx\,dx$$

$$= \frac{1}{\pi} \left\{ \int_0^{2\pi} \frac{\pi}{2} \cdot \sin nx \cdot dx - \int_0^{2\pi} \frac{x}{2} \cdot \sin nx\,dx \right\}$$

$$= \cancel{\frac{1}{\pi} \left\{ \frac{\pi}{2} \left[\frac{-\cos nx}{n} \right]_0^{2\pi} + \left[\frac{x}{2} \cdot \frac{\cos nx}{n} \right]_0^{2\pi} + \frac{1}{2} \right]_0^{2\pi}}$$

$$= \frac{1}{\pi} \cdot \left\{ \frac{-\pi}{2} \left[\frac{\cos nx}{n} \right]_0^{2\pi} + \left[\frac{x \cdot \cos nx}{2\,n} \right]_0^{2\pi} - \frac{1}{2n} \int_0^{2\pi} \cos nx \cdot dx \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{-\pi}{2} \left[\frac{\cos 2n\pi}{n} - \frac{\cos 0}{n} \right] + \left[\frac{2\pi}{2} \cdot \frac{\cos 2\pi}{n} - 0 \right] - \frac{1}{2n^2} \left[\sin nx \right]_0^{2\pi} \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{-\pi}{2} [1 - 1] + \left[\frac{2\pi \times 1}{2} - 0 \right] - 0 \right\}$$

$$= \frac{1}{\pi} \times \frac{2\pi}{2n}$$

$$= \frac{1}{n}$$

Fourier-series is as follows –

$$\frac{\pi - x}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi + b_n \sin n\pi$$

$$\frac{\pi - x}{2} = 0 + \sum_{n=1}^{\infty} \left(0 + \frac{\sin n\pi}{n}\right)$$

$$\frac{\pi - x}{2} = \sin \pi + \frac{\sin 2\pi}{2} + \frac{\sin 3\pi}{3} + \frac{\sin 4\pi}{4} + \text{------}$$

Que. Obtain Fourier Series of $f(x) = \frac{\pi - x}{2}, \; 0 \le x \le 2\pi$

Solution- $f(x) = \frac{\pi - x}{2}$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)\cdot dx$$

$$= \frac{1}{\pi}\left\{\int_0^{2\pi}\frac{\pi}{2}\,dx - \int_0^{2\pi}\frac{x}{2}\,dx\right.$$

$$= \frac{1}{\pi}\left\{\left[\frac{\pi}{2}\cdot[x]_0^{2\pi} - \left[\frac{x^2}{4}\right]_0^{2\pi}\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi}{2}\times 2\pi - 0 - \left(\frac{4\pi^2}{4} - 0\right)\right\}$$

$$= \frac{1}{\pi}\cdot\left\{\frac{2\pi^2}{2} - \frac{4\pi^2}{4}\right\}$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)\cdot\cos nx\cdot dx$$

$$= \frac{1}{\pi}\left\{\int_0^{2\pi}\frac{\pi}{2}\cdot\cos nx\,dx - \int_0^{2\pi}\frac{x}{2}\cdot\cos nx\cdot dx\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi}{2}\left[\frac{\sin nx}{n}\right]_0^{2\pi} - \left[\left(\frac{x}{2}\cdot\frac{\sin nx}{n}\right)_0^{2\pi} - \frac{1}{2}\int_0^{2\pi}\frac{\sin nx\cdot dx}{n}\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{\pi}{2}\cdot\left(\frac{\sin 2\pi n}{n} - \frac{\sin 0}{n}\right) - \left[0 + \frac{1}{2n}\left[\frac{\cos nx}{n}\right]_0^{2\pi}\right]\right\}$$

$$= \frac{1}{\pi}\left\{0 - 0 - \frac{1}{2n^2}\left[\cos 2n\pi - \cos 0\right]\right.$$

$$\Rightarrow \frac{1}{\pi}\left\{\frac{1}{2n^2}\times\left((-1)^{2n} - \frac{1}{\cancel{2n^2}}\right)\right\}$$

$$= \cancel{\frac{((-1)^{2n} - 1)}{2n^2\pi}} \quad \Rightarrow \frac{1}{\pi}\times\frac{1}{2n^2}(1-1) = 0$$

$$\pi - \pi^2 - \pi - \pi^2 = -\frac{2\pi^2}{3} + 2\sum_{n=1}^{\infty}\left(\frac{-4}{n}\right)$$

$$-2\pi^2 = -\frac{2\pi^2}{3} + (-8)\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$$

$$-2\pi^2 + \frac{2\pi^2}{3} = 8\cdot\sum_{n=1}^{\infty}\left(-\frac{1}{n^2}\right)$$

$$-\frac{4\pi^2}{3} = 8\sum_{n=1}^{\infty}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}$$

putting $x=0$ in fourier-series

$$0-0 = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{(-4)(-1)^n\cos 0}{n^2}$$

$$0 = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{(-4)\cdot(-1)^n}{n^2}$$

$$\frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)\cdot(-1)^n}{n^2}$$

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

$$\frac{\pi^2}{12} = \frac{(-1)^{1+1}}{1} + \frac{(-1)^{2+1}}{2^2} + \frac{(-1)^{3+1}}{3^2} + \frac{(-1)^{4+1}}{4^2} + \cdots$$

$$\left[\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$ Ans

Half range sine series -

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{-4}{n\pi} \cos(n\pi) \cdot \sin\left(\frac{n\pi x}{2}\right) \right)$$

$$x = \sum_{n=1}^{\infty} \frac{-4}{n\pi} (-1)^n \cdot \sin\left(\frac{n\pi x}{2}\right)$$

$$x = \frac{-4}{\pi} (-1) \cdot \sin\left(\frac{\pi x}{2}\right) - \frac{4}{2\pi} (-1)^2 \cdot \sin(\pi x) - \frac{4}{3\pi} (-1)^3 \cdot \sin\left(\frac{3\pi x}{2}\right) + \cdots$$

$$x = \frac{4}{\pi} \cdot \sin\left(\frac{\pi x}{2}\right) - \frac{4}{2\pi} \cdot \sin(\pi x) + \frac{4}{3\pi} \cdot \sin\left(\frac{3\pi \cdot x}{2}\right) - \frac{4}{4\pi} \cdot \sin\left(\frac{4\pi x}{2}\right) + \cdots$$

$$x = \frac{4}{\pi} \sin\left(\frac{\pi x}{2}\right) - \frac{2}{\pi} \sin(\pi x) + \frac{4}{3}\pi \cdot \sin\left(\frac{3\pi x}{2}\right) - \frac{1}{\pi} \cdot \sin(2\pi x) + \cdots$$

Que. Obtain the half range sine series for $f(x) = \cos x$ in $0 \le x \le \pi$.

Solution- $f(x) = \cos x.$

$$b_n = \frac{2}{l} \int_0^l f(x) \cdot \sin\left(\frac{n\pi x}{l}\right) \cdot dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \cdot \cos x \cdot \sin(nx) \cdot dx = \frac{2}{\pi} \int_0^{\pi} \sin(nx) \cdot \cos x \cdot dx$$

$$\left[b_n = \frac{2}{\pi} \times \frac{1}{(n+1)} \left[\cos x - \cos x \cdot (-1)^n \right] \right]$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} 2 \sin nx \cdot \cos x \cdot dx$$

$$= \frac{1}{\pi} \int_0^{\pi} [\sin(n+1)x + \sin(n-1)x] \cdot dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin(n+1)x\,dx + \frac{1}{\pi} \int_0^{\pi} \sin(n-1)x\,dx.$$

Half range sine-series –

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{(n\pi x)}{l}$$

$$= \sum_{n=1}^{\infty} \frac{2}{n\pi} \cdot [-(-1)^n + 1] \cdot \sin(nx)$$

$$= \frac{2}{\pi}(1+1)\cdot \sin x + \frac{2}{3\pi}(1+1)\cdot \sin 3x + \frac{2}{5\pi}(1+1)\sin 5x + \cdots$$

$$= \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$$ Answer

Que. Find the half range sine series $f(x) = x$ in $0 < x < 2$.

Solution – $f(x) = x$.

$$b_n = \frac{2}{l}\int_0^l f(x)\cdot \sin\left(\frac{n\pi x}{l}\right)\cdot dx$$

$$b_n = \frac{2}{2}\cdot\int_0^2 x\cdot \sin\left(\frac{n\pi x}{2}\right)\cdot dx$$

$$b_n = \left[x \cdot \frac{-\cos\left(\frac{n\pi x}{2}\right)}{(n\pi)/2}\right]_0^2 + \int_0^2 \frac{\cos\left(\frac{n\pi x}{2}\right)}{(n\pi)/2}\cdot dx$$

$$b_n = \left[\frac{-2x}{n\pi}\cdot\cos\left(\frac{n\pi x}{2}\right)\right]_0^2 + \frac{2}{n\pi}\times\frac{2}{n\pi}\left[\sin\frac{(n\pi x)}{2}\right]_0^2$$

$$b_n = \frac{-2\times 2}{n\pi}\cos(n\pi) + \frac{4}{n^2\pi^2}\left[\sin(n\pi)\right]$$

$$b_n = \frac{-4}{n\pi}\cos(n\pi)$$

→ Half Range sine & cosine Fourier Series

(1) For cosine -

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right).$$

where, $a_0 = \frac{2}{l}\int_0^l f(x)\cdot dx$

$$a_n = \frac{2}{l}\cdot\int_0^l f(x)\cdot\cos\left(\frac{n\pi x}{l}\right)\cdot dx$$

(2) For sine

$$f(x) = \cancel{\frac{a_0}{2}} \cancel{+} \sum_{n=1}^{\infty} b_n\cdot\sin\left(\frac{n\pi x}{l}\right)$$

where, $b_n = \frac{2}{l}\cdot\int_0^l f(x)\cdot\sin\left(\frac{n\pi x}{l}\right)\cdot dx$

Expand $f(x)=1$ in a sine series $0 \le x \le \pi$

Solution- $f(x) = 1$

$$b_n = \frac{2}{l}\int_0^l f(x)\cdot\sin\left(\frac{n\pi x}{l}\right)\cdot dx$$

$$= \frac{2}{l}\int_0^l \sin\left(\frac{n\pi x}{l}\right)\cdot dx$$

$$= \frac{2}{\pi}\int_0^\pi \sin(nx)\cdot dx$$

$$= \frac{2}{\pi}\cdot\left[\frac{-\cos(nx)}{n}\right]_0^\pi \cdot \cancel{dx}$$

$$= \frac{2}{\pi n}\left[-\cos(n\pi) + \cos 0\right]$$

$$= \frac{2}{n\pi}\cdot\left[-(-1)^n + 1\right]$$

Que. $f(x) = c-x$ in $0 \le x \le c$ and find cosine series.

Solution - $f(x) = c-x$

$$\cancel{b_n = \frac{2}{l}\int_0^l f(x)\cdot} \qquad a_0 = \frac{2}{l}\int_0^l f(x)\cdot dx$$

$$a_0 = \frac{2}{c}\int_0^c (c-x)\cdot dx = \frac{2}{c}\left[cx - \frac{x^2}{2}\right]_0^c$$

$$a_0 = \frac{2}{c}\left[c\times c - \frac{c^2}{2} - 0 - 0\right]$$

$$a_0 \Rightarrow \frac{2}{c}\left[\frac{c^2 - c^2}{2}\right] = \frac{2c^2}{2c}$$

$$\left[a_0 = c\right]$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cdot \cos\left(\frac{nx\pi}{l}\right)\cdot dx = \frac{2}{c}\int_0^c (c-x)\cdot \cos\left(\frac{n\pi x}{c}\right)dx$$

$$a_n = \frac{2}{c}\int_0^c c\cdot\cos\left(\frac{n\pi x}{c}\right)\cdot dx - \frac{2}{c}\int_0^c x\cdot\cos\left(\frac{n\pi x}{c}\right)\cdot dx$$

$$a_n = \frac{2\times c}{c}\left[\sin\left(\frac{n\pi x}{c}\right)\times\frac{c}{n\pi}\right]_0^c - \frac{2}{c}\left\{\left[\frac{x c\cdot \sin}{n\pi}\left(\frac{n\pi x}{c}\right)\right]_0^c - \int_0^c \frac{c}{n\pi}\sin\left(\frac{n\pi x}{c}\right)\cdot dx\right\}$$

$$a_n = 2\left[\frac{c\cdot\sin}{n\pi}\left(\frac{n\pi x}{c}\right)\right]_0^c - \frac{2}{c}\left\{0 - 0 - \frac{c}{n\pi}\left[\frac{-c\cos}{n\pi}\left(\frac{n\pi x}{c}\right)\right]_0^c\right\}$$

$$a_n = 2[0-0] - \frac{2}{c}\left\{\frac{c^2}{n^2\pi^2}\left[\cos(n\pi) - \cos 0\right]\right.$$

$$= \frac{-2\times c^2}{c\ n^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = \frac{-2c}{n^2\pi^2}\left[(-1)^n - 1\right]$$

Que. Expand $f(x) = e^x$ in a cosine series over 0 to 1.

Solution- $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{l}\right)$

and $a_0 = \frac{2}{l}\int_0^l f(x)\cdot dx$

$a_n = \frac{2}{l}\int_0^l f(x)\cdot \cos\left(\frac{n\pi x}{l}\right)\cdot dx$

$a_0 = \frac{2}{1}\int_0^1 e^x\, dx = 2\,[e^x]_0^1 \cdot dx$

$a_0 \Rightarrow 2\,[e - e^0] = 2\,[e-1]$

$a_n = \frac{2}{l}\int_0^l \cancel{f}\, e^x \cdot \cos\left(\frac{n\pi x}{1}\right)\cdot dx$

$a_n = \frac{2}{1}\int_0^1 e^x \cdot \cos(n\pi x)\, dx$

$a_n = 2\left[\left[e^x \cdot \left\{\frac{\sin(n\pi x)}{n\pi}\right]_0^1 - \int_0^1 e^x \cdot \left[\frac{\sin(n\pi x)}{n\pi}\right]_0^1 \cdot dx\right]$

$a_n = 2\left[0 - 0 - \int_0^1 e^x \cdot \left(\frac{\sin(n\pi x)}{n\pi}\right)\cdot dx\right]$

$a_n = 2 \times \frac{1}{(n^2\pi^2+1)} \cdot (e(-1)^n - 1)$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$

$= 2[e-1] + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2+1} \cdot [e(-1)^n - 1] \cdot \cos(n\pi x)$ Ans

Que Obtain the half range sine series for $f(x) = \cos$ $0 \le x \le \pi$.

$$b_n = \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1}\right]_0^{\pi} + \frac{1}{\pi}\left[\frac{-\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$b_n = \frac{1}{\pi}\left[\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos 0}{n+1}\right] + \frac{1}{\pi}\left[\frac{-\cos(n-1)\pi}{n-1} + \frac{\cos 0}{n-1}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{-(-1)^{n+1}}{n+1} + \frac{1}{n+1}\right] + \frac{1}{\pi}\left[\frac{-(-1)^{n-1}}{n-1} + \frac{1}{n-1}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{(-1)^{n}}{n+1} + \frac{1}{n+1}\right] + \frac{1}{\pi}\left[\frac{(-1)^{n}}{n+1} + \frac{1}{n-1}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{(-1)^{n}}{n+1} + \frac{1}{n+1} + \frac{(-1)^{n}}{n-1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^{n}(n-1) + n-1 + (-1)^{n}(n+1) + n+1}{(n^2-1)}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^{n} n - (-1)^{n+1} + n-1 + (-1)^{n} n + (-1)^{n-1} + n+1}{(n^2-1)}\right]$$

$$= \frac{1}{\pi}\left[\frac{2(-1)^{n} \cdot n + 2n}{n^2-1}\right]$$

$$b_n = \frac{2n}{\pi(n^2-1)}\left[(-1)^{n} + 1\right]$$

$$f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2n}{\pi(n^2-1)} \cdot \left[(-1)^{n} + 1\right] \cdot \sin nx$$

Ans

taking integual on both side

$$\int \frac{\sec^2 x}{\tan x}\, dx = -\int \frac{\sec^2 y}{\tan y} \cdot dy$$

Let $u = \tan x$, $v = \tan y$

$\frac{du}{dx} = \sec^2 x$, $\frac{dv}{dy} = \sec^2 y$

$du = \sec^2 x \cdot dx$, $dv = \sec^2 y \cdot dy$

$$\int \frac{du}{u} = -\int \frac{dv}{v}.$$

$$\log u = -\log v + C$$

$$\log(\tan x) = -\log(\tan y) + C$$

$$\log(\tan x) + \log(\tan y) = C.$$

$$\log(\tan x \cdot \tan y) = C$$ Answer

Exercise

(1) $y - x \cdot \frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

$$\frac{ydx - x \cdot dy}{dx} = ay^2 + a \cdot \frac{dy}{dx}$$

$$\frac{ydx - x \cdot dy}{dx} = \frac{ay^2 \cdot dx + a \cdot dy}{dx}$$

$$ydx - xdy = ay^2 \cdot dx + a \cdot dy$$

$$ydx - ay^2 dx = xdy + a \cdot dy$$

$$dx(y - ay^2) = (x + a)\, dy$$

$$\frac{dx}{(x+a)} = \frac{dy}{(y - ay^2)}$$

Date - 15/03/2024

UNIT - IV

(1) Seperable Method - $f(x)dx + \phi(y)\cdot dy = 0$

example, Solve $(1-x^2)\cdot(1-y)\,dx = xy(1+y)\cdot dy$

Solution - $\frac{(1-x^2)}{x}\cdot dx = y\left(\frac{1+y}{1-y}\right)\cdot dy$

~~On~~ taking integral on both side

$$\int \frac{(1-x^2)}{x}dx = \int y\left(\frac{1+y}{1-y}\right)\cdot dy$$

$$\int \frac{1}{x}\cdot dx - \int x\,dx = \int\left(-y + \frac{2y}{1-y}\right)\cdot dy$$

$$\int \frac{1}{x}dx - \int x\cdot dx = \int -y\cdot dy + 2\int \frac{y}{1-y}\cdot dy$$

$$\log x - \frac{x^2}{2} = -\frac{y^2}{2} + 2\left\{\int \frac{y-1+1}{1-y}\cdot dy\right\}$$

$$\log x - \frac{x^2}{2} = -\frac{y^2}{2} + 2\left\{\int \frac{-(1-y)}{(1-y)}dy + \int \frac{1}{(1-y)}\cdot dy\right\}$$

$$\log x - \frac{x^2}{2} = -\frac{y^2}{2} + 2\left\{-\int dy + \int \frac{1}{1-y}\cdot dy\right\}$$

$$\log x - \frac{x^2}{2} = -\frac{y^2}{2} + 2\left\{-y - \log(1-y)\right\} + c$$

$$\left[\log x - \frac{x^2}{2} = -\frac{y^2}{2} - 2y - \log(1-y) + c\right]$$ Answer

(2) $\sec^2 x\cdot\tan y\cdot dx + \sec^2 y\cdot\tan x\cdot dy = 0$

Solution - $\sec^2 x\,\tan y\,dx = -\sec^2 y\cdot\tan x\cdot dy$

$$\frac{\sec^2 x}{\tan x}\cdot dx = -\frac{\sec^2 y}{\tan y}\cdot dy$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{l}\right)$$

$(2n-1)$

$$f(x) = \frac{c}{2} + \sum_{n=1}^{\infty}\left[\frac{-2c}{n^2\pi^2}\cdot[(-1)^n - 1]\cdot\cos\left(\frac{n\pi x}{c}\right)\right]$$ Ans

replace $n = 2n-1$

$$f(x) = \frac{c}{2} + \sum_{n=1}^{\infty}\left[\frac{-2c}{(2n-1)^2\pi^2}\cdot[(-1)^{2n-1} - 1]\cdot\cos\frac{(2n-1)\cdot\pi x}{c}\right]$$

$$f(x) = \frac{c}{2} + \sum_{n=1}^{\infty}\left[\frac{4c}{(2n-1)^2\pi^2}\cdot\left[\cos\frac{(2n-1)\pi x}{c}\right]\right]$$

$$\frac{1}{(2n-1)^2}\ \cos\left(\frac{2n\pi x}{c} - \frac{\pi x}{c}\right)$$

$$f(x) = \frac{c}{2} + \sum_{n=1}^{\infty}\left[\frac{4c}{(2n-1)^2\pi^2}\cdot\cos\frac{(2n-1)\,n\pi}{c}\right]$$ Answer

$$\cancel{f(x) = \frac{c}{2} + \frac{4c}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{2m^2}}$$

Que -

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(1) $x\,dx + y\,dy + \dfrac{x\cdot dy - y\,dx}{x^2+y^2} = 0$

$\dfrac{(x\,dx+y\,dy)\cdot(x^2+y^2) + x\,dy - y\,dx}{x^2+y^2} = 0$

$(x^2+y^2)\cdot x\,dx + (x^2+y^2)y\,dy + x\cdot dy - y\,dx = 0$

$dx\left[x^3+y^2x - y\right] + dy\left[y(y^2+x^2)+x\right] = 0$

$dx\left[x^3+y^2x-y\right] + \left[y^3+x^2y+x\right]\cdot dy = 0$

$\cancel{\dfrac{dy}{dx} = -(x^3+y}$

$M = x^3+y^2x-y$, $N = y^3+x^2y+x$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}(x^3+y^2x-y)$, $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}(y^3+x^2y+x)$

$\dfrac{\partial M}{\partial y} = 2yx-1$, $\dfrac{\partial N}{\partial x} = 2xy+1$

II- $x\cdot dx - \dfrac{y\,dx}{x^2+y^2} + y\cdot dy + \dfrac{x\,dy}{x^2+y^2} = 0$

$dx\left[x - \dfrac{y}{x^2+y^2}\right] + dy\left[y + \dfrac{x}{x^2+y^2}\right] = 0$

$M = x - \dfrac{y}{x^2+y^2}$, $N = y + \dfrac{x}{x^2+y^2}$

$\dfrac{\partial M}{\partial y} = 0 - \left[\dfrac{(x^2+y^2)\cdot 1 - y(2y)}{(x^2+y^2)^2}\right]$, $\dfrac{\partial N}{\partial x} = 1 + \dfrac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$

$= \dfrac{2y^2 - x^2 - y^2}{(x^2+y^2)} =$ $\dfrac{\partial N}{\partial x} = \dfrac{x^2+y^2-2x^2}{(x^2+y^2)^2}$

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When $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

then find I.F. to make it $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\cancel{\frac{ax^2}{2} + \frac{hy^2}{2} = c'}$

$\cancel{ax^2 + hy^2 = 2c} \Rightarrow \cancel{[ax^2 + hy^2 = C]}$

$\frac{ax^2}{2} + hxy + gx + \frac{hy^2}{2} + fy = C$

$ax^2 + 2hxy + 2gx + hy^2 + 2fy = 2c$

$ax^2 + 2hxy + 2gx + hy^2 + 2fy = C$ Answer

(2) $(x^3 + 3xy^2)\,dx + (3x^2y + y^3)\,dy = 0$

$M = x^3 + 3xy^2$
$N = 3x^2y + y^3$

$\frac{\partial M}{\partial y} \Rightarrow 0 + 6xy = 6xy$

$\frac{\partial N}{\partial x} = 6xy + 0$

then

$\int (x^3 + 3xy^2)\cdot dx + \int y^3 \cdot dy = C$

$\frac{x^4}{4} + \frac{3y^2 x x^2}{2} + \frac{y^4}{4} = C$

$\frac{x^4 + 6x^2y^2 + y^4}{4} = C$

$[x^4 + 6x^2y^2 + y^4 = C]$

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Exact differential equation-

A diff eq^n $M \cdot dx + N \cdot dy = 0$ is exact

if $\left(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \right)$

Then, its sol^n

$$\left[\int_{(y\ \text{constant})} M\,dx + \int (\text{term in } y \text{ but not containing } x) \cdot N\,dy = c \right]$$

example, $(ax + hy + g)\,dx + (hx + by + f) \cdot dy = 0$

Here $M = ax + hy + g$

$$\frac{\partial M}{\partial xy} = 0 + h + g0 = h$$

and $N = hx + by + f$

$$\frac{\partial N}{\partial x} = h + 0 + 0 = h$$

$\Rightarrow$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Now. $\int M \cdot dx = \int (\text{term in } y \text{ but not containing } x) \cdot N\,dy = C$

$$\int (ax + hy + g) \cdot dx + \int (hy + f)\,dy = C$$

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$$\frac{1}{2}x + \frac{\sin 2x}{4} - \sin a \cdot \sin x \cdot \sin y + \frac{1}{2}y + \frac{\sin 2y}{4} = c$$

$$2x + \sin 2x - 4\sin a \cdot \sin x \cdot \sin y + 2y + \sin 2y = 2c$$

$$2x + 2y + \sin 2x + \sin 2y - 4\sin a \cdot \sin x \cdot \sin y = 4c$$

Answer

→ When $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Meth-(1) $M dx + N \cdot dy = 0$ (Homogeneous)

↓

$Mx + Ny \neq 0$

↓

$$I.F. = \frac{1}{Mx + Ny}$$

$$x^2y \cdot dx - (x^3 + y^3) \cdot dy = 0$$

$$x^2y\, dx + [-(x^3 + y^3)] = 0$$

$M = x^2y$ $\quad$ $N = -(x^3 + y^3)$

When $\frac{\partial M}{\partial y} = x^2$ $\quad$ $\frac{\partial N}{\partial x} = -3x^2$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

~~Ath~~ then, Given eq^n is homogeneous.

$$Mx + Ny = (x^2y) \cdot x + [-(x^3 + y^3) \cdot y]$$

$$= x^3y - x^3y - y^4$$

$$Mx + Ny \cancel{\ } -y^4 \neq 0$$

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(2) $\cos x(\cancel{\sin x}^{\cos x} - \sin a \sin y)\,dx + \cos y(\cos y - \sin a \cdot \sin x)\,dy = 0$

$M = \cos x\,(\sin x - \sin a \cdot \sin y)$

$\frac{\partial M}{\partial y} = \cos x \cdot \frac{\partial}{\partial y}(\sin x - \sin a \cdot \sin y)$

$\frac{\partial M}{\partial y} = \cos x\,(0 - \sin a \cdot \cos y)$

$\frac{\partial M}{\partial y} = -\sin a \cdot \cos x \cdot \cos y$

$N = \cos y\,(\cos y - \sin a \cdot \sin x)$

$\frac{\partial N}{\partial x} = \cos y\,(0 - \sin a \cdot \cos x)$

$\frac{\partial N}{\partial x} = -\sin a \cdot \cos x \cdot \cos y$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\int \cos x(\cancel{\sin x}^{\cos x} - \sin a \cdot \sin y)\cdot dx + \int \cos y(\cos y - \sin a \cdot \sin x)\cdot dy = C$

$\int \cos^2 x \cdot dx - \sin a \sin y \int \cos x \cdot dx + \int \cos^2 y \cdot dy = C$

$\int \frac{1+\cos 2x}{2}\cdot dx - \sin a \cdot \sin y \cdot \sin x + \int \frac{1+\cos 2y}{2} = C$

Meth-

$\int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a)$

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$\frac{\partial M}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$ $\frac{\partial N}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$.

$$\int \left(x - \frac{y}{x^2+y^2}\right) \cdot d x + \int y\,dy = c$$

$$\boxed{\frac{x^2}{2} - \frac{y}{y} \cdot \tan^{-1}\left(\frac{x}{y}\right) + \frac{y^2}{2}} = C.$$

$$\boxed{\frac{x^2}{2} + \frac{y^2}{2} - \tan^{-1}\left(\frac{x}{y}\right) = C}$$ Answer

$$x^2+y^2 - 2\tan^{-1}\frac{y}{x} = 2c$$

$$2x + 2y \cdot \frac{dy}{dx} - 2 \times \frac{1}{1+\frac{y^2}{x^2}} \times \frac{x \cdot dy/dx - y}{x^2} = 0$$

$$2x + 2y \cdot \frac{dy}{dx} - \frac{2 \times x^2 \cdot dx}{x^2+y^2}$$

$$x \cdot dx + y \cdot dy - \frac{x^2 \cdot dx}{x^2+y^2}$$

$$2x + 2y\frac{dy}{dx} - \frac{2x^2(x \cdot dy - y dx)}{(x^2+y^2)x^2 \cdot dx} = 0$$

$$x \cdot dx + y\,dy - \frac{(x \cdot dy - y\,dx)}{(x^2+y^2)} = 0$$

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(iii) If $Mx - Ny = 0$

$$Mx = Ny$$

$$\frac{M}{N} = \frac{y}{x}$$

then, $M \cdot dx + N dy = 0$

$$\frac{M}{N} \cdot dx + dy = 0$$

$$\frac{y}{x} \cdot dx + dy = 0$$

$$y\,dx + x\,dy = 0$$

$$y\,dx = -x\,dy$$

$$\frac{1}{x} \cdot dx = -\frac{1}{y} \cdot dy$$

$$\log x = -\log y + \log c$$

$$\log x + \log y = \log c$$

$$\log xy = \log c$$

$$\boxed{xy = c}$$

Example, $(xy + 2x^2y^2)y\,dx + (xy - x^2y^2) \cdot x\,dy = 0$

$$M = xy^2 + 2x^2y^3$$

$$N = x^2y - x^3y^2$$

$$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2 \qquad \frac{\partial N}{\partial y} = 2xy - 3x^2y^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

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(a) Diff. $M dx + N \cdot dy = 0$.

when $Mx + Ny = 0$

$Mx = -Ny$

$$\boxed{\frac{M}{N} = -\frac{y}{x}}$$

$M \cdot dx + N \cdot dy = 0$

$\frac{M}{N} dx + dy = 0$

$-\frac{y}{x} \cdot dx + dy = 0$

$-y\,dx + x \cdot dy = 0$

$x\,dy = y\,dx$

$\frac{1}{y} \cdot dy = \frac{1}{x} \cdot dx$

$\Rightarrow \log c + \log y = \log x -$

$\log c = \log x - \log y$

$c = \frac{x}{y} \Rightarrow \boxed{\frac{x}{y} = c}$

Method- II - If diff. eq^n $[y f_1(x,y)] \cdot dx + [x f_2(x,y)] \cdot dy = 0$

When, $Mx - Ny \neq 0$

Then, $I.F = \frac{1}{Mx - Ny}$

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$I.F. = \frac{-1}{y^4}$

Now, multiply integrating factor in given eq^n

$\frac{-1}{y^4}\left[x^2y\,dx - (x^3+y^3)\,dy\right] = 0$

$-\frac{x^2}{y^3}dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0$

$M = \frac{-x^2}{y^3}$ and $N = \frac{x^3}{y^4} + \frac{1}{y}$

$\frac{\partial M}{\partial y} = -x^2\left(\frac{-3}{y^4}\right)$ and $N = \frac{3x^2}{y^4}$

$\frac{\partial M}{\partial y} = \frac{3x^2}{y^4}$

then, $\int M \cdot dx + \int N\,dy = C$

$\int \frac{-x^2}{y^3} \cdot dx + \int \frac{1}{y}\,dy = C$

$-\frac{x^3}{3y^3} + \log y = C$

$$\frac{\partial M}{\partial y} = \cancel{y^2}\,\frac{1}{2} + \frac{1}{2x^2}\left(\frac{-1}{y^2}\right) = \frac{1}{2}\,\cancel{0} - \frac{1}{2x^2y^2}$$

$$\frac{\partial N}{\partial x} = \frac{1}{2} + \frac{1}{2xy^2}\left(\frac{-1}{x^2}\right) = \frac{1}{2} - \frac{1}{2x^2y^2}$$

$$\left[\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right]$$

Now

$$\int M\cdot dx + \int N\cdot dy = c$$

$$\int\left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right)\cdot dx + \int\left(\cancel{\frac{x}{2}} - \frac{1}{2y}\right) + \cancel{\frac{1}{2x}}\cdot dy = c$$

$$\left[\frac{xy}{2} + \frac{1}{2}\log x + \frac{1}{2y}\left[\frac{x^{-1}}{-1}\right]\right] + \left[-\frac{1}{2}\log y\right] = c$$

$$\frac{xy}{2} + \frac{1}{2}\log x - \frac{1}{2xy} - \frac{1}{2}\log y = c$$

$$\frac{xy}{2} + \frac{1}{2}\left[\log x - \log y\right] - \frac{1}{2xy} = c$$

$$\frac{xy}{2} + \left[\frac{1}{2}\log\left(\frac{x}{y}\right) - \frac{1}{2xy} = c\right]$$

$=0$

$$xy + \log\left(\frac{x}{y}\right) - \frac{1}{xy} = 2c$$

$=0$ –(ii)

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$$-\frac{1}{xy} + \log x^2 - \log y = 3c$$

$$\log x^2 - \log y = \frac{1}{xy} + 3c$$

$$\left[\log \frac{x^2}{y} = \frac{1}{xy} + C\right]$$ Answer

(2) $(x^2y^2 + xy + 1)y \cdot dx + (x^2y^2 - xy + 1)x \cdot dy = 0$ —(i)

$M = x^2y^3 + xy^2 + y$, $N = x^3y^2 - x^2y + x$

$\frac{\partial M}{\partial y} = 3y^2x^2 + 2xy$, $\frac{\partial N}{\partial x} = 3x^2y^2 - 2xy$

$$\left[\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right]$$

Now, $Mx - Ny = (x^2y^3 + xy^2 + y)x - (x^3y^2 - x^2y + x) \cdot y$

$= x^3y^3 + x^2y^2 + xy - x^3y^3 + x^2y^2 - xy$

$$Mx - Ny = 2x^2y^2 \neq 0$$

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}$$

On multiplying $\frac{1}{2x^2y^2}$ in eq^n (i)

$$\frac{1}{2x^2y^2}\left[(x^2y^3 + xy^2 + y) \cdot dx\right] + \frac{1}{2x^2y^2}\left[(x^3y^2 - x^2y + x) \cdot dy\right] = 0$$

$$\left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right) \cdot dx + \left[\frac{x}{2} - \frac{1}{2y} + \frac{1}{2xy^2}\right] dy = 0$$

$$Mx - Ny$$

$$= (xy^2 + 2x^2y^3)x - (x^2y - x^3y^2)y =$$

$$= x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3$$

$$= 3x^3y^3 \neq 0$$

$$I.F. = \frac{1}{Mx-Ny} = \frac{1}{3x^3y^3}$$

$$\frac{1}{3x^3y^3}\left[(xy + 2x^2y^2)y\,dx + (xy - x^2y^2)x\,dy\right] = 0$$

$$\frac{1}{3x^3y^2}\left[(xy + 2x^2y^2)dx\right] + \frac{1}{3x^2y^3}(xy - x^2y^2)dy = 0$$

$$\left[\frac{xy}{3x^3y^2} + \frac{2x^2y^2}{3x^3y^2}\right] \cdot dx + \left[\frac{xy}{3x^2y^3} - \frac{x^2y^2}{3x^2y^3}\right]dy = 0$$

$$\left[\frac{1}{3x^2y} + \frac{2}{3x}\right] \cdot dx + \left[\frac{1}{3xy^2} - \frac{1}{3y}\right] \cdot dy = 0$$

$$M = \frac{1}{3x^2y} + \frac{2}{3x}, \quad N = \frac{1}{3xy^2} - \frac{1}{3y}$$

$$\frac{\partial M}{\partial y} = \frac{-1}{3x^2y^2}, \quad \frac{\partial N}{\partial y} = \frac{-1}{3x^2y^2}$$

then $\displaystyle\int\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int\left(\frac{1}{3xy^2} - \frac{1}{3y}\right) \cdot dy = C$

$$\frac{1}{3}\left[\frac{-1}{xy} + 2\log x\right] - \frac{1}{3}\log y = C$$

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$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$= \frac{1}{3x^2y^2 + x^2y - 2x^3 + y^2}\left(2xy - 6xy^2 - 2xy + 6x^2\right)$$

$$= \frac{1}{3x^2y^2 + x^2y - 2x^3 + y^2}\left(6x^2 - 6xy^2\right) \neq f(x,y)$$

(1)

(2)

$$\therefore \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{xy^2 - x^2} \cdot \left(6xy^2 + \cancel{2xy} - 6x^2 - \cancel{2xy}\right)$$

(3)

$$= \frac{1}{xy^2 - x^2} \cdot \left(6xy^2 - 6x^2\right)$$

$$= \frac{1}{xy - x^2} \times 6\left(xy^2 - x^2\right)$$

Que.

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = 6$$

Now, $\quad I.F. = e^{\int 6 \cdot dy}$

$= e^{6y}$

Now Multiply the I.F in eq^n (i)

$$e^{6y}(xy^2 - x^2) \cdot dx + (3x^2y^2 + x^2y - 2x^3 + y^2)\,dy =$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) = \frac{-1}{2xy}(2y+2y)$$

$$\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right] = \frac{-1}{2xy}\times(4y)$$

$$\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right] = -\frac{2}{x} = f(x)$$

$$I.F = e^{-2\int \frac{1}{x}dx} = e^{-2\log x} = e^{\log x^{-2}}$$

$$I.F = x^{-2} = \frac{1}{x^2}$$

$\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right] = \lambda$

$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) = \lambda$

$e^{\int \lambda \cdot dx}$

$e^{\int \lambda \cdot dy}$

Now, multiply I.F. in eq^n (i)

$$\frac{1}{x^2}(x^2+y^2)dx - \frac{2xy}{x^2}dy = 0$$

$$\left(1+\frac{y^2}{x^2}\right)dx - \frac{2y}{x}\cdot dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{2y}{x^2}, \quad \frac{\partial N}{\partial x} = \frac{2y}{x^2}$$

So, $\boxed{x - \frac{y^2}{x} = c}$ Proved

Que. $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3 + y^2)dy = 0$

$M = xy^2 - x^2, \quad N = 3x^2y^2 + x^2y - 2x^3 + y^2$

$\frac{\partial M}{\partial y} = 2xy, \quad \frac{\partial N}{\partial x} = 6xy^2 + 2xy - 6x^2$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Method- III - $Mx + Ny = 0$

if $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$

$I.f = e^{\int f(x) \cdot dx}$

if $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y)$

$I.f = e^{\int f(y) dy}$

if $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \lambda$

or $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \lambda$

$I.F = e^{\int \lambda \cdot dx}$,

or $I.F = e^{\int \lambda \cdot dy}$

Questions

(1) $(x^2+y^2) \cdot dx - 2xy \cdot dy = 0$ — (i)

$M = x^2+y^2$, $N = -2xy$

$\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = -2y$

$$\left[\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right]$$

So, using (Method- III)

$N = -2xy$

$$P.I. = \frac{1}{4+10+6} \cdot e^{2x} = \frac{1}{20} e^{2x}$$

$$C.I = C.F. + P.I.$$

$$C.I = C_1 e^{-2x} + C_2 e^{-3x} + \frac{1}{20} e^{2x}$$ Ans

(2) $$\frac{d^2y}{dx^2} - 3\cdot\frac{dy}{dx} + 2y = e^{5x}$$

$$(D^2 - 3D + 2)\, y = e^{5x}$$

$$\Rightarrow M^2 - 3M + 2 = 0$$

$$M^2 - 2M - M + 2 = 0$$

$$M(M-2) - 1\cdot(M-2) = 0$$

$$(M-2)\cdot(M-1) = 0$$

$$M = 2 \text{ or } 1$$

$$C.F = C_1 e^{x} + C_2 e^{2x}$$

$$P.I = \frac{1}{D^2 - 3D + 2} \cdot e^{5x} = \frac{1}{25 - 15 + 2} \cdot e^{5x}$$

$$P.I = \frac{1}{12} e^{5x}$$

$$C.I. = C.F + P.I.$$

$$C.I. = C_1 e^{x} + C_2 e^{2x} + \frac{1}{12} e^{5x}$$ Answer

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

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Que. $\frac{d^2y}{dx^2} - 4\cdot\frac{dy}{dx} + y = 0$

Solution- $(D^2 - 4D + 1)y = 0$

Aux. eqn $M^2 - 4M + 1 = 0$

$a=1,\ b=-4,\ c=1$

By quadratic formula-

$$M = \frac{4 \pm \sqrt{16-4}}{2\times1}$$

$$M = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2}$$

$$M = 2 \pm \sqrt{3}$$

then. $C.F. = C_1 \cdot e^{(2-\sqrt{3})x} + C_2 e^{(2+\sqrt{3})x}$ Ans

$C.F. = C_1 \cdot e^{2x} \cdot e^{-\sqrt{3}x} + C_2 e^{2x} \cdot e^{\sqrt{3}x}$

$C.F. = (C_1 \cdot e^{-\sqrt{3}x} + C_2 e^{\sqrt{3}x}) \cdot e^{2x}$

Que. $(D^2 + 5D + 6)y = e^{2x}$

Let $M^2 + 5M + 6 = 0$

$M^2 + 3M + 2M + 6 = 0$

$M(M+3) + 2(M+3) = 0$

$(M+3)\cdot(M+2) = 0$

$M = -3$ and -2

$CF = C_1 e^{-2x} + C_2 e^{-3x}$

then $F(D) = D^2 + 5D + 6$

$$P.I = \frac{1}{D^2+5D+6} \cdot e^{2x}$$

Date - 19/April/2024 $\frac{dy}{dx} = D$

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Unit - Sol^n of II - Ordered Diff. eq^n

(1) When roots are real and unequal $(M_1, M_2, M_3, \text{-----})$.

$$\Rightarrow \quad C\cdot F = C_1 \cdot e^{M_1 x} + C_2 e^{M_2 x} + C_3 e^{M_3 x} + \text{----}$$

(2) ~~When~~ If roots are real and equal

$$\Rightarrow \quad C\cdot F = (C_1 + C_2 x + C_3 x^2 \text{---}) e^{M_1 x}$$

(3) If roots are complex number $(\alpha \pm i\beta)$.

$$\Rightarrow \quad (C_1 \cdot \cos\beta x + C_2 \sin\beta x) e^{\alpha x} = C.F.$$

Que. $\frac{d^2y}{dx^2} - 7 \cdot \frac{dy}{dx} - 44y = 0$

Solution- $(D^2 - 7D - 44)y = 0$

Aux. equation is - $M^2 - \cancel{9}\,7M - 44 = 0$

$M^2 - 11M + 4M - 44 = 0$

$M(M-11) + 4(M-11) = 0$

$(M-11)\cdot(M+4) = 0$

$M = 11, -4$

$$C\cdot F = C_1 e^{M_1 x} + C_2 e^{M_2 x}$$

$$\left[C\cdot F = C_1 e^{-4x} + C_2 e^{11x} \right]$$

* Rule-3. if $\frac{1}{F(D)}$ (Alge)

Then, use explain of for finding P.I.

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$
$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

Que. $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 2y = 5 + 2x$

$(2D^2 + 5D + 2)y = 5 + 2x$

$$P.I. = \frac{1}{2D^2+5D+2} \cdot (5+2x)$$

$$P.I = \frac{1}{\frac{2D^2+5D}{2}+1} \cdot (5+2x)$$

$$P.I. = \frac{1}{2}\left(1 + \frac{2D^2+5D}{2}\right)^{-1} \cdot (5+2x)$$

$$P.I. = \frac{1}{2}\left[1 - \left(\frac{2D^2+5D}{2}\right) + \left(\frac{2D^2+5D}{2}\right)^2 - \cdots\right] \cdot (5+2x)$$

$$P.I. = \frac{1}{2}\left[1 - \frac{5D}{2}\right] \cdot (5+2x)$$

$$P.I. = \frac{1}{2}\left(5 + 2x - \frac{5}{2}D(5+2x)\right) = \frac{(5+2x)}{2} - \frac{5}{2}\frac{(0+2)}{2}$$

$[P.I. = x]$

*

Que.

$D \to D+3$ $\quad D \quad \frac{1}{D} \quad D^2 \quad \frac{1}{D^2}$

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★ Rule-2- if $\frac{1}{F(D)} \sin(ax)$ or $\cos(ax)$

Then, $P.I = \frac{1}{F(a^2)} \cdot \sin(ax)$ or $\cos(ax)$

$D^2 \to -a^2$

(1) $(D^2 + D + 1)y = \sin 2x$

$M^2 + M + 1 = 0$

$M = \frac{-1 \pm \sqrt{1 - 4 \times 1 \times 1}}{2} = \frac{-1 \pm \sqrt{1-3}}{2}$

$M = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3} \cdot i}{2}$

$\Rightarrow C.F = \left(C_1 \cdot \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3} x}{2}\right) e^{-\frac{1}{2}x}$

$P.I. \Rightarrow \frac{1}{D^2 + D + 1} \cdot \sin 2x = \frac{1}{-2^2 + D + 1} \cdot \sin 2x$

$P.I. = \frac{1}{D-3} \sin 2x \Rightarrow \frac{D+3}{D^2-3} \sin 2x$

$P.I. = \frac{(D+3)}{-4-9} \sin 2x$

$= \frac{D+3}{-13} \cdot \sin 2x$

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(3) $(D^2 - 6D + 9)y = 4e^{3x}$

$\cancel{D^2}.\ M^2 - \cancel{6M}\ 6M + 9 = 0$

$M^2 - 3M - 3M + 9 = 0$

$M(M-3) - 3(M-3) = 0$

$(M-3)\cdot(M-3) = 0$

$M = 3$ or 3

$C.F. = \cancel{C_1e^{3x} + C_2e^{3x}}\ (C_1 + C_2x)e^{3x}$

$F(D) = D^2 - 6D + 9$

$P.I. = \frac{1}{D^2 - 6D + 9} \times 4e^{3x}$

$P.I. = \frac{1}{9 - 18 + 9} \cdot 4e^{3x}$ (undef)

$= \frac{1}{(D-3)^2} 4e^{3x}$

$= \frac{1}{(D+3-3)^2} 4e^{3x}$

$= \frac{1}{D^2} 4e^{3x}$

$= \frac{4 \cdot e^{3x}}{3 \cdot 3} = \frac{4}{9} e^{3x}$

Ans $y = (C_1 + C_2x)e^{3x} + \frac{4}{9} e^{3x}$

D → D+3

★ Rule

Then

(1) (D

M

M

⇒

P

.F

$D^2+9+6D+\ldots$

$3x^2$ $6x$ 6

$$P.I. = \frac{1}{-4}\left[1+\frac{D^2}{4}+\left(\frac{D^2}{4}\right)^2+\left(\frac{D^2}{4}\right)^3+\cdots\right](x^3)$$

$$P.I. = \frac{-1}{4}\left[1+\frac{D^2}{4}\right]\cdot(x^3)$$

$$P.I. = -\frac{1}{4}\left[x^3+\frac{1(6x)}{4}\right]$$

$$= -\frac{1}{16}\left[4x^3+6x\right]$$

$$P.I. = -\frac{1}{8}\left[2x^3+3x\right]$$

Que. $(D^2+2D+1)y = x^2 \cdot e^{3x}$

$$P.I. = \frac{1}{D^2+2D+1}\, x^2 \cdot e^{3x}$$

$$= e^{3x}\left[\frac{1}{D^2+2D+1}\right]x^2$$

$$= e^{3x}\left[\frac{1}{D^2+9+6D+2D+6+1}\right]\cdot x^2$$

$$= e^{3x}\left[\frac{1}{D^2+8D+16}\right]x^2$$

$$= \frac{e^{3x}}{16}\left[\frac{1}{1+\frac{D^2+8D}{16}}\right]\cdot x^2$$

$$= \frac{e^{3x}}{16}\left[1-\left(\frac{D^2+8D}{16}\right)+\left(\frac{D^2+8D}{16}\right)^2-\cdots\right]x^2$$

$$= \frac{e^{3x}}{16}\left[1-\frac{D^2}{16}-\frac{8D}{16}+\frac{\cancel{D^4}+64D^2}{256}\right]x^2$$

$$= \frac{e^{3x}}{16}\left[x^2-\frac{1}{16}(2)-\frac{1\times 2x}{2}+\frac{64\times 2}{256}\right]$$

$3^2=-9$

$3x$

$9(-25-1)$ $\frac{-324}{}$

-26

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Que. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin 3x$

$(D^2 - 5D + 6)y = \sin 3x$

$P.I. = \frac{1}{D^2 - 5D + 6} \times \sin 3x$ $\quad D^2 \to -3^2 = -9$

$= \frac{1}{-9 - 5D + 6} \times \sin 3x$

$= \frac{1}{-5D - 3} \cdot \sin 3x = \frac{-5D + 3}{25D^2 - 9} \cdot \sin 3x$

$= \frac{(-5D + 3)\sin 3x}{25 \times (-9) - 9}$

$= \frac{(-5D + 3)\sin 3x}{-234}$

$= \frac{(-5D\sin 3x + 3\sin 3x)}{-234}$

$= \frac{5 \times 3\cos 3x - 3\sin 3x}{\cancel{216}\ 234}$

$P.I. = \frac{3(5\cos 3x - \sin 3x)}{234}$

$P.I. = \frac{5\cos 3x - \sin 3x}{78}$

Que. $\frac{d^2y}{dx^2} - 4y = x^3$

$(D^2 - 4)y = x^3$

$P.I. = \frac{1}{D^2 - 4} \cdot (x^3) = \frac{1}{4\left(\frac{D^2}{4} - 1\right)} (x^3)$

$P.I. = \frac{1}{-4}\left(1 - \frac{D^2}{4}\right) \cdot (x^3)$

★ Rule-4. if $\frac{1}{F(D)} e^{ax}$ (sinbx or cosbx or Alge)

Then $D \to (D+a)$

Que. $(D^2 - 4D + 3)y = e^{2x} \cdot \sin 3x$

$$P.I. = \frac{1}{F(D)} e^{2x} \sin 3x$$

$$= \frac{1}{D^2 - 4D + 3} e^{2x} \cdot \sin 3x$$

$$= e^{2x} \left[\frac{1}{(D+2)^2 - 4(D+2) + 3} \right] \cdot \sin 3x$$

$$= e^{2x} \left[\frac{1}{D^2 + 4 + 4D - 4D - 8 + 3} \right] \sin 3x$$

$$= e^{2x} \left[\frac{1}{D^2 - 1} \right] \cdot \sin 3x$$

$$\Rightarrow e^{2x} \left[\frac{1}{-9-1} \right] \cdot \sin 3x = \frac{e^{2x} \sin 3x}{-10}$$

$D = \frac{d}{dx}(x^4)$ $\quad (1+x)^{-1} = 1 - x + x^2 - x^3 \dots$

$D^2 = \frac{d^2}{dx^2}$ $\quad (1-x)^{-1}$

$D^4 + 4D^2 + 4D^3$

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$$P.I_2 = \frac{1}{F(D)} \cdot x^2 = \frac{1}{D^2+2D+1} \cdot x^2$$

$$= \frac{1}{1+(D^2+2D)} \cdot x^2$$

$$P.I_2 = (1+(D^2+2D))^{-1} \cdot x^2$$

$$P.I_2 = \left[1-(D^2+2D) + (D^2+2D)^2 - (D^2+2D)^3 \dots\right] \cdot x^2$$

$$P.I_2 = [1 - D^2 - 2D + 4D^2] \cdot x^2$$

$$= x^2 - D^2 \cdot x^2 - 2D \cdot x^2 + 4D^2 x^2$$

$$= x^2 + 3D^2 x^2 - 2Dx^2$$

$$= x^2 + 3 \times 2 - 2 \times 2x$$

$$P.I_2 = x^2 - 4x + 6$$

$$P.I_3 = \frac{1}{F(D)} \cdot \sin x$$

$$= \frac{1}{D^2+2D+1} \cdot \sin x \qquad D^2 \to -(1)^2$$

$$D^2 \to -1$$

$$= \frac{1}{-1+2D+1} \cdot \sin x$$

$$= \frac{1}{2D} \cdot \sin x$$

$$P.I_3 = \frac{1}{2}[-\cos x] = -\frac{\cos x}{2}$$

$$C.I = C.F + P.I$$

$$= (C_1 + C_2 x)e^{-x} + \frac{1}{4}e^{x} + x^2 - 4x + 6 - \frac{\cos x}{2}$$

$(D^2)^3 + (2D)^3 + 3 \times D^2 \times 2D(\ldots)$

$D^2 + 8D^3 + 8D^3$

$D = \frac{d}{dx}$

$D^2 = \frac{d^2}{dx^2}$

Que.

$$P.I. = \frac{e^{3x}}{16}\left[x^2 - \frac{1}{8} - x\right]$$

$$\cancel{P.I. = \frac{e^{3x}}{128}[8x^2 - 8x - 1]}$$

$$P.I = \frac{e^{3x}}{16}\left[x^2 - \frac{1}{8} - x + \frac{1}{2}\right]$$

$$= \frac{e^{3x}}{16}\left[x^2 - x + \frac{1}{2} - \frac{1}{8}\right]$$

$$= \frac{e^{3x}}{16}\left[x^2 - x + \frac{4-1}{8}\right]$$

$$= \frac{e^{3x}}{16}\left[x^2 - x + \frac{3}{8}\right]$$ Answer

Que. $(D^2 + 2D + 1)y = e^x + x^2 - \sin x$

$\Rightarrow$ $M^2 + 2M + 1 = 0$

$M^2 + M + M + 1 = 0$

$M(M+1) + 1\cdot(M+1) = 0$

$(M+1)\cdot(M+1) = 0$

$M = -1$ or -1

$$C\cdot F = (C_1 + C_2 x)\cdot e^{M_1 x}$$

$$= (C_1 + C_2 x)\, e^{-x}$$

$$P\cdot I_1 = \frac{1}{F(D)}\cdot e^x = \frac{1}{D^2 + 2D + 1} \times e^x$$

$$P\cdot I_1 = \frac{1}{1+2+1}\cdot e^x = \frac{1}{4}\cdot e^x$$

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Que.

$$P.I. = \frac{e^{3x}}{16}\left[x^2 - \frac{1}{8} - x\right]$$

$$\cancel{P.I. = \frac{e^{3x}}{128}\left[8x^2 - 8x - 1\right]}$$

$$P.I = \frac{e^{3x}}{16}\left[x^2 - \frac{1}{8} - x + \frac{1}{2}\right]$$

$$= \frac{e^{3x}}{16}\left[x^2 - x + \frac{1}{2} - \frac{1}{8}\right]$$

$$= \frac{e^{3x}}{16}\left[x^2 - x + \frac{4-1}{8}\right]$$

$$= \frac{e^{3x}}{16}\left[x^2 - x + \frac{3}{8}\right] \quad \text{Answer}$$

(1) $(D^2-2D+5)y = e^{2x}\cdot x^2$

$$P.I. = \frac{1}{D^2-2D+5}\cdot e^{2x}\,x^2$$

$$= e^{2x}\cdot\left[\frac{1}{(D+2)^2-2(D+2)+5}\cdot x^2\right]$$

$$= e^{2x}\left[\frac{1}{D^2+4+4D-2D-4+5}\cdot x^2\right]$$

$$= e^{2x}\cdot\left[\frac{1}{D^2+2D+5}\cdot x^2\right]$$

$$= \frac{e^{2x}}{5}\left[\frac{1}{\left(1+\frac{D^2+2D}{5}\right)}\cdot x^2\right]$$

$$= \frac{e^{2x}}{5}\left[1-\left(\frac{D^2+2D}{5}\right)+\left(\frac{D^2+2D}{5}\right)^2-\left(\frac{D^2+2D}{5}\right)^3\right]x^2$$

$$= \frac{e^{2x}}{5}\left[1-\frac{D^2}{5}-\frac{2D}{5}+\frac{4D^2}{25}\right]\cdot x^2$$

$$= \frac{e^{2x}}{5}\left[x^2-\frac{2}{5}-\frac{2\times 2x}{5}+\frac{4}{25}\times 2\right]$$

$$= \frac{e^{2x}}{5}\left[x^2-\frac{4x}{5}+\frac{8}{25}-\frac{2}{5}\right]$$

$$= \frac{e^{2x}}{5}\left[x^2-\frac{4x}{5}-\frac{2}{25}\right]$$ Ans

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* Rule-6- $P.I. = \frac{1}{F(D)} x(\sin ax \text{ or } \cos ax)$

or $P.I. = \frac{1}{F(D)} \cdot x(\text{funct})$

$P.I. = x\frac{1}{F(D)} \cdot (\text{func}) - \left[\frac{d}{dD}(\text{funcD})\right](\text{func})$

(1) $\frac{d^2y}{dx^2} + 2\cdot\frac{dy}{dx} + y = x\cdot\cos x$

$P.I. = \frac{1}{F(D)} \cdot x\cdot\cos x$

$= x\cdot\frac{1}{D^2+2D+1} x\cos x - \left[\frac{d}{dD}(D^2+2D+1)\right]\cdot\cos x$

$= x\cdot\frac{1}{-1+2D+1}\cdot\cos x + \left[\frac{2D+2}{(D^2+2D+1)^2}\right]\cdot\cos x$

$= x\cdot\frac{1}{2D}\cdot\cos x + \left[\frac{2D+2}{(-1+2D+1)^2}\cdot\cos x\right]$

$= \frac{x}{2}\cdot\frac{1}{D}\cdot\cos x + \left[\frac{2D+2}{4D^2}\cdot\cos x\right]$

$= \frac{x}{2}[\sin x] + \frac{1}{2}\left[\frac{D+1}{D^2}\right]\cos x$

$= \frac{x}{2}\sin x + \frac{1}{2}\left[-D-1\right]\cdot\cos x$

$= \frac{x}{2}\sin x + \frac{1}{2}\sin x - \frac{\cos x}{2}$ Answer

x^2