Shape Optimisation with PDE Constraints

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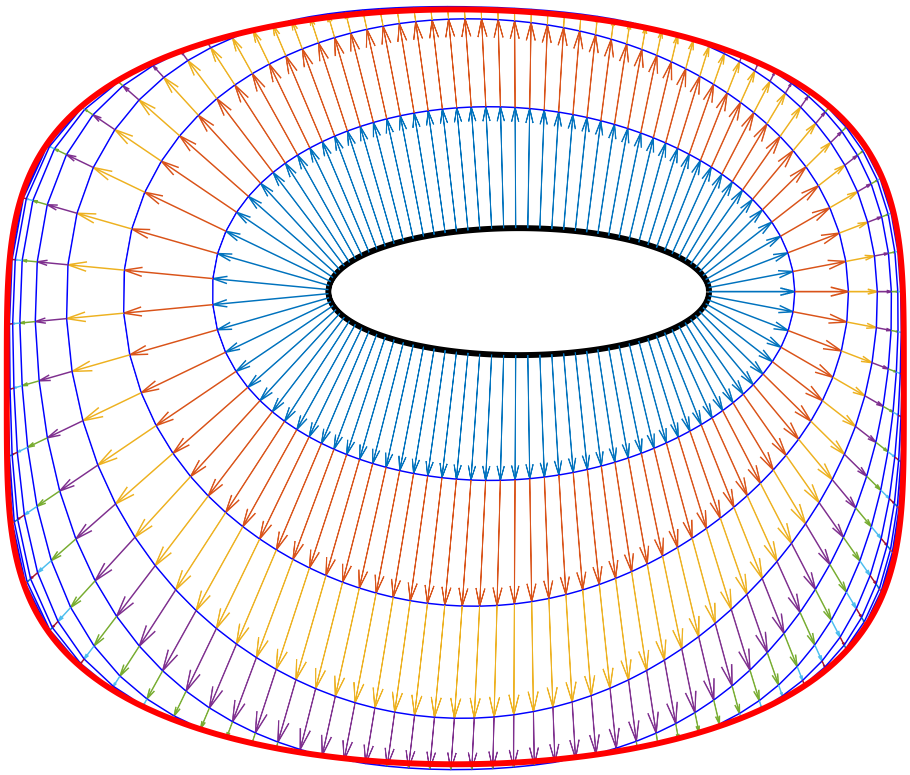


Abstract

Shape optimisation is typically concerned with finding the shape which is optimal in the sense that it minimises a certain cost functional while satisfying given constraints. Often we find that the functional being solved depends on the solution of a PDE constraint.

In this report we briefly discuss unconstrained optimisation methods before focussing on a shape optimisation test case that, despite its relative simplicity, exhibits an unexpected and interesting behaviour: solving this problem via Newton's method with a *truncated* second shape derivative performs better than using full second order information. We will perform numerical and theoretical investigations to try to shed light on this unexpected behaviour.

Contents

1 Unconstrained Optimisation

We first recap on some iterative methods for solving unconstrained optimisation problems.

1.1 Setting the Scene

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Suppose we want to find a value x^* such that f is minimal, i.e

$$x^* \in \arg\min_x f(x).$$

We can find such a value by means of an iterative algorithm. That is, given an initial guess $x^{(0)}$ for x^*, we construct a *minimising sequence* $(x^{(k)})$ of points such that

$$f(x^{(k)}) \longrightarrow f(x^*) \quad \text{as } k \to \infty.$$

Often we only require an approximate solution, so we can reduce computational effort by choosing a tolerance $\varepsilon > 0$ such that when $f(x^{(k)}) - f(x^*) \leqslant \varepsilon$, we terminate the algorithm.

1.2 Descent Methods

The minimising sequences we are going to discuss in this section are all of the form

$$x^{(k+1)} = x^{(k)} + t^{(k)} \Delta x^{(k)},$$

where

- $\Delta x^{(k)}$ represents a *search direction* - which direction we travel in to approach x^*,
- $t^{(k)} \geqslant 0$ represents a *step length* - how far we travel along the vector $\Delta x^{(k)}$.

Recall that our aim is to minimise f, so we would like the sequence to have the property

$$f(x^{(k+1)}) < f(x^{(k)}) \tag{1}$$

for all k, unless $x^{(k)}$ is optimal. If f is differentiable, we can Taylor expand up to first-order and write

$$f(x^{(k+1)}) = f(x^{(k)} + t^{(k)} \Delta x^{(k)}) \approx f(x^{(k)}) + t^{(k)} \mathrm{d}f(x^{(k)}, \Delta x^{(k)}),$$

where $\mathrm{d}f$ denotes the *directional derivative*

$$\mathrm{d}f(x, v) := \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}.$$

Hence, if $x^{(k)}$ is not optimal (so that $t^{(k)} \neq 0$), we can ensure (1) holds up to first-order if

$$\mathrm{d}f(x^{(k)}, \Delta x^{(k)}) < 0. \tag{2}$$

A search direction $\Delta x^{(k)}$ such that (2) holds is called a *descent direction.*

A generic descent method then goes as follows:

Given starting point x
repeat
  (1) Determine *descent direction*, Δx
  (2) Choose *step size*, $t > 0$
  (3) Set $x \leftarrow x + t\Delta x$
until *stopping criterion satisfied*

At the moment this is rather vague. We have a few questions to answer:

- How do we choose the step size?
- What is the descent direction?
- When do we stop the algorithm?

1.3 Line Search

To answer the first question, we discuss two methods of deciding the step size, called *exact line search (ELS)* and *backtracking line search (BLS)*. Here we assume that Δx is a descent direction.

In ELS, we choose t such that f is minimised along the ray $\{x + t\Delta x : t \geqslant 0\}$, i.e

$$t = \arg\min_{s \geqslant 0} f(x + s\Delta x).$$

It may be difficult to find t exactly, so it is generally cost effective to choose a t value such that f is only approximately minimised along the ray. For this reason we may choose to apply BLS, which generates the step size by the following algorithm:

Algorithm 1: Backtracking Line Search

```
1 Given descent direction Δx, α ∈ (0, 0.5), β ∈ (0, 1)
2 t := 1
3 while f(x + tΔx) > f(x) + αt df(x, Δx) do
4 |  Set t ← βt
5 end
```

The resulting value of t is chosen as the step size. The line search is called "backtracking" as it starts with unit step size, then reduces it by the factor β until the stopping criterion

$$f(x + t\Delta x) > f(x) + \alpha t \, \mathrm{d}f(x, \Delta x)$$

holds. Since Δx is a descent direction, we have $\mathrm{d}f(x, \Delta x) < 0$, so

$$f(x + t\Delta x) \approx f(x) + t \, \mathrm{d}f(x, \Delta x) < f(x) + \alpha t \, \mathrm{d}f(x, \Delta x)$$

for small enough t, demonstrating that BLS eventually terminates. The choice of parameters α, β has a noticeable but not dramatic effect on convergence (see [1]); ELS can sometimes improve convergence rate but not dramatically. It is generally not worth the computational effort of using ELS.

1.4 Steepest Descent

We can find a descent direction by analysing the Taylor expansion of $f(x + \Delta x)$ about x up to first-order. The approximation is

$$f(x + \Delta x) \approx f(x) + \mathrm{d}f(x, \Delta x),$$

so we find the direction of *steepest descent* when we minimise the directional derivative. We define a *normalised steepest descent direction* Δx_{nsd} as a descent direction of unit length that minimises $\mathrm{d}f(x, \Delta x)$, i.e

$$\Delta x_{\mathrm{nsd}} \in \arg\min_{||\Delta x||=1} \mathrm{d}f(x, \Delta x)$$

for some norm $|| \cdot ||$ on $\mathbb{R}^n$.

To piece together the *Steepest Descent* method, it will be useful to introduce the concept of inner products and gradients, particularly when it comes to defining stopping criteria. Let V be a real vector space. An *inner product* $(\cdot,\cdot) : V^2 \to \mathbb{R}$ is a function that satisfies the following properties:

- *Linearity:* $(\alpha u + \beta v, w) = \alpha(u, w) + \beta(v, w) \quad \forall u, v, w \in V,\ \alpha, \beta \in \mathbb{R}$
- *Symmetry:* $(u, v) = (v, u) \quad \forall u, v \in V$
- *Positive definitenesss:* $(u, u) \geqslant 0 \quad \forall u \in V, \quad (u, u) = 0 \iff u = 0.$

It follows from the first two properties that $(\cdot,\cdot)$ is *bilinear.*

We call $\nabla f(x)$ the *gradient* of f at x with respect to an inner product $(\cdot,\cdot)$ if

$$(\nabla f(x), v) = \mathrm{d}f(x, v) \quad \forall v.$$

We occasionally write $\nabla_{(\cdot,\cdot)} f(x)$ to make it clear what the associated inner product is.

Lemma 1.1: If a norm $||\cdot||$ is induced by an inner product $(\cdot,\cdot)$, i.e $||v|| = \sqrt{(v, v)}$, then

$$\Delta x_{\text{nsd}} = -\frac{\nabla f(x)}{||\nabla f(x)||}. \tag{3}$$

Proof: We use the method of Lagrange multipliers. Let

$$L(\Delta x, \lambda) := \mathrm{d}f(x, \Delta x) + \frac{\lambda}{2}(||\Delta x||^2 - 1).$$

Then by definition of Δx_{nsd},

$$\begin{aligned} & \partial_{\Delta x} L(\Delta x_{\text{nsd}}, v) = \mathrm{d}f(x, v) + \lambda(\Delta x_{\text{nsd}}, v) = 0 \quad \forall v \\ \iff \quad & (\lambda \Delta x_{\text{nsd}}, v) = -\mathrm{d}f(x, v) \quad \forall v \\ \iff \quad & \lambda \Delta x_{\text{nsd}} = -\nabla f(x). \end{aligned}$$

Taking norms of both sides, and using the fact that $||\Delta x_{\text{nsd}}|| = 1$, we obtain $|\lambda| = ||\nabla f(x)||$. The result follows, using the fact that Δx_{nsd} is a steepest descent direction.

$\square$

Geometrically speaking, we see that Δx_{nsd} is a step in the unit ball of $||\cdot||$ which extends farthest in the direction of $-\nabla f(x)$. It is more convenient in practice to use an unnormalised descent direction

$$\Delta x_{\text{sd}} := ||\nabla f(x)||_* \Delta x_{\text{nsd}},$$

where $||\cdot||_*$ denotes the *dual norm*

$$||v||_* := \sup_{||x|| \leqslant 1} v^T x.$$

We find that Δx_{sd} is simply the unnormalised version of (3), that is

$$\Delta x_{\text{sd}} = -\nabla_{(\cdot,\cdot)} f(x),$$

and we arrive at the *Steepest Descent* method, which uses Δx_{sd} as a descent direction:

Algorithm 2: Steepest Descent

Given starting point x
repeat
    (1) Set $\Delta x \leftarrow \Delta x_{\text{sd}}$
    (2) Choose t via ELS/BLS
    (3) Set $x \leftarrow x + t\Delta x$
until *stopping criterion satisfied*

To define a stopping criterion, recall that when x^* is optimal we have

$$\mathrm{d}f(x^*, v) = 0 \quad \forall v,$$

so for a sufficiently smooth f, we expect the gradient to be small for $x \approx x^*$. For this reason, we can define a stopping criterion of the form

$$||\nabla f(x)|| \leqslant \varepsilon$$

for some chosen small $\varepsilon > 0$. Most practical uses of Steepest Descent check the stopping criterion after the first step.

1.5 Newton's Method

Which inner product do we use to define Δx_{sd}? An interesting choice, which forms the basis of *Newton's Method*, is to consider the inner product induced by the second derivative

$$\mathrm{d}^2 f(x, u, v) := \lim_{t \to 0} \frac{\mathrm{d}f(x + tv, u) - \mathrm{d}f(x, u)}{t}.$$

If it exists, we can extend our Taylor approximation up to second-order to get

$$f(x + \Delta x) \approx f(x) + \mathrm{d}f(x, \Delta x) + \frac{1}{2}\mathrm{d}^2 f(x, \Delta x, \Delta x) =: g(\Delta x).$$

If $\mathrm{d}^2 f(x, \cdot, \cdot)$ is positive definite then g is *strictly convex**, thus has a unique minimiser Δx_{nt}, called the *Newton step*. Since the second derivative exists, $\mathrm{d}^2 f(x, \cdot, \cdot)$ is also symmetric and linear, hence an inner product. By minimality, the Newton step must satisfy

$$\begin{aligned} & \mathrm{d}g(\Delta x_{\mathrm{nt}}, v) = 0 \quad \forall v \\ \iff \quad & \mathrm{d}f(x, v) + \mathrm{d}^2(x, \Delta x_{\mathrm{nt}}, v) = 0 \quad \forall v \\ \iff \quad & \mathrm{d}^2(x, \Delta x_{\mathrm{nt}}, v) = -\mathrm{d}f(x, v) \quad \forall v \end{aligned}$$

so Δx_{nt} is the negative gradient with respect to the inner product induced by $\mathrm{d}^2 f$, that is,

$$\Delta x_{\mathrm{nt}} = -\nabla_{\mathrm{d}^2 f(x, \cdot, \cdot)} f(x).$$

If f is quadratic, then $f(\Delta x_{\mathrm{nt}}) = g(\Delta x)$ exactly, so $x + \Delta x_{\mathrm{nt}}$ is the *exact* minimiser of f. Hence it should also be a very good estimate for x^* if f is approximately quadratic. Since f is assumed to be twice continuously differentiable, the quadratic model of f is very accurate for x near x^*, so $x + \Delta x_{\mathrm{nt}}$ is a very good estimate for x^*.

Before we introduce *Newton's method*, we will construct a stopping criterion using the directional derivative, as demonstrated in the following lemma:

Lemma 1.2: $\frac{1}{2}\mathrm{d}f(x, \Delta x_{\mathrm{nt}})$ estimates the error $f(x) - f(x^*)$, based on the quadratic approximation of f.

Proof: The error based on the quadratic approximation is

$$\begin{aligned} f(x) - \inf_y g(y) = \; f(x) - g(\Delta x_{\mathrm{nt}}) &= f(x) - \left(f(x) + \underbrace{\mathrm{d}f(x, \Delta x_{\mathrm{nt}})}_{=0} + \frac{1}{2}\mathrm{d}^2 f(x, \Delta x_{\mathrm{nt}}, \Delta x_{\mathrm{nt}}) \right) \\ &= f(x) - \left(f(x) - \frac{1}{2}\mathrm{d}f(x, \Delta x_{\mathrm{nt}}) \right) \\ &= \frac{1}{2}\mathrm{d}f(x, \Delta x_{\mathrm{nt}}). \end{aligned}$$

□

*A function g is said to be *strictly convex* if the line segment connecting any two distinct points on the surface of g lies strictly above g, except at the endpoints.

We use this as a stopping criterion for Newton's Method, which goes as follows:

Algorithm 3: Newton's Method

Given starting point x, tolerance $\varepsilon > 0$
repeat
  (1) Set $\Delta x \leftarrow \Delta x_{\mathrm{nt}}$
  (2) **Stop** if $\mathrm{d}f(x, \Delta x)/2 \leqslant \varepsilon$
  (3) Choose t via BLS
  (4) Set $x \leftarrow x + t\Delta x$

Observe that this is more or less a general descent method, with the difference that the stopping criterion is checked after computing Δx_{nt}, rather than after updating the value of x.

Newton's Method has *quadratic convergence* near x^* [1]. Roughly speaking, this means that the number of correct digits doubles after each iteration. Newton's Method also scales with the size of the problem: its performance in $\mathbb{R}^{10000}$ is similar to problems in $\mathbb{R}^{10}$ say, with only a modest increase in the number of iterations.

A pitfall of Newton's Method is the cost of computing Δx_{nt}, which requires solving a system of linear equations. *Quasi-Newton* methods require less cost to form the search direction, sharing some advantages of Newton's Method such as rapid convergence near x^*, but we will not discuss such methods here.

2 Introduction to Shape Optimisation

In the previous section, we had a function $f : \mathbb{R}^n \to \mathbb{R}$ and we used its derivatives to find the points x^* which minimised $f(x)$. What if we were instead given a cost functional $\mathcal{J}[\Omega]$, with the aim of minimising $\mathcal{J}$ over bounded domains (shapes) Ω? We would need a notion of *shape differentiation.*

In this section we define the *shape derivative*. We will ultimately use such derivatives to find a domain Ω^* in a collection of admissible shapes $\mathcal{U}_{\text{ad}}$ which minimises a given cost functional $\mathcal{J}$. For simplicity, we restrict ourselves to functionals of the form

$$\mathcal{J}[\Omega] := \int_\Omega f \, \mathrm{d}x.$$

In the style of Section 1.1, we write

$$\Omega^* \in \underset{\Omega \in \mathcal{U}_{\text{ad}}}{\arg\min} \, \mathcal{J}[\Omega].$$

2.1 Defining the Shape Derivative

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently smooth vector field. Then

$$\mathcal{J}[T(\Omega)] = \int_{T(\Omega)} f \, \mathrm{d}x = \int_\Omega (f \circ T) |\det \mathbf{D}T| \, \mathrm{d}x,$$

where $\mathbf{D}T$ denotes the *Jacobian* of T. Note that $T(\Omega) \neq \Omega$ in general, and similarly $\mathcal{J}[T(\Omega)] \neq \mathcal{J}[\Omega]$.

Furthermore let $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, and define a duality pairing

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}T} \mathcal{J}[T(\Omega)], \mathcal{V} \right\rangle := \lim_{t \to 0} \frac{\mathcal{J}[(T + t\mathcal{V})(\Omega)] - \mathcal{J}[T(\Omega)]}{t}.$$

We then define the *Eulerian (shape) derivative* of $\mathcal{J}$ in the direction $\mathcal{V}$ as

$$\mathrm{d}\mathcal{J}(\Omega, \mathcal{V}) := \left\langle \frac{\mathrm{d}}{\mathrm{d}T} \mathcal{J}[T(\Omega)], \mathcal{V} \right\rangle \Big|_{T=I},$$

where I is the identity map $I(\Omega) = \Omega$. Therefore, we can write

$$\mathrm{d}\mathcal{J}(\Omega, \mathcal{V}) = \left\langle \frac{\mathrm{d}}{\mathrm{d}T} \left(\int_\Omega (f \circ T) |\det \mathbf{D}T| \, \mathrm{d}x \right), \mathcal{V} \right\rangle \Big|_{T=I} = \int_\Omega \left\langle \frac{\mathrm{d}}{\mathrm{d}T} ((f \circ T) \det \mathbf{D}T), \mathcal{V} \right\rangle \Big|_{T=I} \mathrm{d}x.$$

Ideally we want $\mathrm{d}\mathcal{J}$ to exist for all $\mathcal{V}$, so we say that $\mathcal{J}$ is *shape differentiable* at Ω if the map

$$\mathcal{V} \mapsto \mathrm{d}\mathcal{J}(\Omega, \mathcal{V})$$

is linear and bounded on $C^1(\mathbb{R}^n, \mathbb{R}^n)$, the set of continuously differentiable maps from $\mathbb{R}^n$ to $\mathbb{R}^n$.

2.2 Computing $\mathrm{d}\mathcal{J}(\Omega, \mathcal{V})$

Using the above, we prove two formulae for the shape derivative $\mathrm{d}\mathcal{J}$, which will help us to compute it in practice.

Theorem 2.1: Let $\nabla \cdot \mathcal{V}$ denote the *divergence* of $\mathcal{V}$. Then

$$\mathrm{d}\mathcal{J}(\Omega, \mathcal{V}) = \int_\Omega \nabla f \cdot \mathcal{V} + f \nabla \cdot \mathcal{V} \, \mathrm{d}x.$$

Proof: By the product rule,

$$\mathrm{d}\mathcal{J}(\Omega, \mathcal{V}) = \int_\Omega \left\{ \left\langle \frac{\mathrm{d}}{\mathrm{d}T} (f \circ T), \mathcal{V} \right\rangle \det \mathbf{D}T + \left\langle \frac{\mathrm{d}}{\mathrm{d}T} \det \mathbf{D}T, \mathcal{V} \right\rangle (f \circ T) \right\} \Big|_{T=I} \mathrm{d}x.$$

Considering the first term, we have

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}T}(f \circ T), \mathcal{V} \right\rangle = (\nabla f \circ T) \cdot \mathcal{V},$$

hence

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}T}(f \circ T), \mathcal{V} \right\rangle \det \mathbf{D}T\Big|_{T=I} = \nabla f \cdot \mathcal{V}.$$

Considering the second term and using Jacobi's formula [2], we have

$$\begin{aligned}
\left\langle \frac{\mathrm{d}}{\mathrm{d}T} \det \mathbf{D}T, \mathcal{V} \right\rangle &= \lim_{t \to 0} \frac{\det \mathbf{D}(T + t\mathcal{V}) - \det \mathbf{D}T}{t} \\
&= \lim_{t \to 0} \frac{\det(\mathbf{D}T + t\mathbf{D}\mathcal{V}) - \det \mathbf{D}T}{t} \\
&= \lim_{t \to 0} \frac{(\det \mathbf{D}T + t \operatorname{Tr}(\operatorname{adj}(\mathbf{D}T)\mathbf{D}\mathcal{V}) + O(t^2)) - \det \mathbf{D}T}{t} = \operatorname{Tr}(\operatorname{adj}(\mathbf{D}T)\mathbf{D}\mathcal{V}),
\end{aligned}$$

where $\operatorname{Tr}(A), \operatorname{adj}(A)$ denote the *trace* and *adjugate*† of A respectively. Hence

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}T} \det \mathbf{D}T, \mathcal{V} \right\rangle (f \circ T)\Big|_{T=I} = \operatorname{Tr}(\operatorname{adj}(\mathrm{Id})\mathbf{D}\mathcal{V})f = \operatorname{Tr}(\mathbf{D}\mathcal{V})f = f\nabla \cdot \mathcal{V}$$

since $\operatorname{adj}(\mathrm{Id}) = \mathrm{Id}$ is the identity matrix, and the result follows. □

Theorem 2.2: The shape derivative is equal to the surface integral

$$\mathrm{d}\mathcal{J}(\Omega, \mathcal{V}) = \int_{\partial\Omega} f(\mathcal{V} \cdot n) \, \mathrm{d}S,$$

where n is the outward pointing unit normal of the boundary $\partial\Omega$.

Proof: Using the Divergence Theorem,

$$\begin{aligned}
\mathrm{d}\mathcal{J}(\Omega, \mathcal{V}) &= \int_{\Omega} \nabla f \cdot \mathcal{V} + f\nabla \cdot \mathcal{V} \, \mathrm{d}x \\
&= \int_{\Omega} \nabla \cdot (f\mathcal{V}) \, \mathrm{d}x \\
&= \int_{\partial\Omega} (f\mathcal{V}) \cdot n \, \mathrm{d}S
\end{aligned}$$

where the second equality follows from vector calculus. The result follows. □

2.3 Descent Methods for Shapes

To construct a Steepest Descent algorithm for shapes, we need a minimising sequence of shapes. But how do we define the next shape in the sequence, in a similar manner to Section 1.2? We could define the shape by its boundary, then *add* shape boundaries together by taking their parametrisations and summing component-wise. It then makes sense to write

$$\partial\Omega^{(k+1)} = \partial\Omega^{(k)} + t^{(k)} \Delta\partial\Omega^{(k)},$$

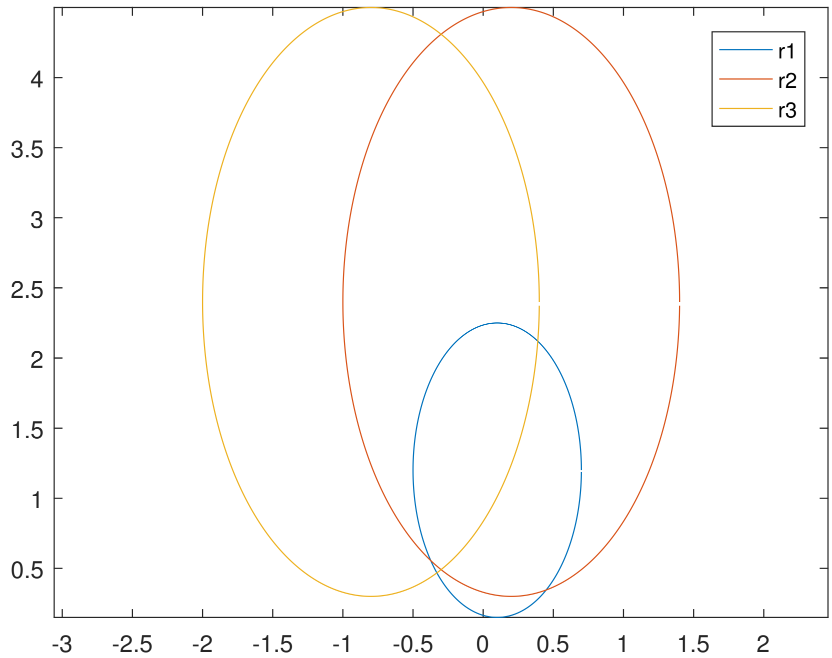
where

- $\Delta\partial\Omega^{(k)}$ represents a *shape search direction*,
- $t^{(k)} \geqslant 0$ represents a *step length.*

†For a matrix A, the *adjugate* of A is the transpose of the *cofactor matrix* C, where $c_{ij} = (-1)^{i+j} \det A_{ij}$, and A_{ij} is the matrix A with row i and column j removed. For an invertible matrix, this is simply $\operatorname{adj} A = (\det A)A^{-1}$.

In MATLAB, we can illustrate this with the help of the *Chebfun* package. For a 2-dimensional problem, define the boundary $\partial\Omega$ as being in the complex plane, and parametrise as follows:

```
r = @(t) 0.1+1.2i + 0.6*cos(t) + 1.05*1i*sin(t);
t = linspace(0,2*pi);
r1 = chebfun(@(t) r(t), [0, 2*pi], 'trig'); %Parametrised boundary
r2 = 2*r1; %Scaled boundary
r3 = r2-[1+0i]; %Scaled and shifted boundary
```



Plotting **r1**, **r2** and **r3** gives the (intuitive) figure above.

We can find a suitable shape search direction using the gradient definition in Section 1.4. If the shape derivative is linear with respect to $\mathcal{V}$, there is a unique element of $\nabla\mathcal{J} \in L^2(\partial\Omega)$ [3] and

$$\mathrm{d}\mathcal{J}(\Omega, \mathcal{V}) = (\nabla\mathcal{J}, \mathcal{V})_{\partial\Omega} := \int_{\partial\Omega} \nabla\mathcal{J} \cdot \mathcal{V} \,\mathrm{d}S,$$

where $\nabla\mathcal{J}$ is the *shape gradient* of $\mathcal{J}$, and $(\cdot,\cdot)_{\partial\Omega}$ is the L^2-inner product on $\partial\Omega$.

Using Theorem 2.2, we deduce that

$$\nabla\mathcal{J}\Big|_{\partial\Omega} = fn,$$

so in line with the Steepest Descent Method, we now choose our shape search direction to be the negative gradient

$$\Delta\partial\Omega := -\nabla\mathcal{J}\Big|_{\partial\Omega} = -fn.$$

Note that if $f \neq 0$ on $\partial\Omega$, we have

$$\mathrm{d}\mathcal{J}(\Omega, \Delta\partial\Omega) = -\int_{\partial\Omega} f(fn \cdot n)\,\mathrm{d}S = -\int_{\partial\Omega} f^2 \,\mathrm{d}S < 0, \tag{4}$$

since n is a unit normal, so $\Delta\partial\Omega$ is indeed a descent direction.

From (4), we deduce that the shape derivative vanishes when f vanishes on $\partial\Omega$. This intuitively makes sense; to minimise $\mathcal{J}$, we want to integrate over the shape $\Omega^* = \{x : f(x) \leqslant 0\}$ - any points outside of Ω^* can only increase the value of $\mathcal{J}$.

Example 2.1: Take $f : \mathbb{R}^2 \to \mathbb{R}$ to be the function defined by

$$f(x) = x_1^2 + x_2^2 - 1.$$

We know that $\mathcal{J}(\Omega)$ is minimised when integrated over the domain $\Omega = \{x : x_1^2 + x_2^2 \leqslant 1\}$.

We implement a basic shape optimisation method in MATLAB, with initial guess

$$\Omega^{(0)} := \left\{ x : \frac{(x_1 - 0.2)^2}{0.6^2} + \frac{(x_2 - 0.3)^2}{0.2^2} \leqslant 1 \right\},$$

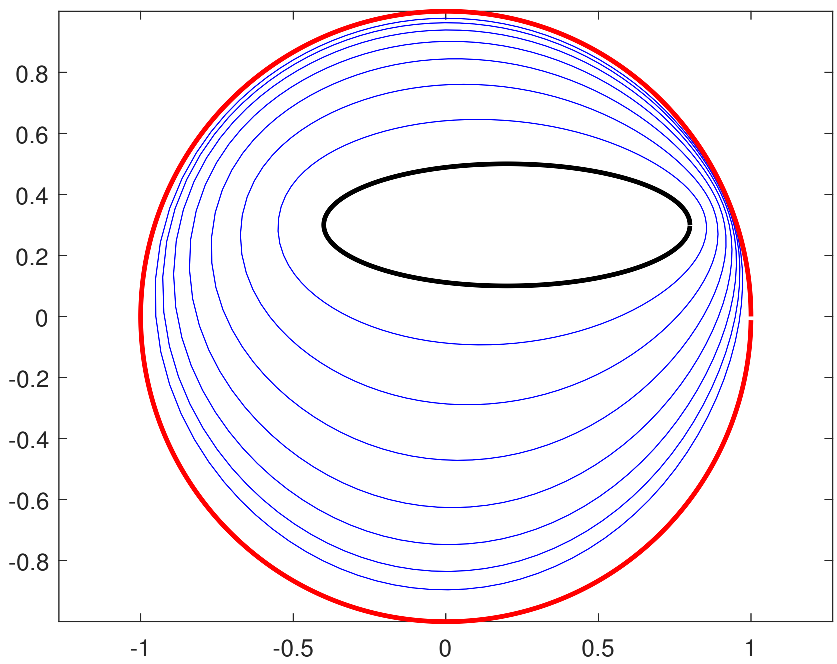
shape search direction $\Delta\partial\Omega$ and fixed step size $t = 0.2$.

```
g_ = @(t) [0.2+0.3i] + 0.6*cos(t) + 0.2*1i*sin(t);
t = linspace(0,2*pi);
g = chebfun(@(t) g_(t), [0, 2*pi], 'trig'); %Initial boundary
init = plot(g, 'k') %Plot of initial boundary
set(init,'LineWidth',2);
dg = diff(g); n_ = -1i*dg;
n = n_./abs(n_); %Unit normal to boundary

f_ = @(x,y) x.^2 + y.^2 - 1;
f = chebfun2(@(x,y) f_(x,y), [-5 5 -5 5]); %Integrand, f

for k=1:7
    hold on
    fn = chebfun(@(t) n(t).*f(real(g(t)), imag(g(t))), [0 2*pi]);
    g = g - 0.2*fn; %Updated boundary
    plot(real(g(t)), imag(g(t)), 'b')
end

exact = fimplicit(f_, 'r')
set(exact,'LineWidth',2);
axis equal
```



The above figure shows the first 7 iterations $\partial\Omega^{(k)}$ in blue, together with the initial shape boundary $\partial\Omega^{(0)}$ in black and the optimum shape boundary $\partial\Omega^{*}$ in red.

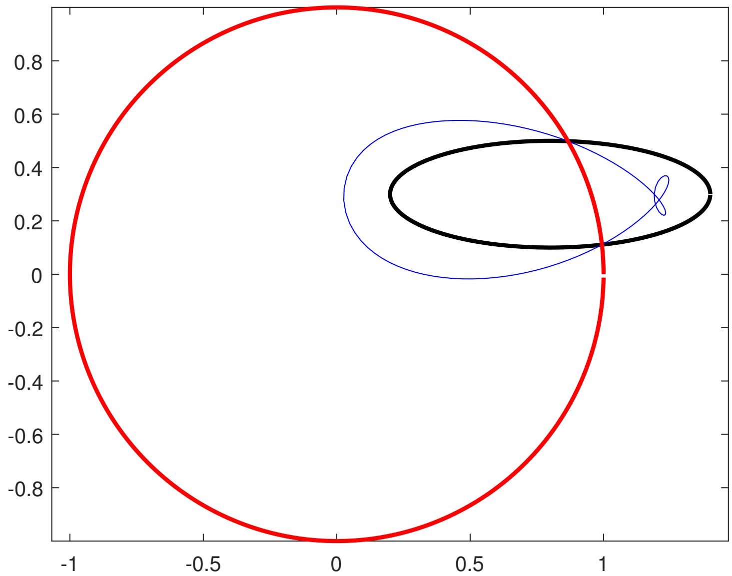
Remark: Recall that the optimum shape boundary is the set of points such that f vanishes. If the initial shape boundary intersects the optimum boundary, then

$$\Delta\partial\Omega = -fn = 0,$$

so each boundary iteration will always include those points. To visualise this, suppose we instead took the initial boundary in Example 2.1 to be

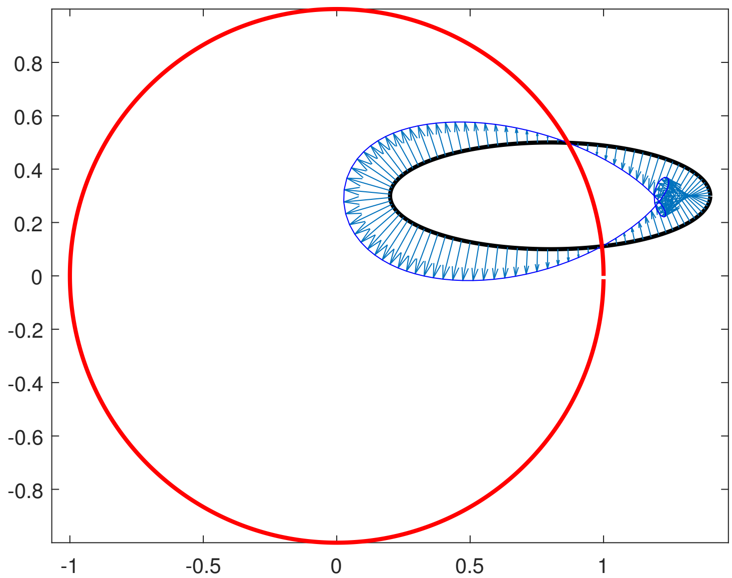
$$\Omega^{(0)} := \left\{ x : \frac{(x_1 - 0.8)^2}{0.6^2} + \frac{(x_2 - 0.3)^2}{0.2^2} \leqslant 1 \right\},$$

which intersects the optimum boundary at two points. The first iteration would look like this:



A problem arises in that the curve is no longer simple. Why does this happen? By plotting the descent direction of the initial boundary using `quiver`, we can visualise the direction that points of the initial boundary are being transformed to.

```
quiver(real(g(t)), imag(g(t)), −0.2*real(fn(t)), −0.2*imag(fn(t)),0)
```



In this example, points further away from the optimum boundary have a greater value of f, so they are being transformed towards the red circle with a greater scale factor than those points closer to the 0-level set of f. It follows that this descent method will *always* fail for optimisation problems with simple boundary solutions, if the initial curve intersects, or even lies outside of the initial shape.

2.4 Line Search for Shapes

In the previous example we fixed the step size t. If t is too large, there is danger of transforming the boundary so much that the next iterationg intersects the optimum boundary, so we need to find a better way to choose t. We will consider an ELS/BLS-like algorithm to find the step size, and observe the improvements (if any) on our shape optimisation method. *BLS for Shapes* goes as follows:

Algorithm 4: Backtracking Line Search for Shapes

Given shape descent direction $\Delta\partial\Omega$, $\alpha \in (0, 0.5), \beta \in (0, 1)$
$t := 1$
while $\mathcal{J}(\Omega + t\Delta\Omega) > \mathcal{J}(\Omega) + \alpha t \, \mathrm{d}\mathcal{J}(\Omega, \Delta\partial\Omega)$ **do**
    Set $t \leftarrow \beta t$
end

Note that in the above algorithm, $\Omega + t\Delta\Omega$ denotes the shape with boundary $\partial\Omega + t\Delta\partial\Omega$.

Steepest Descent for Shapes follows in a similar manner:

Algorithm 5: Steepest Descent for Shapes

Given starting shape Ω
repeat
    (1) Set $\Delta\partial\Omega \leftarrow -fn$
    (2) Choose t via ELS/BLS
    (3) Set $\partial\Omega \leftarrow \partial\Omega + t\Delta\partial\Omega$
until *stopping criterion satisfied*

3 Shape Optimisation Test Case

4 Conclusion

References

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