AMAT 581 Nonparametric Statistics – Fall 2024
Homework 1 Solution Reference
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Problems

- E1.11, P1.1, E2.11, E2.17, E3.6, P3.1
- E4.10, E4.12, E5.5, E5.9

1.E1.11 page12

$$\sum_{i=3}^{5} \binom{6}{i} \left(\frac{1}{3}\right)^{i} \left(\frac{2}{3}\right)^{6-i} =$$

$$\binom{6}{3} \left(\frac{1}{3}\right)^{3} \left(\frac{2}{3}\right)^{3} + \binom{6}{4} \left(\frac{1}{3}\right)^{4} \left(\frac{2}{3}\right)^{2} + \binom{6}{5} \left(\frac{1}{3}\right)^{5} \left(\frac{2}{3}\right) =$$

$$= \frac{6!}{3!(6-3)!} \left(\frac{1}{3}\right)^{3} \left(\frac{2}{3}\right)^{3} + \frac{6!}{4!(6-4)!} \left(\frac{1}{3}\right)^{4} \left(\frac{2}{3}\right)^{2} + \frac{6!}{5!(6-5)!} \left(\frac{1}{3}\right)^{5} \left(\frac{2}{3}\right) =$$

$$= 20 \left(\frac{1}{3}\right)^{3} \left(\frac{2}{3}\right)^{3} + 15 \left(\frac{1}{3}\right)^{4} \left(\frac{2}{3}\right)^{2} + 6 \left(\frac{1}{3}\right)^{5} \left(\frac{2}{3}\right) = \frac{232}{729}$$

2.P1.1, Page13

1. :

$$\binom{N_1}{n_1}\binom{N_2}{n_2}\cdots\binom{N_k}{n_k} \text{if } N_i \geq n_i \text{ for all } i = 1, \ldots, k$$

2. :If $n_i > N_i$ for some i, then:

$$\binom{N_1}{n_1}\binom{N_2}{n_2}\cdots\binom{N_k}{n_k} = 0$$

3. E2.11 page21

Let X be the number of heads, where $n = 3$.

$$P(X = 3|X \geq 1) = \frac{P(X = 3)}{P(X \geq 1)}$$

First, calculate the probabilities:

$$P(X = 3) = \frac{1}{8}$$

$$P(X \geq 1) = P(X = 1) + P(X = 2) + P(X = 3)$$

$$P(X = 1) = \binom{3}{1}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 = \frac{3}{8}$$

$$P(X = 2) = \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{3}{8}$$

$$P(X = 3) = \frac{1}{8}$$

Thus:

$$P(X \geq 1) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$$

Now:

$$P(X = 3|X \geq 1) = \frac{\frac{1}{8}}{\frac{7}{8}} = \frac{1}{7}$$

4. E2.17 page 22

(a) If 3 digits are distinct, then:

$$P(\text{Predict correctly}) = \frac{3!}{10 \times 10 \times 10} = \frac{6}{1000}$$

If 2 digits are the same, then:

$$P(\text{Predict correctly}) = \frac{\frac{3!}{2!1!}}{1000} = \frac{3}{1000}$$

If 3 digits are the same

$$P(\text{predict correctly}) = \frac{1}{1000}$$

(b) No. see the part (a)

5.E3.6

(a) The sample space consists of all possible combinations of pass and fail for 17 students.

$$S = \{(x_1, x_2, \ldots, x_{17}) | X_i \in \{P, F\}, \quad \forall i = 1, \ldots, 17\}$$

where $P =$ pass, $F =$ fail.

(b)

$$\frac{\binom{7}{3} 0.2^3 0.8^4 \binom{10}{10} 0.2^0 0.8^{10}}{\binom{17}{3} 0.2^3 0.8^{14}} = \frac{7}{136}$$

(c) Binomial distribution.

(d) Same with part (b), $\frac{7}{136}$.

6.P3.1

(c) is the correct answer.

Since

$$\sum_{x=1}^{4} \frac{1}{6} = \frac{4}{6} \neq 1 \quad \text{for (a)}$$

$$\sum_{x=1}^{\infty} \left(\frac{1}{4}\right)^x = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3} \neq 1 \text{for (b)}$$

But

$$\sum_{x=0}^{\infty} (1-p)p^x = (1-p)\frac{1}{1-p} = 1 \quad \text{for (c)} \quad (Note: \sum_{x=0}^{\infty} p^x = \frac{p^0}{1-p} = \frac{1}{1-p})$$

7.E4.10 page51

Using theorem 5, let Y be the sum of the numbers of the customers selected for the interview:

$$X_1, X_2, \ldots, X_{12}$$

$$P(X_i = k) = \frac{1}{100}, \quad k = 1, \ldots, 100$$

$$E(X_i) = \sum_{k=1}^{100} k \cdot \frac{1}{100} = \frac{100 + 1}{2}$$

Thus, the mean of Y:

$$E(Y) = E(X_1) + E(X_2) + \ldots + E(X_{12}) = 12 \cdot \frac{100 + 1}{2} = 606$$

The variance:

$$\text{Var}(Y) = \frac{12 \cdot (100+1)(100-12)}{12} = 8888$$

The range:

Smallest Y = 1+2+3...+12 = 78,   Largest Y = 89+90+91+...+100 = 1134,

Thus, the range is:

$$\text{Range} = 1134 - 78 = 1056$$

8.E4.12,page 51

(1) since $F_X(1) = P(X \leq 1) = 0.25 + 0.25 = 0.5$, so the median of X is 1.

(2) The pmf of of X, Y:

X	$f_X(x)$
1	0.75
2	0.25

Table 1: pmf of X

Y	$f_Y(y)$
1	0.5
2	0.5

Table 2: pmf of Y

We can get:

$$E(Y) = 1.05 + 2.05 = 1.5$$
$$E(Y^2) = 1^2 \cdot 0.5 + 2^2 \cdot 0.5 = 2.5$$
$$Var(Y) = E(Y^2) - (E(Y))^2 = 0.25$$

X	$f(xy)$
1	0.25
2	0.75

Table 3: pmf of XY

(3)

$$E(X) = 1 \cdot 0.75 + 2 \cdot 0.25 = 1.25$$
$$E(Y) = 1 \cdot 0.5 + 2 \cdot 0.5 = 1.5$$

Covariance between X and Y:

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

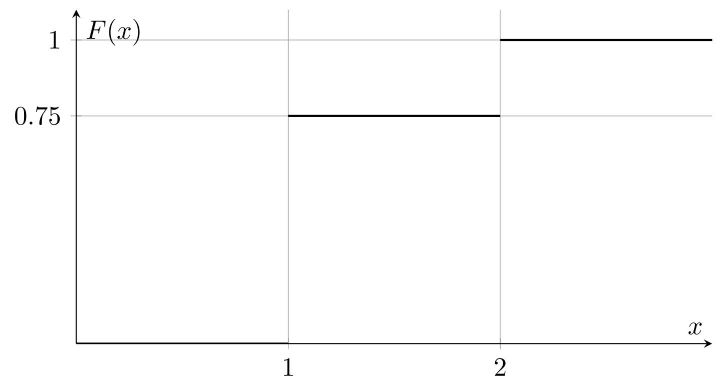
$$E(XY) = 1 \cdot 0.25 + 2 \cdot 0.75 = 1.75$$

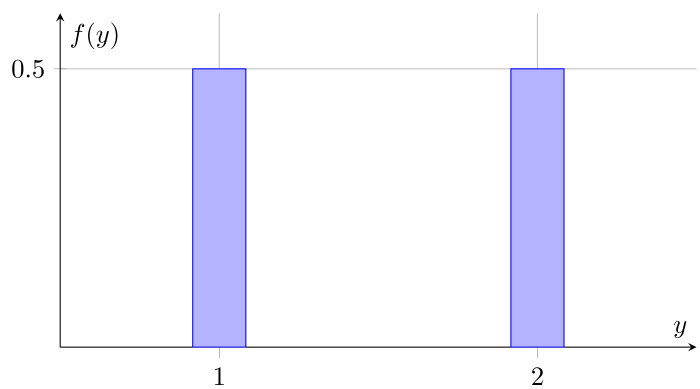
$$\text{Cov}(X,Y) = 1.75 - 1.25 \cdot 1.5 = -0.125$$

(4)

$$F(1,2) = P(X \leq 1, Y \leq 2) = 0.25 + 0.5 = 0.75$$

(5) The graphs below represent $F(x)$and $f(y)$:





9.E5.5,page 63

Given that $X \sim N(160, 20^2)$, we need to compute the value such that the cumulative probability is 0.99.

From the table A1, we have $z_{0.99} = 2.3263$.

Now, using the formula for a normal distribution:

$$X = \sigma z_{0.99} + \mu = 20 \times 2.3263 + 160 = 206.526$$

Thus, the required value of X is approximately 206.526.

10.E5.9

Using the Central Limit Theorem (CLT):

$$\frac{X - np}{\sqrt{np(1-p)}} \sim N(0,1)$$

For $n = 500$,

$$P(X > 80) = P\left(Z > \frac{80 - 500 \cdot 0.15}{\sqrt{500 \cdot 0.15 \cdot 0.85}}\right) =$$

$$P(Z > 0.6262) = 1 - P(Z \leq 0.6262)1 - 0.734 = 0.266$$

Since from the Table A1,

$$P(Z \leq 0.6250) = 0.734$$