AMAT583 (8434) Midterm I

Name .... Score: .../24

Problem 1. (2 points) Find all the subsets of $S = \{A, B, C\}$.

Solution: $\emptyset, \{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}, S$.

Problem 2. (2 points) Find sets S and T such that $S \cup T$ and $S \cap T$ have 6 elements each.

Solution: Here we can pick any set S with 6 elements and let $S = T$. For instance, $S = T = \{1, 2, 3, 4, 5, 6\}$.

Problem 3. (3 points) For each of the following, decide if the set is an interval or not. If it is an interval, write it on standard interval notation.

a. $[1, 5] \cup (0, 3) = (0, 5]$,

b. $[1, 5] \cap (0, 3) = [1, 3)$,

c. $[0, 1) \cup (1, 4]$ is not an interval, as it is not connected. (It is missing 0.)

d. $[0, 1) \cap (1, 4]$ is empty, and therefore not an interval.

e. $[-1, 0] \cup (0, 4) = [-1, 4)$,

f. $[-1, 0] \cap (0, 4)$ is empty, and therefore not an interval.

Problem 4. (3 points) Let $f\colon [0, \infty) \to [0, \infty)$, $f(x) = x^2$.

a. What is $\operatorname{im}(f)$? $[0, \infty)$, since for all $x \in [0, \infty)$, $f(\sqrt{x}) = x$. (NB: $[0, \infty)$ is the set of all nonnegative <u>real</u> numbers, not $\mathbb{N}$.)

b. Is f injective? Yes: $f(x) = f(y)$ gives $x^2 = y^2$, which implies $x = \pm y$. Since x and y are nonnegative, this means that $x = y$. (NB: Many wrote something ambiguous like "each element is mapped to a unique element". I chose to give points for this now, but I will require more precise language on the final exam.)

c. Is f surjective? Yes, since $\operatorname{im}(f)$ is equal to the codomain $[0, \infty)$.

d. Is f bijective? Yes, since it is both injective and surjective.

Problem 5. (3 points) In Fig. 1, three graphs G_1, G_2 and G_3 are drawn. Two of them are isomorphic, and one is not isomorphic to the others.

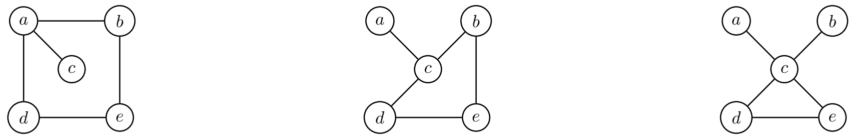
a. Find an isomorphism between the two isomorphic graphs.

b. Pick two of the graphs and prove that they are not isomorphic.

Solution:

a. G_1 and G_2 are isomorphic. An isomorphism from G_1 to G_2 is given by

$$\begin{aligned} a &\mapsto c, & b &\mapsto b, \\ c &\mapsto a, & d &\mapsto d, \\ e &\mapsto e, & f &\mapsto f. \end{aligned}$$



(a) G_1 (b) G_2 (c) G_3

Figure 1

(NB: This is asking you to find an isomorphism, not to show that two graphs are isomorphic. Besides, showing that two graphs are isomorphic without finding an isomorphism is usually very difficult. It is not enough to check that degrees, lengths of cycles, etc. are the same.)

b. G_1 and G_3 are not isomorphic, since G_3 has a vertex of degree 4 (namely, c), and G_1 does not. (NB: It is not enough to say that c has degree 1 in G_1 and 4 in G_3. Such an argument could also be used for G_1 and G_2, but they are isomorphic.)

Problem 6. (2 points) Let $\sim$ be the relation on $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ defined by $x \sim y$ if and only if $x - y$ is divisible by 4. Write the equivalence classes of $\sim$ explicitly.

Solution: The equivalence classes are

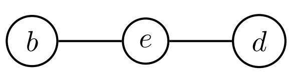
$$\{4x \mid x \in \mathbb{Z}\},$$
$$\{4x + 1 \mid x \in \mathbb{Z}\},$$
$$\{4x + 2 \mid x \in \mathbb{Z}\},$$
$$\{4x + 3 \mid x \in \mathbb{Z}\}.$$

Problem 7. (3 points) Let $G_1 = (V, E)$ be the graph from Fig. 3a.

a. Write V and E explicitly.

b. Draw the subgraph of G_1 induced by $\{b, d, e\}$.

Solution: a. $V = \{a, b, c, d, e\}$, $E = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, e\}, \{d, e\}\}$.

b. 

Problem 8. (2 points) Find all the minimum spanning trees of G in Fig. 2.

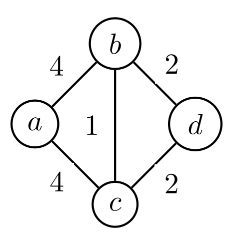


Figure 2: G

Problem 9. (2 points) Let $d\colon \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be defined by $d(x, y) = (x - y)^2$. Show that d is not a metric on $\mathbb{R}$.

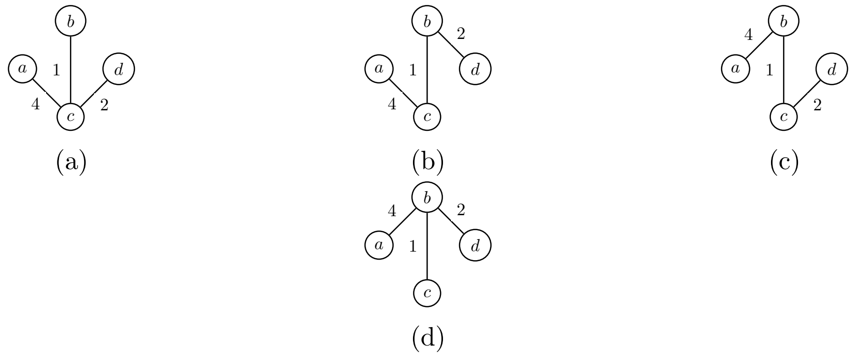


Figure 3: Solution to Problem 8.

Solution: We have $d(0,1) = d(1,2) = 1^2 = 1$, but $d(0,2) = 2^2 = 4$. This means that $d(0,2) \not\leq d(0,1) + d(1,2)$. Thus, d does not satisfy the triangle inequality, so it is not a metric.

Problem 10. (2 points) Define two different metrics on $\{a, b, c\}$ with $d(a,b) = 1$. (Correction: Define different metrics d and d' on $\{a, b, c\}$ such that $d(a,b) = d'(a,b) = 1$.)

Solution: Let $d(x,y) = 1$ for all $x \neq y$, and $d(x,x) = 0$ for all x.
Let $d'(b,c) = d'(c,b) = 2$, and let $d'(x,y) = d(x,y)$ for all other choices of x and y.