Questions for Part I of the exam (solutions)

Exercise 1. How many elements does the set $\{\{1,2\},3,\{4\}\}$ contain?

Solution: 3. (The elements are $\{1,2\}$, 3 and $\{4\}$.)

Exercise 2. How many elements does the set $\{1,2,3\} \times \{1,2,3,4,5\}$ contain?

Solution: 15. (In general, $S \times T$ contains $|S| \cdot |T|$ elements.)

Exercise 3. List all the subsets of $\{1,2\}$.

Solution: $\{1,2\}$, $\{1\}$, $\{2\}$, $\emptyset$. (Remember that both the set itself and the empty set are subsets!)

Exercise 4. Write two sets S and T such that $S \cap T = \{1,4\}$ and $S \cup T = \{1,2,3,4\}$.

Solution: E.g., $S = \{1,4\}$ and $T = \{1,2,3,4\}$.

Exercise 5. Write an interval I such that $(1,2) \subseteq I \subseteq [1,2]$, but $I \neq (1,2)$ and $I \neq [1,2]$.

Solution: There are two solutions: $[1,2)$ and $(1,2]$.

Exercise 6. Write two intervals I and J such that $I \cap J$ contains exactly one element.

Solution: E.g., $I = [0,1]$ and $J = [1,2)$. (A minimalistic solution is $I = J = [0,0]$, since $[0,0] = \{0\}$.)

Exercise 7. Find a nonempty set $S \subseteq \mathbb{R}$ that is not an interval.

Solution: E.g., $\{0,1\}$. (Anything that is disconnected will work.)

Exercise 8. Write an interval I with $\mathbb{N} \subseteq I$. (Recall that $\mathbb{N} = \{0,1,2,3,\dots\}$.)

Solution: E.g., $[0,\infty)$.

Exercise 9. Define a function $f\colon \{1,2\} \to \{A,B,C\}$ that is not injective.

Solution: There are three solutions: Either $f(1) = f(2) = A$, or $f(1) = f(2) = B$, or $f(1) = f(2) = C$.

Exercise 10. Define a function $f\colon \{1,2,3\} \to \{A,B\}$ that is not surjective.

Solution: There are two solutions: Either $f(1) = f(2) = f(3) = A$, or $f(1) = f(2) = f(3) = B$.

Exercise 11. Write all the partitions of $\{2,4\}$.

Solution: There are two partitions: $\{\{2,4\}\}$ and $\{\{2\},\{4\}\}$.

(a) (b)

Figure 1: Solution to Exercise 14.

Exercise 12. Give an example of a bijective function f.

Solution: E.g., $f\colon \{1\} \to \{1\}$ given by $f(1) = 1$ (which is the only function from $\{1\}$ to $\{1\}$).

Exercise 13. Give an example of a function f that is surjective, but not injective.

Solution: E.g., $f\colon \{1, 2\} \to \{1\}$ given by $f(1) = f(2) = 1$.

Exercise 14. Draw a graph with vertex set $\{a, b, c, d\}$ that is connected, and draw a graph with vertex set $\{1, 2, 3, 4\}$ that is not connected.

Solution: See Figure 1 for one solution.

Exercise 15. Draw the graph $G = (V, E)$ where $V = \{a, b, c, d\}$ and $E = \{\{a, c\}, \{b, c\}\}$.

Solution: See Figure 2.

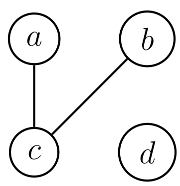

Figure 2: Solution to Exercise 15.

Exercise 16. Draw a tree that has a vertex with degree exactly 4, and write which vertex this is.

Solution: In the graph in Figure 3, e has degree 4.

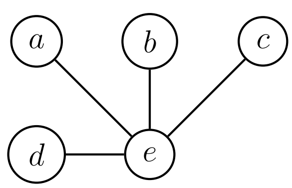

Figure 3: A solution to Exercise 16.

Exercise 17. Draw a connected graph (with at least one vertex) that is not a tree.

Solution: See Figure 4.

Exercise 18. Draw a graph with three connected components.

Solution: See Figure 5.

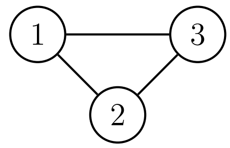

Figure 4: A solution to Exercise 17.

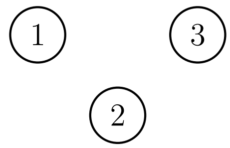

Figure 5: A solution to Exercise 18.

Exercise 19. Draw a spanning tree of the graph G in Figure 6a.

Solution: See Figure 6b for one solution.

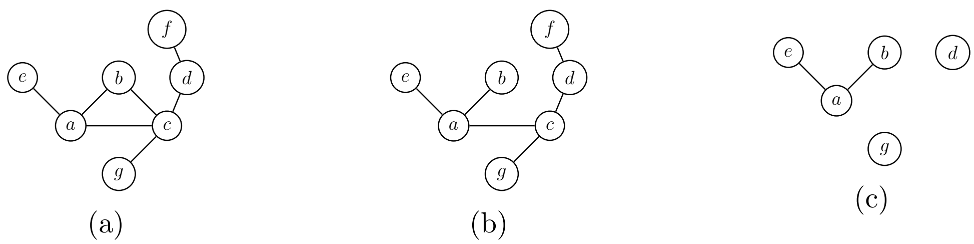

Figure 6: The graph G of Exercise 19 in (a) and a spanning tree of G in (b). The solution to Exercise 20 in (c).

Exercise 20. Draw the subgraph of G in Figure 6a induced by $\{a, b, d, e, g\}$.

Solution: See Figure 6c.

Exercise 21. Draw two non-isomorphic trees with 4 vertices each.

Solution: See Figure 7.

Figure 7: A solution to Exercise 21.

Exercise 22. Draw two graphs with vertex set $\{1, 2, 3, 4\}$ that are isomorphic, but not equal.

Solution: See Figure 8.

Exercise 23. What is the edit distance between ABC and BAC?

Solution: 2. One optimal edit sequence is $ABC \to BC \to BAC$.

(a) (b)

Figure 8: A solution to Exercise 22.

Exercise 24. Let d_E be the edit distance. Write three words u, v, w such that $d_E(u,v) = d_E(u,w) = d_E(v,w) = 1$.

Solution: E.g., $u = 01$, $v = 1$, $w = 0$.

Exercise 25. Let $\sim$ be the equivalence relation on $\{-1,1,3,5\}$ defined by $x \sim y$ if $xy > 0$. Write the equivalence classes of $\sim$.

Solution: The equivalence classes of $\sim$ are $\{-1\}$ and $\{1,3,5\}$. (We have $1 \sim 3 \sim 5$ because $1 \cdot 3 > 0$ and $3 \cdot 5 > 0$, and $1 \nsim -1$ because $-1 \cdot 1 \not> 0$.)

Exercise 26. What is the Wasserstein distance between $(3,0,0)$ and $(0,0,3)$?

Solution: 6. (We have to move 3 units a distance of 2, which costs $3 \cdot 2 = 6$.)

Exercise 27. What is the 3-means clustering of $\{0,1,1.1,2\}$ equipped with the standard metric?

Solution: $\{\{0\},\{1,1.1\},\{2\}\}$. (This can be verified with computations, but we know that we need a partition of $\{0,1,1.1,2\}$ with three sets, which means that we have to put two elements in the same set, and putting the closest points together has the lowest cost.)

Exercise 28. Find $a,b,c,d,e \in \mathbb{R}$ such that the 3-means clustering of $\{a,b,c,d,e\}$ is $\{\{a\},\{b,c,d\},\{e\}\}$.

Solution: E.g., $a = -100$, $b = -1$, $c = 0$, $d = 1$, $e = 100$. (Just put b,c,d very close together and a and e very far away.)

Exercise 29. What is the barcode of the dendrogram in Figure 9?

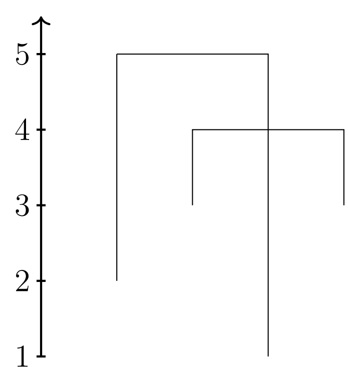

Figure 9: A dendrogram.

Solution: The barcode is $\{[1,\infty),[2,5),[3,4),[3,4)\}$.

Exercise 30. Write all the paths from a to g in the graph G in Figure 6a.

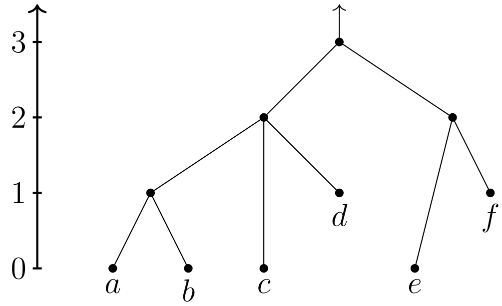

Figure 10: A dendrogram D.

Solution: There are two paths: a, c, g and a, b, c, g. (Remember that in a path, we are not allowed to repeat vertices.)

Exercise 31. Find the partition obtained by the ToMATo clustering of D in Figure 10 with $\tau = 2.9$.

Solution: The partition is $\{\{a, b, c, d\}, \{e, f\}\}$. (Only the branch with e and f is longer than 2.9.)

Exercise 32. Write a function f such that $f(x) = o(x)$ and $\log x = o(f(x))$

Solution: E.g., $f(x) = \sqrt{x}$. (Or any other x^a for $0 < a < 1$, or $(\log x)^2$, etc. But not $\log(x^2)$, since this is equal to $2 \log x = \Theta(\log x)$.)

Exercise 33. Write a function f such that $f(x) = \Theta(e^x)$, but $f(x) < e^x$ for all x.

Solution: E.g., $f(x) = \frac{1}{2}e^x$.

Exercise 34. Write a function f such that $f(x) = O(e^x)$, but $f(x) > e^x$ for all x, or state that it is impossible.

Solution: E.g., $f(x) = 2e^x$.

Exercise 35. Write a function f such that $f(x) = o(e^x)$, but $f(x) > e^x$ for all x, or state that it is impossible.

Solution: Impossible. (Since $f(x) = o(e^x)$, we have $\lim_{x\to\infty} \frac{f(x)}{e^x} = 0$, which means that for large enough x, $f(x) < e^x$.)