AMAT583 (8432) Midterm I

Name .... Score: .../26

Problem 1. (2 points) Let $S = \{1, 2\}$ and $T = \{1, 3, 4\}$. Write the following sets explicitly:

a. $S \cup T = \{1, 2, 3, 4\}$

b. $S \cap T = \{1\}$

c. $S \setminus T = \{2\}$

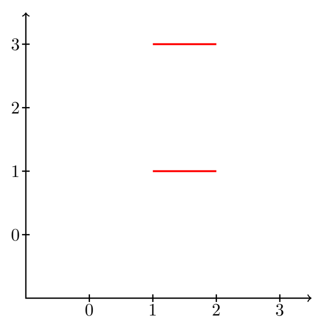
d. $S \times T = \{(1, 1), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4)\}$

Problem 2. (3 points) Draw

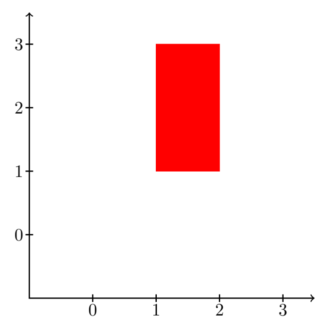
a. $[1, 2] \times \{1, 3\}$,

b. $[1, 2] \times [1, 3]$.

Solution:



(a) $[1, 2] \times \{1, 3\}$



(b) $[1, 2] \times [1, 3]$

Figure 1

Problem 3. (3 points) Say if the following statements are true or false:

a. $1 \in \{\{1\}\}$: False

b. $\{\{0\}, \{b\}\} = \{\{0\}, b\}$: False

c. $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$: True

d. $\{1\} \cap \{2\} = 0$: False (the intersection is equal to $\emptyset$)

e. $1 \subseteq \{1, 2\}$: False

f. $1 \in \{1, 2\}$: True

g. $\{1\}$ is an interval: True (it can be written as $[1, 1]$)

Problem 4. (2 points) Find two intervals I and J such that $I \cup J = [0, 2]$ and $I \cap J = \emptyset$. (Recall: $\emptyset$ is not an interval.)

Solution: One solution is $I = [0, 1]$ and $J = (1, 2]$.

Problem 5. (3 points) Let $f\colon \mathbb{Z} \to \mathbb{R}$ be given by $f(x) = x + 2$.

a. What is $\mathrm{im}(f)$?

b. Is f injective? Explain why/why not.

c. Is f surjective? Explain why/why not.

d. Is f bijective? Explain why/why not.

Solution:

a. $\mathbb{Z}$

b. Yes. If $x \neq y$, then $f(x) = x + 2 \neq y + 2 = f(y)$, so $f(x) \neq f(y)$.

c. No. For instance, $\frac{1}{2}$ is in the image of f, but not in the codomain $\mathbb{R}$.

d. No. f is not surjective by (c), so it cannot be bijective.

Problem 6. (3 points)

a. Draw a graph $G = (V, E)$ with five vertices, three edges and three connected components. (If you cannot solve this, draw a graph and solve b and c with the graph you drew.)

b. Write V and E explicitly.

c. Write a cycle of G.

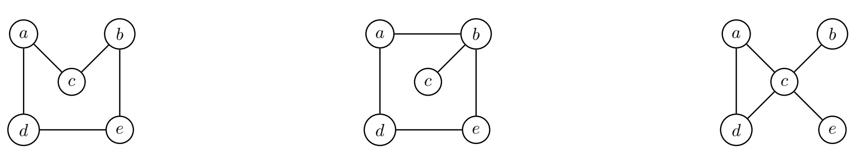
Solution:

a. Draw a graph $G = (V, E)$ with five vertices, three edges and three connected components. (If you cannot solve this, draw a graph and solve b and c with the graph you drew.)

b. $V = \{a, b, c, d, e\}$, $E = \{\{a, b\}, \{b, c\}, \{a, c\}\}$.

c. a, b, c, a.

Problem 7. (3 points) Show that none of the graphs G_1, G_2 and G_3 in Fig. 2 are isomorphic to each other.



(a) G_1 (b) G_2 (c) G_3

Figure 2

Solution: G_1 has one cycle (up to switching directions and changing starting point), and it has length 5. G_2 has one cycle, and it has length 4. G_3 has one cycle, and it has length 3. Since lengths of cycles are preserved by isomorphisms, none of the graphs can be isomorphic. (Another way to solve it is to observe that the number of vertices of degree 1 is different in each graph. You can also look at the largest vertex degree in each graph and observe that this is 2, 3 and 4, respectively.)

NB: It is not enough to check that c (or some other vertex) has different degree in the graphs. For instance, the two middle graphs in Fig. 3 are isomorphic, but a has different degrees in the two graphs.

Problem 8. (2 points) Draw all the spanning trees of G_2 in Fig. 2.

Solution: See Fig. 3.

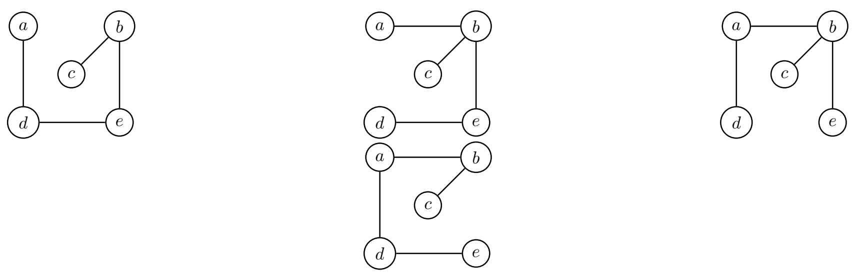


Figure 3

Problem 9. (2 points) Let $S = \{(x, y) \in \mathbb{R}^2 \mid x = y\} \subseteq \mathbb{R}^2$. Define $d\colon S \times S \to [0, \infty)$ by

$$d((x, y), (x', y')) = |x - x'|.$$

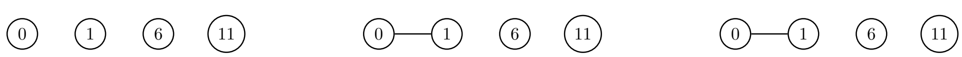
Show that (S, d) is a metric space.

Solution: We check the three conditions for a metric:

1. $d((x, y), (x', y')) = 0$ if and only if $|x - x'| = 0$, which holds if and only if $x = x'$. Since $y = x$ and $y' = x'$, this is the same as saying that $(x, y) = (x', y')$. (NB: It is not enough to say that $d(a, b) = 0$, you also need that $d(a, b) = 0$ implies $a = b$!)
2. $d((x, y), (x', y')) = |x - x'| = |x' - x| = d((x', y'), (x, y))$.
3. $d((x, y), (x', y')) + d((x', y'), (x'', y'')) = |x - x'| + |x' - x''| \geq |x - x''| = d((x, y), (x'', y''))$.

Problem 10. (3 points) For $x, y \in \mathbb{R}$, let $d(x, y) = |x - y|$. Find a set S of four points in $\mathbb{R}$ such that both $N(S)_1$ and $N(S)_4$ have three connected components. Draw $N(S)_0$, $N(S)_1$ and $N(S)_4$.

Solution: We need to find four points where two are close to each other, but the two other are far away, and also far away from each other. An example is $S = \{0, 1, 6, 11\}$.



(a) $N(S)_0$ (b) $N(S)_1$ (c) $N(S)_4$

Figure 4