# AMAT583 (8433) Midterm I

Name .............................................................. Score: .../25

**Problem 1. (3 points)** Let $S = \{A, B\}$. Write the following sets explicitly.

a. $S \cup \emptyset = S = \{A, B\}$

b. $S \cap S = S = \{A, B\}$

c. $S \cap \emptyset = \emptyset$

d. $S \cup S = S = \{A, B\}$

e. $S \times S = \{(A, A), (A, B), (B, A), (A, B)\}$

**Problem 2. (2 points)** Let $S$ and $T$ be sets with $S \cap T = \{2, 7\}$, $S \setminus T = \{1, 8\}$ and $S \cup T = \{1, 2, 3, 7, 8\}$. Find $S$ and $T$.

Solution: $S = \{1, 2, 7, 8\}$ and $T = \{2, 3, 7\}$. (2 and 7 are in both of $S$ and $T$, while 1, 3 and 8 have to be in exactly one of $S$ and $T$. Since $S \setminus T = \{1, 8\}$, 1 and 8 are the elements that are only in $S$, so 3 is only in $T$.)

**Problem 3. (2 points)** Find three intervals $I$, $J$ and $K$ such that $I \cap J = \{1\}$ and $J \cap K = \{2\}$.

Solution: $J$ has to be $[1, 2]$, and we can for instance choose $I = [0, 1]$ and $K = [2, 3]$.

**Problem 4. (2 points)** Write out the power set $\mathcal{P}(\{1, 3, 4\})$ explicitly.

Solution:

$$\mathcal{P}(\{1, 3, 4\}) = \{\{1, 3, 4\}, \{1, 3\}, \{1, 4\}, \{3, 4\}, \{1\}, \{3\}, \{4\}, \emptyset\}$$

**Problem 5. (3 points)** Let $f : \mathbb{Z} \to \mathbb{Z}$ be given by $f(x) = x^3$.

a. What is $\mathrm{im}(f)$?

b. Is $f$ injective? Why/why not?

c. Is $f$ surjective? Why/why not?

d. Is $f$ bijective? Why/why not?

Solution:

a. $\mathrm{im}(f) = \{x^3 \mid x \in \mathbb{Z}\}$

b. Yes, because if $x \neq y$, then $f(x) \neq f(y)$. (This follows for example from $x^3$ being strictly increasing in $x$.)

c. No. 2 is in the codomain $\mathbb{Z}$, but not in $\mathrm{im}(f)$, since it is not equal to $x^3$ for any integer $x$.

d. No. It is not surjective, so it is not bijective.

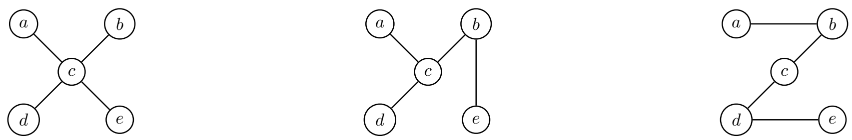
**Problem 6. (2 points)** Define an equivalence relation on $\mathbb{Z}$ with two equivalence classes.

Solution: If $\sim$ is defined by $x \sim y$ if $x - y$ is even, then $\sim$ has two equivalence classes: the set of even integers, and the set of odd integers.

Alternatively, one can simply define the equivalence classes $S$ and $T := \mathbb{Z} \setminus S$ of $\sim$ directly. Then $\sim$ is defined by $x \sim y$ if $x$ and $y$ are either both in $S$ or both in $T$. For example: $S = \{0\}$ and $T = \mathbb{Z} \setminus \{0\}$, or $S = \mathbb{N}$ and $T = \mathbb{Z} \setminus \mathbb{N}$.

**Problem 7. (3 points)** Draw three trees with five vertices each such that none of the trees are isomorphic.

Solution: See the trees in Fig. 1. To see that these are not isomorphic, note that the maximal degree of a vertex is 4 in $T_1$, 3 in $T_2$ and 2 in $T_3$.




(a) $T_1$ (b) $T_2$ (c) $T_3$

Figure 1

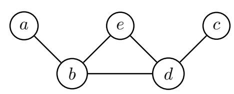



Figure 2: $G$

**Problem 8. (3 points)** Let $G$ be the graph in Fig. 2.

a. What is the diameter of $G$?

b. What is the closeness centrality of $b$?

c. Draw all the spanning trees of $G$.

Hint: Closeness centrality was defined using a formula of this form:

$$\frac{n-1}{\sum \cdots}.$$

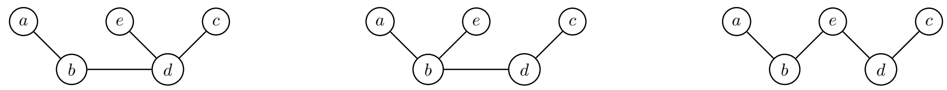
Solution: Let $d(u, v)$ be the length of the shortest path between two vertices $u$ and $v$.

a. 3. We have $d(a, c) = 3$ (the shortest path is $a, b, d, c$), and there is no other pair of vertices with a larger distance.

b. What is the closeness centrality of $b$?

$$\frac{5-1}{d(b,a) + d(b,e) + d(b,d) + d(b,c)} = \frac{4}{1+1+1+2} = \frac{4}{5}.$$

c. See Fig. 3.




(a) $T_1$ (b) $T_2$ (c) $T_3$

Figure 3

**Problem 9. (2 points)** Let $(S, d)$ be a metric space, and let $d' \colon S \times S \to [0, \infty)$ be defined by $d'(x, y) = 3d(x, y)$ for all $x, y \in S$. Show that $d'$ is a metric on $S$.

Solution: Using $d(x, y) = 0 \Leftrightarrow x = y$ for the metric $d$, we get

$$\begin{aligned} d'(x, y) &= 0 \\ \Leftrightarrow 3d'(x, y) &= 0 \\ \Leftrightarrow d(x, y) &= 0 \\ \Leftrightarrow x &= y. \end{aligned}$$

Using $d(x, y) = d(y, x)$ for the metric $d$, we get

$$d'(x, y) = 3d(x, y) = 3d(y, x) = d'(y, x).$$

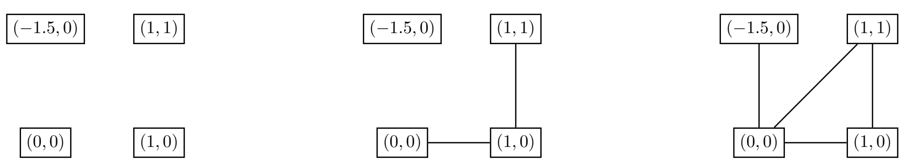
Using the triangle equality for $d$, we get

$$d'(x, y) + d'(y, z) = 3(d(x, y) + d(y, z)) \leq 3d(x, z) = d'(x, z).$$

Thus, all the conditions for $d'$ being a metric are satisfied.

**Problem 10. (3 points)** Let $S = \{(-1.5, 0), (0, 0), (1, 0), (1, 1)\} \subseteq \mathbb{R}^2$, and let $d$ be the Euclidean metric on $\mathbb{R}^2$. Draw $N(S)_\epsilon$ for $\epsilon = 0, 1, 2$.

Solution:




(a) $N(S)_0$ (b) $N(S)_1$ (c) $N(S)_2$

Figure 4