# Vector Calculus

Que 1) Find the directional derivative of $f(x,y,z) = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of vector $I + 2J + 2K$

Sol$^n$ Given that

$$f(x,y,z) = xy^2 + yz^3 \quad \text{--- (1)}$$

Let $\vec{d}$ be the vector. Then

$$\vec{d} = I + 2J + 2K$$

Now $$\nabla f = I \frac{\partial}{\partial x}(xy^2 + yz^3) + J \frac{\partial}{\partial y}(xy^2 + yz^3) + K \frac{\partial}{\partial z}(xy^2 + yz^3)$$

$$\nabla f = I(y^2) + J(2xy + z^3) + K(3z^2y)$$

At $(2,-1,1)$

$$\nabla f = I - 3J - 3K$$

direction derivative of f in direction of vector $\vec{d} = \nabla f \cdot \hat{d}$

$$= \nabla f \cdot \frac{\vec{d}}{|\vec{d}|} = (I - 3J - 3K) \cdot \frac{(I + 2J + 2K)}{\sqrt{1+4+4}}$$

$$= \frac{1-6-6}{3} = \frac{-11}{3}$$

Que) What is the directional derivative of $\phi = xy^2 + yz^3$ at the point $(2,-1,1)$ in the direction of the normal to the surface $x\log z - y^2 = -4$ at $(-1,2,1)$

Soln - Given surface is

$$\phi = xy^2 + yz^3 \quad \text{------} \quad (1)$$

and

$$f \equiv x\log z - y^2 + 4 = 0 \quad \text{---} \quad (2)$$

$$\nabla\phi = I\frac{\partial}{\partial x}(xy^2 + yz^3) + J\frac{\partial}{\partial y}(xy^2 + yz^3)$$
$$+ K\frac{\partial}{\partial z}(xy^2 + yz^3)$$

$$= I(y^2) + J(2xy + z^3) + K(3z^2y)$$

At $(2,-1,1)$

$$\nabla\phi = I + J(-4+1) + K(-3)$$

$$\nabla\phi = I - 3J - 3K$$

Now

$$\nabla f = I\frac{\partial}{\partial x}(x\log z - y^2 + 4)$$
$$+ J\frac{\partial}{\partial y}(x\log z - y^2 + 4)$$
$$+ K\frac{\partial}{\partial z}(x\log z - y^2 + 4)$$

$\nabla f = I(\log z) + J(-2y) + K\left(\frac{x}{z}\right)$

At $(-1, 2, 1)$

$\nabla f = I \log I + J(-4) + K\left(\frac{-1}{1}\right)$

$\nabla f = 0I - 4J - K$

Direction derivative of $\phi$ in the direction of normal to the surface f

$= \nabla \phi \cdot \frac{\nabla f}{|\nabla f|}$

$= (I - 3J - 3K) \cdot \frac{(0I - 4J - K)}{\sqrt{0 + 16 + 1}}$

$= \frac{0 + 12 + 3}{\sqrt{17}} = \frac{15}{\sqrt{17}}$

Que) Find the angle between the surfaces $x^2+y^2+z^2 = 9$ and $z = x^2+y^2-3$ at the point $(2, -1, 2)$

Soln — Let $f_1 \equiv x^2+y^2+z^2-9 = 0$

$f_2 \equiv x^2+y^2-z-3 = 0$

Then

$N_1 = \nabla f_1 = 2x I + 2y J + 2z K$

At $(2,-1,2)$

$N_1 = 4I - 2J + 4K$

$N_2 = \nabla f_2 = 2x I + 2y J - K$

At $(2,-1,2)$

$N_2 = 4I - 2J - K$

Since the angle $\theta$ between the two surfaces at a point is the angle between their normals at that point and $N_1, N_2$ are the normals at $(2,-1,2)$ to the given surfaces, therefore

$$N_1 \cdot N_2 = |N_1|\,|N_2| \cos\theta$$

$$\cos\theta = \frac{N_1 \cdot N_2}{|N_1|\,|N_2|} = \frac{(4J - 2J + 4K)\cdot(4I - 2J - K)}{\sqrt{16+4+16}\;\sqrt{16+4+1}}$$

$$= \frac{16+4-4}{6\cdot\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Que) Find the values of a and b such that the surface $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1,-1,2)$

Solⁿ-: let

$$f_1 \equiv ax^2 - byz - (a+2)x = 0 \quad \text{--- (i)}$$

$$f_2 \equiv 4x^2y + z^3 - 4 = 0 \quad \text{--- (ii)}$$

$$\nabla f_1 = I \frac{\partial}{\partial x}(ax^2 - byz - (a+2)x) + J \frac{\partial}{\partial y}(ax^2 - byz - (a+2)x) + K \frac{\partial}{\partial z}(ax^2 - byz - (a+2)x)$$

$$= I(2ax - a - 2) + J(-bz) + K(-by)$$

At $(1,-1,2)$

$$\nabla f_1 = (a-2)I - 2bJ + bK \quad \text{--- (iii)}$$

$$\nabla f_2 = I \frac{\partial}{\partial x}(4x^2y + z^3 - 4) + J \frac{\partial}{\partial y}(4x^2y + z^3 - 4) + K \frac{\partial}{\partial z}(4x^2y + z^3 - 4)$$

$$= I(8xy) + J(4x^2) + K(3z^2)$$

At $(1,-1,2)$

$$\nabla f_2 = -8I + 4J + 12K$$

The surface (i) and (ii) ~~cut~~ will cut orthogonally if $\nabla f_1 \cdot \nabla f_2 = 0$

$$\Rightarrow [(a-2)I - 2bJ + bK] \cdot [-8I + 4J + 12K] = 0$$

$$\Rightarrow -8(a-2) - 8b + 12b = 0$$

$$\Rightarrow -8a + 16 + 4b = 0$$

$$\Rightarrow \cancel{8a} \; -2a + 4 + b = 0$$

$$\Rightarrow 2a - b = 4 \quad --- \text{(iv)}$$

Also since the point $(1,-1,2)$ lies on (i) and (ii), we have

from (i) $\Rightarrow a + 2b - (a+2) = 0$

$$\Rightarrow a + 2b - a - 2 = 0$$

$$\Rightarrow 2b = 2 \Rightarrow b = 1$$

from (iv) $\Rightarrow 2a - 1 = 4$

$$2a = 5$$

$$a = 5/2$$

Hence $a = 5/2$, $b = 1$

Que) Find div F and curl F where

$$F = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz)$$

Solⁿ :-

$$\text{grad}\,(x^3 + y^3 + z^3 - 3xyz) = \nabla\,(x^3+y^3+z^3-3xyz)$$

$$= I\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz) + J\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)$$

$$+ K\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$= I\,(3x^2-3yz) + J(3y^2-3xz)$$

$$+ K(3z^2-3xy)$$

since $F = \text{grad}\,(x^3+y^3+z^3-3xyz)$

$$\therefore\; F = (3x^2-3yz)I + (3y^2-3xz)J + (3z^2-3xy)K$$

$$\text{div } F = \nabla \cdot F$$

$$= \left(I\frac{\partial}{\partial x} + J\frac{\partial}{\partial y} + K\frac{\partial}{\partial z}\right)\cdot\left\{(3x^2-3yz)I + (3y^2-3xz)J + (3z^2-3xy)K\right\}$$

$$= \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$$

$$= 6x + 6y + 6z = 6(x+y+z)$$

Curl F = $\nabla \times F$

$$= \begin{vmatrix} I & J & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= I(-3x+3x) - J(-3y+3y)$$
$$+ K(-3z+3z)$$

$$= 0I - J(0) + K(0) = 0$$

Que) If $f = (x^2+y^2+z^2)^{-n}$ find div grad f and determine n if div grad f = 0

Soln

$$r = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2$$

$$f = (x^2+y^2+z^2)^{-n} = (r^2)^{-n} = r^{-2n}$$

$$\text{grad } f = \Sigma I \frac{\partial f}{\partial x} = \Sigma I \frac{\partial}{\partial x}(r^{-2n})$$

$$= \Sigma I(-2n) r^{-2n-1} \frac{\partial r}{\partial x}$$

$$= \Sigma I(-2n) r^{-2n-1} \frac{x}{r}$$

$$= \Sigma I(-2n) r^{-2n-2} x$$

$$= (-2n)\, r^{-2n-2}\, \Sigma x I$$

$$= (-2n)\, r^{-2n-2}\, R \qquad [\Sigma x I = R]$$

~~div grad f = $\nabla \cdot$ grad f~~

$$\operatorname{div}(\operatorname{grad} f) = \operatorname{div}\left(-2n\, r^{-2n-2} R\right)$$

$$= (-2n)\left[\operatorname{grad}\left(r^{-2n-2}\right)\cdot R + r^{-2n-2} \operatorname{div} R\right]$$

$$= (-2n)\left[\Sigma(-2n-2)\, r^{-2n-3} \frac{\partial r}{\partial x}\cdot R + 3\, r^{-2n-2}\right]$$

$$[\operatorname{div} R = 3]$$

$$= (-2n)\left[\Sigma(-2n-2)\, r^{-2n-3} \frac{x}{r}\cdot R + 3\, r^{-2n-2}\right]$$

$$= (-2n)\left[(-2n-2)\, r^{-2n-4}\, \Sigma x I \cdot R + 3\, r^{-2n-2}\right]$$

$$= (-2n)\left[(-2n-2)\, r^{-2n-4}\, R\cdot R + 3\, r^{-2n-2}\right]$$

$$= (-2n)\left[(-2n-2)\, r^{-2n-4}\, r^2 + 3\, r^{-2n-2}\right]$$

$$[R\cdot R = r^2]$$

$$= (-2n)\left[(-2n-2)\, r^{-2n-2} + 3\, r^{-2n-2}\right]$$

$$= (-2n)\, r^{-2n-2}\left[-2n-2+3\right]$$

$$= (2n)(2n-1)\, r^{-2n-2}$$

~~Suppose~~

Now if div grad $f = 0$

$$(2n)(2n-1)\, r^{-2n-2} = 0$$

$$\Rightarrow \quad (2n-1) = 0 \quad , \qquad n \neq 0$$

$$\Rightarrow \quad n = 1/2 \quad , \qquad r \neq 0$$

Que) Prove that

$$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$$

Soln

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z^2}\right)$$

$$\nabla^2 f(r) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(r)$$

$$= \frac{\partial^2}{\partial x^2}\{f(r)\} + \frac{\partial^2}{\partial y^2}\{f(r)\} + \frac{\partial^2}{\partial z^2}\{f(r)\} \quad \text{--- (1)}$$

Now $\frac{\partial}{\partial x}[f(r)] = f'(r)\frac{\partial r}{\partial x}$

$$\frac{\partial}{\partial x}(f(r)) = f'(r)\frac{x}{r} \qquad \left(\frac{\partial r}{\partial x} = \frac{x}{r}\right)$$

$$= x\left(\frac{f'(r)}{r}\right)$$

$\cancel{\frac{\partial f}{\partial x x}} =$

$$\frac{\partial^2}{\partial x^2} f(r) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} f(r)\right]$$

$$= \frac{\partial}{\partial x}\left[x \frac{f'(r)}{r}\right]$$

$$= 1 \cdot \frac{f'(r)}{r} + x\left\{\frac{r f''(r) - f'(r) \cdot 1}{r^2}\right\} \frac{\partial r}{\partial x}$$

$$= \frac{f'(r)}{r} + x\left\{\frac{r f''(r) - f'(r)}{r^2}\right\} \frac{x}{r}$$

$$= \frac{f'(r)}{r} + \frac{x^2}{r^3}\left(r f''(r) - f'(r)\right)$$

$$= \frac{f'(r)}{r} + \frac{x^2}{r^2} f''(r) - \frac{x^2}{r^3} f'(r) \quad \text{--- (2)}$$

Similarly

$$\frac{\partial^2}{\partial y^2}\{f(r)\} = \frac{f'(r)}{r} + \frac{y^2}{r^2} f''(r) - \frac{y^2}{r^3} f'(r) \quad \text{--- (3)}$$

&

$$\frac{\partial^2}{\partial z^2}[f(r)] = \frac{f'(r)}{r} + \frac{z^2}{r^2} f''(r) - \frac{z^2}{r^3} f'(r) \quad \text{--- (4)}$$

Adding (2), (3) & (4) we get

$$\frac{\partial^2}{\partial x^2}\{f(r)\} + \frac{\partial^2}{\partial y^2}\{f(r)\} + \frac{\partial^2}{\partial z^2}\{f(r)\}$$

$$= \frac{3f'(r)}{r} + \frac{(x^2+y^2+z^2)}{r^2} f''(r)$$

$$- \frac{(x^2+y^2+z^2)}{r^3} f'(r)$$

$$= \frac{3f'(r)}{r} + \frac{r^2}{r^2} f''(r) - \frac{r^2}{r^3} f'(r)$$

$$= \frac{3f'(r)}{r} + f''(r) - \frac{1}{r} f'(r)$$

$$= \frac{2f'(r)}{r} + f''(r)$$

$$\Rightarrow \nabla^2 f(r) = \frac{2f'(r)}{r} + f''(r)$$

Hence proved.