Unit – 3.

Sequence.

An ordered set of real numbers $a_1, a_2, \ldots$ is called a sequence and is denoted by (a_n) or $\{a_n\}$.

If the numbers of terms is unlimited, then the sequence is said to be an infinite sequence, and a_n is its general term.

eg: $1, 3, 5, 7 \ldots, (2n-1), \ldots$ 2) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{n}, \ldots$

3) $1, -1, 1, -1, \ldots$ are infinite sequence.

Limit of a sequence: A sequence is said to tend to a limit l, if for every $\epsilon > 0$, a value N of n can be found such that $|a_n - l| < \epsilon \;\forall\; n \geq N$

Symbolically $\lim\limits_{n \to \infty} a_n = l$ or $(a_n) \to l$ as $n \to \infty$.

Convergence. If a sequence (a_n) has a finite limit, it is called a convergence sequence. If (a_n) is not convergent, it is said to be divergent.

$1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, \ldots \to 0$ as $n \to \infty$ $\therefore$ convergent

$1, 3, 5, 7 \ldots \not\to$ finite limit $\therefore$ divergent

$1, -1, 1, -1, \ldots \not\to$ finite limit $\therefore$ divergent.

Bounded Sequence: A sequence $\langle a_n \rangle$ or (a_n) is said to be bounded, if $\exists$ a number k such that $a_n < k$ for every n.

Monotonic Sequence.

$\langle a_n \rangle$ is said to increase steadily or to decrease steadily according as $a_{n+1} \geq a_n$ or $a_{n+1} \leq a_n$ for all values of n

Note:

i) If $\lim_{n\to\infty} (a_n) = l$ is finite and unique, then the seq is convergent

ii) If $\lim_{n\to\infty} a_n$ is infinite $(\pm\infty)$, then the seq is said to be divergent.

iii) If $\lim_{n\to\infty} a_n$ is not unique, then (a_n) is said to be oscillatory $-1, +1, -1, +1 \cdots\cdots \to -1$ or $+1$

Examine the following seq for convergence.

1) $a_n = \dfrac{n^2-2n}{3n^2+n}$ 2) 2^n 3) $3+(-1)^n$

$$\lim_{n\to\infty} \frac{n^2-2n}{3n^2+n} = \lim_{n\to\infty} \frac{1-\frac{2}{n}}{3+\frac{1}{n}} = \frac{1}{3} \text{ finite and unique.}$$

$\therefore \langle a_n \rangle$ — convergent.

2). $\lim_{n\to\infty} 2^n = \infty$. Hence seq$(2^n)$ is divergent.

3. $\lim_{n\to\infty} 3+(-1)^n = 3+1 = 4$ if n - even

$3-1 = 2$ if n - odd.

That is seq does not have a unique limit.

Hence it oscillates

1) $\lim_{n\to\infty} \dfrac{3n-1}{1+2n} = \lim_{n\to\infty} \dfrac{3-\frac{1}{n}}{\frac{1}{n}+2} = \dfrac{3}{2}$

2) $\lim_{n\to\infty} 1+\dfrac{2}{n} = 1$ 4) $\lim_{n\to\infty} 1+\dfrac{(-1)^n}{n} = 1$

3) $\lim_{n\to\infty} \dfrac{1}{2n} = 0$ 5) $\lim_{n\to\infty} \dfrac{1}{2n} = 0$ 6) $\lim_{n\to\infty} \left(\dfrac{n}{n-1}\right)^2 = 1$

Series:

Def: If $u_1, u_2, \dots u_n, \dots$ be an infinite seq of real numbers, then $u_1+u_2+u_3+\dots$ is called an infinite series. An infinite series is denoted by $\sum u_n$ and sum of its first n-terms is denoted by S_n.

Convergence divergence and oscillation of a series.

Consider the infinite series

$\sum u_n = u_1+u_2+u_3+\dots+u_n+\dots$ and let sum of first n terms be $S_n = u_1+u_2+\dots+u_n$.

i) If S_n tends to a finite limit as $n \to \infty$, the series $\sum u_n$ is said to be convergent.

ii) If S_n tends to $\pm\infty$ as $n \to \infty$, the series $\sum u_n$ is said to be divergent.

iii) If S_n does not tend to a unique limit as $n \to \infty$ then the series $\sum u_n$ is said to be oscilatory or non convergent.

1. Examine the convergence of the series

i) $1+2+3+\dots+n+\dots\infty$ ii) $5-4-1+5-4-1\dots$

i) $S_n = 1+2+\dots+n = \frac{n(n+1)}{2}$

$\lim_{n\to\infty} S_n = \frac{1}{2}\lim_{n\to\infty}(n^2+n) \to \infty$ as $n \to \infty$

Hence this series is divergent.

ii) $S_n = 5 - 4 - 1 + 5 - 4 - 1 + 5 - 4 - 1 + \cdots$ n terms

$$= \begin{cases} 0 & \text{no of terms is } 3m \\ 5 & \text{no of terms is } 3m+1 \\ 1 & \text{no of terms is } 3m+2 \end{cases}$$

S_n does not tend to a unique limit.

Hence this series is oscillatory.

Geometric series.

Show that the series $1 + r + r^2 + r^3 + \cdots \infty$

i) convergent if $|r| < 1$ ii) diverges if $r \geq 1$ and

iii) oscillates if $r \leq -1$.

Solution.

$$S_n = 1 + r + r^2 + \cdots + r^{n-1}$$

case i) If $|r| < 1$, $\lim_{n \to \infty} r^n = 0$.

$$S_n = \frac{1 - r^n}{1 - r} = \frac{1}{1-r} - \frac{r^n}{1-r}$$

$$\lim_{n \to \infty} S_n = \frac{1}{1-r} \quad \left(\because \lim_{n \to \infty} r^n = 0 \right).$$

case ii) when $r > 1$, $\lim_{n \to \infty} r^n = \infty$.

$$S_n = \frac{r^n - 1}{r - 1} = \frac{r^n}{r-1} - \frac{1}{r-1}$$

$$\lim_{n \to \infty} S_n = \infty$$

$\therefore$ series is divergent.

case iii) when $r = 1$ then $S_n = 1 + 1 + 1 + \cdots = n$.

$\therefore \lim_{n \to \infty} S_n = \infty$. $\therefore$ series is divergent.

Case iv) when $r = -1$.

Then the series becomes $1 - 1 + 1 - 1 \dots$ which is an oscillatory series.

Necessary condition for convergence.

If a positive term series $\sum u_n$ is convergent, then $\lim_{n \to \infty} u_n = 0$.

$u_n = S_n - S_{n-1}$

$\lim_{n \to \infty} u_n = 0$.

Note:

If $\lim_{n \to \infty} u_n \neq 0$, the series $\sum u_n$ must be divergent.

Comparison test

1. If two positive terms series $\sum u_n$ and $\sum v_n$ be such that

i) $\sum v_n$ converges ii) $u_n \le v_n$ for all values of n, then $\sum u_n$ also converges.

ii) $\sum v_n$ diverges, ii) $u_n \ge v_n$ for all values of n, then $\sum u_n$ also diverges.

1. Show that the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \dots$$

i) converges for $p > 1$ ii) diverges for $p \le 1$.

Comparison test (Limit form)

If two positive term series $\sum u_n$ and $\sum v_n$ be such that $\lim_{n\to\infty} \frac{u_n}{v_n}$ = finite quantity $(\neq 0)$

then $\sum u_n$ and $\sum v_n$ converge or diverge together.

D'Alembert's Ratio test

In a positive term series $\sum u_n$, if

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda$, then the series converges for $\lambda < 1$ and diverges for $\lambda > 1$. (Ratio test fails when $\lambda = 1$).

Test for convergence the series.

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \cdots \infty . \qquad n=1$$

we have $u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$ $\qquad u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$.

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^{2n-2}}{(n+1)\sqrt{n}} \times \frac{(n+2)\sqrt{n+1}}{x^{2n}}.$$

$$= \lim_{n\to\infty} \frac{n+2}{n+1}\left(\frac{n+1}{n}\right)^{1/2} x^{-2}$$

$$= \lim_{n\to\infty} \frac{1+\frac{2}{n}}{1+\frac{1}{n}}\left(1+\frac{1}{n}\right)^{1/2} x^{-2}.$$

$$= x^{-2}.$$

Hence $\sum u_n$ converges if $x^{-2} > 1$ ie $x^2 < 1$ and

$\sum u_n$ diverges if $x^{-2} < 1$ ie $x^2 > 1$.

If $x^2 = 1$ $\qquad u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}} \cdot \frac{1}{1+\frac{1}{n}}$ $\qquad v_n = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1,$$ a finite quantity.

$\therefore \sum u_n$ and $\sum v_n$ converge or diverge together

But $\sum v_n = \sum \frac{1}{n^{3/2}}$ is a convergent series.

$\therefore \sum u_n$ is also convergent.

$\therefore$ Given series converges if $x^2 \leq 1$ and diverges if $x^2 > 1$.

Comparison test (limit form).

1. Test the convergence of the series

$$\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \cdots$$

$$u_n = \frac{2n-1}{n(n+1)(n+2)} \qquad n=1$$

$$= \frac{n\left(2-\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = \frac{1}{n^2} \frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}.$$

$$v_n = \frac{1}{n^2}.$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{n^2} \frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} \times \frac{n^2}{1} = \lim_{n\to\infty} \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$= 2$ which is finite and non zero.

$\therefore$ both $\sum u_n$ and $\sum v_n$ converge or diverge together

But $\sum v_n = \sum \frac{1}{n^2}$ $p = 2 > 1$ is convergent by p-series test. $\therefore \sum u_n$ also converges

2. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

$$\left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots$$

$$= 1 + 1 + \frac{\left(1-\frac{1}{n}\right)}{2!} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!} + \cdots$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$= e.$$

2. Test for convergence the series.

$$1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \cdots$$

$$u_n = \frac{n^n}{(n+1)^{n+1}} = \frac{1}{(n+1)}\left(\frac{n}{n+1}\right)^n \text{ (ignoring the first term)}$$

$$v_n = \frac{1}{n}.$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{n}{n+1}\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}\cdot\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$= 1\cdot\frac{1}{e} \neq 0.$$

Now since $\sum v_n$ is divergent, $\therefore \sum u_n$ is also divergent.

3. Test the conver $\sqrt{\frac{1}{2^3}} + \sqrt{\frac{2}{3^3}} + \sqrt{\frac{3}{4^3}} + \cdots$

$$u_n = \sqrt{\frac{n}{(n+1)^3}} = \frac{n^{1/2}}{n^{3/2}\left(1+\frac{1}{n}\right)^{3/2}} \quad v_n = \frac{n^{1/2}}{n^{3/2}} = \frac{1}{n}, \quad \frac{u_n}{v_n} = \frac{1}{\left(1+\frac{1}{n}\right)^{3/2}}$$

$\lim_{n\to\infty}\frac{u_n}{v_n} = 1$ $\quad \sum v_n = \frac{1}{n}$ is a divergent series

$\therefore \sum u_n$ also diverge.

Cauchy's root test

In a positive series $\sum u_n$ if $\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lambda$

then the series converges for $\lambda<1$ and diverges for $\lambda>1$. [Cauchy's root test fails when $\lambda = 1$].

1. Test the convergence of the series

i) $\sum \frac{n^3}{3^n}$

$$u_n = \frac{n^3}{3^n}$$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{n^3}{3^n}\right)^{1/n} = \cancel{\left(n^{\frac{1}{n}}\right)^3}$$

$$= \lim_{n\to\infty} \frac{n^{3/n}}{3} = \lim_{n\to\infty}\frac{\left(n^{1/n}\right)^3}{3} = \frac{1}{3} < 1.$$

$\therefore$ given series converges.

$y = \cancel{\lim_{n\to\infty}} n^{\frac{1}{n}}$.

$\log y = \cancel{\frac{1}{n}\log n}$.

$\cancel{n^\circ}$

$0 = \infty^0$

$y = \lim_{n\to\infty} n^{\frac{1}{n}}$.

$\log y = \lim_{n\to\infty} \frac{1}{n}\log n$

$= \lim_{n\to\infty}\frac{\log n}{n}$

$= \lim_{n\to\infty}\frac{\frac{1}{n}}{1}$

$= 0$.

$\boxed{y = 1}$

2. $\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \cdots$

$$u_n = \left(\frac{n+1}{n+2}\right)^n x^n \quad \text{(Ignoring the first term)}.$$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left(\frac{n+1}{n+2}\right)^n x^n\right]^{1/n} = \lim_{n\to\infty}\frac{n(1+\frac{1}{n})}{n(1+\frac{2}{n})}x = x.$$

the series converges for $x<1$ and diverges for $x>1$.

when $x=1$ $u_n = \left(\frac{n+1}{n+2}\right)^n = \frac{1}{\left(\frac{n+2}{n+1}\right)^n}$

$$n+1\overline{)\,n+2\,}\,1$$
$$\frac{n+1}{+1}$$
$$1+\frac{1}{n+1}.$$

$$= \frac{1}{\left(1+\frac{1}{n+1}\right)^n} = \frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}\cdot\left(1+\frac{1}{n+1}\right) \qquad \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right)^n$$

$\lim_{n\to\infty} u_n = \frac{1}{e} \neq 0 \therefore \sum u_n$ ~~converges~~ diverges.

Discuss the convergence of the following series.

i) $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots$

$$u_n=\left(\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{n+1}{n}\right)^{-n}$$

$$u_n^{\frac{1}{n}}=\left(\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{n+1}{n}\right)^{-1}$$

$$=\left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^{n}-1\right]^{-1}$$

$$=\left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^{n}-1\right]^{-1}$$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}}=(e-1)^{-1}=\frac{1}{e-1}<1.$$

$\therefore$ given series converges.

Convergence of exponential series.

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$$

$$u_n=\frac{x^n}{n!} \quad u_{n+1}=\frac{x^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\frac{x^{n+1}}{(n+1)!}\times\frac{n!}{x^n}=\frac{1}{n+1}\cdot x=0 \cdot \text{ as } n\to\infty \cdot \text{ i.e } 0<1$$

$\therefore e^x$ converges for all values of x.

2) ~~Comparisom~~ Comparison test

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\cdots\infty$$

$$\cancel{\frac{(n+1)^{n+1}}{(n+2)^{n+2}} = \frac{1}{(n+2)}\left(\frac{n+1}{n+2}\right)^{n+1}}$$

$$\frac{1}{4\cdot7\cdot10}+\frac{4}{7\cdot10\cdot13}+\frac{9}{10\cdot13\cdot16}+\cdots$$

$$u_n = \frac{n^2}{(3n+1)(3n+4)(3n+7)}$$

$$= \frac{n^2}{n^3(3+\frac{1}{n})(3+\frac{4}{n})(3+\frac{7}{n})}$$

$$= \frac{1}{n}\left(\frac{1}{\quad}\right).$$

$$v_n = \frac{1}{n}.$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \frac{1}{27} \neq 0$$

$\sum v_n = \sum \frac{1}{n^1}$ divergent $(p=1)$

2) $$\sum_{n=1}^{\infty}\sqrt{\frac{3^n-1}{2^n+1}} =$$

$$u_n = \frac{(3^n-1)^{1/2}}{(2^n+1)^{1/2}} = \frac{3^{n/2}\left(1-\frac{1}{3^n}\right)^{1/2}}{2^{n/2}\left(1+\frac{1}{2^n}\right)^{1/2}} = \left(\frac{3}{2}\right)^{n/2}$$

$v_n = \left(\frac{3}{2}\right)^n$ $\qquad \lim_{n\to\infty}\frac{u_n}{v_n} = 1$ ~~f~~ finite

$\sum v_n = \sum\left(\frac{3}{2}\right)^n$ G. seri $r=\frac{3}{2}>1$ diverges

3) $\frac{\sqrt{2}-1}{3^3-1}+\frac{\sqrt{3}-1}{4^3-1}+\frac{\sqrt{4}-1}{5^3-1}+\cdots$

$$u_n=\frac{\sqrt{n+1}-1}{(n+2)^3-1}=\frac{\sqrt{n}\left(\frac{\sqrt{n+1}}{\sqrt{n}}-\frac{1}{\sqrt{n}}\right)}{n^3\left[\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right]}$$

$$=\frac{\sqrt{n}\left(\sqrt{\frac{n+1}{n}}-\frac{1}{\sqrt{n}}\right)}{n^3\left(\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right)}$$

$$=\frac{1}{n^{5/2}}\left[\frac{\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}}{\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}}\right].$$

$\Sigma\frac{1}{n^2}$

$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}$

$\sqrt{n}^{1/2}$

$\frac{1}{2^{1/2}}+\frac{1}{3^{1/2}}$

$\Sigma\frac{1}{n^p}$

$\Sigma\frac{1}{}$

$\frac{1}{1^p}+\frac{1}{2^p}+$

$\cdots$

Ratio test.

i) $\Sigma\frac{n!}{(n^n)^2}$.

$u_n=\frac{n!}{(n^n)^2}\qquad u_{n+1}=\frac{(n+1)!}{[(n+1)^{n+1}]^2}$.

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{n!}{n^{2n}}\times\frac{(n+1)^{2n+2}}{(n+1)!}=\lim_{n\to\infty}\frac{n!}{(n+1)!}\cdot\frac{(n+1)^{2(n+1)}}{n^{2n}}$$

$$=\lim_{n\to\infty}\frac{1}{(n+1)}\cdot\frac{(n+1)^{2(n+1)}}{n^{2n}}$$

$$=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{2n}\cdot\frac{(n+1)^2}{(n+1)}$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}\cdot(n+1).$$

$$=\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\right]^2(n+1)$$

$$=e\cdot\lim_{n\to\infty}(n+1)\Rightarrow\infty.$$

$\therefore\lim_{n\to\infty}\frac{u_{n+1}}{u_n}\to 0$ as $n\to\infty$.

$\therefore\lim_{n\to\infty}\frac{u_{n+1}}{u_n}<1\Rightarrow$ series converges.

ii) $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \cdots$

$$u_n = \frac{n!}{(n^n)} \qquad u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}.$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{e} \not< 1 \quad \underline{\text{convergent}}$$

3) $\frac{x}{1+x} + \frac{x^2}{1+x^2} + \cdots$

$$u_n = \frac{x^n}{1+x^n} \qquad u_{n+1} = \frac{x^{n+1}}{1+x^{n+1}}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{1+x^{n+1}} \times \frac{1+x^n}{x^n}$$

$$= \left(\frac{x}{1+x^{n+1}}\right)\left(1+x^n\right) = \frac{x + x^{n+1}}{1+x^{n+1}}$$

$$= \frac{\cancel{x^{n+1}}\left(1 + \frac{x}{x^{n+1}}\right)}{\cancel{x^{n+1}}\left(1 + \frac{1}{x^{n+1}}\right)}$$

$$\begin{array}{r} 1 \\ x^{n+1}+1 \,\big)\, x^{n+1} + x \\ x^{n+1} + 1 \\ \hline x \end{array}$$

$$1 + \frac{x}{x^{n+1}+1}.$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x \text{ if } x < 1.$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u} = 1 \text{ if } x \geq 1.$$

Hence the given series converges for $x < 1$ and fails for $x \geq 1$

when $x = 1$, $\sum u_n = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \infty$ which is divergent. $\therefore$ $\underline{\sum u_n \text{ diverges for } x \geq 1}$.

4) $\frac{x}{1\cdot 2}+\frac{x^2}{2\cdot 3}+\cdots$

$$u_n=\frac{x^n}{n(n+1)} \qquad u_{n+1}=\frac{x^{n+1}}{(n+1)(n+2)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{x^{n+1}}{\cancel{(n+1)}(n+2)}\times\frac{n\cancel{(n+1)}}{x^n}=\left(\frac{n}{n+2}\right)x$$

$$\lim_{n\to\infty}\left(\frac{n}{n+2}\right)x=\lim_{n\to\infty}\left(\frac{1}{1+\frac{2}{n}}\right)x=x.$$

If $x<1$ — converges

$x>1$ diverges

$x=1$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots$$

$$u_n=\frac{1}{n(n+1)}=\frac{1}{n^2(1+\frac{1}{n})}$$

$$v_n=\frac{1}{n^2}.$$

$\sum\frac{1}{n^2}$ converges.

$\therefore$ $x=1$ series converges

5) $2+\frac{3}{2}x+\frac{4}{3}x^2+\cdots$

$$u_n=\frac{n+1}{n}x^{n-1} \qquad u_{n+1}=\frac{n+2}{n+1}x^n.$$

$$\lim_{n\to\infty}\frac{u_{n+2}}{u_{n+1}}=\lim_{n\to\infty}\frac{n(n+2)}{(n+1)^2}=x.$$

$x=1$

$$u_n=\frac{n+1}{n}=1+\frac{1}{n}.$$

$\lim_{n\to\infty}u_n=1\neq 0$ $\therefore$ series diverges

Convergence of Logarithmic series.

P.T $\log x = \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots (-1)^n \frac{x^n}{n} + \cdots$

is convergent for $-1 < x < 1$.

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(-1)^{n+1} x^{n+1}}{n+1} \times \frac{n}{(-1)^n x^n}$$

$$= \lim_{n\to\infty} \left(-\frac{n}{n+1} x \right)$$

$$= \lim_{n\to\infty} -x\left(\frac{1}{1+\frac{1}{n}}\right) = -x.$$

$\lambda < 1$

$-x < 1$ or

$x > -1$

$-1 \quad 1$

hence the series converges for

30025

$|x| < 1$ and diverges for $|x| > 1$.

when $x = 1$ the series being $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$ is

convergent (by Leibnitz series).

when $x = -1$ the series $-(1 + \frac{1}{2} + \frac{1}{3} \cdots)$ is divergent

Hence the series converges for $-1 < x < 1$.

Alternating series.

A series in which the terms are alternately +ve or -ve is called an alternating series.

Leibnitz series: An alternating series $u_1 - u_2 + u_3 - u_4 \cdots$ converges i) each term is numerically less than the preceding term and ii) $\lim_{n\to\infty} u_n = 0$. or i) $u_{n+1} - u_n < 0$

eg: $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots \quad \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}} < 0$ & $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

$\therefore$ converges.

1. Test the convergence of the series

$$\sqrt{\frac{1}{2^3}} + \sqrt{\frac{2}{3^3}} + \sqrt{\frac{3}{4^3}} + \cdots$$

Here $u_n = \sqrt{\frac{n}{(n+1)^3}} = \frac{n^{\frac{1}{2}}}{n^{3/2}(1+\frac{1}{n})^{3/2}} =$

$$v_n = \frac{n^{\frac{1}{2}}}{n^{3/2}} = \frac{1}{n}.$$

$$\frac{u_n}{v_n} = \frac{n^{\frac{1}{2}}.n}{(n+1)^{3/2}} = \frac{n^{\frac{3}{2}}}{(n+1)^{3/2}} = \left(\frac{n}{n+1}\right)^{3/2} = \frac{1}{\left(\frac{n+1}{n}\right)^{3/2}}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^{3/2}}.$$

$\lim_{n\to\infty} \frac{u_n}{v_n} = 1$ (finite non zero number).

$\therefore$ ~~Σu_n~~ by comparision test Σu_n and Σv_n converge or diverge together.

but by p-series test, $\Sigma \frac{1}{n^1} = \Sigma v_n$ is divergent

Hence by comparison test Σu_n is also divergent.