Module 1.1. Introductory Concepts: On Sequences and Recurrence Relation

Some of the best ways to describe events is through identifying similarities (or differences) and significant patterns between them. In this section, you will be introduced or refreshed to two necessary ideas for better understanding of the Fibonacci sequence – sequences and recurrence relation.

Sequences

A good colloquialism for the definition of a sequence is that a sequence is an ordered list of number $a_1, a_2, a_3, \dots, a_n, \dots$ The integer n is called the index of a_n. For example, $3, 6, 9, 12, \dots, 3n, \dots$ where $a_1 = 3, a_2 = 6, a_3 = 9, a_4 = 12, \dots a_n = 3n, \dots$ Sequences can be described by writing rules that specify their terms such as $a_n = n^2$ or $a_n = (-1)^n$.

Definition. A *sequence* is a function from a subset of integer $\mathbb{Z}$ to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.

Example 1. Write the first five terms of the sequence $b_n = \frac{(-1)^n}{n}$.

Answer. Evaluate the expression b_n for $n = 1$, we have, $b_1 = \frac{(-1)^1}{1} = \frac{-1}{1} = -1$. So, $b_1 = -1$. Similarly, we have the following:

$b_2 = \frac{(-1)^2}{2} = \frac{1}{2}$ $\quad b_3 = \frac{(-1)^3}{3} = \frac{-1}{3} = -\frac{1}{3}$ $\quad b_4 = \frac{(-1)^4}{4} = \frac{1}{4}$ $\quad b_5 = \frac{(-1)^5}{5} = \frac{-1}{5} = -\frac{1}{5}$

So the first five terms of the sequence $b_n = \frac{(-1)^n}{n}$ are $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}$.

Example 2. Consider the sequence $\{a_n\}$ where $a_n = \frac{1}{n}$. List the terms $\{a_1, a_2, a_3, a_4, \dots\}$

Answer. Starting with $n = 1, a_1 = \frac{1}{1} = 1$. For $n = 2, a_2 = \frac{1}{2}$. For $n = 3, a_3 = \frac{1}{3}$. For $n = 4, a_4 = \frac{1}{4}$. The sequence $\{a_n\}$ is $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$.

Recurrence Relation

From the discussion above, sequences were specified or described by providing explicit formulas for their terms. There are many other ways to specify a sequence. For example, we can provide one or more initial terms together with a rule for determining subsequent terms from those that proceed them. For example, we have an equation of the form $a_n = 7 + a_{n-1}$. Notice we have the term a_{n-1} on the right side of the equation. This indicates that a_n can be expressed in terms of the previous term a_{n-1} added to the constant 7.

Given an initial term of the sequence we later define to be as the *initial condition*, we can derive all the terms of the sequences by recursively define a sequence using the previous terms. The process of recursively defining a sequence is what we call a *recurrence relation.*

Definition. A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, a_2, \dots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. The *initial conditions* for a sequence specify the terms before n_0 (before the recurrence relation takes effect).
The recurrence relations together with the initial conditions uniquely determines the sequence.

Example 3. Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n \geq 1$ with $a_0 = 2$. Find the terms a_1, a_2, a_3.

Answer. Start with $n = 1$ to find a_1 using the initial condition $a_0 = 2$

$a_1 = a_{1-1} + 3 = a_0 + 3 = 2 + 3 = 5$ so we have $a_1 = 5$.

Now, use $n = 2$ to find a_2 using the previous term $a_1 = 5$.

$a_2 = a_{2-1} + 3 = a_1 + 3 = 5 + 3 = 8$ so we have $a_2 = 8$.

Lastly, use $n = 3$ to find a_3 using the previous term $a_2 = 8$.

$a_3 = a_{3-1} + 3 = a_2 + 3 = 8 + 3 = 11$ so we have $a_3 = 11$.

The terms are: $a_1 = 5, a_2 = 8$ and $a_3 = 11$.

Example 4. Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n \geq 2$ and suppose $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

Answer. Start with $n = 2$ to find a_2 with initial conditions $a_0 = 3$ and $a_1 = 5$,

$a_2 = a_{2-1} - a_{2-2} = a_1 - a_0 = 5 - 3 = 2$ so we have $a_2 = 2$.

Lastly, use $n = 3$ to find a_3 using the previous term $a_2 = 2$.

$a_3 = a_{3-1} - a_{3-2} = a_2 - a_1 = 2 - 5 = -3$ so we have $a_3 = -3$.

The terms are: $a_2 = 2$ and $a_3 = -3$.