# Chapter 9: Tests of Hypotheses for a Single Sample

## LEARNING OBJECTIVES

1. Introduction
2. Test on the μ of NORMDIST
   - $\sigma^2$ known
   - $\sigma^2$ unknown
3. Test on the *p*: Large-sample

Introduction

Test on the $\mu$ of NORMDIST

$\sigma^2$ known

$\sigma^2$ unknown

Test on the p

Summary

**Definition** — Statistical Hypotheses

A *statistical hypothesis* is a statement about the parameters of one or more populations.

**Example**

Suppose that we are interested in the burning rate of a solid propellant used to power aircrew escape systems.

- Burning rate is a random variable
- We are interested in deciding whether or not the mean burning rate is 50 centimeters per second.

Introduction

Test on the μ of NORMDIST

$\sigma^2$ known

$\sigma^2$ unknown

Test on the p

Summary

## Two-sided Alternative Hypothesis

$$H_0: \ \mu = 50 \ ; \ H_1: \ \mu \neq 50$$

Null hypothesis

Alternative hypothesis

## One-sided Alternative Hypotheses

$$H_0: \ \mu = 50 \ ; \ H_1: \ \mu > 50$$

$$H_0: \ \mu = 50 \ ; \ H_1: \ \mu < 50$$

Introduction

Test on the μ of NORMDIST

$\sigma^2$ known

$\sigma^2$ unknown

Test on the p

Summary

## Test of a Hypothesis

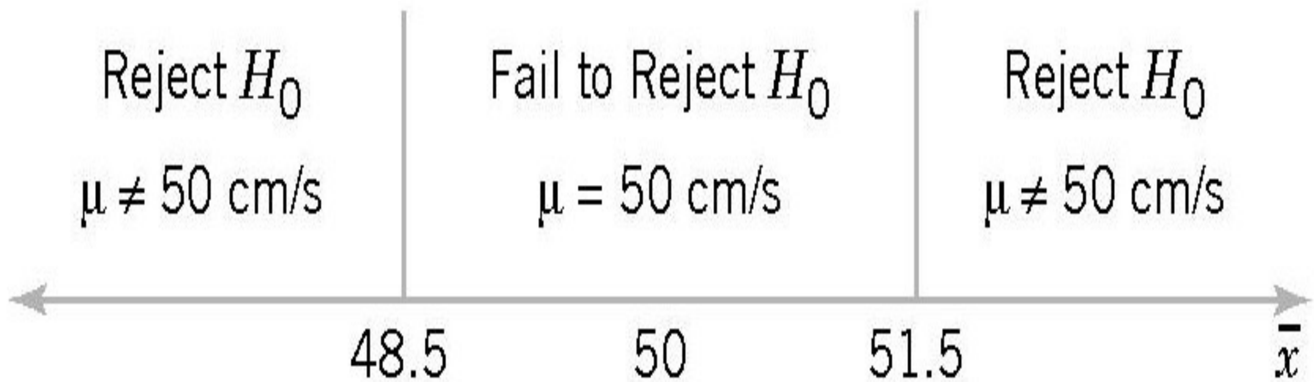
- A procedure leading to a decision about a particular hypothesis

- Hypothesis-testing procedures rely on using the information in a random sample from the population of interest.

- If this information is *consistent* with the hypothesis, then we will conclude that the hypothesis is true; if this information is *inconsistent* with the hypothesis, we will conclude that the hypothesis is false.

## Example 1

$$H_0: \ \mu = 50\,; \ H_1: \ \mu \neq 50$$




Decision criteria for testing $H_0$: $\mu = 50$; $H_1$: $\mu \neq 50$

# ERRORS

Introduction

Test on the μ of NORMDIST

$\sigma^2$ known

$\sigma^2$ unknown

Test on the p

Summary

**Table 9-1 Decisions in Hypothesis Testing**

| Decision | $H_0$ Is True | $H_0$ Is False |
|---|---|---|
| Fail to reject $H_0$ | no error | type II error |
| Reject $H_0$ | type I error | no error |

$\alpha = P(\text{ type I error }) = P(\text{ reject } H_0 \text{ when } H_0 \text{ is true })$

Significance level

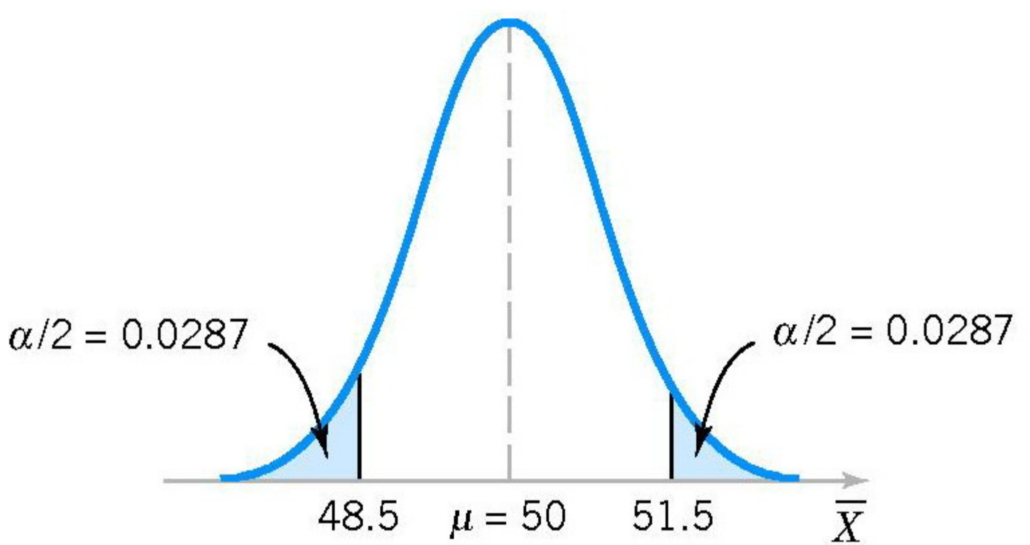
$\beta = P(\text{ type II error}) = P(\text{ fail to reject } H_0 \text{ when } H_0 \text{ is false})$

Introduction
Test on the μ of NORMDIST
$\sigma^2$ known
$\sigma^2$ unknown
Test on the p
Summary

To calculate α, we assume that σ = 2.5 and n = 10.

$$\alpha = P(\overline{X} < 48.5 \ when \ \ \mu = 50) + P(\overline{X} > 51.5 \ when \ \ \mu = 50)$$
$$= P(Z < -1.90) + P(Z > 1.90) \simeq 0.0574$$

$$\boxed{= \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} = \frac{\overline{X} - 50}{2.5 / \sqrt{10}}}$$



$$\beta = P(48.5 < \overline{X} < 51.5 \ \ when \ \ \mu = 52)$$
$$= P(-4.43 < Z < -0.63) \simeq 0.2643$$

$$\beta = P(48.5 < \overline{X} < 51.5 \ \ when \ \ \mu = 50.5)$$
$$= P(-2.53 < Z < 1.27) \simeq 0.8923$$

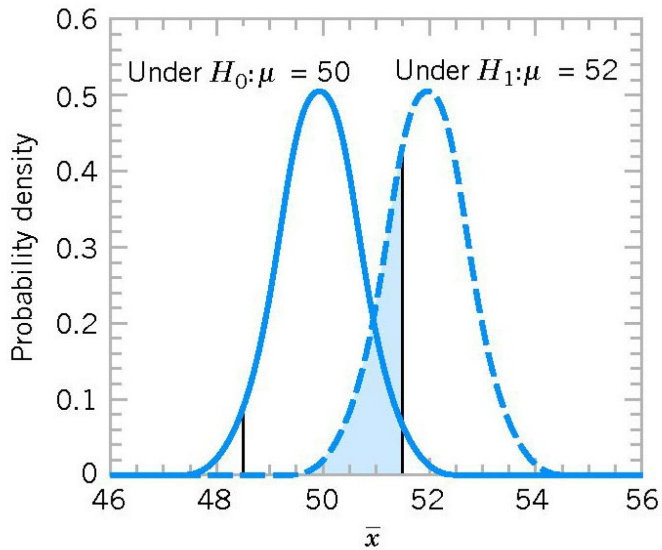



Figure 9-3 The probability of type II error when $\mu = 52$ and $n = 10$.

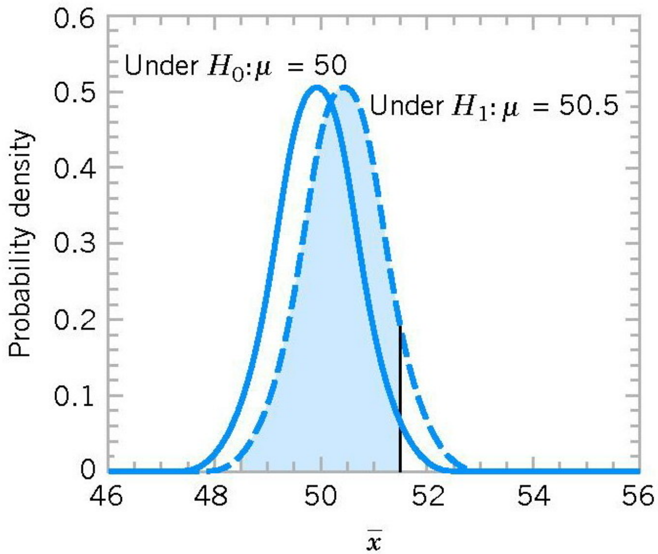



Figure 9-4 The probability of type II error when $\mu = 50.5$ and $n = 10$.

# ERRORS

Introduction
Test on the μ of NORMDIST
$\sigma^2$ known
$\sigma^2$ unknown
Test on the p
Summary

| Acceptance Region | Sample Size | $\alpha$ | $\beta$ at $\mu = 52$ | $\beta$ at $\mu = 50.5$ |
|---|---|---|---|---|
| $48.5 < \bar{x} < 51.5$ | 10 | 0.0576 | 0.2643 | 0.8923 |
| $48 < \bar{x} < 52$ | 10 | 0.0114 | $\boxed{0.5000}$ | $\boxed{0.9705}$ |
| $48.5 < \bar{x} < 51.5$ | 16 | 0.0164 | 0.2119 | $\boxed{0.9445}$ |
| $48 < \bar{x} < 52$ | 16 | $\boxed{0.0014}$ | $\boxed{0.5000}$ | $\boxed{0.9918}$ |

Introduction

Test on the μ of NORMDIST

$\sigma^2$ known

$\sigma^2$ unknown

Test on the p

Summary

1. The size of the critical region, and consequently the probability of a type I error , can always be reduced by appropriate selection of the critical values.
2. Type I and type II errors are related: $\beta \downarrow$ then $\alpha \uparrow$ and $\alpha \downarrow$ then $\beta \uparrow$ , provided that $n$ fix.
3. An increase in sample size will generally reduce both $\alpha$ and $\beta$, provided that the critical values are held constant.
4. When the null $H_0$ is false, $\beta \uparrow$ as the true value of the parameter approaches the value hypothesized in $H_0$,$\beta \downarrow$ as the difference between the true mean and the hypothesized value increases.

# POWER OF A TESTING

Introduction

Test on the μ of NORMDIST

$\sigma^2$ known

$\sigma^2$ unknown

Test on the p

Summary

**Definition** power of a statistical test

The **power** of a statistical test is the probability of rejecting the null hypothesis $H_0$ when the alternative hypothesis is true.

- The power is computed as 1 - b, and power can be interpreted as *the probability of correctly rejecting a false null hypothesis*.
- We often compare statistical tests by comparing their power properties.

## Hypothesis-testing problems

Two-sided test: $H_0$: $\mu = \mu_0$; $H_1$: $\mu \neq \mu_0$

One-sided test: $H_0$: $\mu = \mu_0$; $H_1$: $\mu > \mu_0$

$H_0$: $\mu = \mu_0$; $H_1$: $\mu > \mu_0$

# TEST ON THE μ OF NORMDIST: $\sigma^2$ KNOWN



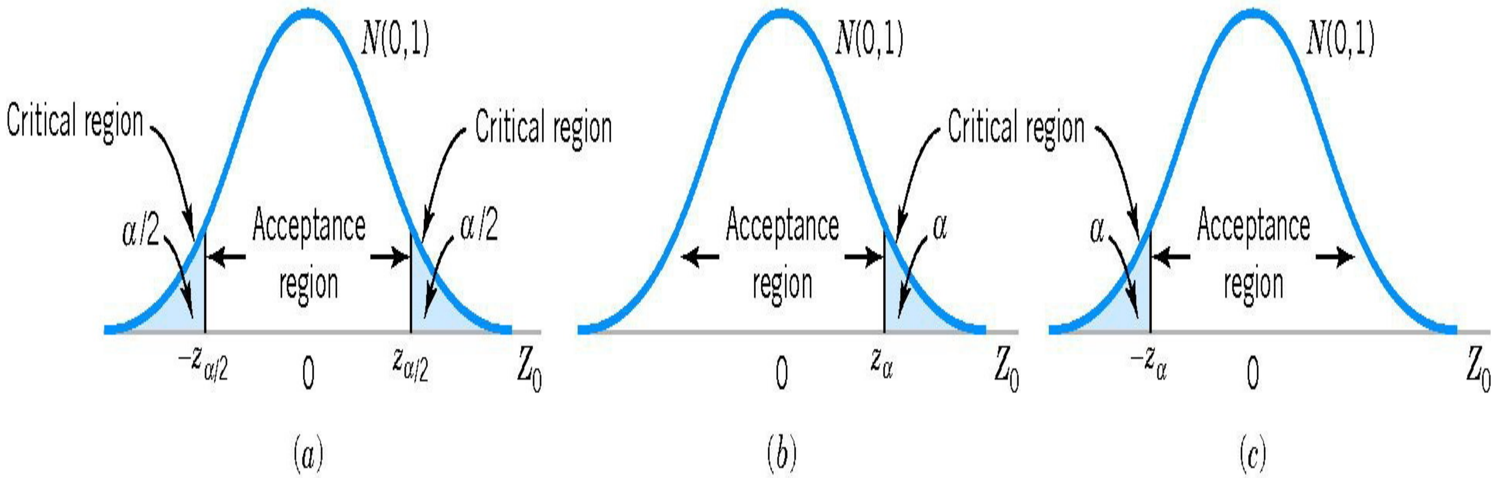



**Figure 9-6** The distribution of $Z_0$ when $H_0$: $\mu = \mu_0$ is true, with critical region for (a) the two-sided alternative $H_1: \mu \neq \mu_0$, (b) the one-sided alternative $H_1: \mu > \mu_0$, and (c) the one-sided alternative $H_1: \mu < \mu_0$.

Introduction

Test on the μ of NORMDIST

$\sigma^2$ known

$\sigma^2$ unknown

Test on the p

Summary

## Hypothesis Tests on the Mean — *z*-test

Null hypothesis: $H_0$: $\mu = \mu_0$

Test statistic: $$Z_0 = \frac{\overline{X} - \mu_0}{\sigma / \sqrt{n}}$$

| Alternative hypothesis | Rejection criteria |
| --- | --- |
| $H_1$: $\mu \neq \mu_0$ | $\lvert z_0 \rvert > z_{\alpha/2}$ |
| $H_1$: $\mu > \mu_0$ | $z_0 > z_\alpha$ |
| $H_1$: $\mu < \mu_0$ | $z_0 < -z_\alpha$ |

# TEST ON THE μ OF NORMDIST: $\sigma^2$ KNOWN

Introduction

Test on the μ of NORMDIST

$\sigma^2$ known

$\sigma^2$ unknown

Test on the p

Summary

## Example 2

A melting point test of $n = 10$ samples of a binder used in manufacturing a rocket propellant resulted in $\bar{x} = 154.2^0$. Assume that melting point is normally distributed with $\sigma = 1.5^0$.

Test $H_0$: $\mu = 155$; $H_1$: $\mu \neq 155$ using $\alpha = 0.05$.

Solution: Test statistic:

$$z_0 = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} = \frac{154.2 - 155}{1.5 / \sqrt{10}} \simeq -1.69$$

$z_{\alpha/2} = z_{0.025} = 1.96.$

$|z_0| < z_{\alpha/2}.$

Fail to reject $H_0$ at $\alpha = 0.05$

Introduction

Test on the μ of NORMDIST

$\sigma^2$ known

$\sigma^2$ unknown

Test on the p

Summary

## *P*-Values in Hypothesis Tests

**Definition**

The ***P*-value** is the smallest level of significance that would lead to rejection of the null hypothesis $H_0$ with the given data.

$$P = \begin{cases} 2[1 - \Phi(|z_0|)] & \text{for a two-tailed test: } H_0\text{: } \mu = \mu_0 \quad H_1\text{: } \mu \neq \mu_0 \\ 1 - \Phi(z_0) & \text{for a upper-tailed test: } H_0\text{: } \mu = \mu_0 \quad H_1\text{: } \mu > \mu_0 \\ \Phi(z_0) & \text{for a lower-tailed test: } H_0\text{: } \mu = \mu_0 \quad H_1\text{: } \mu < \mu_0 \end{cases}$$

Introduction

Test on the μ of NORMDIST

σ² known

σ² unknown

Test on the p

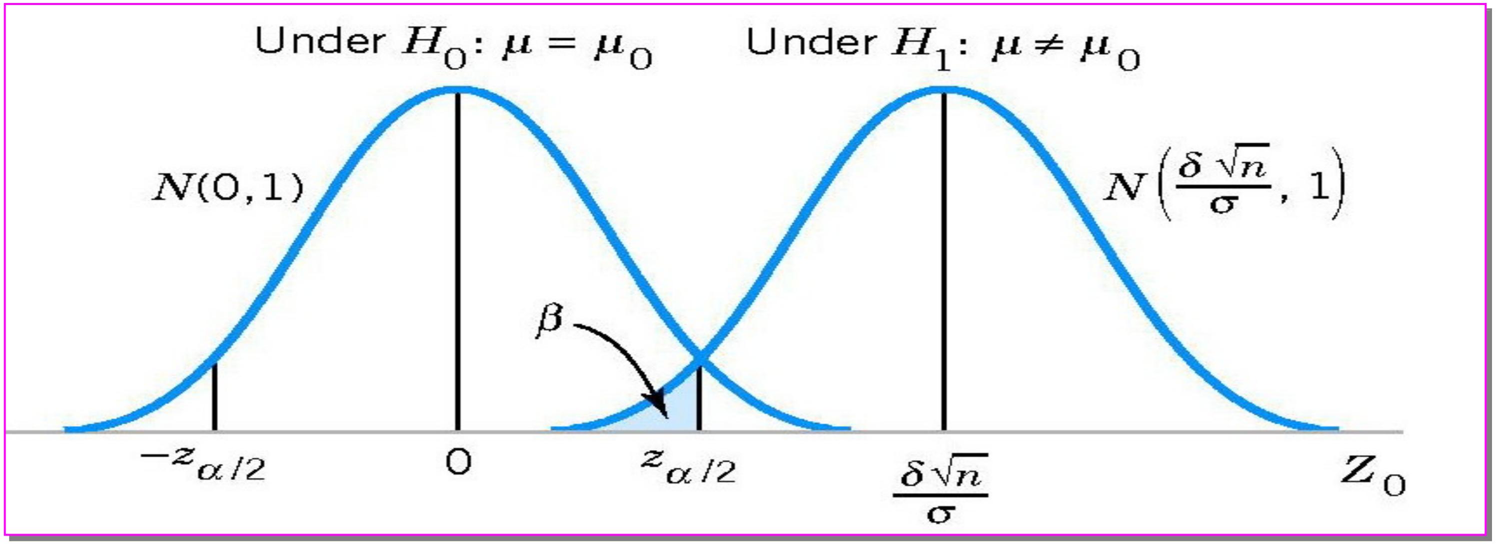
Summary

## Type II Error and Choice of Sample Size

**Probability of type II error β for a two-sided test**

$$\beta = \Phi\left(z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right) - \Phi\left(-z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right)$$

where $\delta = \mu - \mu_0$



Introduction

Test on the μ of NORMDIST

$\sigma^2$ known

$\sigma^2$ unknown

Test on the p

Summary

## Type II Error and Choice of Sample Size

### Sample Size Formulas

Two-sided test

$$n \simeq \frac{(z_{\alpha/2} + z_{\beta})^2 \sigma^2}{\delta^2}$$

One-sided test

$$n \simeq \frac{(z_{\alpha} + z_{\beta})^2 \sigma^2}{\delta^2}$$

where $\delta = \mu - \mu_0$.

# TEST ON THE $\mu$ OF NORMDIST: $\sigma^2$ KNOWN



## Exercise

The life in hours of a battery is known to be approximately normally distributed, with standard deviation $\sigma = 1.25$ hours. A random sample of 10 batteries has a mean life of $\bar{x} = 40.5$ hours.

(a) Is there evidence to support the claim that battery life exceeds 40 hours? Use $\alpha = 0.05$.

(b) What is the *P*-value for the test in part (a)?

(c) What is the $\beta$-error for the test in part (a) if the true mean life is 42 hours?

(d) What sample size would be required to ensure that  does not exceed 0.10 if the true mean life is 44 hours?

Introduction

Test on the μ of NORMDIST

$\sigma^2$ known

$\sigma^2$ unknown

Test on the p

Summary

## Hypothesis Tests on the Mean — *t*-test

Null hypothesis: $H_0$: $\mu = \mu_0$

Test statistic: $$T_0 = \frac{\bar{X} - \mu_0}{S / \sqrt{n}}$$

| Alternative hypothesis | Rejection criteria |
|---|---|
| $H_1$: $\mu \neq \mu_0$ | $\lvert t_0 \rvert > t_{\alpha/2,\, n-1}$ |
| $H_1$: $\mu > \mu_0$ | $t_0 > t_{\alpha,\, n-1}$ |
| $H_1$: $\mu < \mu_0$ | $t_0 < - t_{\alpha,\, n-1}$ |

# TEST ON THE μ OF NORMDIST: $\sigma^2$ UNKNOWN

Introduction

Test on the μ of NORMDIST

$\sigma^2$ known

$\sigma^2$ unknown

Test on the p

Summary

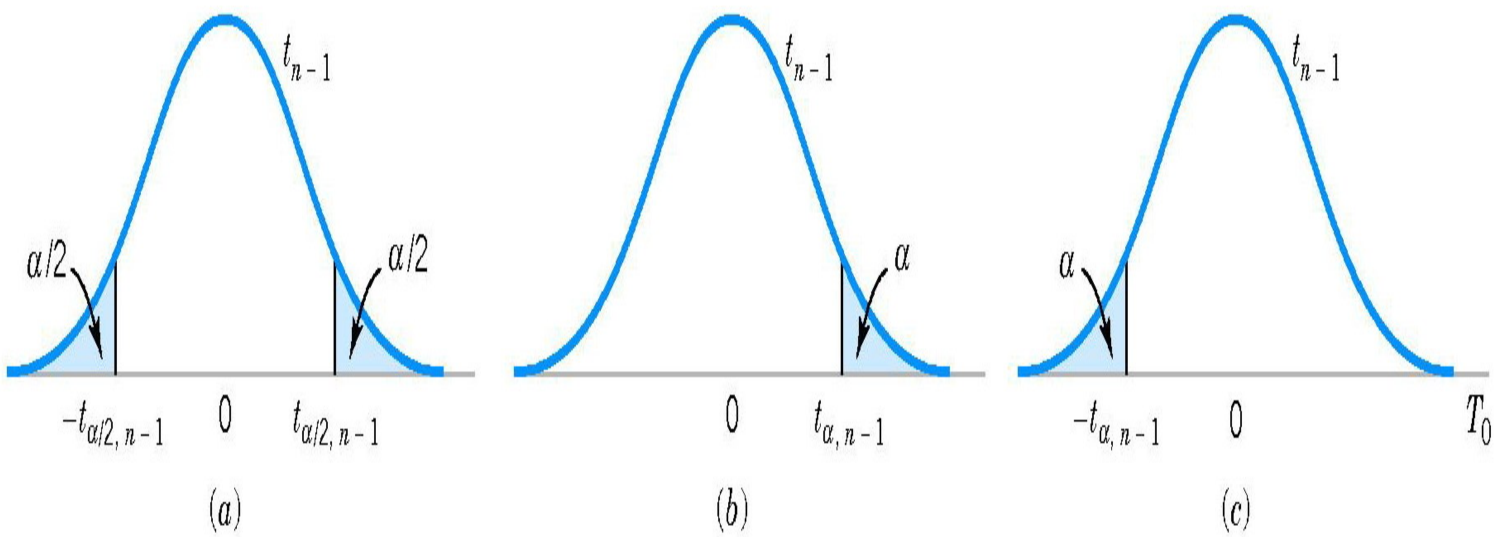



**Figure 9-8** The reference distribution for $H_0$: $\mu = \mu_0$ with critical region for (a) $H_1$: $\mu \neq \mu_0$, (b) $H_1$: $\mu > \mu_0$, and (c) $H_1$: $\mu < \mu_0$.

Introduction

Test on the μ of NORMDIST

$\sigma^2$ known

$\sigma^2$ unknown

Test on the p

Summary

**Exercise**

Consider the dissolved oxygen concentration at TVA dams. The observations are (in milligrams per liter): 5.0, 3.4, 3.9, 1.3, 0.2, 0.9, 2.7, 3.7, 3.8, 4.1, 1.0, 1.0, 0.8, 0.4, 3.8, 4.5, 5.3, 6.1, 6.9, and 6.5.

(a) Test the hypotheses $H_0$: $\mu = 4$; $H_1$: $\mu \neq 4$. Use $\alpha = 0.01$.

(b) What is the $P$-value in part (a)?

(c) Compute the power of the test if the true mean $\mu = 3$.

Introduction

Test on the μ of NORMDIST

$\sigma^2$ known

$\sigma^2$ unknown

Test on the $p$

Summary

## Hypothesis Tests on the Mean

Null hypothesis: $H_0: p = p_0$

Test statistic: $$Z_0 = \frac{\hat{P} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

| Alternative hypothesis | Rejection criteria |
|---|---|
| $H_1: p \neq p_0$ | $\lvert z_0 \rvert > z_{\alpha/2}$ |
| $H_1: p > p_0$ | $z_0 > z_\alpha$ |
| $H_1: p < p_0$ | $z_0 < - z_\alpha$ |

## Example

In a random sample of 85 automobile engine crankshaft bearings, 10 have a surface finish roughness that exceeds the specifications. Does this data present strong evidence that the proportion of crankshaft bearings exhibiting excess surface roughness exceeds 0.10? State and test the appropriate hypotheses using α = 0.05.

Solution: $H_0$: $p = 0.10$; $H_1$: $p > 0.10$

$$\hat{p} = \frac{10}{85} \simeq 0.12 \Rightarrow z_0 = \frac{0.12 - 0.10}{\sqrt{\frac{0.10(1-0.10)}{85}}} = 0.61 < z_\alpha \boxed{=1.65}$$

Fail to reject $H_0$ at $\alpha = 0.05$.

Introduction
Test on the μ of NORMDIST
$\sigma^2$ known
$\sigma^2$ unknown
Test on the $p$
Summary

## Type II Error and Choice of Sample Size

### Probability of type II error β

Two-sided test

$$\beta = \Phi\left(\frac{p_0 - p + z_{\alpha/2}\sqrt{p_0(1-p_0)/n}}{\sqrt{p(1-p)/n}}\right) - \Phi\left(\frac{p_0 - p - z_{\alpha/2}\sqrt{p_0(1-p_0)/n}}{\sqrt{p(1-p)/n}}\right)$$

One-sided test: $p > p_0$

$$\beta = \Phi\left(\frac{p_0 - p + z_{\alpha}\sqrt{p_0(1-p_0)/n}}{\sqrt{p(1-p)/n}}\right)$$

One-sided test: $p < p_0$

$$\beta = 1 - \Phi\left(\frac{p_0 - p - z_{\alpha}\sqrt{p_0(1-p_0)/n}}{\sqrt{p(1-p)/n}}\right)$$

Introduction

Test on the μ of NORMDIST

$\sigma^2$ known

$\sigma^2$ unknown

Test on the $p$

Summary

## Type II Error and Choice of Sample Size

### Sample Size Formulas

Two-sided test

$$n = \left[ \frac{z_{\alpha/2}\sqrt{p_0(1-p_0)} + z_\beta\sqrt{p(1-p)}}{p - p_0} \right]^2$$

One-sided test

$$n = \left[ \frac{z_{\alpha}\sqrt{p_0(1-p_0)} + z_\beta\sqrt{p(1-p)}}{p - p_0} \right]^2$$

# SUMMARY

We have studied:

1. Test on the μ of NORMDIST

- $\sigma^2$ known
- $\sigma^2$ unknown

2. Test on the *p*: Large-sample

Homework: Read slides of the next lecture.