

4

Continuous Random Variables and Probability Distributions

CHAPTER OUTLINE

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|-----|-------------------------------------------------------------|-----|----------------------------------------------------------------|
| 4-1 | CONTINUOUS RANDOM VARIABLES | 4-6 | NORMAL DISTRIBUTION |
| 4-2 | PROBABILITY DISTRIBUTIONS AND PROBABILITY DENSITY FUNCTIONS | 4-7 | NORMAL APPROXIMATION TO THE BINOMIAL AND POISSON DISTRIBUTIONS |
| 4-3 | CUMULATIVE DISTRIBUTION FUNCTIONS | 4-8 | EXPONENTIAL DISTRIBUTION |
| 4-4 | MEAN AND VARIANCE OF A CONTINUOUS RANDOM VARIABLE | | |
| 4-5 | CONTINUOUS UNIFORM DISTRIBUTION | | |

LEARNING OBJECTIVES

After careful study of this chapter you should be able to do the following:

1. Determine probabilities from probability density functions.
2. Determine probabilities from cumulative distribution functions and cumulative distribution functions from probability density functions, and the reverse.
3. Calculate means and variances for continuous random variables.
4. Understand the assumptions for each of the continuous probability distributions presented.
5. Select an appropriate continuous probability distribution to calculate probabilities in specific applications.
6. Calculate probabilities, determine means and variances for each of the continuous probability distributions presented.
7. Standardize normal random variables.
8. Use the table for the cumulative distribution function of a standard normal distribution to calculate probabilities.
9. Approximate probabilities for some binomial and Poisson distributions.

4-1 Continuous Random Variables

Previously, we discussed the measurement of the current in a thin copper wire. We noted that the results might differ slightly in day-to-day replications because of small variations in variables that are not controlled in our experiment—changes in ambient temperatures, small impurities in the chemical composition of the wire, current source drifts, and so forth.

Another example is the selection of one part from a day's production and very accurately measuring a dimensional length. In practice, there can be small variations in the actual measured lengths due to many causes, such as vibrations, temperature fluctuations, operator differences, calibrations, cutting tool wear, bearing wear, and raw material changes. Even the measurement procedure can produce variations in the final results.

4-2 Probability Distributions and Probability Density Functions

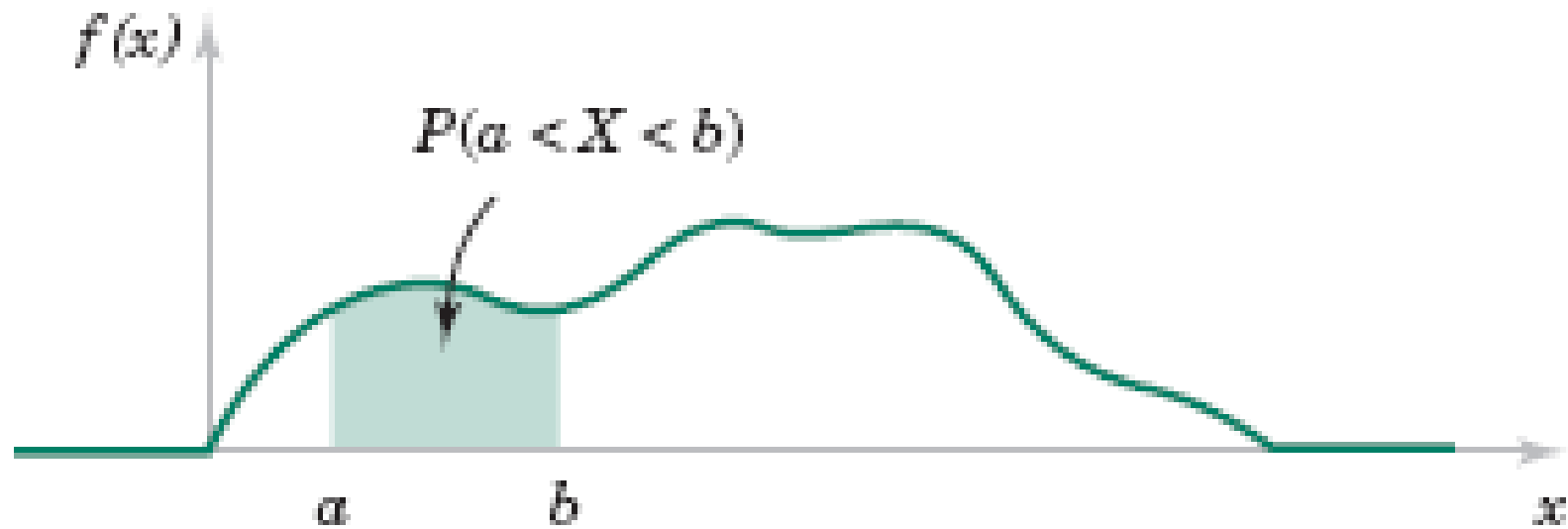


Figure 4-2 Probability determined from the area under $f(x)$.

4-2 Probability Distributions and Probability Density Functions

Definition

For a continuous random variable X , a **probability density function** is a function such that

$$(1) \quad f(x) \geq 0$$

$$(2) \quad \int_{-\infty}^{\infty} f(x) dx = 1$$

$$(3) \quad P(a \leq X \leq b) = \int_a^b f(x) dx = \text{area under } f(x) \text{ from } a \text{ to } b$$

for any a and b

(4-1)

4-2 Probability Distributions and Probability Density Functions

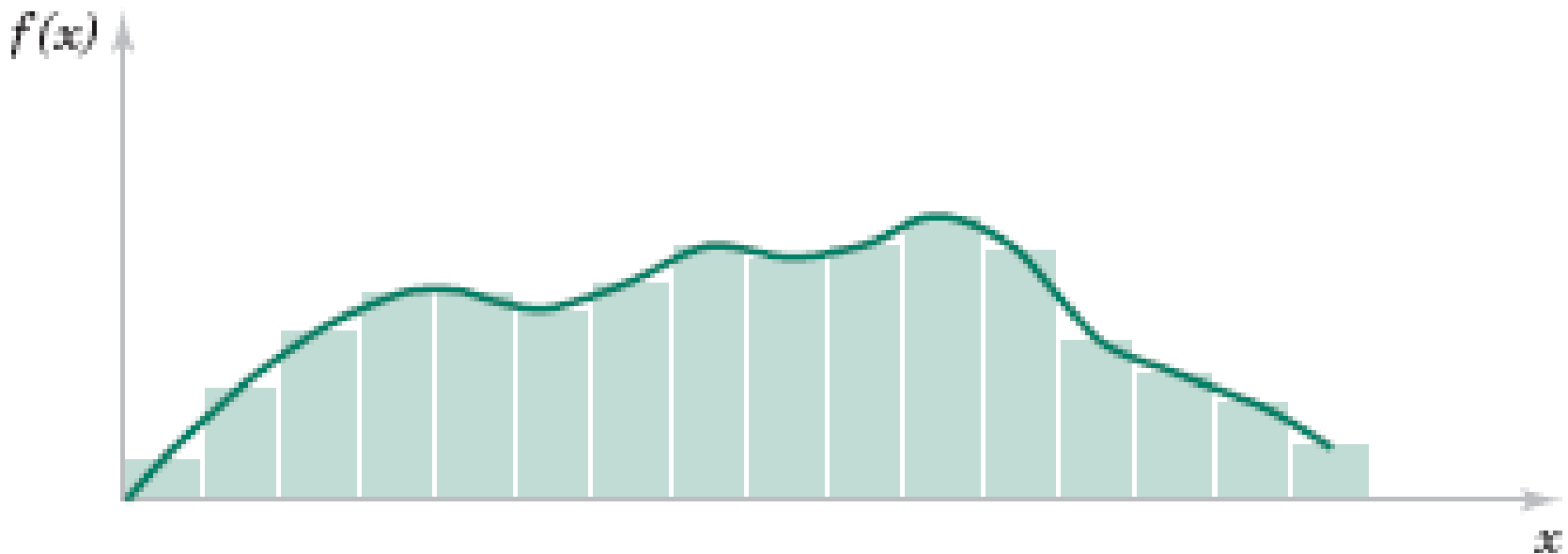


Figure 4-3 Histogram approximates a probability density function.

4-2 Probability Distributions and Probability Density Functions

If X is a **continuous random variable**, for any x_1 and x_2 ,

$$P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X < x_2) \quad (4-2)$$

4-2 Probability Distributions and Probability Density Functions

Example 4.1

Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the range of X is $[0, 20 \text{ mA}]$, and assume that the probability density function of X is $f(x) = 0.05$ for $0 \leq x \leq 20$.

What is the probability that a current measurement is less than 10 milliamperes?

And the probability that a current measurement is more than 5 and less than 20?

4-2 Probability Distributions and Probability Density Functions

Example 4-2

Let the continuous random variable X denote the diameter of a hole drilled in a sheet metal component. The target diameter is 12.5 millimeters. Most random disturbances to the process result in larger diameters. Historical data show that the distribution of X can be modeled by a probability density function $f(x) = 20e^{-20(x-12.5)}, x \geq 12.5$.

If a part with a diameter larger than 12.60 millimeters is scrapped, what proportion of parts is scrapped? The density function and the requested probability are shown in Fig. 4-5. A part is scrapped if $X > 12.60$. Now,

- Solution:

$$P(X > 12.60) = \int_{12.6}^{\infty} f(x) dx = \int_{12.6}^{\infty} 20e^{-20(x-12.5)} dx = -e^{-20(x-12.5)} \Big|_{12.6}^{\infty} = 0.135$$

4-2 Probability Distributions and Probability Density Functions

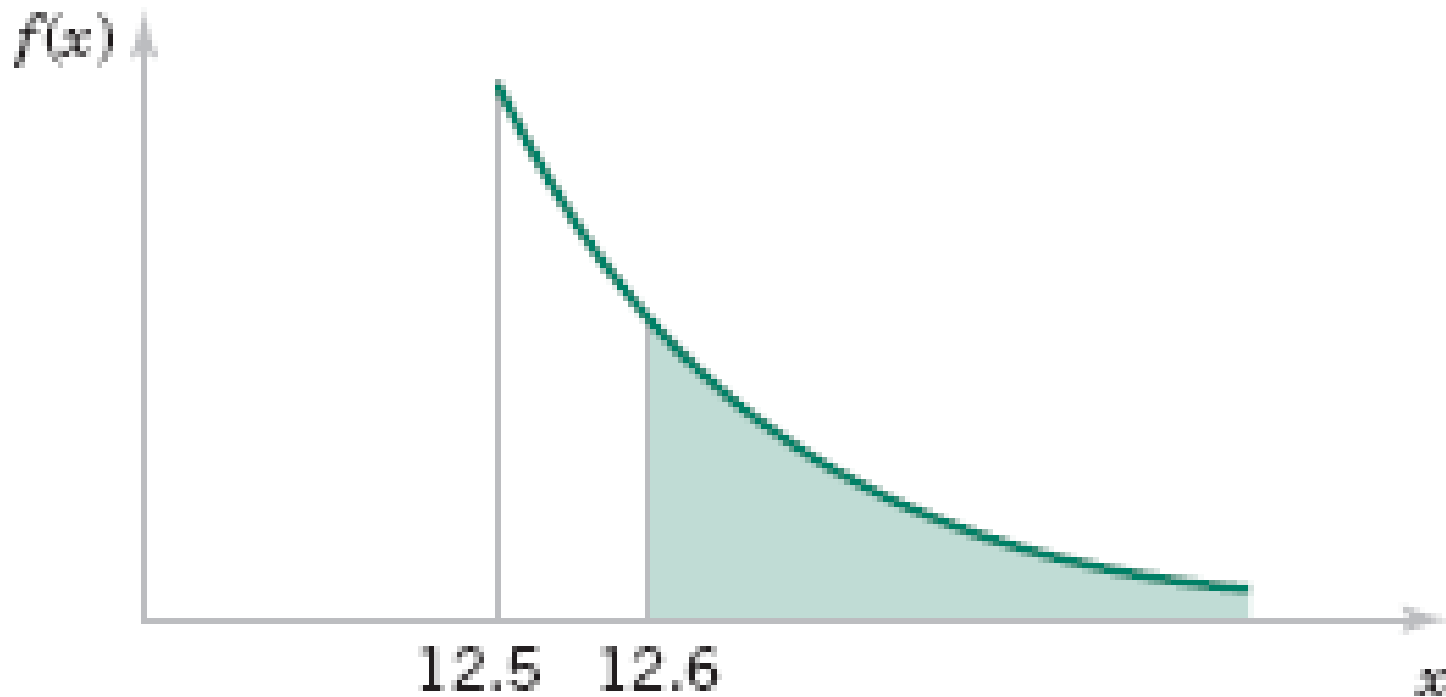


Figure 4-5 Probability density function for Example 4-2.

4-2 Probability Distributions and Probability Density Functions

Example 4-2 (continued)

What proportion of parts is between 12.5 and 12.6 millimeters? Now,

- Solution:

$$P(12.5 < X < 12.6) = \int_{12.5}^{12.6} f(x) dx = -e^{-20(x-12.5)} \Big|_{12.5}^{12.6} = 0.865$$

Because the total area under $f(x)$ equals 1, we can also calculate $P(12.5 < X < 12.6) = 1 - P(X > 12.6) = 1 - 0.135 = 0.865$.

4-3 Cumulative Distribution Functions

Definition

The **cumulative distribution function** of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(u) du \quad (4-3)$$

for $-\infty < x < \infty$.

4-3 Cumulative Distribution Functions

Example 4.3

For the copper current measurement in Ex 4.1, the cumulative distribution function of the random variable X consists of three expressions. If $x < 0$, $f(x) = 0$. Therefore,

$$F(x) = \begin{cases} 0 & , \quad x < 0 \\ 0.05x & , \quad 0 \leq x < 20 \\ 1 & , \quad 20 \leq x \end{cases}$$

4-3 Cumulative Distribution Functions

Example 4-4

For the drilling operation in Example 4-2, $F(x)$ consists of two expressions.

$$F(x) = 0 \quad \text{for } x < 12.5$$

and for $12.5 \leq x$

$$\begin{aligned} F(x) &= \int_{12.5}^x 20e^{-20(u-12.5)} du \\ &= 1 - e^{-20(x-12.5)} \end{aligned}$$

Therefore,

$$F(x) = \begin{cases} 0 & x < 12.5 \\ 1 - e^{-20(x-12.5)} & 12.5 \leq x \end{cases}$$

Figure 4-7 displays a graph of $F(x)$.

4-3 Cumulative Distribution Functions

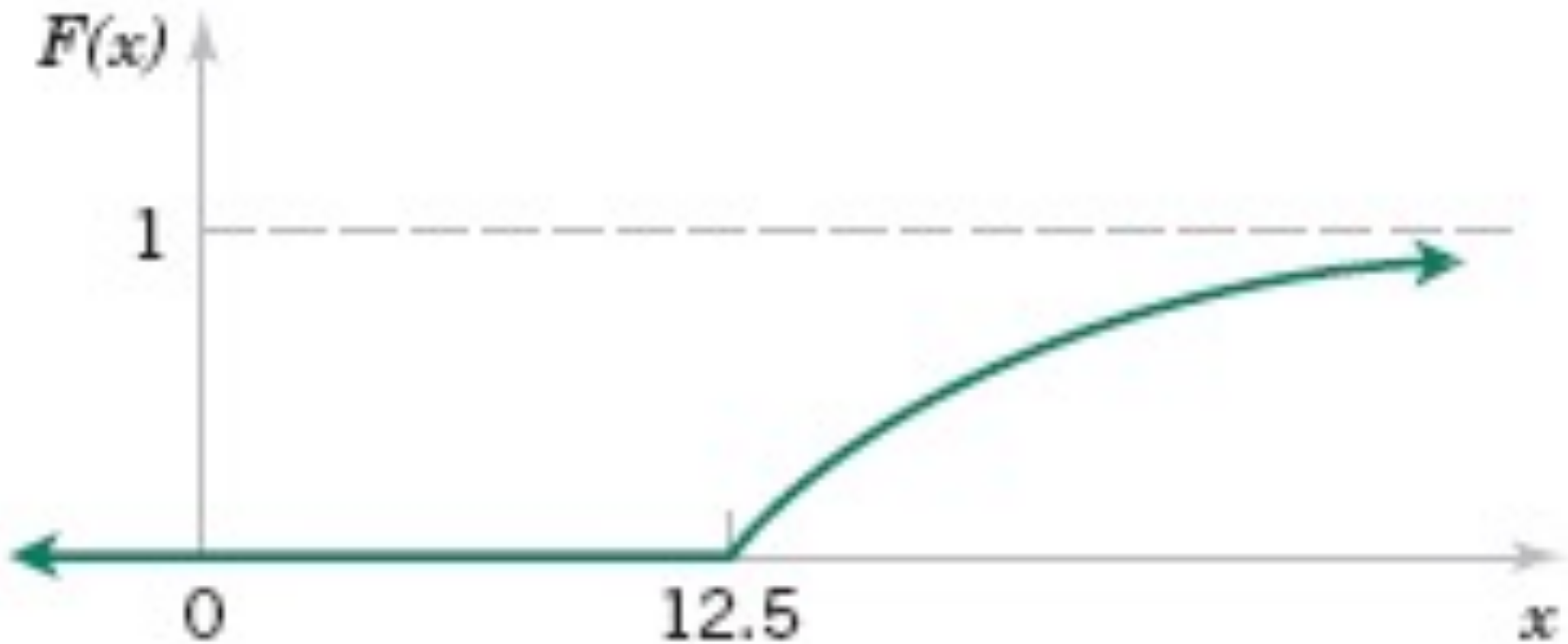


Figure 4-7 Cumulative distribution function for Example 4-4.

4-3 Cumulative Distribution Functions

Example 4.5

The time until a chemical reaction (in milliseconds) is approximated by the cumulative distribute function:

$$F(x) = \begin{cases} 0 & , x < 0 \\ 1 - e^{-0.01x} & , 0 \leq x \end{cases}$$

Find the probability density function of X. What proportion of reactions is complete within 200 milliseconds.

Solution:

Using the result that the probability density function is the derivative of $F(x)$, we obtain

$$f(x) = \begin{cases} 0 & x < 0 \\ 0.01 e^{-0.01x} & 0 \leq x \end{cases}$$

The probability that a reaction completes within 200 milliseconds is

$$P(X < 200) = F(200) = 1 - e^{-2} = 0.8647$$

4-4 Mean and Variance of a Continuous Random Variable

Definition

Suppose X is a continuous random variable with probability density function $f(x)$. The **mean** or **expected value** of X , denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x) dx \quad (4-4)$$

The **variance** of X , denoted as $V(X)$ or σ^2 , is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

4-4 Mean and Variance of a Continuous Random Variable

Example 4-6

For the copper current measurement in Example 4-1, the mean of X is

$$E(X) = \int_0^{20} xf(x) dx = 0.05x^2/2 \Big|_0^{20} = 10$$

The variance of X is

$$V(X) = \int_0^{20} (x - 10)^2 f(x) dx = 0.05(x - 10)^3/3 \Big|_0^{20} = 33.33$$

4-4 Mean and Variance of a Continuous Random Variable

Example 4-8

For the drilling operation in Example 4-2, the mean of X is

$$E(X) = \int_{12.5}^{\infty} xf(x) dx = \int_{12.5}^{\infty} x 20e^{-20(x-12.5)} dx$$

Integration by parts can be used to show that

$$E(X) = -xe^{-20(x-12.5)} - \frac{e^{-20(x-12.5)}}{20} \Big|_{12.5}^{\infty} = 12.5 + 0.05 = 12.55$$

The variance of X is

$$V(X) = \int_{12.5}^{\infty} (x - 12.55)^2 f(x) dx$$

Although more difficult, integration by parts can be used two times to show that $V(X) = 0.0025$.

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4-5 Continuous Uniform Random Variable

Definition

A continuous random variable X with probability density function

$$f(x) = 1/(b - a), \quad a \leq x \leq b \quad (4-6)$$

is a **continuous uniform random variable**.

4-5 Continuous Uniform Random Variable

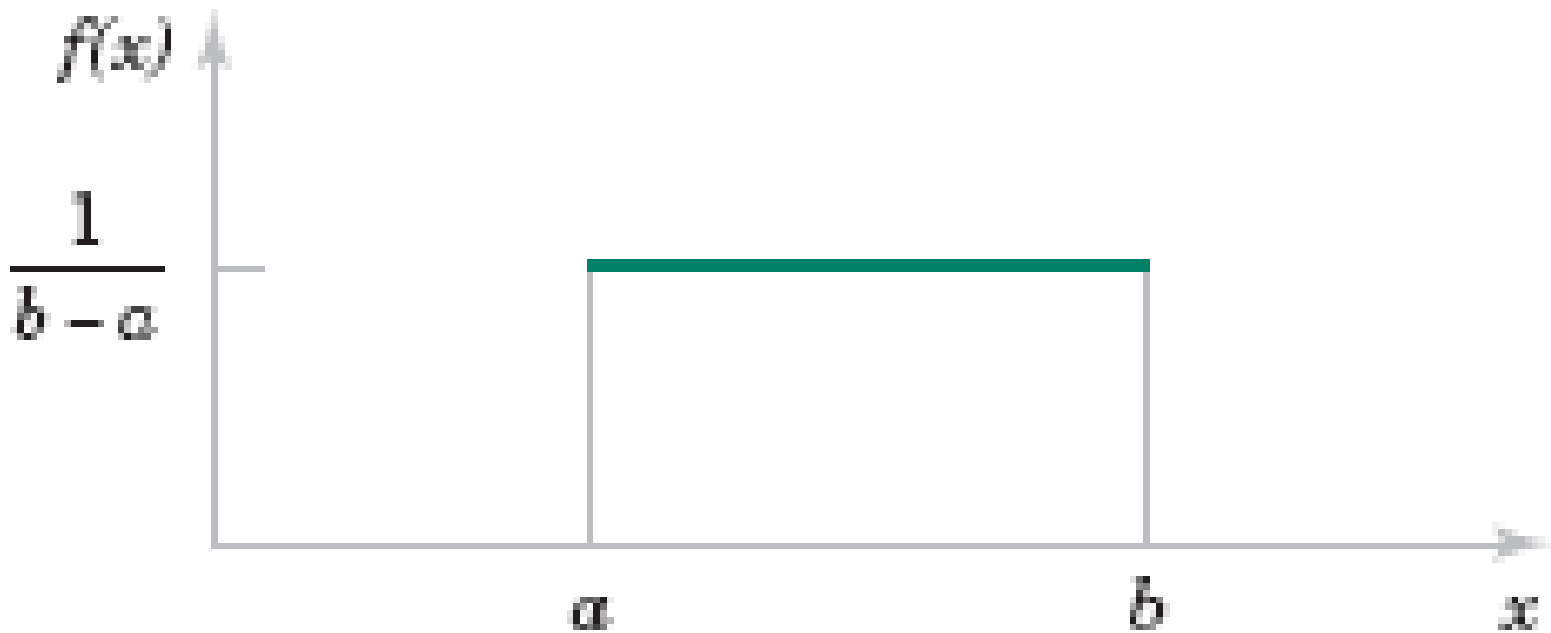


Figure 4-8 Continuous uniform probability density function.

4-5 Continuous Uniform Random Variable

Mean and Variance

If X is a continuous uniform random variable over $a \leq x \leq b$,

$$\mu = E(X) = \frac{(a + b)}{2} \quad \text{and} \quad \sigma^2 = V(X) = \frac{(b - a)^2}{12} \quad (4-7)$$

4-5 Continuous Uniform Random Variable

Example 4-9

Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the range of X is $[0, 20 \text{ mA}]$, and assume that the probability density function of X is $f(x) = 0.05, 0 \leq x \leq 20$.

What is the probability that a measurement of current is between 5 and 10 milliamperes? The requested probability is shown as the shaded area in Fig. 4-9.

$$\begin{aligned} P(5 < X < 10) &= \int_5^{10} f(x) dx \\ &= 5(0.05) = 0.25 \end{aligned}$$

The mean and variance formulas can be applied with $a = 0$ and $b = 20$. Therefore,

$$E(X) = 10 \text{ mA} \quad \text{and} \quad V(X) = 20^2/12 = 33.33 \text{ mA}^2$$

Consequently, the standard deviation of X is 5.77 mA.

Solution:

The mean and variance formulas can be applied with $a = 0$ and $b = 20$. Therefore,

$$E(X) = 10 \text{ mA} \quad \text{and} \quad V(X) = 20^2/12 = 33.33 \text{ mA}^2$$

Consequently, the standard deviation of X is 5.77 mA.

4-5 Continuous Uniform Random Variable

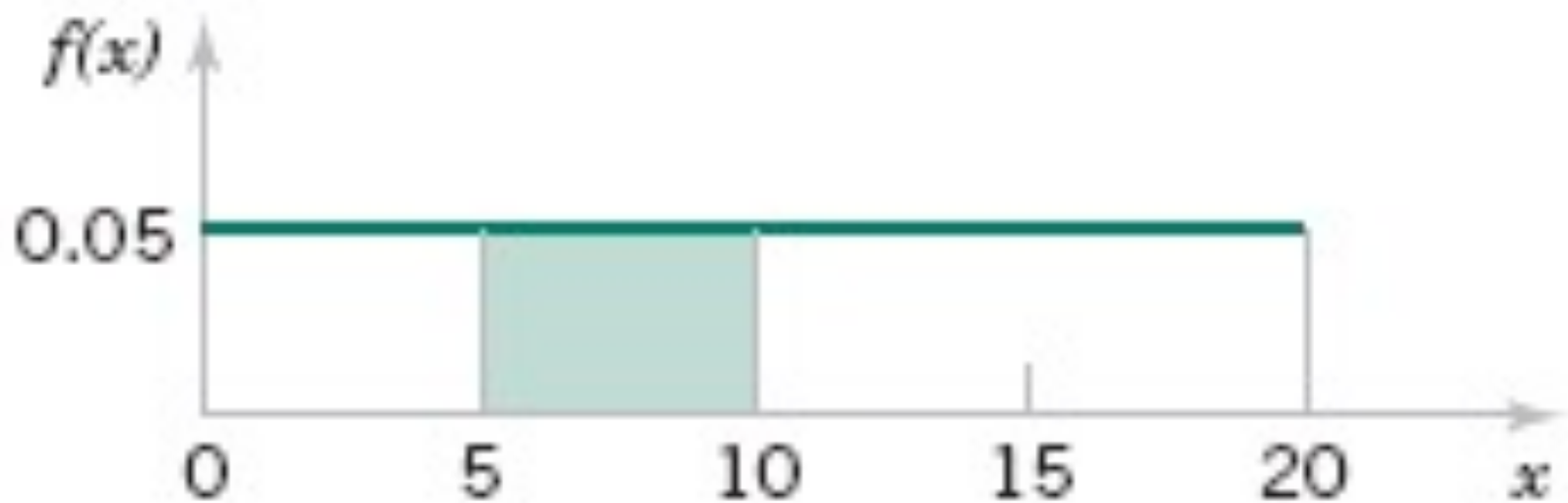


Figure 4-9 Probability for Example 4-9.

4-5 Continuous Uniform Random Variable

The cumulative distribution function of a continuous uniform random variable is obtained by integration. If $a < x < b$,

$$F(x) = \int_a^x 1/(b - a) du = x/(b - a) - a/(b - a)$$

Therefore, the complete description of the cumulative distribution function of a continuous uniform random variable is

$$F(x) = \begin{cases} 0 & x < a \\ (x - a)/(b - a) & a \leq x < b \\ 1 & b \leq x \end{cases}$$

4-6 Normal Distribution

Definition

A random variable X with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty \quad (4-8)$$

is a **normal random variable** with parameters μ , where $-\infty < \mu < \infty$, and $\sigma > 0$. Also,

$$E(X) = \mu \quad \text{and} \quad V(X) = \sigma^2 \quad (4-9)$$

and the notation $N(\mu, \sigma^2)$ is used to denote the distribution. The mean and variance of X are shown to equal μ and σ^2 , respectively, at the end of this Section 5-6.

4-6 Normal Distribution

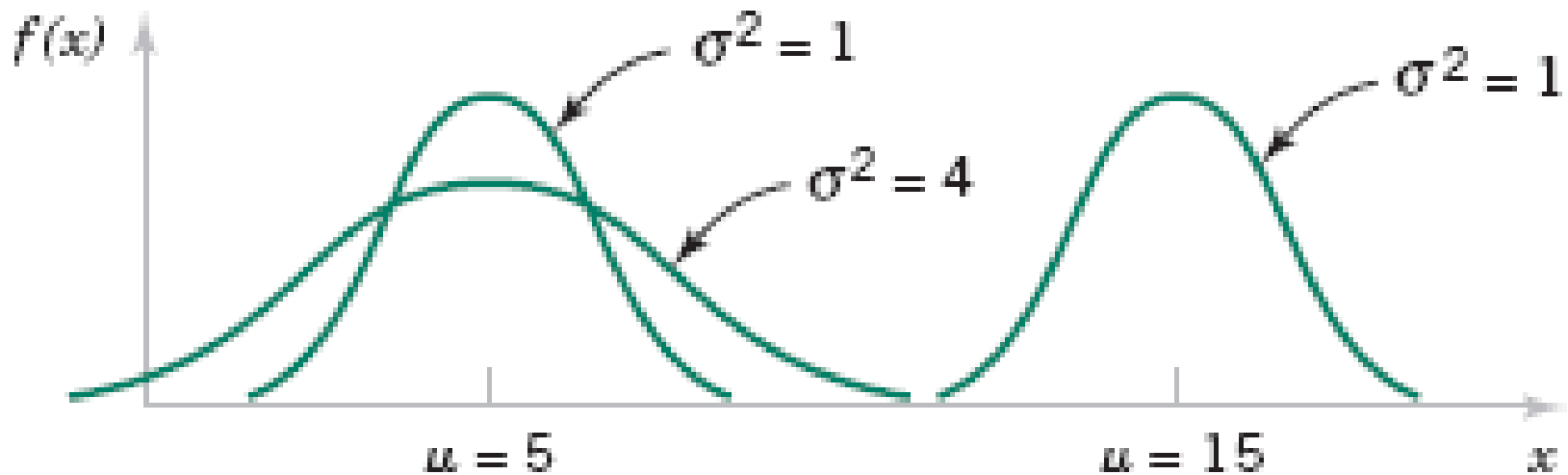


Figure 4-10 Normal probability density functions for selected values of the parameters μ and σ^2 .

4-6 Normal Distribution

Example 4 – 10

Assume that the current measurement is a strip of wire follow a normal distribution with a mean of 10 milliamperes and a variance of 4 (milliamperes)². What is the probability that a measurement exceeds 13 milliamperes?

Let X be the current in milliamperes. The requested probability can be represented as $P(X > 13)$. This probability is shown as the shaded area under the normal probability density function in Fig. 4 – 11.

4-6 Normal Distribution

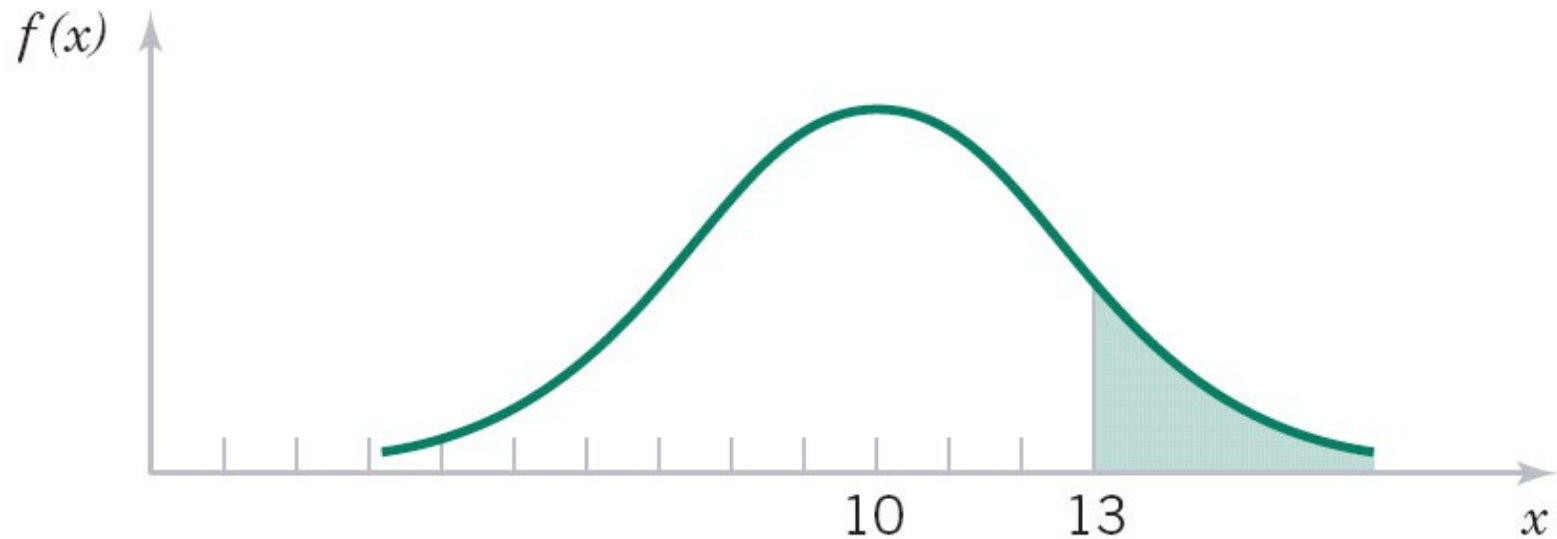


Figure 4-11 Probability that $X > 13$ for a normal random variable with $\mu = 10$ and $\sigma^2 = 4$.

4-6 Normal Distribution

Some useful results concerning the normal distribution

For any normal random variable,

$$P(\mu - \sigma < X < \mu + \sigma) = 0.6827$$

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9545$$

$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$$

4-6 Normal Distribution

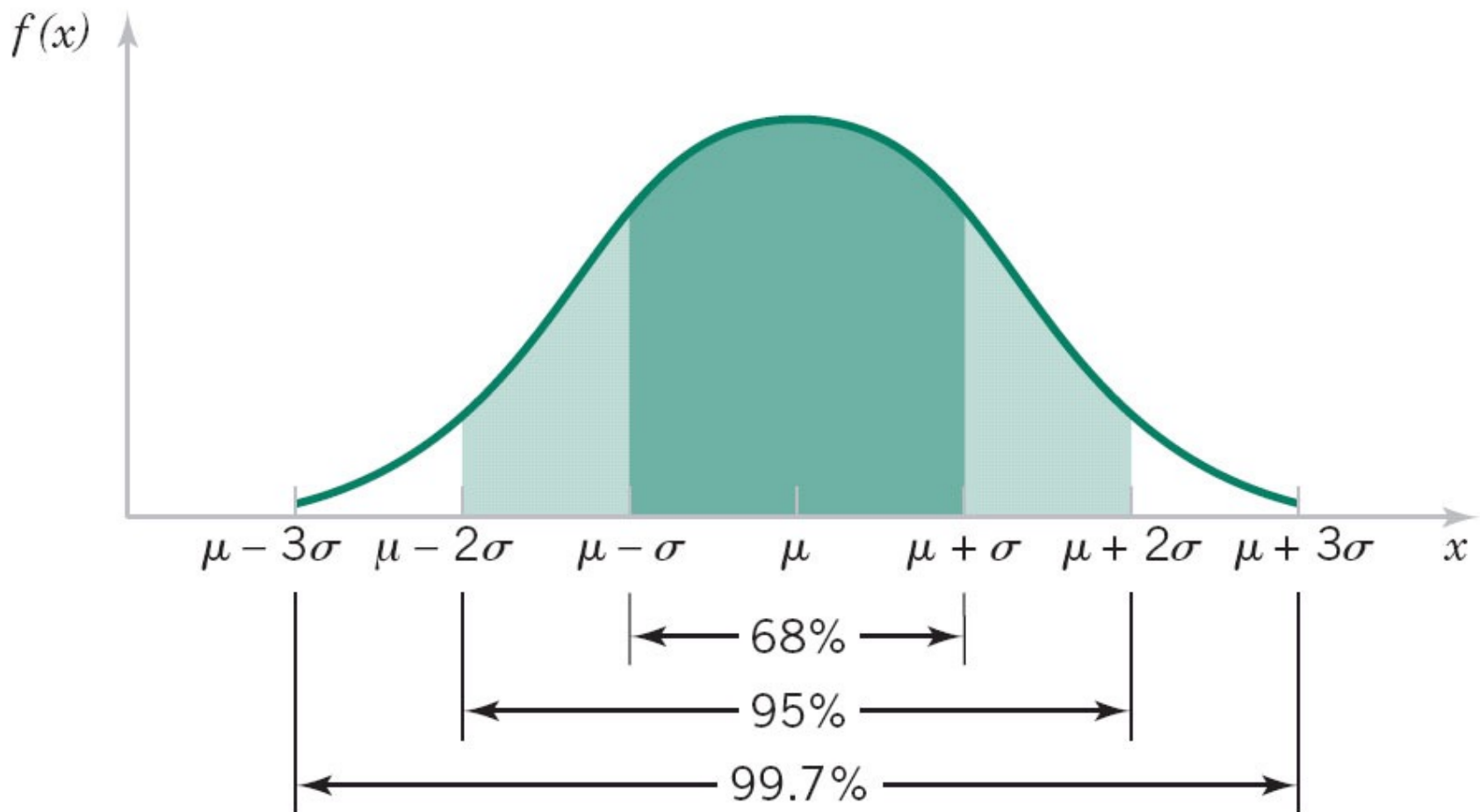


Figure 4-12 Probabilities associated with a normal distribution.

4-6 Normal Distribution

Definition : Standard Normal

A normal random variable with

$$\mu = 0 \quad \text{and} \quad \sigma^2 = 1$$

is called a **standard normal random variable** and is denoted as Z .

The cumulative distribution function of a standard normal random variable is denoted as

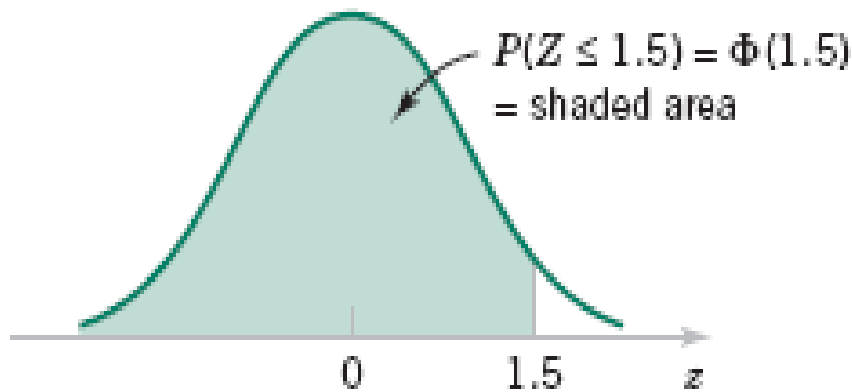
$$\Phi(z) = P(Z \leq z)$$

4-6 Normal Distribution

Example 4-11

Assume Z is a standard normal random variable. Appendix Table II provides probabilities of the form $P(Z \leq z)$. The use of Table II to find $P(Z \leq 1.5)$ is illustrated in Fig. 4-13. Read down the z column to the row that equals 1.5. The probability is read from the adjacent column, labeled 0.00, to be 0.93319.

The column headings refer to the hundredth's digit of the value of z in $P(Z \leq z)$. For example, $P(Z \leq 1.53)$ is found by reading down the z column to the row 1.5 and then selecting the probability from the column labeled 0.03 to be 0.93699.



z	0.00	0.01	0.02	0.03
0	0.50000	0.50399	0.50398	0.51197
\vdots		\vdots		
1.5	0.93319	0.93448	0.93574	0.93699

Figure 4-13 Standard normal probability density function.

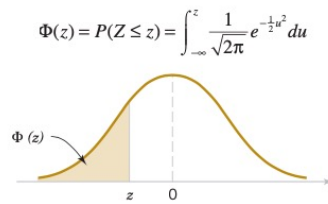


TABLE • III Cumulative Standard Normal Distribution

z	-0.09	-0.08	-0.07	-0.06	-0.05	-0.04	-0.03	-0.02	-0.01	-0.00
-3.9	0.000033	0.000034	0.000036	0.000037	0.000039	0.000041	0.000042	0.000044	0.000046	0.000048
-3.8	0.000050	0.000052	0.000054	0.000057	0.000059	0.000062	0.000064	0.000067	0.000069	0.000072
-3.7	0.000075	0.000078	0.000082	0.000085	0.000088	0.000092	0.000096	0.000100	0.000104	0.000108
-3.6	0.000112	0.000117	0.000121	0.000126	0.000131	0.000136	0.000142	0.000147	0.000153	0.000159
-3.5	0.000165	0.000172	0.000179	0.000185	0.000193	0.000200	0.000208	0.000216	0.000224	0.000233
-3.4	0.000242	0.000251	0.000260	0.000270	0.000280	0.000291	0.000302	0.000313	0.000325	0.000337
-3.3	0.000350	0.000362	0.000376	0.000390	0.000404	0.000419	0.000434	0.000450	0.000467	0.000483
-3.2	0.000501	0.000519	0.000538	0.000557	0.000577	0.000598	0.000619	0.000641	0.000664	0.000687
-3.1	0.000711	0.000736	0.000762	0.000789	0.000816	0.000845	0.000874	0.000904	0.000935	0.000968
-3.0	0.001001	0.001035	0.001070	0.001107	0.001144	0.001183	0.001223	0.001264	0.001306	0.001350
-2.9	0.001395	0.001441	0.001489	0.001538	0.001589	0.001641	0.001695	0.001750	0.001807	0.001866
-2.8	0.001926	0.001988	0.002052	0.002118	0.002186	0.002256	0.002327	0.002401	0.002477	0.002555
-2.7	0.002635	0.002718	0.002803	0.002890	0.002980	0.003072	0.003167	0.003264	0.003364	0.003467
-2.6	0.003573	0.003681	0.003793	0.003907	0.004025	0.004145	0.004269	0.004396	0.004527	0.004661
-2.5	0.004799	0.004940	0.005085	0.005234	0.005386	0.005543	0.005703	0.005868	0.006037	0.006210
-2.4	0.006387	0.006569	0.006756	0.006947	0.007143	0.007344	0.007549	0.007760	0.007976	0.008198
-2.3	0.008424	0.008656	0.008894	0.009137	0.009387	0.009642	0.009903	0.010170	0.010444	0.010724
-2.2	0.011011	0.011304	0.011604	0.011911	0.012224	0.012545	0.012874	0.013209	0.013553	0.013903
-2.1	0.014262	0.014629	0.015003	0.015386	0.015778	0.016177	0.016586	0.017003	0.017429	0.017864
-2.0	0.018309	0.018763	0.019226	0.019699	0.020182	0.020675	0.021178	0.021692	0.022216	0.022750
-1.9	0.023295	0.023852	0.024419	0.024998	0.025588	0.026190	0.026803	0.027429	0.028067	0.028717
-1.8	0.029379	0.030054	0.030742	0.031443	0.032157	0.032884	0.033625	0.034379	0.035148	0.035930
-1.7	0.036727	0.037538	0.038364	0.039204	0.040059	0.040929	0.041815	0.042716	0.043633	0.044565
-1.6	0.045514	0.046479	0.047460	0.048457	0.049471	0.050503	0.051551	0.052616	0.053699	0.054799
-1.5	0.055917	0.057053	0.058208	0.059380	0.060571	0.061780	0.063008	0.064256	0.065522	0.066807
-1.4	0.068112	0.069437	0.070781	0.072145	0.073529	0.074934	0.076359	0.077804	0.079270	0.080757
-1.3	0.082264	0.083793	0.085343	0.086915	0.088508	0.090123	0.091759	0.093418	0.095098	0.096801
-1.2	0.098525	0.100273	0.102042	0.103835	0.105650	0.107488	0.109349	0.111233	0.113140	0.115070
-1.1	0.117023	0.119000	0.121001	0.123024	0.125072	0.127143	0.129238	0.131357	0.133500	0.135666
-1.0	0.137857	0.140071	0.142310	0.144572	0.146859	0.149170	0.151505	0.153864	0.156248	0.158655
-0.9	0.161087	0.163543	0.166023	0.168528	0.171056	0.173609	0.176185	0.178786	0.181411	0.184060
-0.8	0.186733	0.189430	0.192150	0.194894	0.197662	0.200454	0.203269	0.206108	0.208970	0.211855
-0.7	0.214764	0.217695	0.220650	0.223627	0.226627	0.229650	0.232695	0.235762	0.238852	0.241964
-0.6	0.245097	0.248252	0.251429	0.254627	0.257846	0.261086	0.264347	0.267629	0.270931	0.274253
-0.5	0.277595	0.280957	0.284339	0.287740	0.291160	0.294599	0.298056	0.301532	0.305026	0.308538
-0.4	0.312067	0.315614	0.319178	0.322758	0.326355	0.329969	0.333598	0.337243	0.340903	0.344578
-0.3	0.348268	0.351973	0.355691	0.359424	0.363169	0.366928	0.370700	0.374484	0.378281	0.382089
-0.2	0.385908	0.389739	0.393580	0.397432	0.401294	0.405165	0.409046	0.412936	0.416834	0.420740
-0.1	0.424655	0.428576	0.432505	0.436441	0.440382	0.444330	0.448283	0.452242	0.456205	0.460172
0.0	0.464144	0.468119	0.472097	0.476078	0.480061	0.484047	0.488033	0.492022	0.496011	0.500000

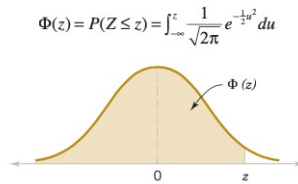


TABLE • III Cumulative Standard Normal Distribution (*Continued*)

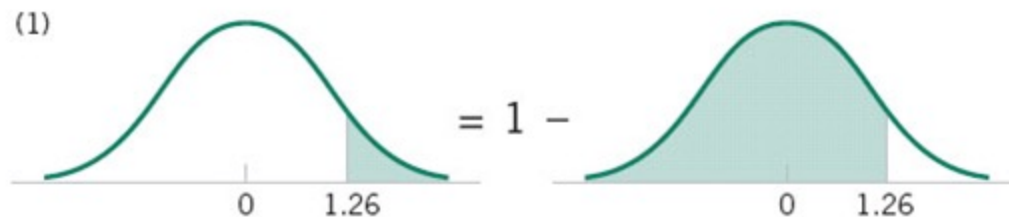
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.523922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555670	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273
1.8	0.964070	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002	0.975581	0.976148	0.976705
2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989
2.3	0.989276	0.989556	0.989830	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365
2.8	0.997445	0.997523	0.997599	0.997673	0.997744	0.997814	0.997882	0.997948	0.998012	0.998074
2.9	0.998134	0.998193	0.998250	0.998305	0.998359	0.998411	0.998462	0.998511	0.998559	0.998605
3.0	0.998650	0.998694	0.998736	0.998777	0.998817	0.998856	0.998893	0.998930	0.998965	0.998999
3.1	0.999032	0.999065	0.999096	0.999126	0.999155	0.999184	0.999211	0.999238	0.999264	0.999289
3.2	0.999313	0.999336	0.999359	0.999381	0.999402	0.999423	0.999443	0.999462	0.999481	0.999499
3.3	0.999517	0.999533	0.999550	0.999566	0.999581	0.999596	0.999610	0.999624	0.999638	0.999650
3.4	0.999663	0.999675	0.999687	0.999698	0.999709	0.999720	0.999730	0.999740	0.999749	0.999758
3.5	0.999767	0.999776	0.999784	0.999792	0.999800	0.999807	0.999815	0.999821	0.999828	0.999835
3.6	0.999841	0.999847	0.999853	0.999858	0.999864	0.999869	0.999874	0.999879	0.999883	0.999888
3.7	0.999892	0.999896	0.999900	0.999904	0.999908	0.999912	0.999915	0.999918	0.999922	0.999925
3.8	0.999928	0.999931	0.999933	0.999936	0.999938	0.999941	0.999943	0.999946	0.999948	0.999950
3.9	0.999952	0.999954	0.999956	0.999958	0.999959	0.999961	0.999963	0.999964	0.999966	0.999967

4-6 Normal Distribution

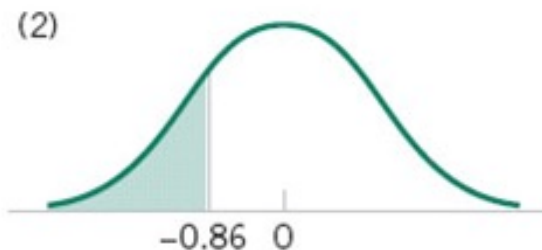
Example 4 – 12

The following calculations are shown pictorially in Fig. 4 – 14. In practice, a probability is often rounded to one or two significant digits.

$$(1) P(Z > 1.26) = 1 - P(Z \leq 1.26) = 1 - 0.89616$$



$$(2) P(Z < -0.86) = 0.19490$$



4-6 Normal Distribution

Example 4 – 12

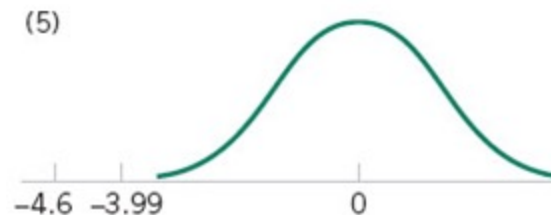
$$(3) P(Z > -1.37) = P(Z < 1.37) = 0.91465$$



$$(4) P(-1.25 < Z < 0.37) = P(Z < 0.37) - P(Z < -1.25) = 0.53866$$



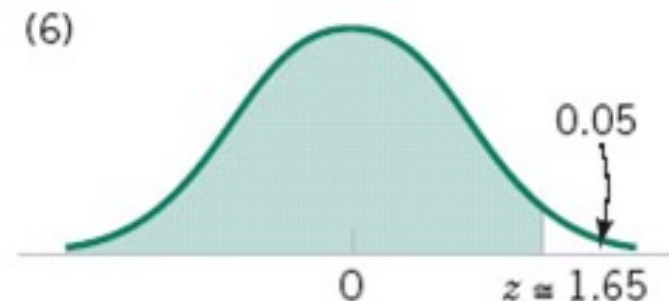
$$(5) P(Z < -4) \approx 0$$



4-6 Normal Distribution

Example 4 – 12

(6) Find the value z such that $P(Z > z) = 0.05$. This probability expression can be written as $P(Z \leq z) = 0.95$. We search the probabilities to find the value that corresponds to 0.95. We do not find 0.95 exactly, the nearest value is 0.9503, corresponding $z = 1.65$.

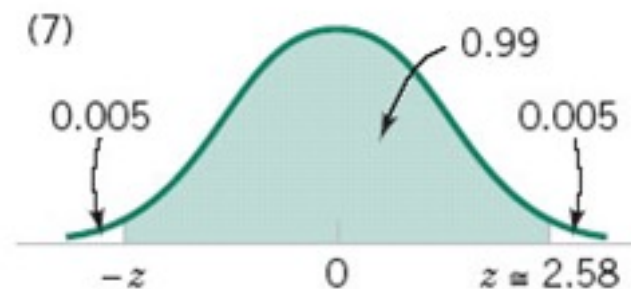


4-6 Normal Distribution

Example 4 – 12

(7) Find the value z such that $P(-z < Z < z) = 0.99$.

Because of the symmetry of the normal distribution, if the area of the shaded region in Fig. 4 – 14 (7) is to equal 0.99, the area in each tail of the distribution must equal 0.005. So, the value for z corresponds to a probability of 0.995 in Table III. The nearest probability is 0.99506, then $z=2.58$.



4-6 Normal Distribution

Standardizing

If X is a normal random variable with $E(X) = \mu$ and $V(X) = \sigma^2$, the random variable

$$Z = \frac{X - \mu}{\sigma} \quad (4-10)$$

is a normal random variable with $E(Z) = 0$ and $V(Z) = 1$. That is, Z is a standard normal random variable.

4-6 Normal Distribution

Example 4-13

Suppose the current measurements in a strip of wire are assumed to follow a normal distribution with a mean of 10 milliamperes and a variance of 4 (milliamperes)². What is the probability that a measurement will exceed 13 milliamperes?

Let X denote the current in milliamperes. The requested probability can be represented as $P(X > 13)$. Let $Z = (X - 10)/2$. The relationship between the several values of X and the transformed values of Z are shown in Fig. 4-15. We note that $X > 13$ corresponds to $Z > 1.5$. Therefore, from Appendix Table II,

$$P(X > 13) = P(Z > 1.5) = 1 - P(Z \leq 1.5) = 1 - 0.93319 = 0.06681$$

Rather than using Fig. 4-15, the probability can be found from the inequality $X > 13$. That is,

$$P(X > 13) = P\left(\frac{(X - 10)}{2} > \frac{(13 - 10)}{2}\right) = P(Z > 1.5) = 0.06681$$

4-6 Normal Distribution

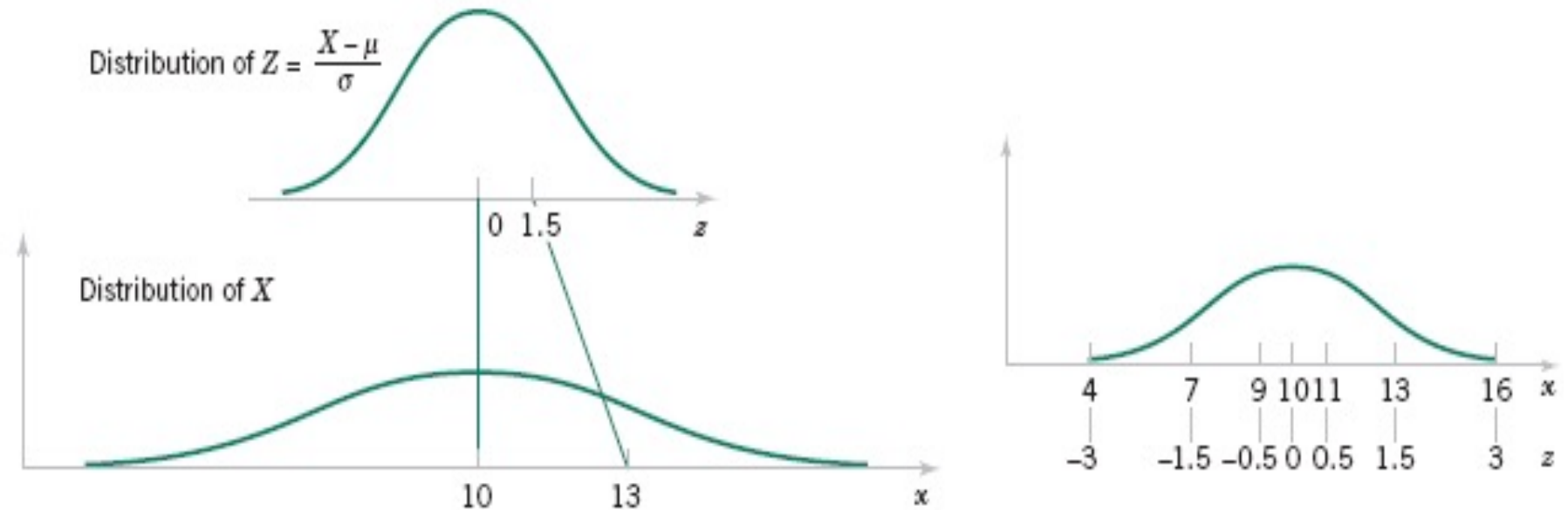


Figure 4-15 Standardizing a normal random variable.

4-6 Normal Distribution

To Calculate Probability

Suppose X is a normal random variable with mean μ and variance σ^2 . Then,

$$P(X \leq x) = P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) = P(Z \leq z) \quad (4-11)$$

where Z is a **standard normal random variable**, and $z = \frac{(x - \mu)}{\sigma}$ is the z -value obtained by **standardizing** X .

The probability is obtained by entering Appendix Table II with $z = (x - \mu)/\sigma$.

4-6 Normal Distribution

Example 4-14

Continuing the previous example, what is the probability that a current measurement is between 9 and 11 milliamperes? From Fig. 4-15, or by proceeding algebraically, we have

- Solution:

$$\begin{aligned}P(9 < X < 11) &= P((9 - 10)/2 < (X - 10)/2 < (11 - 10)/2) \\&= P(-0.5 < Z < 0.5) = P(Z < 0.5) - P(Z < -0.5) \\&= 0.69146 - 0.30854 = 0.38292\end{aligned}$$

4-6 Normal Distribution

Example 4-14 (continued)

Determine the value for which the probability that a current measurement is below this value is 0.98. The requested value is shown graphically in Fig. 4-16. We need the value of x such that $P(X < x) = 0.98$. By standardizing, this probability expression can be written as

$$\begin{aligned} P(X < x) &= P((X - 10)/2 < (x - 10)/2) \\ &= P(Z < (x - 10)/2) \\ &= 0.98 \end{aligned}$$

Appendix Table II is used to find the z -value such that $P(Z < z) = 0.98$. The nearest probability from Table II results in

$$P(Z < 2.05) = 0.97982$$

Therefore, $(x - 10)/2 = 2.05$, and the standardizing transformation is used in reverse to solve for x . The result is

$$x = 2(2.05) + 10 = 14.1 \text{ milliamperes}$$

4-6 Normal Distribution

Example 4-14 (continued)

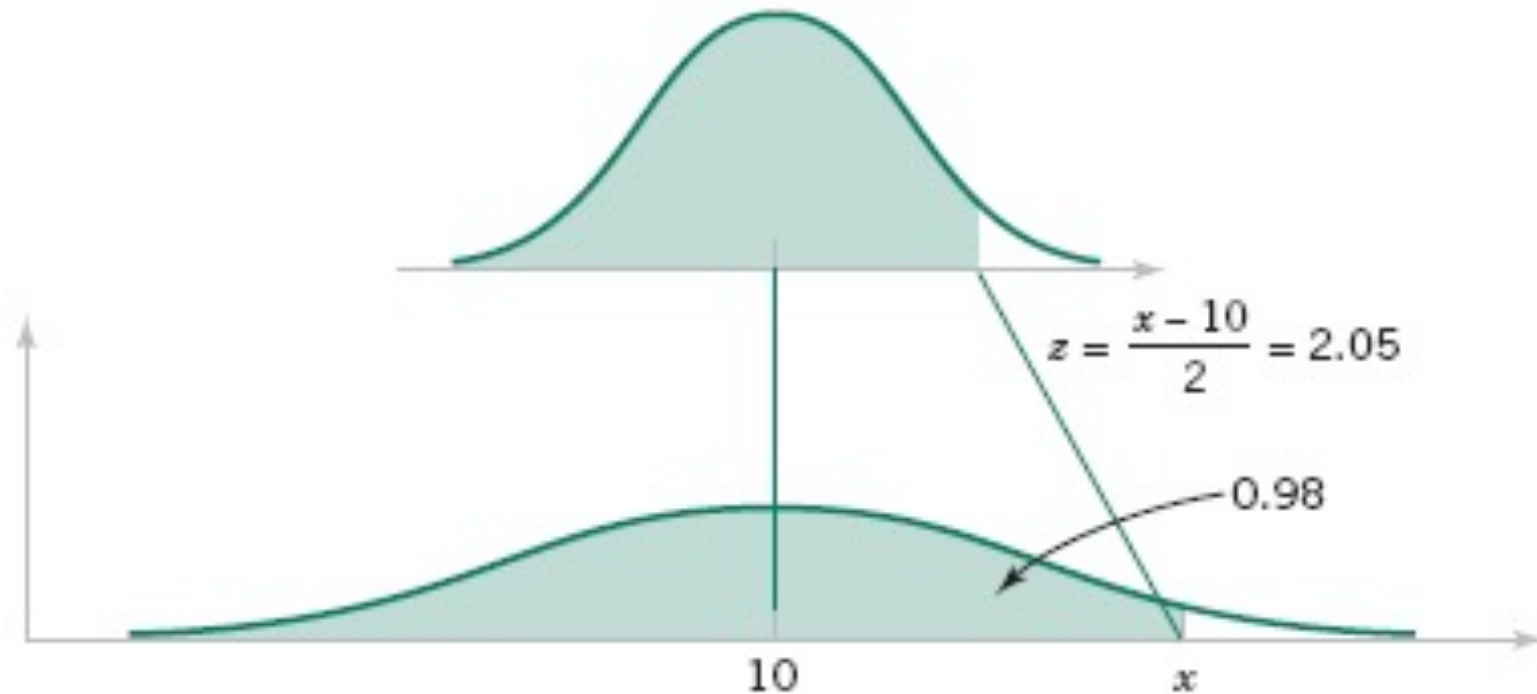


Figure 4-16 Determining the value of x to meet a specified probability.

4-6 Normal Distribution

Example 4 – 15 Signal Detection

Assume that in the detection of a digital signal the background noise follows a normal distribution with a mean of 0 volt and standard deviation of 0.45 volt. The system assumes a digital 1 has been transmitted when the voltage exceed 0.9. What is the probability of detecting a digit 1 when none was sent?

Solution:

Let the random variable N denote the voltage of noise. The requested probability is

$$P(N > 0.9) = P\left(\frac{N}{0.45} > \frac{0.9}{0.45}\right) = P(Z > 2) = 1 - 0.97725 = 0.02275.$$

4-6 Normal Distribution

Example 4 – 16 Shaft Diameter

The diameter of a shaft in an optical storage drive is normal distributed with mean 0.2508 inch and standard deviation 0.0005 inch. The specifications on the shaft are 0.2500 ± 0.0015 inch. What proportion of shafts conforms to specifications?

Let X denote the shaft diameter in inches. The requested probability is shown in Fig.4 – 18

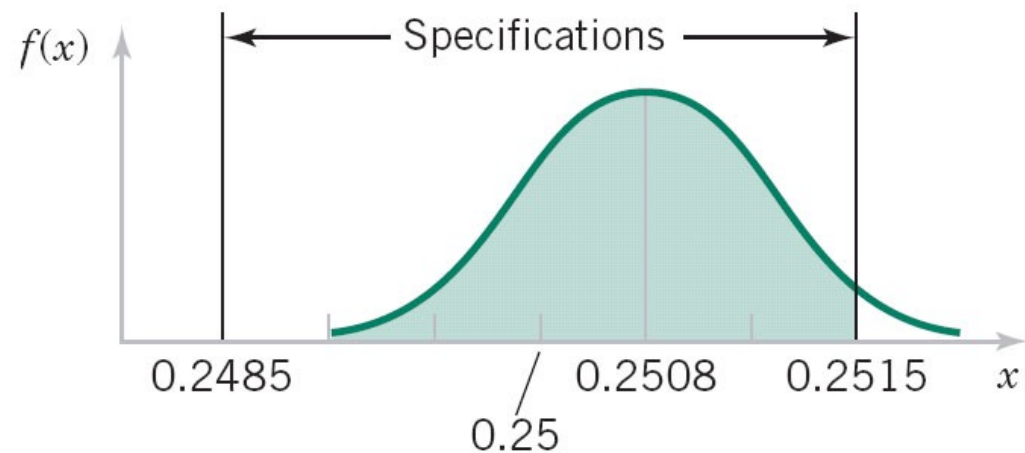


Figure 4-18
Distribution for
Example 4-16.

4-6 Normal Distribution

and

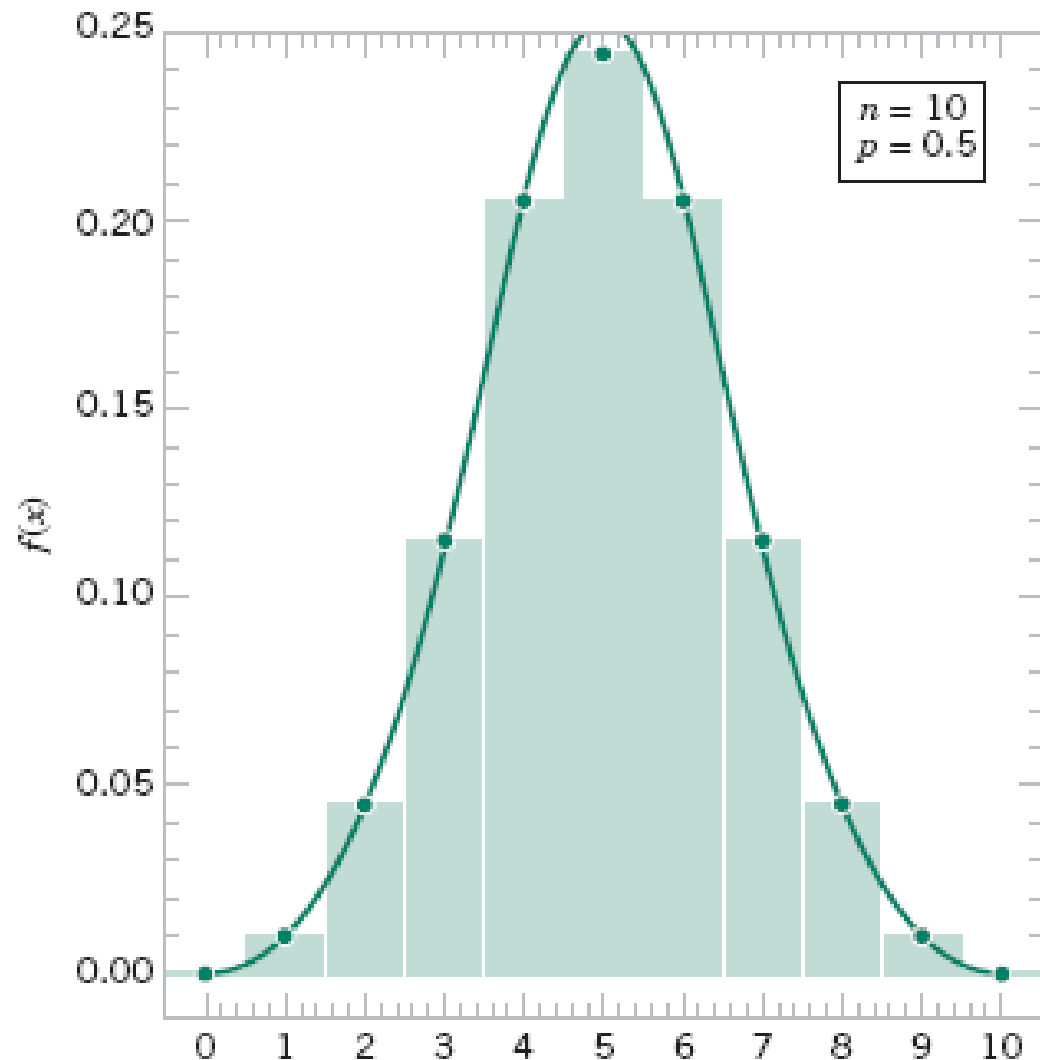
$$\begin{aligned}
 P(0.2485 < X < 0.2515) &= P\left(\frac{0.2485 - 0.2508}{0.0005} < Z < \frac{0.2515 - 0.2508}{0.0005}\right) \\
 &= P(-4.6 < Z < 1.4) = 0.91924.
 \end{aligned}$$

4-7 Normal Approximation to the Binomial and Poisson Distributions

- Under certain conditions, the normal distribution can be used to approximate the binomial distribution and the Poisson distribution.

4-7 Normal Approximation to the Binomial and Poisson Distributions

Figure 4-19 Normal approximation to the binomial.



4-7 Normal Approximation to the Binomial and Poisson Distributions

Example 4-17

In a digital communication channel, assume that the number of bits received in error can be modeled by a binomial random variable, and assume that the probability that a bit is received in error is 1×10^{-5} . If 16 million bits are transmitted, what is the probability that more than 150 errors occur?

Let the random variable X denote the number of errors. Then X is a binomial random variable and

$$P(X > 150) = 1 - P(x \leq 150) = 1 - \sum_{x=0}^{150} \binom{16,000,000}{x} (10^{-5})^x (1 - 10^{-5})^{16,000,000-x}$$

Clearly, the probability in Example 4-17 is difficult to compute. Fortunately, the normal distribution can be used to provide an excellent approximation in this example.

4-7 Normal Approximation to the Binomial and Poisson Distributions

Normal Approximation to the Binomial Distribution

If X is a binomial random variable, with parameters n and p

$$Z = \frac{X - np}{\sqrt{np(1 - p)}} \quad (4-12)$$

is approximately a standard normal random variable. To approximate a binomial probability with a normal distribution a **continuity correction** is applied as follows

$$P(X \leq x) = P(X \leq x + 0.5) \cong P\left(Z \leq \frac{x + 0.5 - np}{\sqrt{np(1 - p)}}\right)$$

and

$$P(x \leq X) = P(x - 0.5 \leq X) \cong P\left(\frac{x - 0.5 - np}{\sqrt{np(1 - p)}} \leq Z\right)$$

The approximation is good for $np > 5$ and $n(1 - p) > 5$.

4-7 Normal Approximation to the Binomial and Poisson Distributions

Example 4-18

The digital communication problem in the previous example is solved as follows:

$$\begin{aligned}
 P(X > 150) &= P\left(\frac{X - 0.5 - 160}{\sqrt{160(1 - 10^{-5})}} > \frac{151 - 0.5 - 160}{\sqrt{160(1 - 10^{-5})}}\right) \\
 &= P(Z > -0.75) = P(Z < 0.75) = 0.7733
 \end{aligned}$$

Because $np = (16 \times 10^6)(1 \times 10^{-5}) = 160$ and $n(1 - p)$ is much larger, the approximation is expected to work well in this case.

4-7 Normal Approximation to the Binomial and Poisson Distributions

Example 4 – 19 Normal Approximation to Binomial

Again consider the transmission of bits in Example 4 – 18. To judge how well the normal approximation works, assume only $n = 50$ bits are to be transmitted and that the probability of an error is $p = 0.1$. the exact probability that 2 or less errors occur is

$$P(X \leq 2) = \binom{50}{0} 0.9^{50} + \binom{50}{1} 0.9^{49} + \binom{50}{2} 0.9^{48} = 0.112$$

4-7 Normal Approximation to the Binomial and Poisson Distributions

Example 4 – 19 Normal Approximation to Binomial

Based on the normal approximation

$$P(X \leq 2) = P\left(\frac{X - 5}{\sqrt{50(0.1)(0.9)}} < \frac{2.5 - 5}{\sqrt{50(0.1)(0.9)}}\right) = P(Z < -1.18) = 0.119$$

As another example, $P(8 < X) = P(9 \leq X)$ and this better approximated as

$$\begin{aligned} P(9 \leq X) &\approx P\left(\frac{8.5 - 5}{2.12} \leq Z\right) = P(1.65 \leq Z) = 0.05 \\ &= P(Z > -0.79) = P(Z < 0.79) = 0.785 \end{aligned}$$

4-7 Normal Approximation to the Binomial and Poisson Distributions

Example 4 – 19 Normal Approximation to Binomial

We can approximate $P(X = 5) = P(5 \leq X \leq 5)$ as

$$P(5 \leq X \leq 5) = P\left(\frac{5 - 0.5 - 5}{2.12} \leq Z \leq \frac{5 + 0.5 - 5}{2.12}\right) = P(-0.24 \leq Z \leq 0.24) = 0.19.$$

4-7 Normal Approximation to the Binomial and Poisson Distributions

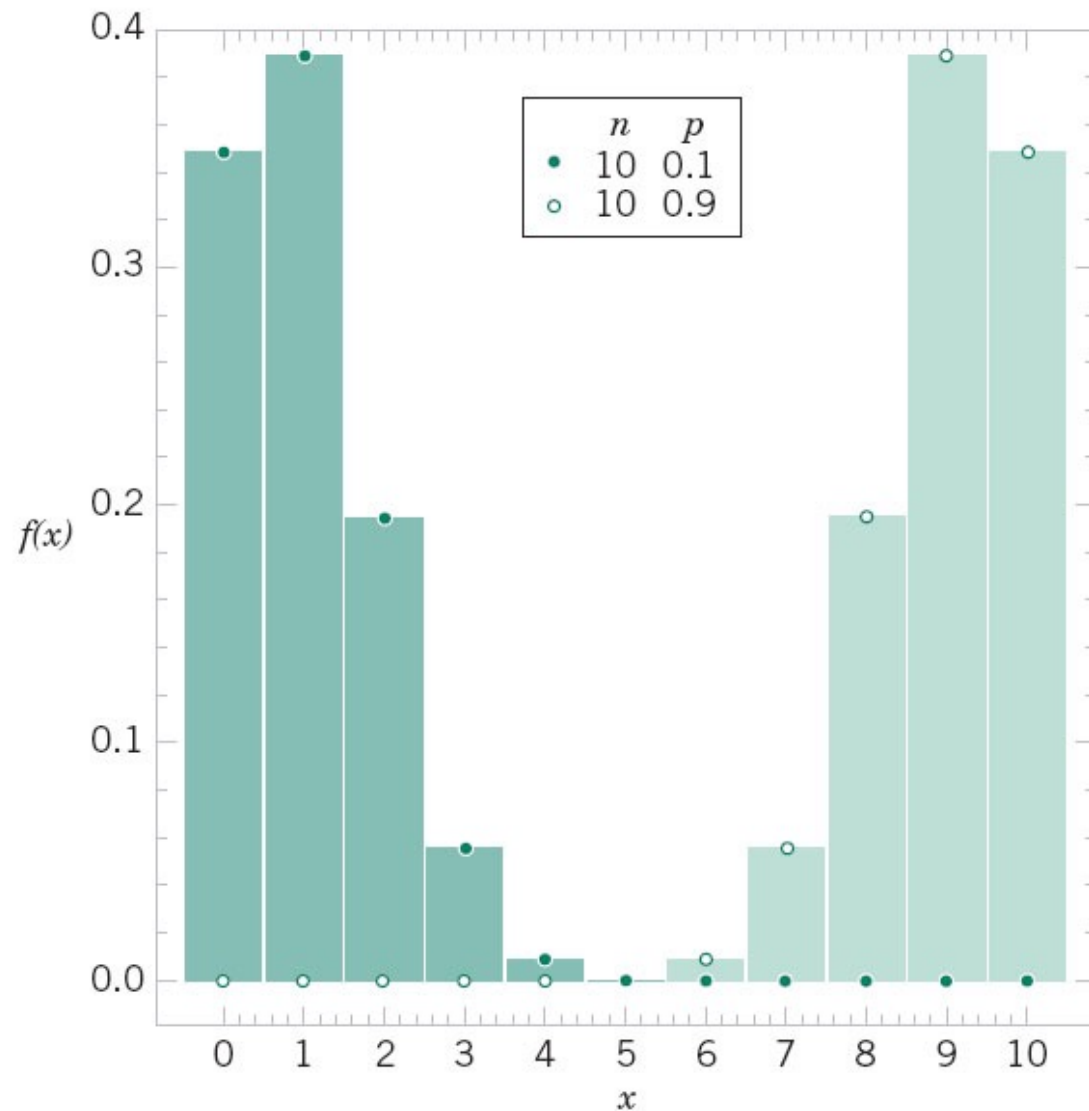


Figure 4-20 Binomial distribution is not symmetrical if p is near 0 or 1.

4-7 Normal Approximation to the Binomial and Poisson Distributions

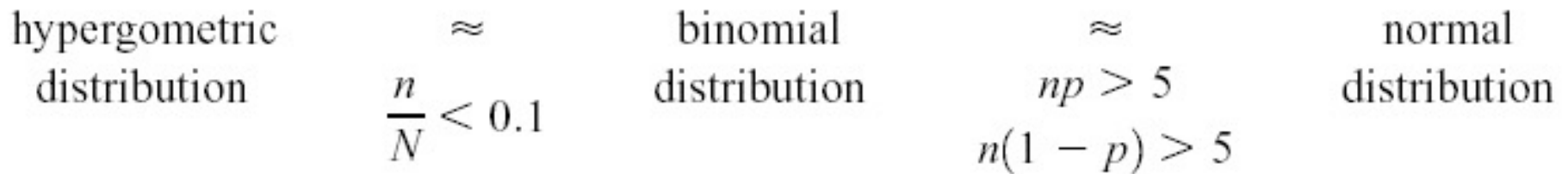


Figure 4-21 Conditions for approximating hypergeometric and binomial probabilities.

4-7 Normal Approximation to the Binomial and Poisson Distributions

Normal Approximation to the Poisson Distribution

If X is a Poisson random variable with $E(X) = \lambda$ and $V(X) = \lambda$,

$$Z = \frac{X - \lambda}{\sqrt{\lambda}} \quad (4-13)$$

is approximately a standard normal random variable. The approximation is good for

$$\lambda > 5$$

4-7 Normal Approximation to the Binomial and Poisson Distributions

Example 4-20

Assume that the number of asbestos particles in a squared meter of dust on a surface follows a Poisson distribution with a mean of 1000. If a squared meter of dust is analyzed, what is the probability that less than 950 particles are found?

This probability can be expressed exactly as

$$P(X < 950) = \sum_{x=0}^{949} \frac{e^{-1000} \cdot 1000^x}{x!}$$

The computational difficulty is clear. The probability can be approximated as

$$P(X \leq x) = P\left(Z \leq \frac{950 - 1000}{\sqrt{1000}}\right) = P(Z \leq -1.58) = 0.057$$

4-8 Exponential Distribution

Definition

The random variable X that equals the distance between successive events of a Poisson process with mean $\lambda > 0$ is an **exponential random variable** with parameter λ . The probability density function of X is

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } 0 \leq x < \infty \quad (4-14)$$

4-8 Exponential Distribution

Mean and Variance

If the random variable X has an exponential distribution with parameter λ ,

$$\mu = E(X) = \frac{1}{\lambda} \quad \text{and} \quad \sigma^2 = V(X) = \frac{1}{\lambda^2} \quad (4-15)$$

It is important to **use consistent units** in the calculation of probabilities, means, and variances involving exponential random variables. The following example illustrates unit conversions.

4-8 Exponential Distribution

Example 4-21

In a large corporate computer network, user log-ons to the system can be modeled as a Poisson process with a mean of 25 log-ons per hour. What is the probability that there are no log-ons in an interval of 6 minutes?

Let X denote the time in hours from the start of the interval until the first log-on. Then, X has an exponential distribution with $\lambda = 25$ log-ons per hour. We are interested in the probability that X exceeds 6 minutes. Because λ is given in log-ons per hour, we express all time units in hours. That is, 6 minutes = 0.1 hour. The probability requested is shown as the shaded area under the probability density function in Fig. 4-23. Therefore,

$$P(X > 0.1) = \int_{0.1}^{\infty} 25e^{-25x} dx = e^{-25(0.1)} = 0.082$$

4-8 Exponential Distribution

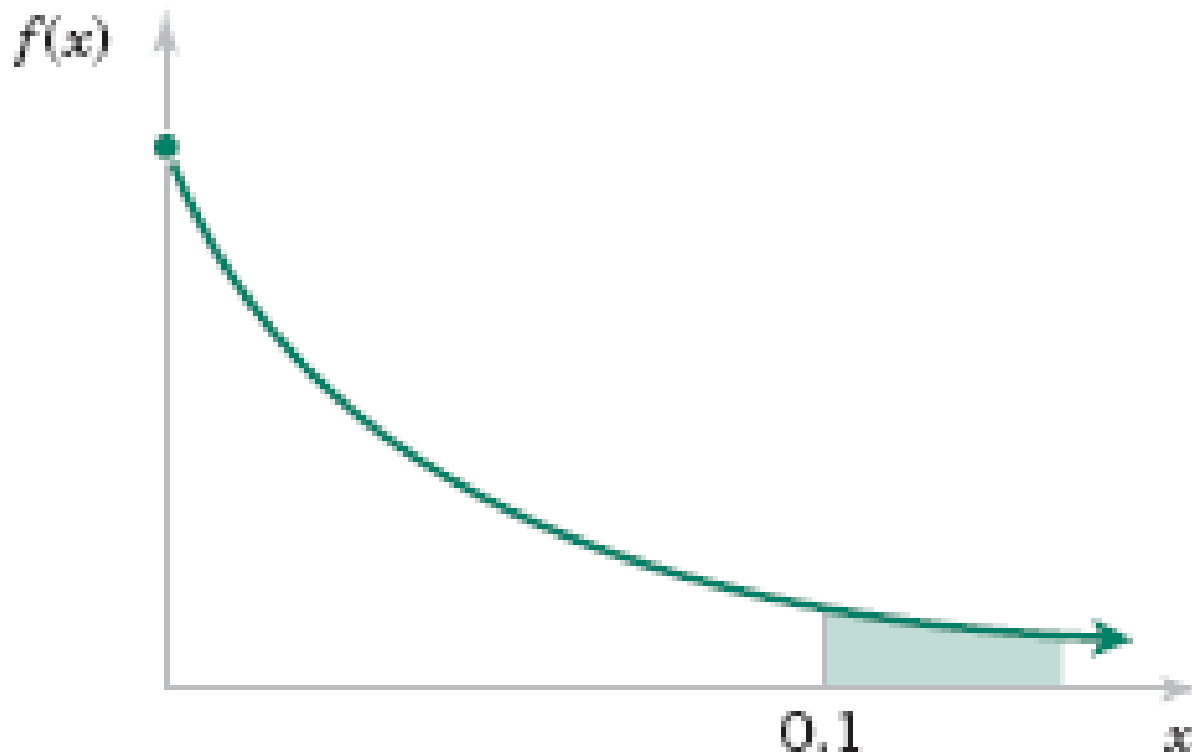


Figure 4-23 Probability for the exponential distribution in Example 4-21.

4-8 Exponential Distribution

Example 4-21 (continued)

Also, the cumulative distribution function can be used to obtain the same result as follows:

$$P(X > 0.1) = 1 - F(0.1) = e^{-25(0.1)}$$

An identical answer is obtained by expressing the mean number of log-ons as 0.417 log-ons per minute and computing the probability that the time until the next log-on exceeds 6 minutes. Try it.

What is the probability that the time until the next log-on is between 2 and 3 minutes? Upon converting all units to hours,

$$P(0.033 < X < 0.05) = \int_{0.033}^{0.05} 25e^{-25x} dx = -e^{-25x} \Big|_{0.033}^{0.05} = 0.152$$

4-8 Exponential Distribution

Example 4-21 (continued)

An alternative solution is

$$P(0.033 < X < 0.05) = F(0.05) - F(0.033) = 0.152$$

Determine the interval of time such that the probability that no log-on occurs in the interval is 0.90. The question asks for the length of time x such that $P(X > x) = 0.90$. Now,

$$P(X > x) = e^{-25x} = 0.90$$

Take the (natural) log of both sides to obtain $-25x = \ln(0.90) = -0.1054$. Therefore,

$$x = 0.00421 \text{ hour} = 0.25 \text{ minute}$$

4-8 Exponential Distribution

Example 4-21 (continued)

Furthermore, the mean time until the next log-on is

$$\mu = 1/25 = 0.04 \text{ hour} = 2.4 \text{ minutes}$$

The standard deviation of the time until the next log-on is

$$\sigma = 1/25 \text{ hours} = 2.4 \text{ minutes}$$

IMPORTANT TERMS AND CONCEPTS

Chi-squared distribution	Gamma distribution	Normal approximation to binomial and Poisson probabilities	Standard deviation-continuous random variable
Continuous uniform distribution	Lack of memory property-continuous random variable	Normal distribution	Standard normal distribution
Continuity correction	Lognormal distribution	Probability density function	Standardizing
Cumulative probability distribution function-continuous random variable	Mean-continuous random variable	Probability distribution-continuous random variable	Variance-continuous random variable
Erlang distribution	Mean-function of a continuous random variable		Weibull distribution
Exponential distribution			