Chapter 7: Point Estimation of Parameters

Learning objectives

1. Introduction
2. Sampling Distributions
3. General Concepts Of Point
4. Estimation Methods Of Point Estimation

SET UP PROBLEM

Let X is a random variable with probability distribution $f(x)$, which is characterized by the unknown $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$

For example, $X \sim N(\mu, \sigma^2)$ then $\theta = (\mu, \sigma^2)$.

How to “determine” the values of θ?

In engineering, we often need to estimate

1. The mean μ of a single population.
2. The variance σ^2 (or standard deviation) of a single population.
3. The proportion p of items in a population that belong to a class of interest.
4. The difference in means of two populations, $\mu_1 - \mu_2$.
5. The difference in two population proportions, $p_1 - p_2$.

The important results on point estimation

1. For μ , the estimate is the $\hat{\mu} = \bar{x}$, sample mean.
2. For σ^2 , the estimate is $\hat{\sigma}^2 = s^2$, the sample variance.
3. For p, the estimate is $\hat{p} = x/n$, the sample proportion.
4. For $\mu_1 - \mu_2$, the estimate is $\hat{\mu}_1 - \hat{\mu}_2 = \bar{x}_1 - \bar{x}_2$
5. For $p_1 - p_2$, the estimate is $\hat{p}_1 - \hat{p}_2$

Definition — Random Sample

The random variables $X_1, \ldots, X_n$ are called a random sample of size n if

- The X_i's are independent
- Every X_i has the same probability distribution

Definition — Statistic

- A statistic $\hat{\Theta}$ is any function of the observations $X_1, \ldots, X_n$:

$$\hat{\Theta} = h(X_1, \ldots, X_n)$$

- The probability distribution of a statistic is called a sampling distribution.

Example Two important statistic

- Sample mean $\overline{X}$

$$\overline{X} = \frac{X_1 + \ldots + X_n}{n}$$

- Sample variance S^2

$$S^2 = \frac{(X_1 - \overline{X})^2 + \ldots + (X_n - \overline{X})^2}{n-1}$$

Theorem

In above example, if $(X_1, \ldots., X_n)$ is a random sample of size n take from a normal distribution $N(\mu, \sigma^2)$ then

- $\bar{X}$ has a normal distribution $N(\mu, \sigma^2/n)$
- $\dfrac{(n-1)S^2}{\sigma^2}$ has a chi-square distribution with n-1 degrees of freedom (see pages 273-274).

Theorem — **Central Limit Theorem**

Let $(X_1, \ldots, X_n)$ is a random sample of size n take from a population with mean μ and finite variance σ^2, and if $\overline{X}$ is the sample mean, the limiting form of the distribution of

$$Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$$

As $n \to \infty$, is the standard normal distribution.

Remark: The normal approximation for $\overline{X}$ depends on the sample size n.

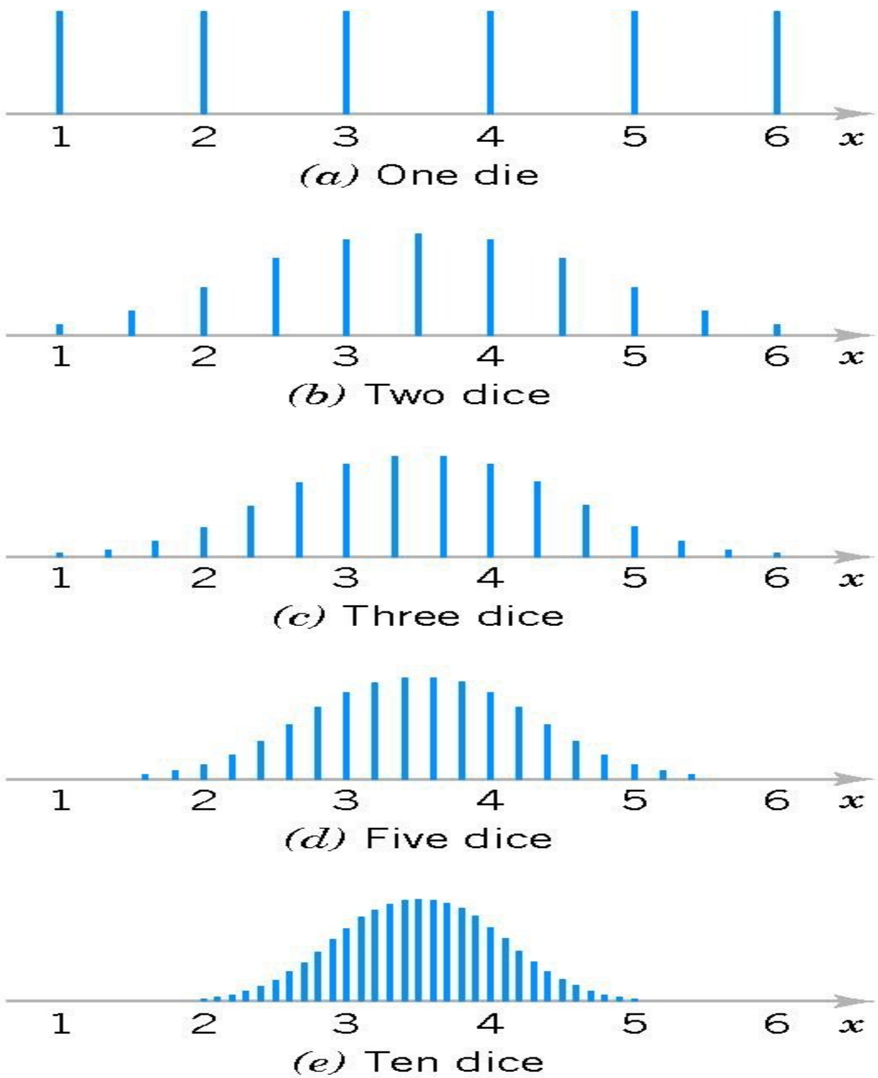


Figure 7-1 Distributions of average scores from throwing dice.

Definition — Point Estimate

- A **point estimate** of some population parameter θ is a single numerical value $\hat{\theta}$ of a statistic $\hat{\Theta}$.
- The statistic $\hat{\Theta}$ is called the **point estimator**.

Two steps to find point estimation:

Step 1. Determine $\hat{\Theta}$ by using the theoretical results.

Steps 2. Calculate $\hat{\theta}$ from the experimental data.

Example

Market researchers use the number of sentences per advertisement as a measure of readability for magazine advertisements. The following represents a random sample of the number of sentences found in 15 advertisements. Find a point estimate of the population mean, μ.

9 20 18 16 9 9 11 13 22 16 5 18 6 6 5

Step 1. $$\hat{\Theta} = \frac{X_1 + \dots + X_{15}}{15}$$

Step 2. A point estimate for μ is

$$\hat{\mu} = \frac{9+20+\dots+6+5}{15} = \frac{183}{15} = 12.2$$

Definition — Unbiased Estimator

The point estimator $\hat{\Theta}$ is an **unbiased estimator** for the parameter θ if

$$E(\hat{\Theta}) = \theta$$

If the estimator is not unbiased, then the difference

$$E(\hat{\Theta} - \theta)$$

is called the **bias** of the estimator $\hat{\Theta}$.

Example

Suppose that X is a random variable with mean μ and variance σ^2. Let $X_1, \ldots, X_n$ be a random sample of size n from the population represented by X. Show that the sample mean $\bar{X}$ and sample variance S^2 are unbiased estimators of μ and σ^2, respectively.

For the sample mean.

$$E(\bar{X}) = E\frac{X_1 + \ldots + X_n}{n} = \frac{EX_1 + \ldots + EX_n}{n} = \mu$$

Now consider the sample variance. We have

$$E(S^2) = E\left[\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}\right] = \frac{1}{n-1}E\sum_{i=1}^{n}(X_i - \overline{X})^2$$

$$= \frac{1}{n-1}E\sum_{i=1}^{n}(X_i^2 + \overline{X}^2 - 2\overline{X}X_i) = \frac{1}{n-1}E\left(\sum_{i=1}^{n}X_i^2 - n\overline{X}^2\right)$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}E(X_i^2) - nE(\overline{X}^2)\right]$$

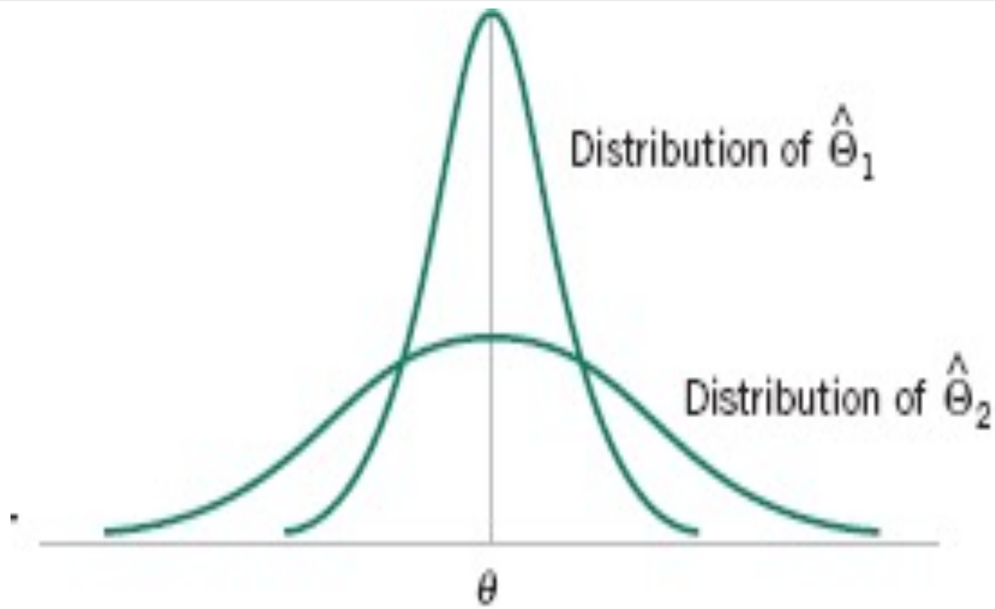
Hence $E(S^2) = \sigma^2$

MINIMUM VARIANCE UNBIASED ESTIMATOR

Definition Minimum Variance Unbiased Estimator

If we consider all unbiased estimators of θ, the one with the smallest variance is called the **minimum variance unbiased estimator** (MVUE).

Figure 7-5
The sampling distributions of two unbiased estimators $\hat{\Theta}_1$ and $\hat{\Theta}_2$.



Theorem

Let $(X_1, \ldots., X_n)$ is a random sample of size n take from a normal distribution $N(\mu, \sigma^2)$, the sample mean $\overline{X}$ is the MVUE for μ.

Definition Minimum Variance Unbiased Estimator

The **standard error** of an estimator $\hat{\Theta}$ is its standard deviation, given by

$$\sigma_{\hat{\Theta}} = \sqrt{V(\hat{\Theta})}$$

If the standard error involves unknown parameters that can be estimated, substitution of those values into $\sigma_{\hat{\Theta}}$ produces an **estimated standard error,** denoted by $\hat{\sigma}_{\hat{\Theta}}$

Remark

If the random sample of size n take from a normal distribution $N(\mu, \sigma^2)$ then $\overline{X}$ has a normal distribution $N(\mu, \sigma^2/n)$.

STANDARD ERROR

Standard error of $\overline{X}$ is

$$\sigma_{\hat{X}} = \frac{\sigma}{\sqrt{n}}$$

If we did not know σ then the estimated standard error of $\overline{X}$ would be

$$\hat{\sigma}_{\hat{X}} = \frac{s}{\sqrt{n}}$$

Exercise

A new method of measuring the thermal conductivity of Armco iron: Using a temperature of 100^0F and a power input of 550 watts, the following 10 measurements of thermal conductivity (in Btu/hr-ft-^{0}F) were obtained:

$$41.60, 41.48, 42.34, 41.95, 41.86,$$

$$42.18, 41.72, 42.26, 41.81, 42.04$$

a, Find a point estimate of the mean thermal conductivity at 100^0F and 550 watts.

b, Find the estimated standard error of $\overline{X}$

Definition — The kth moments

Let $(X_1, \ldots, X_n)$ be a random sample from random variable X with the probability distribution $f(x)$

- The kth moment of X are denoted by $M_k := E(X^k)$
- The corresponding kth **sample moment** is

$$\overline{X}^k = \frac{X_1^k + \ldots + X_n^k}{n}$$

Definition — The *k*th moments

Let $(X_1, \ldots, X_n)$ be a random sample from either a probability mass function or probability density function with m unknown parameters $(\theta_1, \theta_2, \ldots, \theta_m)$. The moment estimators $\hat{\Theta}_1, \ldots, \hat{\Theta}_m$ are found the following system

$$\begin{cases} M_1 = \overline{X}^1 \\ \ldots\ldots\ldots \quad \ldots \\ M_m = \overline{X}^m \end{cases}$$

Example

Suppose that $X_1, \ldots, X_n$ be a random sample of size n from a normal distribution $N(\mu, \sigma^2)$. Find the moment estimators of μ and σ^2.

We have to solve following system

$$\begin{cases} M_1 = \overline{X}^1 \\ M_2 = \overline{X}^2 \end{cases} \quad \text{or} \quad \begin{cases} EX = \dfrac{X_1 + \ldots + X_n}{n} \\ EX^2 = \dfrac{X_1^2 + \ldots + X_n^2}{n} \end{cases}$$

Here $E(X) = \mu$ and $E(X^2) = V(X) + (EX)^2 = \sigma^2 + \mu^2$

Hence, by solving the above system we have

$$\mu = \overline{X}, \qquad \sigma^2 = \frac{\sum_{i=1}^{n} X_i^2 - n(\overline{X})^2}{n} = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n}$$

Finally, the moment estimators are

$$\hat{\mu} = \overline{X}, \qquad \hat{\sigma}^2 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n}$$

Unbiased estimator

Biased estimator

Definition Maximum likelihood Estimator

Suppose that X is a random variable with probability distribution $f(x, \theta)$, where θ is a single unknown parameter. Let $x_1, x_2, , x_n$ be the observed values in a random sample of size n. Then the **likelihood function** of the sample is

$$L(\theta) = f(x_1, \theta)\, f(x_2, \theta) \ldots f(x_n, \theta)$$

The maximum likelihood estimator (MLE) of θ is the value of θ that maximizes the $L(\theta)$.

Example

Let X is a Bernoulli random variable. The probability mass function is

$$f(x,p)=\begin{cases} p^{x}(1-p)^{1-x}, & x=0,1 \\ 0 & otherwise \end{cases}$$

Find the maximum likelihood estimator of p.

The likelihood function of a random sample of size n is

$$L(p)=p^{x_1}(1-p)^{1-x_1}p^{x_2}(1-p)^{1-x_2}\cdots p^{x_n}(1-p)^{1-x_n}$$
$$=\prod_{i=1}^{n}p^{x_i}(1-p)^{1-x_i}=p^{\sum_{i=1}^{n}x_i}(1-p)^{n-\sum_{i=1}^{n}x_i}$$

Introduction

Sampling Distributions

General concepts of point estimation

Methods of point estimation

Method of moments

Method of maximum likelihood

Summary

$$\ln L(p) = \left(\sum_{i=1}^{n} x_i\right)\ln p + \left(n - \sum_{i=1}^{n} x_i\right)\ln(1-p)$$

$$\frac{d\ln L(p)}{dp} = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{\left(n - \sum_{i=1}^{n} x_i\right)}{1-p}$$

The maximum likelihood estimator of p is

$$\hat{P} = \frac{X_1 + \ldots + X_n}{n}$$

Is this a unbiased estimator?

Exercise

Let X is a random variable, which has probability mass function is

$$f(x) = \begin{cases} p & , \quad x = -1 \\ 1-p & , \quad x = 1 \\ 0 & , \quad otherwise \end{cases}$$

Find the maximum likelihood estimator of *p*.

Solution:

$$f(x) = \begin{cases} p^{\frac{1-x}{2}}(1-p)^{\frac{1+x}{2}} & , \quad x = -1,1 \\ 0 & , \quad otherwise \end{cases}$$

$$\hat{P} = \frac{1}{2} - \frac{X_1 + \ldots + X_n}{2n}$$

Invariance Property

Let $\hat{\Theta}_1,\ldots,\hat{\Theta}_k$ be the maximum likelihood estimators of the parameters $\theta_1,\ldots,\theta_k$. Then the maximum likelihood estimator of any function $h(\theta_1,\ldots,\theta_k)$ of these parameters is the same function $h(\hat{\Theta}_1,\ldots,\hat{\Theta}_k)$ of the estimators $\hat{\Theta}_1,\ldots,\hat{\Theta}_k$.

Exercise

Let X be normally distributed $N(\mu, \sigma^2)$, where both μ and σ^2 are unknown.

(a) Find the maximum likelihood estimator of μ and σ^2.

(b) Find the maximum likelihood estimator of $\theta = \mu^2\sigma^2$.

Solution:

$$\hat{\mu} = \overline{X}, \qquad \hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n}$$

$$\hat{\theta} = (\overline{X})^2 \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n}$$

SUMMARY

We have studied:

1. Sampling distributions
2. General concepts of point estimation: Unbiased Estimator
3. Methods of point estimation

- Method of Moments
- Method of Maximum Likelihood

Homework: Read slides of the next lecture.