FPT UNIVERSITY
DREAM OF INNOVATION

8 Statistical Intervals for a Single Sample

CHAPTER OUTLINE

LEARNING OBJECTIVES

After careful study of this chapter, you should be able to do the following:

1. Construct confidence intervals on the mean of a normal distribution, using either the normal distribution or the t distribution method
2. Construct confidence intervals on the variance and standard deviation of a normal distribution
3. Construct confidence intervals on a population proportion
4. Use a general method for constructing an approximate confidence interval on a parameter

8-1 Introduction

- In the previous chapter we illustrated how a parameter can be estimated from sample data. However, it is important to understand how good is the estimate obtained.
- Bounds that represent an interval of plausible values for a parameter are an example of an interval estimate.
- Three types of intervals will be presented:
    - Confidence intervals
    - Prediction intervals
    - Tolerance intervals

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

8-2.1 Development of the Confidence Interval and its Basic Properties

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution with unknown mean μ and known variance σ^2. From the results of Chapter 5 we know that the sample mean $\overline{X}$ is normally distributed with mean μ and variance σ^2/n. We may **standardize** $\overline{X}$ by subtracting the mean and dividing by the standard deviation, which results in the variable

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \tag{8-3}$$

Now Z has a standard normal distribution.

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

8-2.1 Development of the Confidence Interval and its Basic Properties

A **confidence interval** estimate for μ is an interval of the form $l \leq \mu \leq u$, where the end-points l and u are computed from the sample data. Because different samples will produce different values of l and u, these end-points are values of random variables L and U, respectively. Suppose that we can determine values of L and U such that the following probability statement is true:

$$P\{L \leq \mu \leq U\} = 1 - \alpha \tag{8-4}$$

where $0 \leq \alpha \leq 1$. There is a probability of $1 - \alpha$ of selecting a sample for which the CI will contain the true value of μ. Once we have selected the sample, so that $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$, and computed l and u, the resulting **confidence interval** for μ is

$$l \leq \mu \leq u \tag{8-5}$$

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

8-2.1 Development of the Confidence Interval and its Basic Properties

The endpoints or bounds *l* and *u* are called lower- and upper-confidence limits, respectively.

• Since Z follows a standard normal distribution, we can write:

$$P\left\{-z_{\alpha/2} \le \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \le z_{\alpha/2}\right\} = 1-\alpha$$

Now manipulate the quantities inside the brackets by (1) multiplying through by $\sigma/\sqrt{n}$, (2) subtracting $\overline{X}$ from each term, and (3) multiplying through by -1. This results in

$$P\left\{\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right\} = 1-\alpha \qquad (8\text{-}6)$$

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

8-2.1 Development of the Confidence Interval and its Basic Properties

Definition

If $\overline{x}$ is the sample mean of a random sample of size n from a normal population with known variance σ^2, a $100(1-\alpha)\%$ CI on μ is given by

$$\overline{x} - z_{\alpha/2}\sigma/\sqrt{n} \leq \mu \leq \overline{x} + z_{\alpha/2}\sigma/\sqrt{n} \qquad (8\text{-}7)$$

where $z_{\alpha/2}$ is the upper $100\alpha/2$ percentage point of the standard normal distribution.

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

Example 8-1

ASTM Standard E23 defines standard test methods for notched bar impact testing of metallic materials. The Charpy V-notch (CVN) technique measures impact energy and is often used to determine whether or not a material experiences a ductile-to-brittle transition with decreasing temperature. Ten measurements of impact energy (J) on specimens of A238 steel cut at 60°C are as follows: 64.1, 64.7, 64.5, 64.6, 64.5, 64.3, 64.6, 64.8, 64.2, and 64.3. Assume that impact energy is normally distributed with $\sigma = 1J$. We want to find a 95% CI for μ, the mean impact energy. The required quantities are $z_{\alpha/2} = z_{0.025} = 1.96$, $n = 10$, $\sigma = 1$, and $\bar{x} = 64.46$. The resulting 95% CI is found from Equation 8-7 as follows:

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$64.46 - 1.96\frac{1}{\sqrt{10}} \le \mu \le 64.46 + 1.96\frac{1}{\sqrt{10}}$$

$$63.84 \le \mu \le 65.08$$

That is, based on the sample data, a range of highly plausible vaules for mean impact energy for A238 steel at 60°C is $63.84J \le \mu \le 65.08J$.

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

Confidence Level and Precision of Error

The length of a confidence interval is a measure of the precision of estimation.

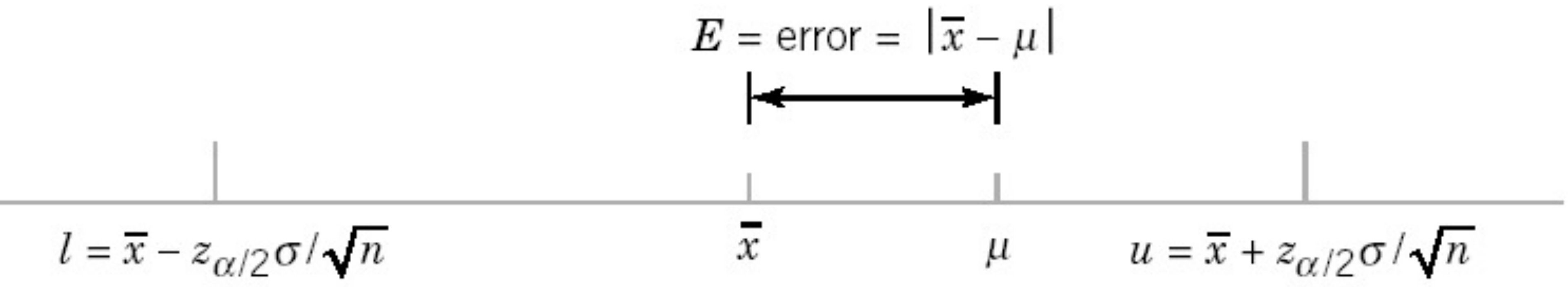


Figure 8-2 Error in estimating μ with $\bar{x}$.

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

8-2.2 Choice of Sample Size

Definition

If $\bar{x}$ is used as an estimate of μ, we can be $100(1-\alpha)\%$ confident that the error $|\bar{x}-\mu|$ will not exceed a specified amount E when the sample size is

$$n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 \tag{8-8}$$

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

Example 8-2

To illustrate the use of this procedure, consider the CVN test described in Example 8-1, and suppose that we wanted to determine how many specimens must be tested to ensure that the 95% CI on μ for A238 steel cut at 60°C has a length of at most 1.0J. Since the bound on error in estimation E is one-half of the length of the CI, to determine n we use Equation 8-8 with $E = 0.5$, $\sigma = 1$, and $z_{\alpha/2} = 0.025$. The required sample size is 16

$$n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 = \left[\frac{(1.96)1}{0.5}\right]^2 = 15.37$$

and because n must be an integer, the required sample size is $n = 16$.

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

8-2.3 One-Sided Confidence Bounds

Definition

A $100(1-\alpha)\%$ upper-confidence bound for μ is

$$\mu \leq u = \bar{x} + z_{\alpha}\sigma/\sqrt{n} \tag{8-9}$$

and a $100(1-\alpha)\%$ lower-confidence bound for μ is

$$\bar{x} - z_{\alpha}\sigma/\sqrt{n} = l \leq \mu \tag{8-10}$$

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

8-2.5 A Large-Sample Confidence Interval for μ

Definition

When n is large, the quantity

$$\frac{\overline{X} - \mu}{S/\sqrt{n}}$$

has an approximate standard normal distribution. Consequently,

$$\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}} \qquad (8\text{-}13)$$

is a **large sample confidence interval** for μ, with confidence level of approximately 100(1 − α)%.

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

Example 8-4

An article in the 1993 volume of the *Transactions of the American Fisheries Society* reports the results of a study to investigate the mercury contamination in largemouth bass. A sample of fish was selected from 53 Florida lakes and mercury concentration in the muscle tissue was measured (ppm). The mercury concentration values are

1.230	0.490	0.490	1.080	0.590	0.280	0.180	0.100	0.940
1.330	0.190	1.160	0.980	0.340	0.340	0.190	0.210	0.400
0.040	0.830	0.050	0.630	0.340	0.750	0.040	0.860	0.430
0.044	0.810	0.150	0.560	0.840	0.870	0.490	0.520	0.250
1.200	0.710	0.190	0.410	0.500	0.560	1.100	0.650	0.270
0.270	0.500	0.770	0.730	0.340	0.170	0.160	0.270	

Example 8-4 (continued)

The summary statistics from Minitab are displayed below:

Descriptive Statistics: Concentration

Variable	N	Mean	Median	TrMean	StDev	SE Mean
Concentration	53	0.5250	0.4900	0.5094	0.3486	0.0479

Variable	Minimum	Maximum	Q1	Q3
Concentration	0.0400	1.3300	0.2300	0.7900

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

Example 8-4 (continued)

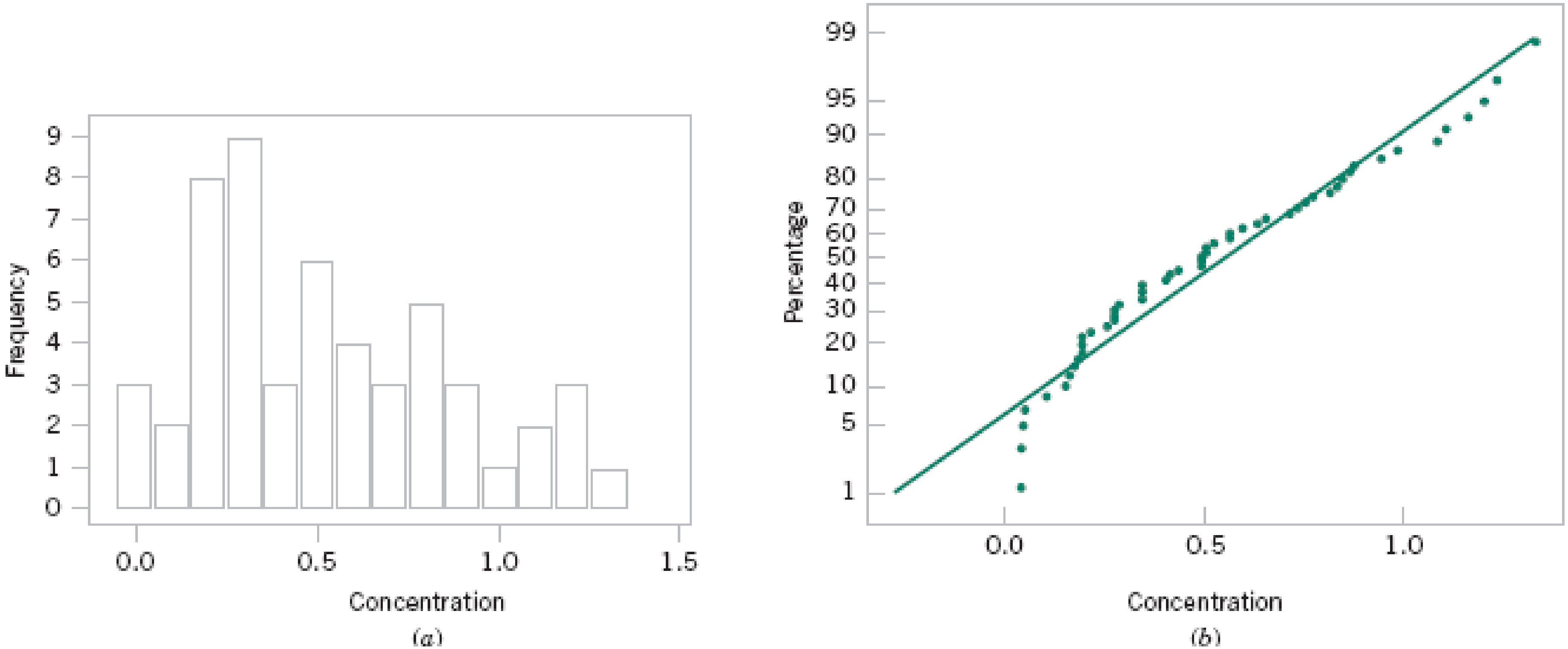


Figure 8-3 Mercury concentration in largemouth bass (a) Histogram. (b) Normal probability plot

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

Example 8-4 (continued)

Figure 8-3(a) and (b) presents the histogram and normal probability plot of the mercury concentration data. Both plots indicate that the distribution of mercury concentration is not normal and is positively skewed. We want to find an approximate 95% CI on μ. Because $n > 40$, the assumption of normality is not necessary to use Equation 8-13. The required quantities are $n = 53$, $\bar{x} = 0.5250$, $s = 0.3486$, and $z_{0.025} = 1.96$. The approximate 95% CI on μ is

$$\bar{x} - z_{0.025}\frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{0.025}\frac{s}{\sqrt{n}}$$

$$0.5250 - 1.96\frac{0.3486}{\sqrt{53}} \leq \mu \leq 0.5250 + 1.96\frac{0.3486}{\sqrt{53}}$$

$$0.4311 \leq \mu \leq 0.6189$$

This interval is fairly wide because there is a lot of variability in the mercury concentration measurements.

8-3 Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

8-3.1 The *t* distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with unknown mean μ and unknown variance σ^2. The random variable

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}} \tag{8-15}$$

has a t distribution with $n - 1$ degrees of freedom.

8-3.1 The *t* distribution

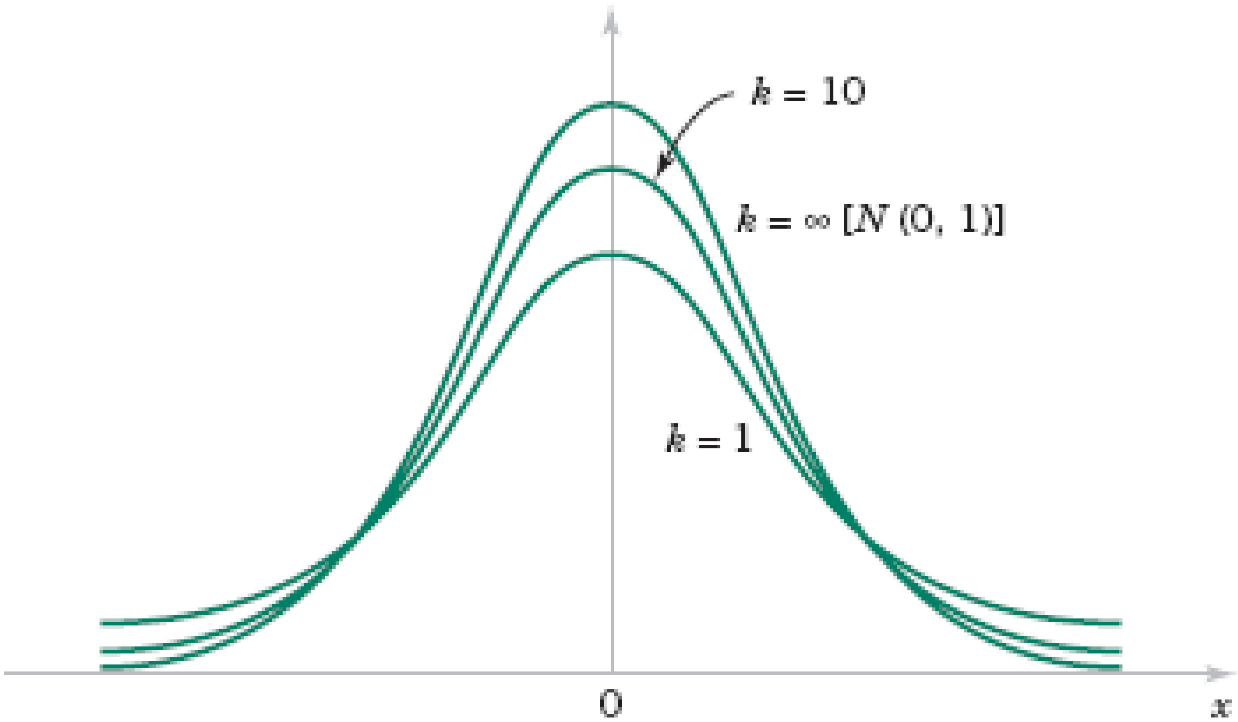


Figure 8-4 Probability density functions of several *t* distributions.

8-3 Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

8-3.1 The *t* distribution

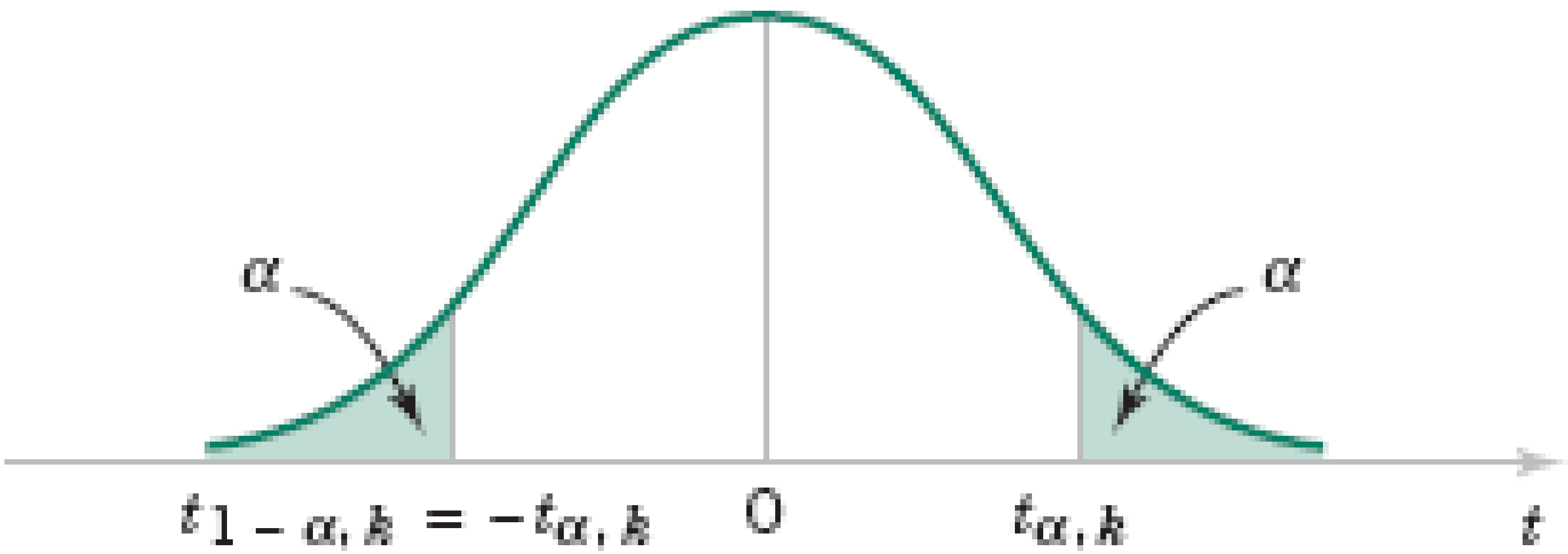


Figure 8-5 Percentage points of the *t* distribution.

8-3 Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

8-3.2 The *t* Confidence Interval on μ

If $\bar{x}$ and s are the mean and standard deviation of a random sample from a normal distribution with unknown variance σ^2, a **100(1 − α) percent confidence interval on μ** is given by

$$\bar{x} - t_{\alpha/2,n-1}s/\sqrt{n} \le \mu \le \bar{x} + t_{\alpha/2,n-1}s/\sqrt{n} \qquad (8\text{-}18)$$

where $t_{\alpha/2,n-1}$ is the upper $100\alpha/2$ percentage point of the t distribution with $n-1$ degrees of freedom.

One-sided confidence bounds on the mean are found by replacing $t_{\alpha/2,n-1}$ in Equation 8-18 with $t_{\alpha,n-1}$.

8-3 Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

Example 8-5

An article in the journal *Materials Engineering* (1989, Vol. II, No. 4, pp. 275–281) describes the results of tensile adhesion tests on 22 U-700 alloy specimens. The load at specimen failure is as follows (in megapascals):

19.8	10.1	14.9	7.5	15.4	15.4
15.4	18.5	7.9	12.7	11.9	11.4
11.4	14.1	17.6	16.7	15.8	
19.5	8.8	13.6	11.9	11.4	

The sample mean is $\bar{x} = 13.71$, and the sample standard deviation is $s = 3.55$. Figures 8-6 and 8-7 show a box plot and a normal probability plot of the tensile adhesion test data, respectively. These displays provide good support for the assumption that the population is normally distributed. We want to find a 95% CI on μ. Since $n = 22$, we have $n - 1 = 21$ degrees of freedom for t, so $t_{0.025,21} = 2.080$. The resulting CI is

$$\bar{x} - t_{\alpha/2,n-1}s/\sqrt{n} \leq \mu \leq \bar{x} + t_{\alpha/2,n-1}s/\sqrt{n}$$

$$13.71 - 2.080(3.55)/\sqrt{22} \leq \mu \leq 13.71 + 2.080(3.55)/\sqrt{22}$$

$$13.71 - 1.57 \leq \mu \leq 13.71 + 1.57$$

$$12.14 \leq \mu \leq 15.28$$

The CI is fairly wide because there is a lot of variability in the tensile adhesion test measurements.

8-3 Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

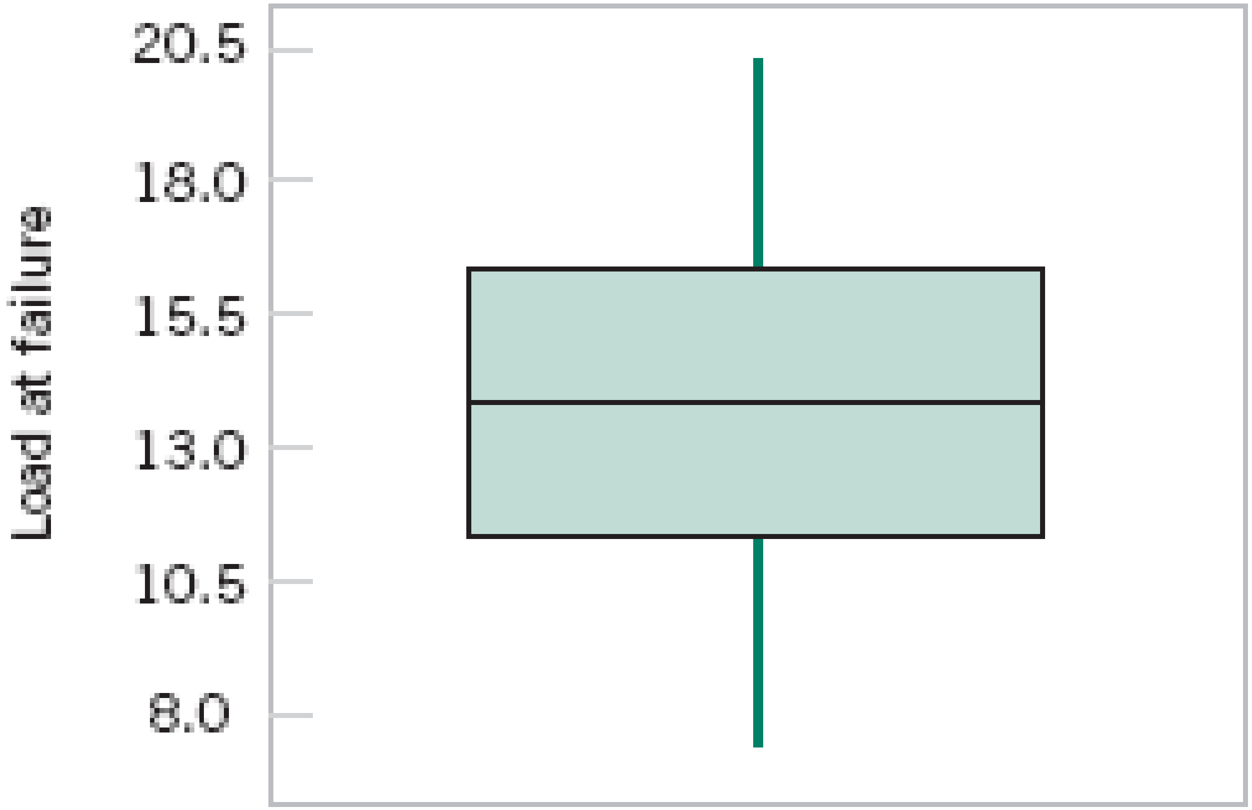


Figure 8-6 Box and Whisker plot for the load at failure data in Example 8-5.

8-3 Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

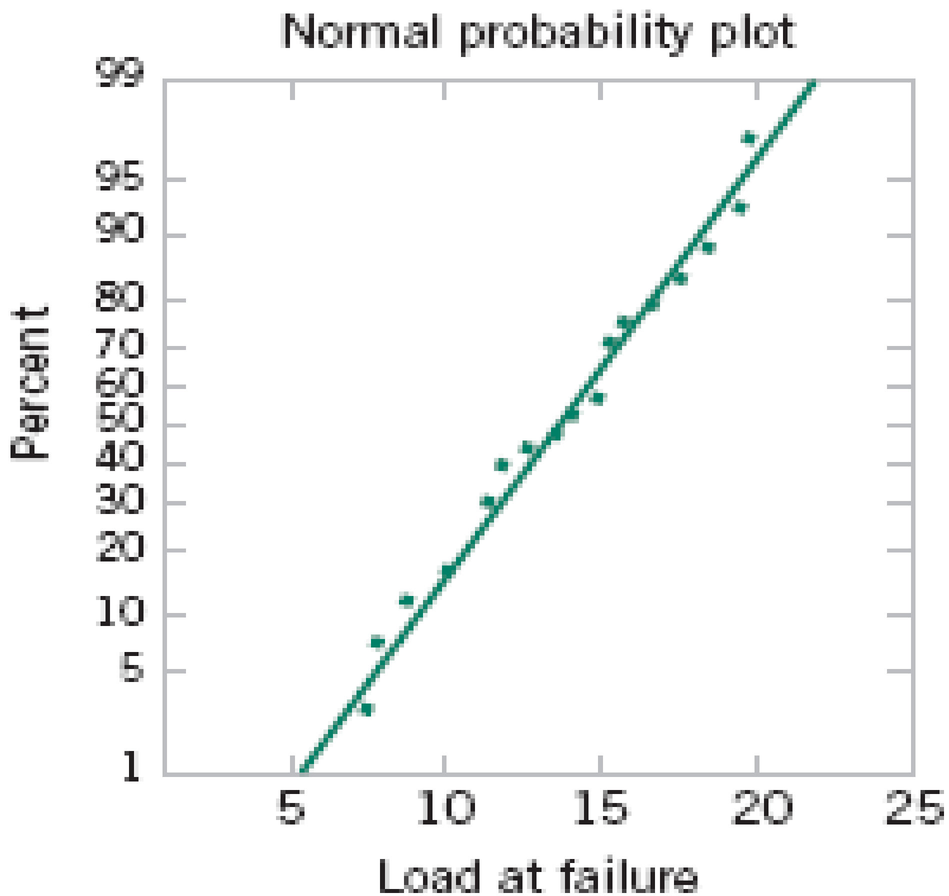


Figure 8-7 Normal probability plot of the load at failure data in Example 8-5.

8-4 Confidence Interval on the Variance and Standard Deviation of a Normal Distribution

Definition

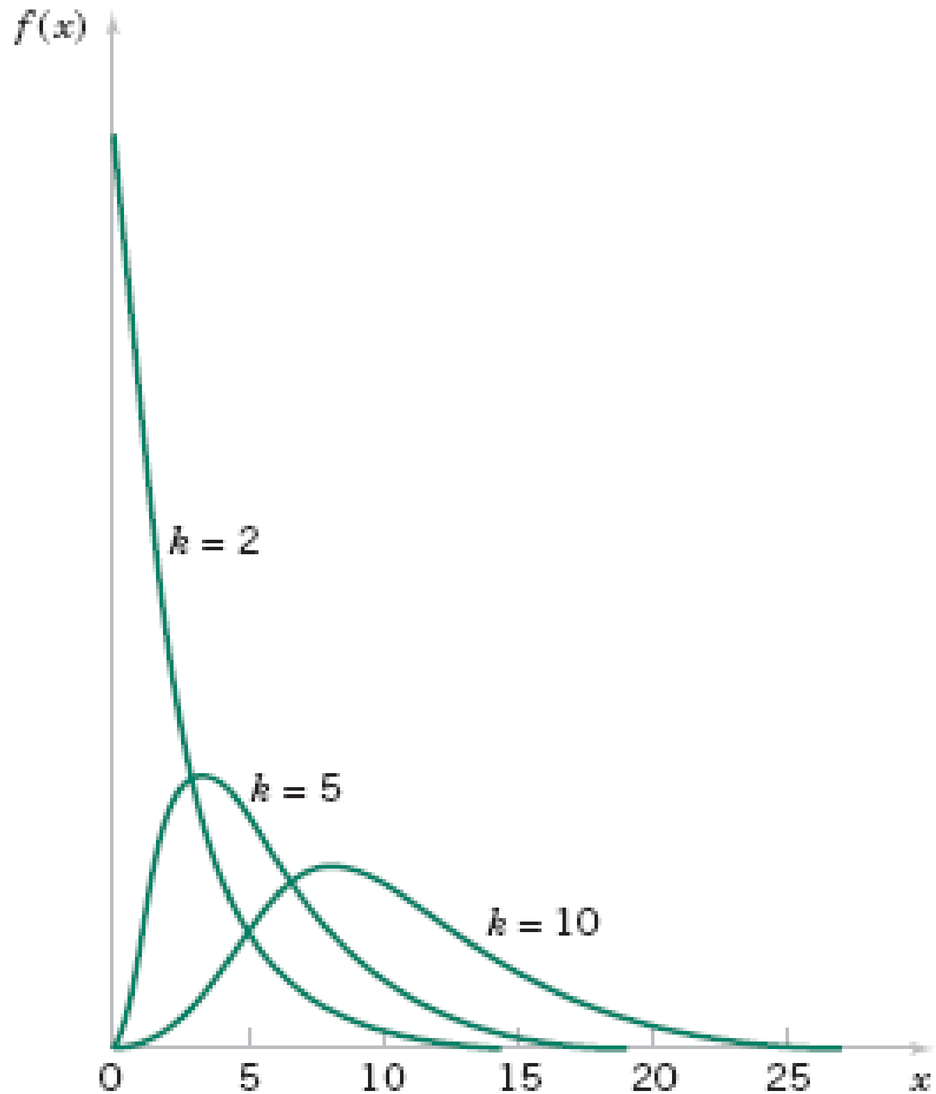
Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2, and let S^2 be the sample variance. Then the random variable

$$X^2 = \frac{(n-1)\,S^2}{\sigma^2} \qquad (8\text{-}19)$$

has a chi-square (χ^2) distribution with $n - 1$ degrees of freedom.

8-4 Confidence Interval on the Variance and Standard Deviation of a Normal Distribution

Figure 8-8 Probability density functions of several χ^2 distributions.



8-4 Confidence Interval on the Variance and Standard Deviation of a Normal Distribution

Definition

If s^2 is the sample variance from a random sample of n observations from a normal distribution with unknown variance σ^2, then **a $100(1-\alpha)\%$ confidence interval on σ^2** is

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}} \tag{8-21}$$

where $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-\alpha/2,n-1}$ are the upper and lower $100\alpha/2$ percentage points of the chi-square distribution with $n-1$ degrees of freedom, respectively. A **confidence interval for σ** has lower and upper limits that are the square roots of the corresponding limits in Equation 8-21.

8-4 Confidence Interval on the Variance and Standard Deviation of a Normal Distribution

One-Sided Confidence Bounds

The $100(1 - \alpha)\%$ lower and upper confidence bounds on σ^2 are

$$\frac{(n-1)s^2}{\chi^2_{\alpha,n-1}} \le \sigma^2 \quad \text{and} \quad \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha,n-1}} \qquad (8\text{-}22)$$

respectively.

8-4 Confidence Interval on the Variance and Standard Deviation of a Normal Distribution

Example 8-6

An automatic filling machine is used to fill bottles with liquid detergent. A random sample of 20 bottles results in a sample variance of fill volume of $s^2 = 0.0153$ (fluid ounces)^2. If the variance of fill volume is too large, an unacceptable proportion of bottles will be under- or overfilled. We will assume that the fill volume is approximately normally distributed. A 95% upper-confidence interval is found from Equation 8-22 as follows:

$$\sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{0.95,19}}$$

or

$$\sigma^2 \leq \frac{(19)0.0153}{10.117} = 0.0287 \text{ (fluid ounce)}^2$$

This last expression may be converted into a confidence interval on the standard deviation σ by taking the square root of both sides, resulting in

$$\sigma \leq 0.17$$

Therefore, at the 95% level of confidence, the data indicate that the process standard deviation could be as large as 0.17 fluid ounce.

8-5 A Large-Sample Confidence Interval For a Population Proportion

Normal Approximation for Binomial Proportion

If n is large, the distribution of

$$Z = \frac{X - np}{\sqrt{np(1-p)}} = \frac{\hat{P} - p}{\sqrt{\dfrac{p(1-p)}{n}}}$$

is approximately standard normal.

The quantity $\sqrt{p(1-p)/n}$ is called the standard error of the point estimator $\hat{P}$.

8-5 A Large-Sample Confidence Interval For a Population Proportion

If $\hat{p}$ is the proportion of observations in a random sample of size n that belongs to a class of interest, an approximate $100(1-\alpha)\%$ confidence interval on the proportion p of the population that belongs to this class is

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \qquad (8\text{-}25)$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution.

8-5 A Large-Sample Confidence Interval For a Population Proportion

Example 8-7

In a random sample of 85 automobile engine crankshaft bearings, 10 have a surface finish that is rougher than the specifications allow. Therefore, a point estimate of the proportion of bearings in the population that exceeds the roughness specification is $\hat{p} = x/n = 10/85 = 0.12$. A 95% two-sided confidence interval for p is computed from Equation 8-25 as

$$\hat{p} - z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

or

$$0.12 - 1.96\sqrt{\frac{0.12(0.88)}{85}} \le p \le 0.12 + 1.96\sqrt{\frac{0.12(0.88)}{85}}$$

which simplifies to

$$0.05 \le p \le 0.19$$

8-5 A Large-Sample Confidence Interval For a Population Proportion

Choice of Sample Size

The sample size for a specified value E is given by

$$n = \left(\frac{z_{\alpha/2}}{E}\right)^2 \hat{p}(1 - \hat{p}) \qquad (8\text{-}26)$$

$$E = z_{\alpha/2}\sqrt{p(1-p)/n}$$

An upper bound on n is given by

$$n = \left(\frac{z_{\alpha/2}}{E}\right)^2 (0.25) \qquad (8\text{-}27)$$

8-5 A Large-Sample Confidence Interval For a Population Proportion

Example 8-8

Consider the situation in Example 8-7. How large a sample is required if we want to be 95% confident that the error in using $\hat{p}$ to estimate p is less than 0.05? Using $\hat{p} = 0.12$ as an initial estimate of p, we find from Equation 8-26 that the required sample size is

$$n = \left(\frac{z_{0.025}}{E}\right)^2 \hat{p}(1 - \hat{p}) = \left(\frac{1.96}{0.05}\right)^2 0.12(0.88) \cong 163$$

If we wanted to be *at least* 95% confident that our estimate $\hat{p}$ of the true proportion p was within 0.05 regardless of the value of p, we would use Equation 8-27 to find the sample size

$$n = \left(\frac{z_{0.025}}{E}\right)^2 (0.25) = \left(\frac{1.96}{0.05}\right)^2 (0.25) \cong 385$$

8-5 A Large-Sample Confidence Interval For a Population Proportion

One-Sided Confidence Bounds

The approximate $100(1-\alpha)\%$ lower and upper confidence bounds are

$$\hat{p} - z_\alpha \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \quad \text{and} \quad p \le \hat{p} + z_\alpha \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \tag{8-28}$$

respectively.

Example

A manufacturer of electric calculators takes a random sample of 1200 calculators and finds that there are eight defective units. Construct a 96% lower confident bound for the population proportion about defective units.

IMPORTANT TERMS AND CONCEPTS

- Confidence coefficient
- Confidence interval
- Confidence interval for a population proportion
- Chi-squared distribution
- Confidence intervals for the mean of a normal distribution
- Confidence interval for the variance of a normal distribution
- Confidence level
- Error in estimation
- Large sample confidence interval
- One-sided confidence bounds
- Precision of parameter estimation
- Prediction interval
- Tolerance interval
- Two-sided confidence interval
- t distribution