Monash University
Faculty of Information Technology

Lecture 7
Kleene's Theorem.

Slides by David Albrecht (2011) and Graham Farr (2013).

FIT2014 Theory of Computation

Overview

- Questions
- Kleene's Theorem
- Convert Regular Expressions to NFA
- Convert NFA to FA
- Convert FA to Regular Expression

Stephen Cole Kleene
(1909-1994)
http://www-history.mcs.st-and.ac.uk/Biographies/Kleene.html

Questions

- Can every language which is represented by a **regular expression** be described by a **finite automaton**?
- Can every language which is described by a **finite automaton** be represented by a **regular expression**?
- Can every language be represented by a **regular expression** or a **finite automaton**?

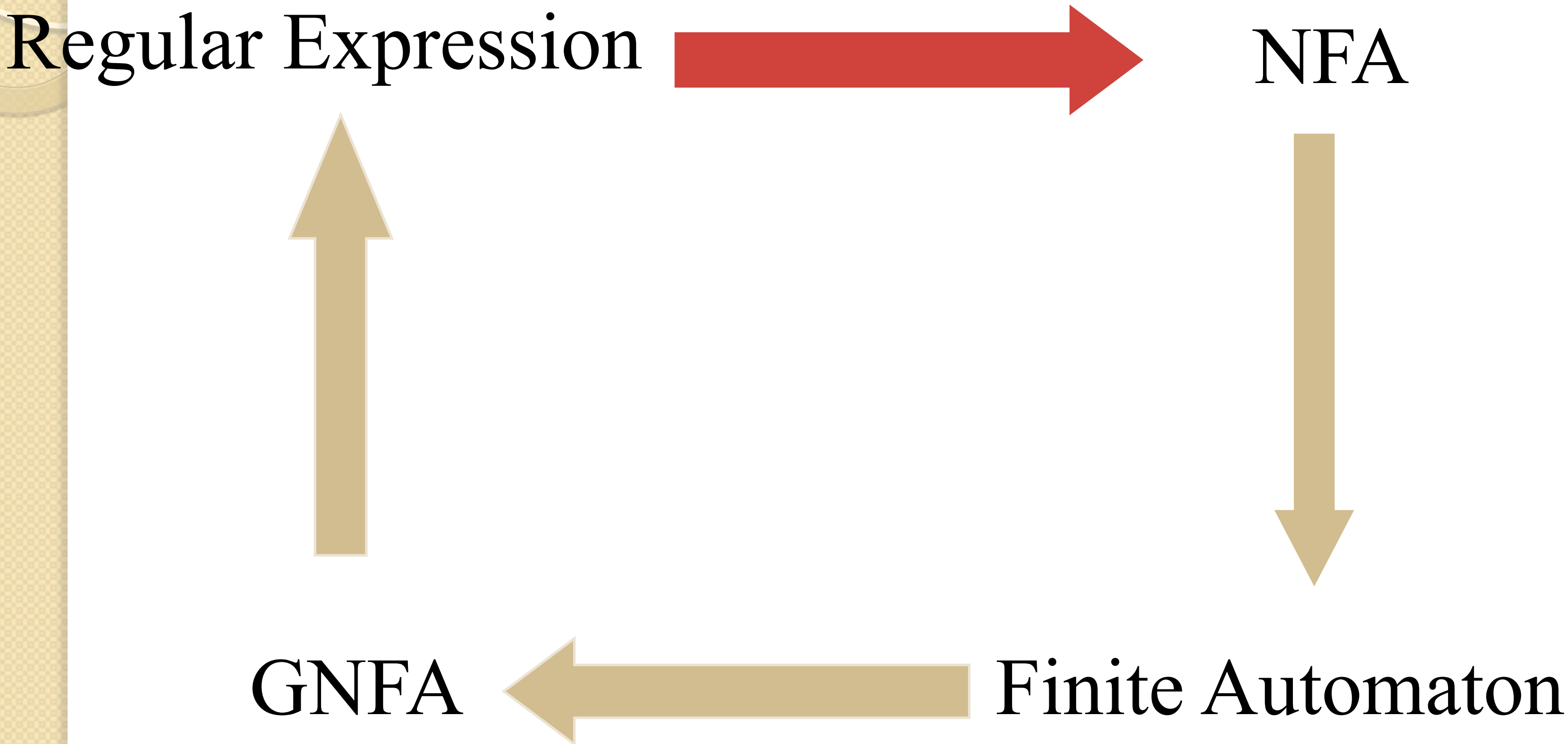
Kleene's Theorem

Any language which can be defined by

- **Regular Expressions**
- **Finite Automata**
- **Nondeterministic Finite Automata (NFA)**
- **Generalized Nondeterministic Finite Automata (GNFA)**

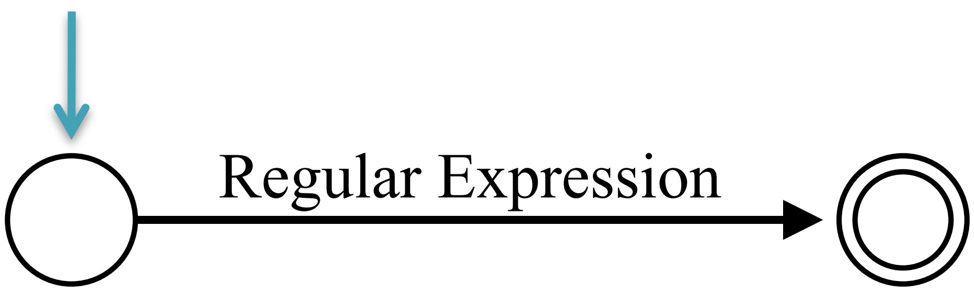
can be defined by any of the other methods.

Kleene's Theorem



Converting Regular Expression to NFA

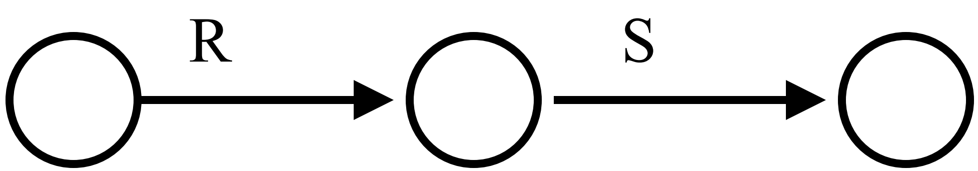
Start with the graph.



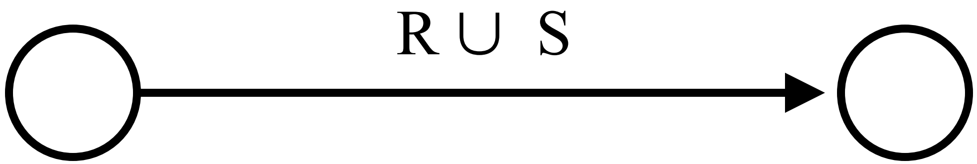
Apply the following rules until all edges are labelled with a letter or ε .

1. Delete any edge labelled with ϕ.
2. Transform any edge like

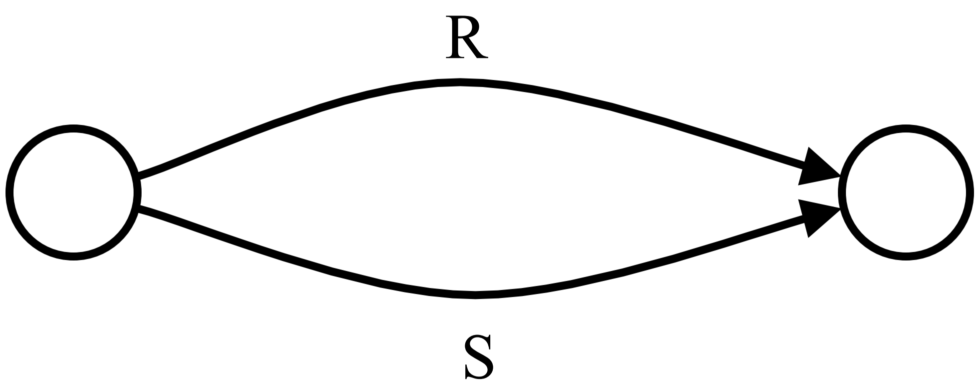
into



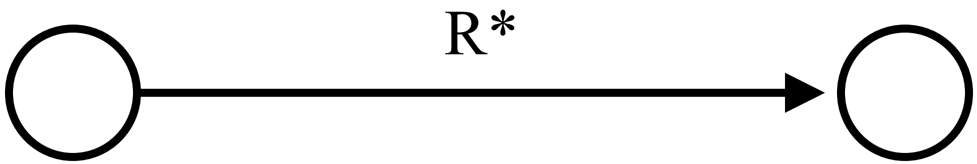
3. Transform any edge like:



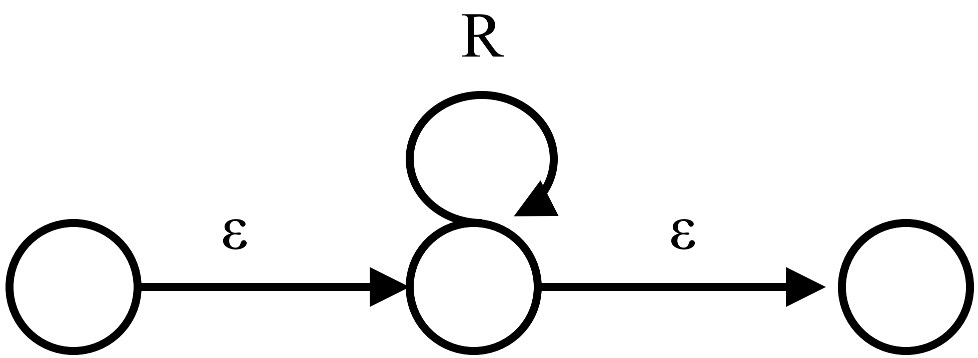
into



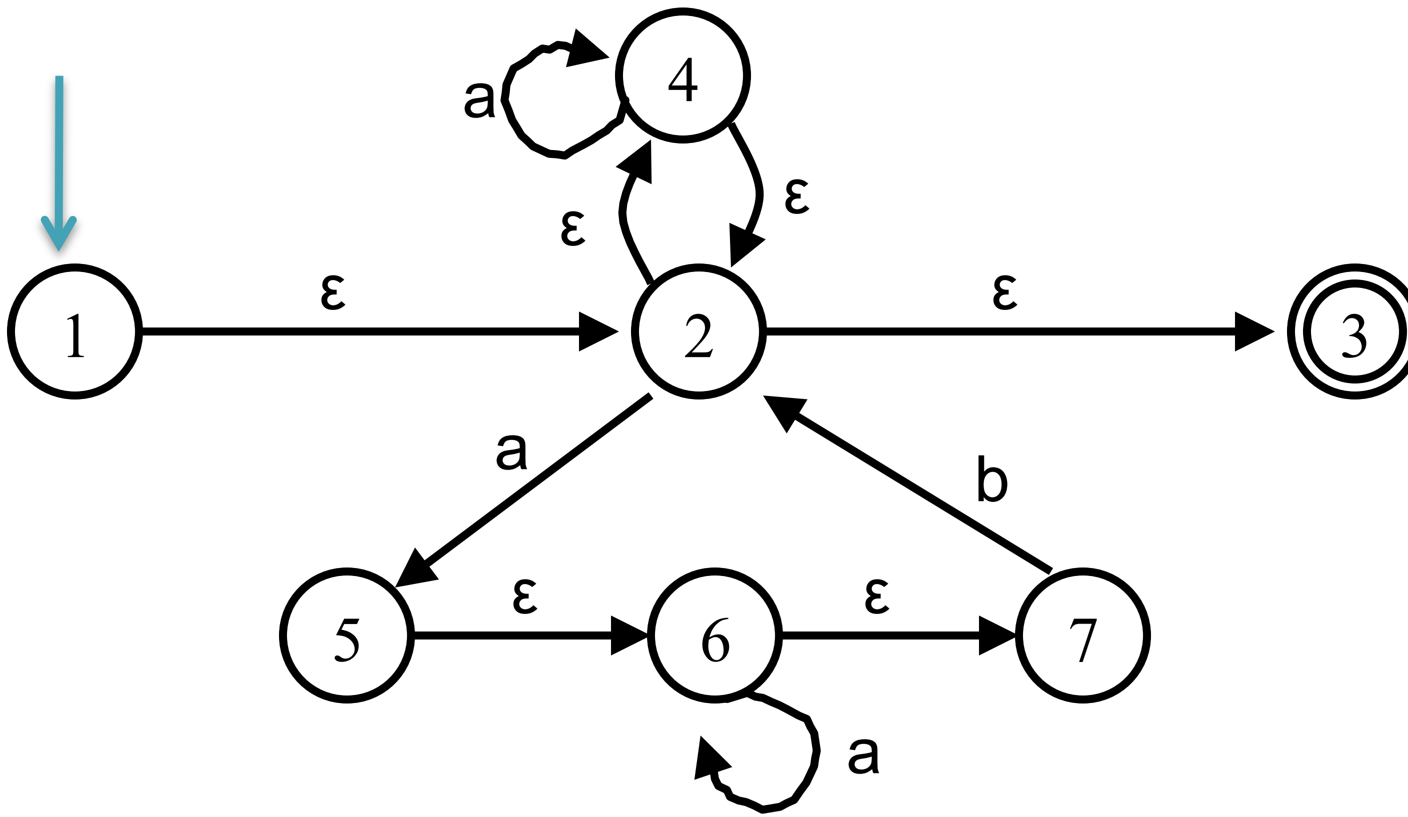
4. Transform any edge like:



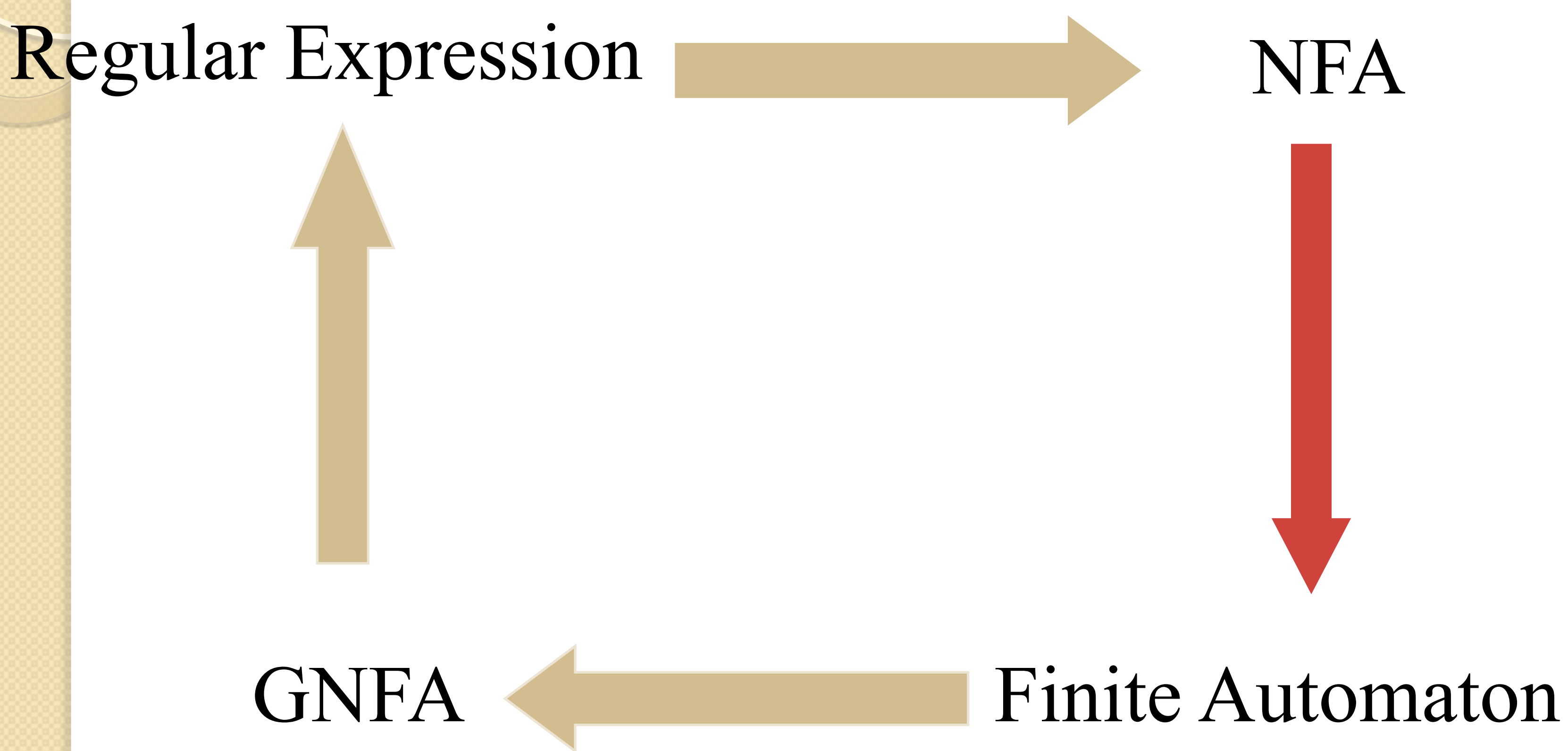
into



(a* ∪ aa*b)*



Kleene's Theorem



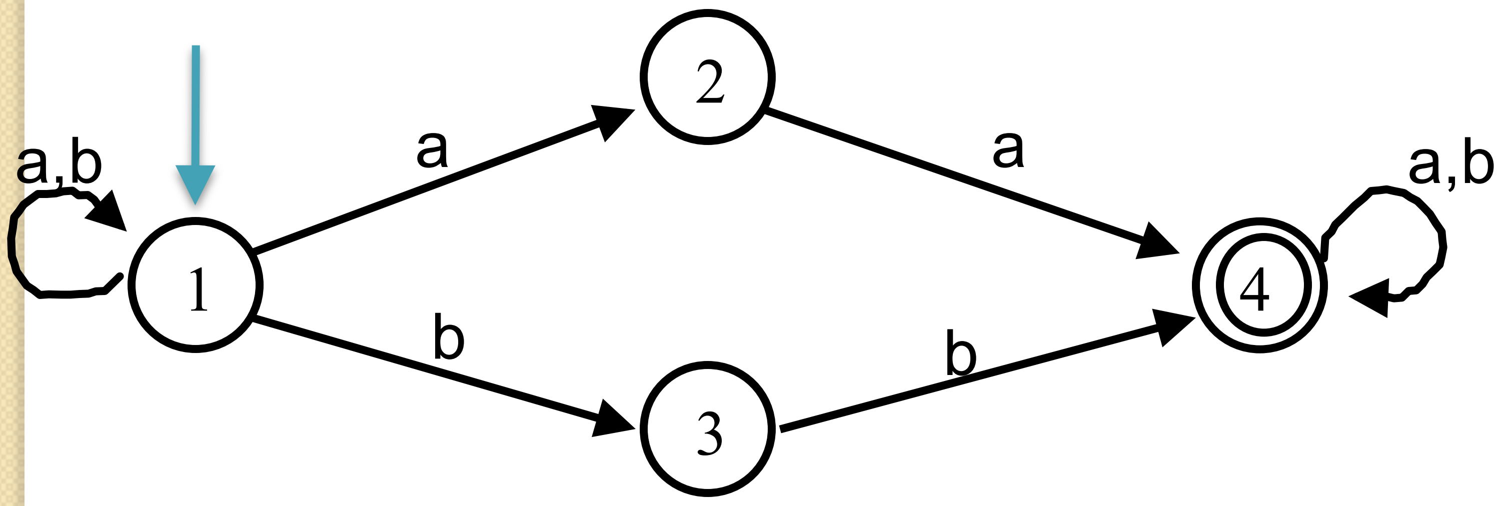
Converting a NFA to a FA

In a FA:

- Any string *w* traces a ***unique path***, starting from the Start State and ending at ***some unique state***, which we'll call endState(*w*).
- The string *w* is accepted if endState(*w*) ***is*** a Final State, otherwise it is rejected.

In a NFA:

- Any string *w* traces a ***set of paths***, starting from the Start State and ending at some ***set of states***, which we'll call endState***s***(*w*).
  - The set might have zero, one or more members.
- The string *w* is accepted endStates(*w*) ***contains*** a Final State, otherwise it is rejected.



endStates(**ab**) = $\{1,3\}$
endStates(**aba**) = $\{1,2\}$

In general, if w is a string and x is a single letter, then

endStates(wx) =

$\{\, q$: for some state p in endStates(w),

there is a transition $p \xrightarrow{x} q \,\}$

... ***provided there are no empty string transitions*** $q_1 \xrightarrow{\varepsilon} q_2$

Converting a NFA to a FA

This suggests an algorithm for constructing all possible
endStates(w) for all strings w.

w := ε

endStates(ε) := { Start State }

For all strings w in order of increasing length:

For each x in {a,b}

For each p in endStates(w), and each transition $p \xrightarrow{x} q$

Add q to the set endStates(wx).

… until we keep getting the same endStates(…) sets all the time.

Again, assumes ***there are no empty string transitions*** $q_1 \xrightarrow{\varepsilon} q_2$

Converting a NFA to a FA

Use the sets endStates(…) as the states of a new FA.

Transitions: endStates(w) $\xrightarrow{x}$ endStates(wx)

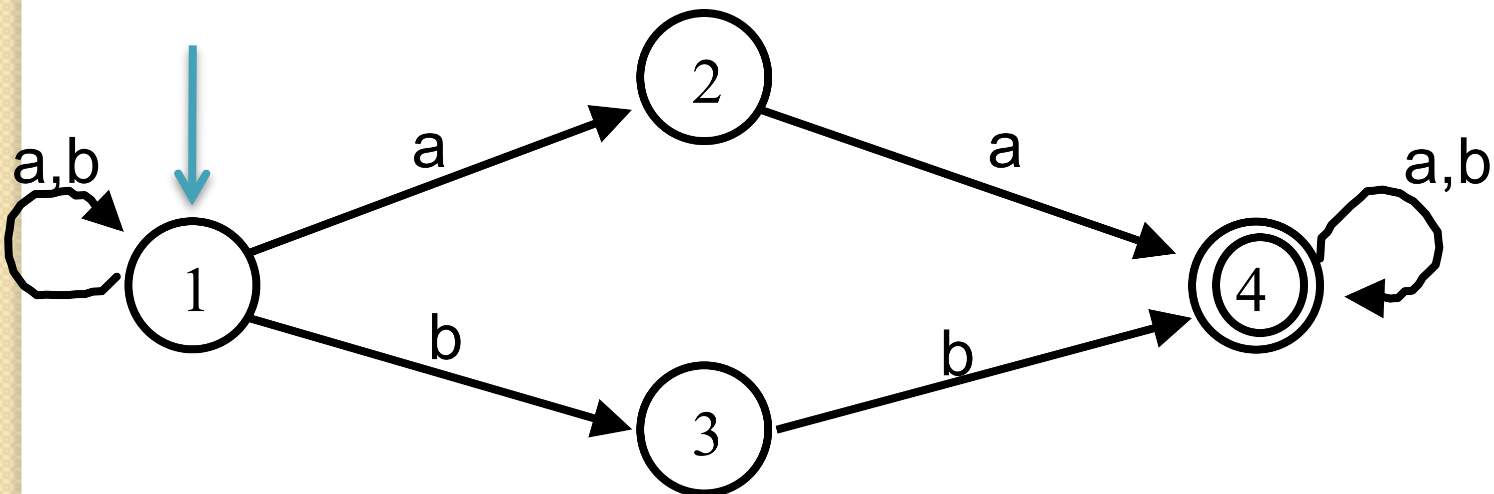
Start State of the new FA = { Start State of the NFA }

Final States of the new FA = any set endStates(…) that contains a Final State of the NFA.

Algorithm is only an outline.

Some things to think about:

- loop through the p outside, *then* loop through x ?
- How do we know it stops?
- Complexity?
- How to deal with empty string transitions?



	a	**b**
Start {1}	**{1,2}**	**{1,3}**
{1,2}	**{1,2,4}**	**{1,3}**
{1,3}	**{1,2}**	**{1,3,4}**
Final {1,2,4}	**{1,2,4}**	**{1,3,4}**
Final {1,3,4}	**{1,2,4}**	**{1,3,4}**

Converting a NFA to a FA

Now suppose that the FA might have empty string transitions, $q_1 \xrightarrow{\varepsilon} q_2$.

These allow change of state without reading any letter of the input string.

Every time we include a new state q in some endStates(…) , we also need to include any state we can reach *from it* along empty string transitions.

Look at all paths from q that just use ε transitions …

$$q \xrightarrow{\varepsilon} q_1 \xrightarrow{\varepsilon} q_2 \xrightarrow{\varepsilon} \cdots \xrightarrow{\varepsilon} q_i$$

… and include all states on such paths.

Modify earlier algorithm, for constructing the sets endStates(…) , to take account of empty string transitions.

Earlier algorithm:

w := ε

endStates(ε) := { Start State }

For all strings w in order of increasing length:

For each x in {a,b}

For each p in endStates(w), and each transition $p \xrightarrow{x} q$

Add q to the set endStates(wx).

… until we keep getting the same endStates(…) sets all the time.

Modify earlier algorithm, for constructing the sets endStates(…) , to take account of empty string transitions.

Modified algorithm:

w := ε

endStates(ε) := { Start State }

Add, to endStates(ε), states reachable from **Start State**

along ε-transitions.

For all strings *w* in order of increasing length:

For each *x* in {a,b}

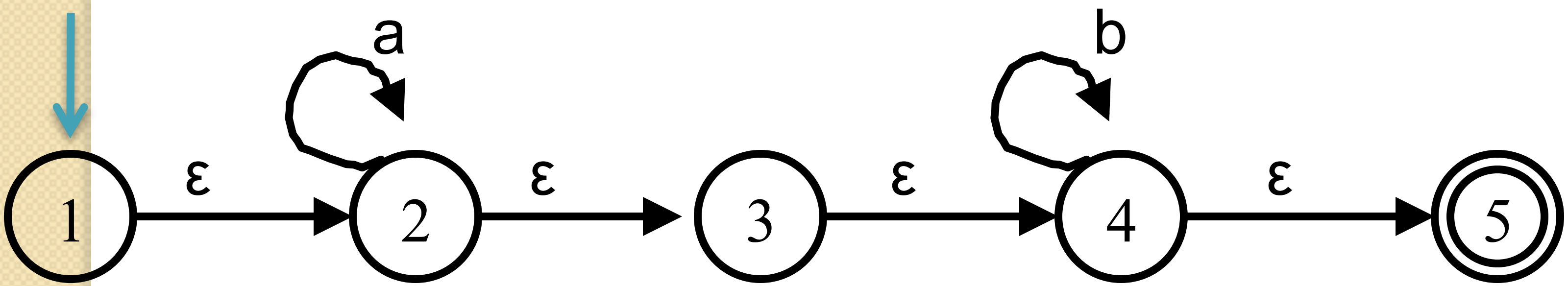
For each *p* in endStates(*w*), and each transition $p \xrightarrow{x} q$

Add *q* to the set endStates(*wx*).

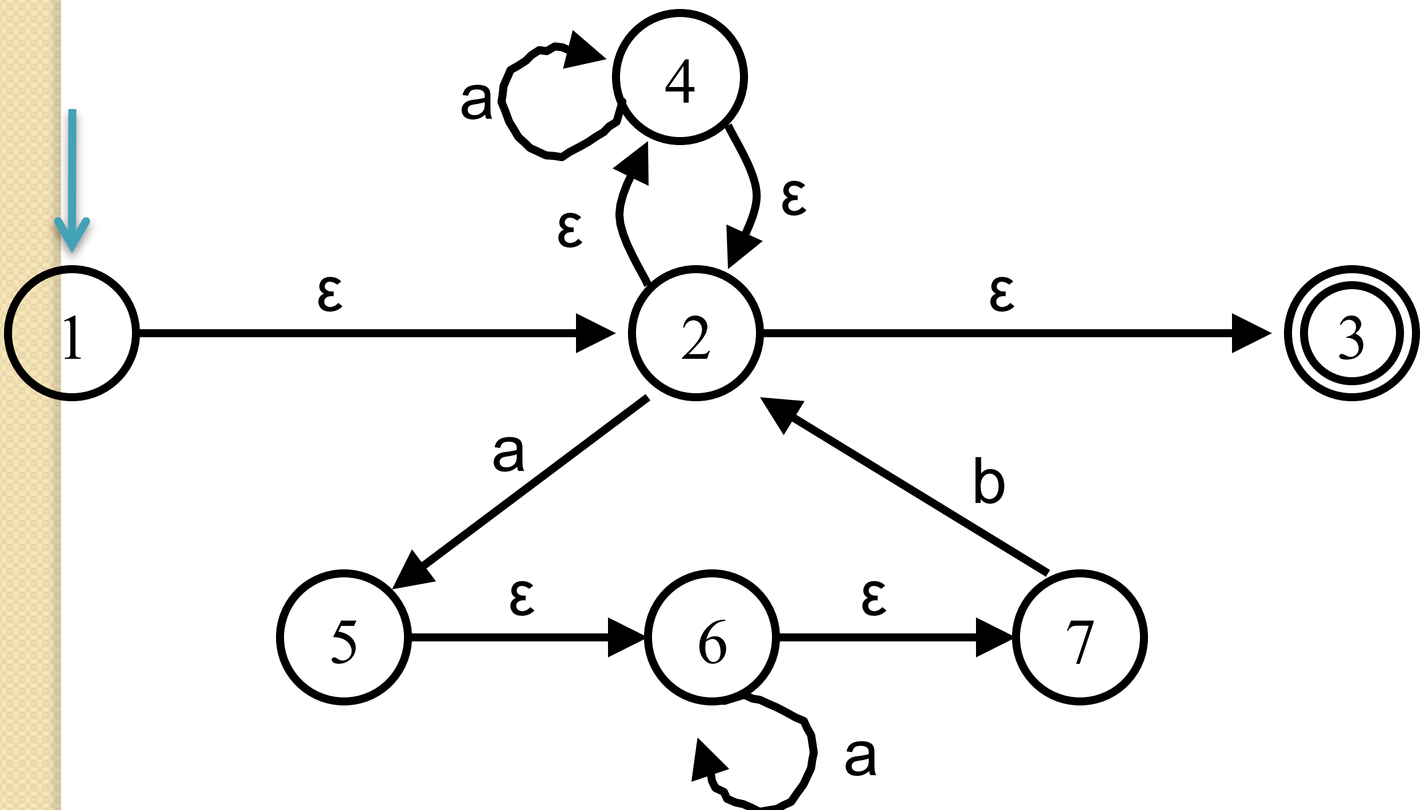
Add, to endStates(*wx*), states reachable from ***q***

along ε-transitions.

… until we keep getting the same endStates(…) sets all the time.

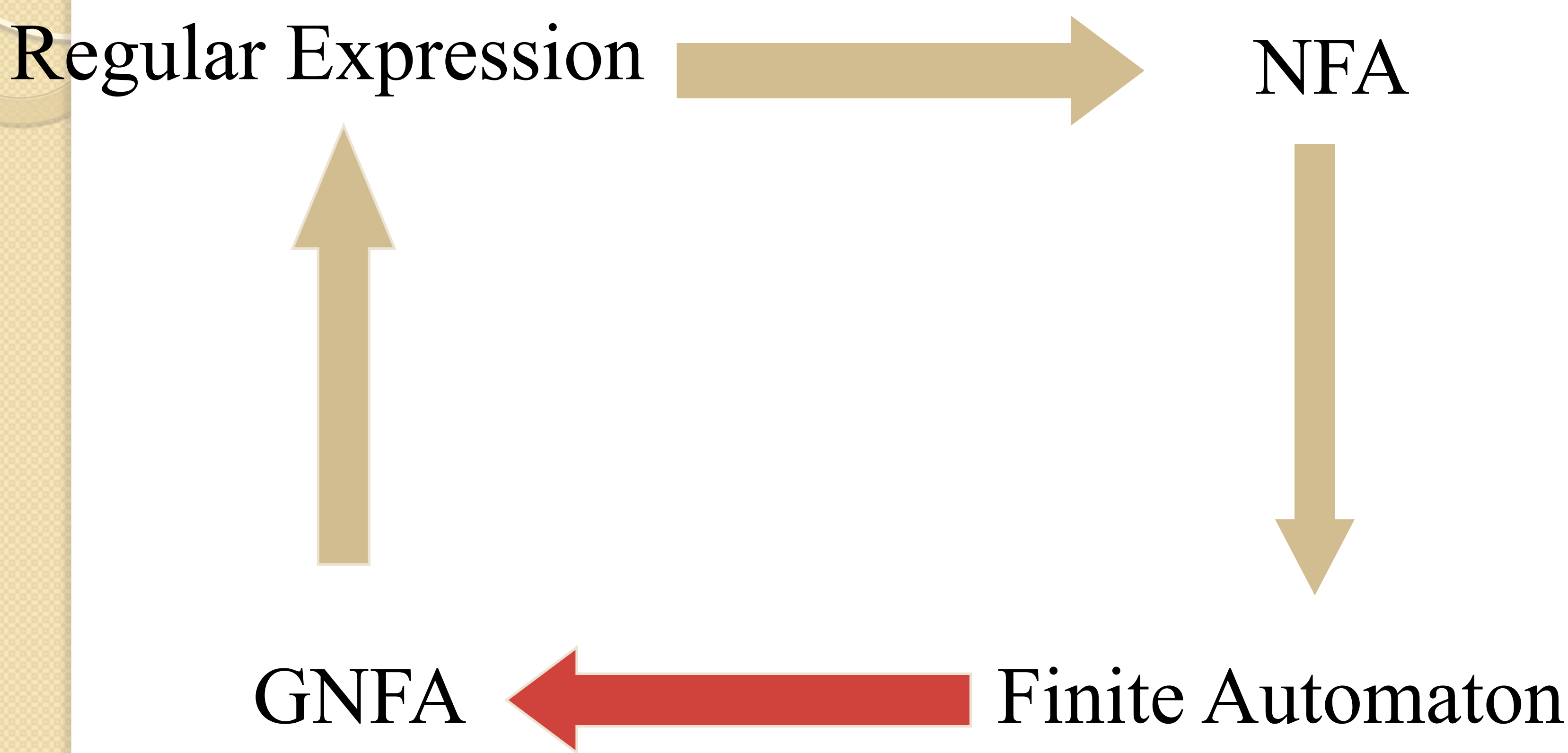


		a	b
Start/Final	{1,2,3,4,5}	{2,3,4,5}	{4,5}
Final	{2,3,4,5}	{2,3,4,5}	{4,5}
Final	{4,5}	ϕ	{4,5}
	ϕ	ϕ	ϕ



		a	b
Start /Final	{1,2,3,4}	{5,4,6,7,2,3}	ϕ
Final	{2,3,4,5,6,7}	{5,4,6,7,2,3}	{2,3,4}
Final	{2,3,4}	{5,4,6,7,2,3}	ϕ
	ϕ	ϕ	ϕ

Kleene's Theorem



Generalised Nondeterministic Finite Automaton (GNFA)

Definition

A **Generalised Nondeterministic Finite Automaton** (GNFA) is a NFA in which:

- transitions are labelled by *regular expressions*, not just by single letters;
- there is just one Final State, and it is not the Start State;
- there are transitions from every state to every other state (including itself), except that the Start State has no incoming transitions and the Final State has no outgoing transitions.

(Note: you can just label a transition by ∅ if you don't want to use it.)

Generalised Nondeterministic Finite Automaton (GNFA)

Definition

A string w is **accepted** by a given GNFA if it can be divided into substrings, $w = w_1 \ldots w_k$, such that there is some sequence of transitions, starting at the Start State, finishing at the Final State, and labelled by regular expressions $R_1, \ldots, R_k$, such that, for all i, w_i matches R_i .

If a string w is not accepted by the GNFA, then it is **rejected**.

From FA to GNFA

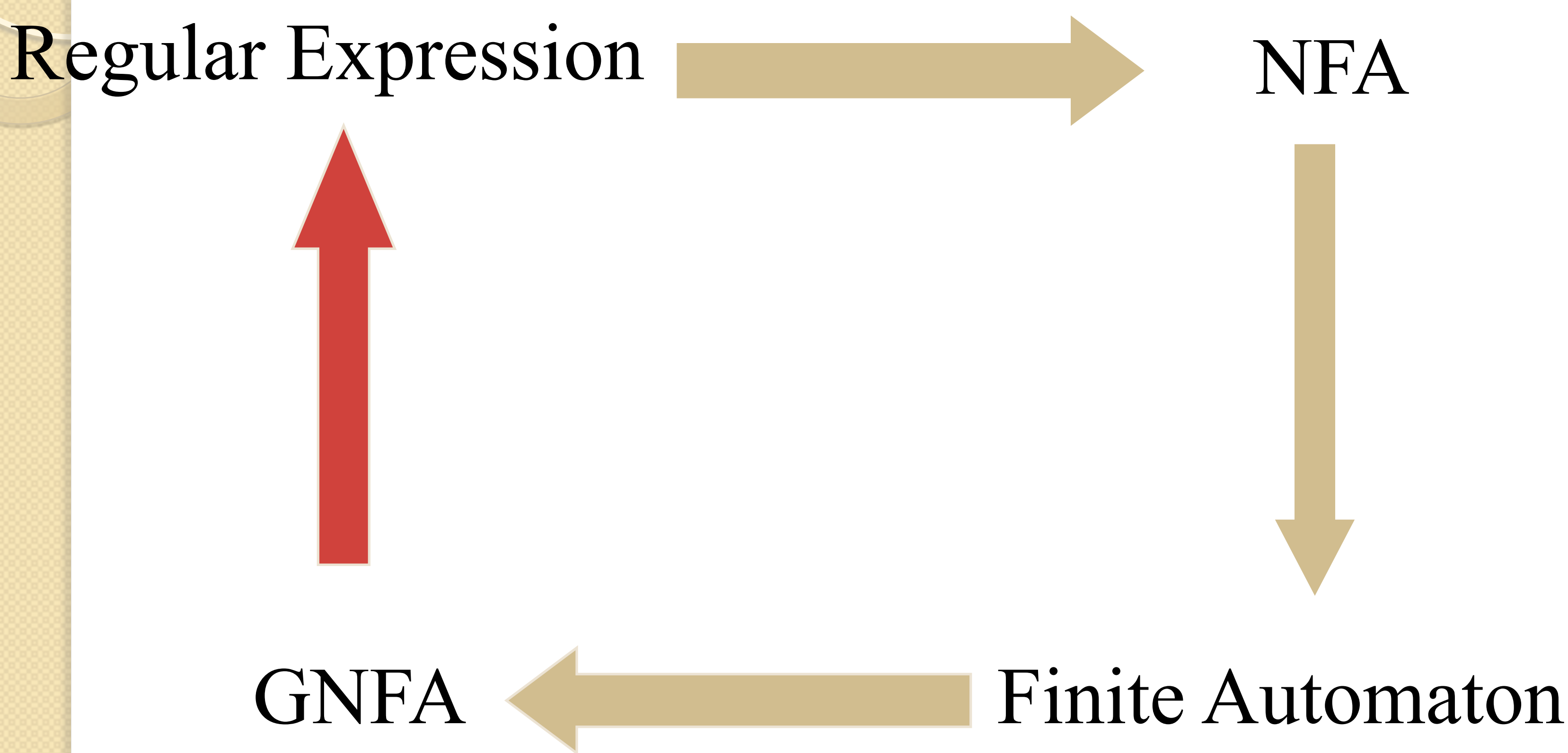
Given a FA:

1. Add a new Final State, add new transitions labelled ε from the previous Final States to this new one, and make those states no longer Final. So now there is just one Final State.
2. Add new arcs "everywhere", labelled $\emptyset$.

The letters on the arcs are already regular expressions in their own right.

Now it's a GNFA, accepting the same language as the original FA.

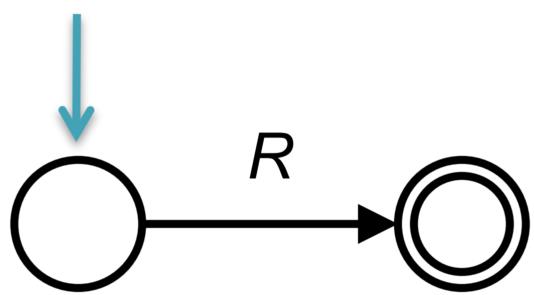
Kleene's Theorem



From GNFA to Regular Expression

Starting with a GNFA, we convert it to an equivalent GNFA with one fewer state.

We keep doing this until we have a GNFA with just one transition:



If our initial GNFA has only a Start State and a Final State, we are done.

From GNFA to Regular Expression

So, let q be some non-Start, non-Final state.

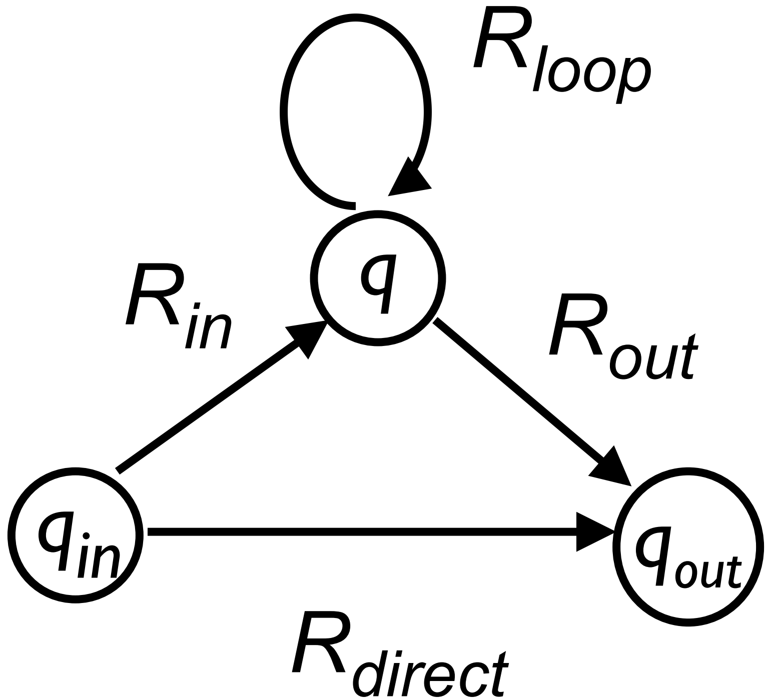
Let q_{in} be any non-Final state. Let R_{in} be the regular expression on the transition from q_{in} to q.

Let q_{out} be any non-Start state. Let R_{out} be the regular expression on the transition from q to q_{out}.

Let R_{loop} be the regular expression on the transition from q to itself.

Let R_{direct} be the regular expression on the transition from q_{in} to q_{out}.

From GNFA to Regular Expression



becomes …

$$R_{direct} \cup (R_{in} (R_{loop})^* R_{out})$$

From GNFA to Regular Expression

Ensure this replacement is done for all q_{in} , q_{out} .

Then q is removed, and we have an equivalent GNFA to the one we started with.

Keep doing this whole procedure, removing one state at a time, until you are left with just the Start State and the Final State, with a single transition between them.

The regular expression on this transition is the one you want. It matches precisely those strings accepted by the original GNFA.

Examples: Sipser, pp 75-76.

From GNFA to Regular Expression

For FIT2004 students:

Compare this algorithm with the Floyd-Warshall algorithm for the All Pairs Shortest Path problem.

Revision

- Understand Kleene's Theorem
- Be able to convert Regular Expressions into NFA
- Be able to convert NFA into a Finite Automaton
- Be able to convert a FA into a Regular Expression

Reference

- Sipser, Ch 1, especially pp 54-56, 66-76.