MAT1830

Lecture 9: Mathematical induction

Induction - why should you care?

- Induction is a vital technique for proofs in CS and maths
- It's particularly useful for proving things about:
  - algorithms that involve recursion (loops)
  - strings and similar data structures
  - trees and similar data structures
- Understanding induction can help you better understand these recursive algorithms and recursive data structures.

Question Prove that, for each integer $n \geq 3$, the sum of the angles of a convex n-sided polygon is $180n - 360$ degrees.

Polygon: A 2D shape with straight sides.

Convex: Any line between two corners is completely inside the shape.

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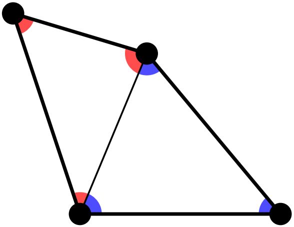
The sum of the angles of a convex 4-sided polygon is 360 degrees so the statement is true for $n = 4$.



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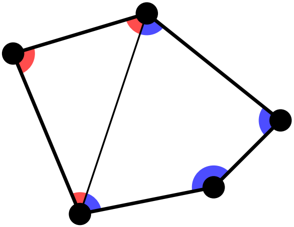
The sum of the angles of a convex 4-sided polygon is 360 degrees so the statement is true for $n = 4$.

The sum of the angles of a convex 5-sided polygon is 540 degrees so the statement is true for $n = 5$.



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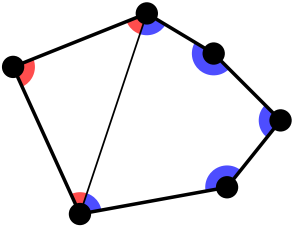
The sum of the angles of a convex 4-sided polygon is 360 degrees so the statement is true for $n = 4$.

The sum of the angles of a convex 5-sided polygon is 540 degrees so the statement is true for $n = 5$.

The sum of the angles of a convex 6-sided polygon is 720 degrees so the statement is true for $n = 6$.



$180 + 540 = 720$

Question Prove that, for each integer $n \geq 3$, the sum of the angles of a convex n-sided polygon is $180n - 360$ degrees.

Solution Let $P(n)$ be the statement "the sum of the angles of a convex n-sided polygon is $180n - 360$ degrees".

First we show that the statement is true for $n = 3$.

Base step. The sum of the angles of a convex 3-sided polygon is 180 degrees so $P(3)$ is true.

Now we show that **if** $P(k)$ is true for some integer $k \geq 3$, **then** $P(k+1)$ is also true.

Induction step.

- Suppose that $P(k)$ is true.
- Any convex $(k+1)$-sided polygon can be "split" into a k-sided polygon and a triangle.
- The sum of the angles of a triangle is 180 degrees.
- The sum of the angles of a k-sided polygon is $180k - 360$ degrees (by $P(k)$).
- So the sum of the angles of a $(k+1)$-sided polygon is $180 + (180k - 360) = 180(k+1) - 360$ degrees. So $P(k+1)$ is true.

This proves the original statement!

Since the natural numbers $0, 1, 2, 3, \ldots$ are generated by a process which begins with 0 and repeatedly adds 1, we have the following.

Property P is true for all natural numbers if
1. $P(0)$ is true.
2. $P(k) \Rightarrow P(k+1)$ for all $k \in \mathbb{N}$.

This is called the *principle of mathematical induction.*

It is used in a style of proof called *proof by induction*, which consists of two steps.

Base step: Proof that the required property P is true for 0.

Induction step: Proof that **if** $P(k)$ is true **then** $P(k+1)$ is true, for each $k \in \mathbb{N}$.

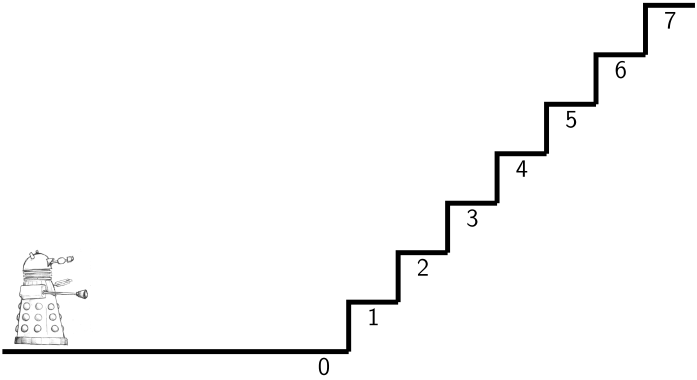
To prove that some statement $P(n)$ is true for all integers $n \geq 0$:

- prove $P(0)$ is true (called the base step); then
- prove that, for each integer $k \geq 0$, **if** $P(k)$ is true **then** $P(k+1)$ is true (called the induction step).

We get:

n	0	1	2	3	4	5	6	7	8	$\cdots$
$P(n)$	T	T	T	T	T	T	T	T	T	$\cdots$

We usually prove the induction step by **assuming that** $P(k)$ **is true** for an arbitrary k and then using this to prove that $P(k+1)$ is true.


0
1
2
3
4
5
6
7

Example 1. Prove that 3 divides $n^3 + 2n$ for each integer $n \geq 0$.

Solution Let $P(n)$ be the statement "3 divides $n^3 + 2n$".

Base step. $0^3 + 0 = 0$ and 3 divides 0, so $P(0)$ is true.

Induction step. Suppose that $P(k)$ is true for some integer $k \geq 0$. This means that $k^3 + 2k = 3a$ for some integer a.

We want to prove that $P(k+1)$ is true. We want to show that $(k+1)^3 + 2(k+1) = 3b$ for some integer b.

$$\begin{aligned}
(k+1)^3 + 2(k+1) &= (k^3 + 3k^2 + 3k + 1) + 2k + 2 \\
&= k^3 + 3k^2 + 5k + 3 \\
&= (k^3 + 2k) + 3k^2 + 3k + 3 \\
&= 3a + 3k^2 + 3k + 3 && \text{(by } P(k)\text{)} \\
&= 3(a + k^2 + k + 1)
\end{aligned}$$

Because $(a + k^2 + k + 1)$ is an integer, 3 divides $(k+1)^3 + 2(k+1)$. So $P(k+1)$ is true.

This proves that $P(n)$ is true for each integer $n \geq 0$.

Example 2. Prove that there are 2^n n-letter words using the letters A and B for each integer $n \geq 1$.

Solution Let $P(n)$ be the statement "there are 2^n n-letter words using the letters A and B".

Base step. There are two 1-letter words: 'A' and 'B'. So $P(1)$ is true.

Induction step. Suppose that $P(k)$ is true for some integer $k \geq 1$. This means that there are 2^k k-letter words using the letters A and B.

We want to prove that $P(k+1)$ is true. We want to show that there are 2^{k+1} $(k+1)$-letter words using the letters A and B.

Every $(k+1)$-letter word can be written as WA or WB for some k-letter word W.
By $P(k)$ there are 2^k words that can be written WA.
By $P(k)$ there are 2^k words that can be written WB.
So in total there are $2^k + 2^k = 2^{k+1}$ $(k+1)$-letter words. So $P(k+1)$ is true.

This proves that $P(n)$ is true for each integer $n \geq 1$.

9.2 Starting the base step higher

It is not always appropriate to start the induction at 0. Some properties are true only from a certain positive integer upwards, in which case the induction starts at that integer.

Question 9.1 Guess what x stands for in the following.

x divides $n^2 + n$

n	1	2	3	4	5	6	7	8	9
$n^2 + n$	2	6	12	20	30	42	56	72	90

$x = 2$

The sum of the first n odd numbers is x.

n	sum	value
1	1	1
2	$1 + 3$	4
3	$1 + 3 + 5$	9
4	$1 + 3 + 5 + 7$	16
5	$1 + 3 + 5 + 7 + 9$	25
6	$1 + 3 + 5 + 7 + 9 + 11$	36

$x = n^2$

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{n\times(n+1)}=1-x$$

n	sum	value
1	$\frac{1}{1\times 2}$	$\frac{1}{2}$
2	$\frac{1}{1\times 2}+\frac{1}{2\times 3}$	$\frac{2}{3}$
3	$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}$	$\frac{3}{4}$
4	$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}$	$\frac{4}{5}$
5	$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\frac{1}{5\times 6}$	$\frac{5}{6}$
6	$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\frac{1}{5\times 6}+\frac{1}{6\times 7}$	$\frac{6}{7}$

$$x=\frac{1}{n+1}$$

Example 3. Prove that $n! > 2^n$ for each integer $n \geq 4$.

Solution Let $P(n)$ be the statement "$n! > 2^n$".

Base step. $4! = 24$ and $2^4 = 16$. So $P(4)$ is true.

Induction step. Suppose that $P(k)$ is true for some integer $k \geq 4$. This means that $k! > 2^k$.

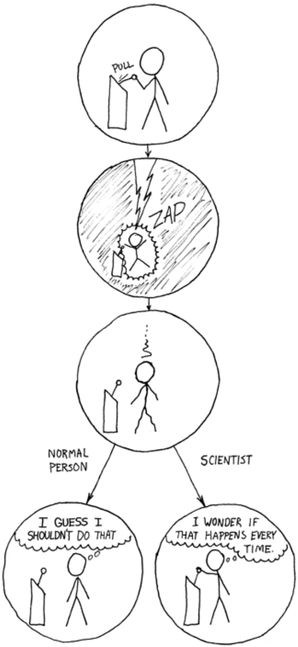
We want to prove that $P(k+1)$ is true. We want to show that $(k+1)! > 2^{k+1}$.

$$(k+1)! = (k+1) \times k! > (k+1) \times 2^k > 2 \times 2^k = 2^{k+1}$$

$(k+1) \times k! > (k+1) \times 2^k$ is true by $P(k)$.
$(k+1) \times 2^k > 2 \times 2^k$ is true because $k \geq 4$.

So $P(k+1)$ is true.

This proves that $P(n)$ is true for each integer $n \geq 4$.


PULL
ZAP
NORMAL PERSON
SCIENTIST
I GUESS I SHOULDN'T DO THAT
I WONDER IF THAT HAPPENS EVERY TIME.

Example 4. Prove that n cents can be made from 3c and 5c stamps for each integer $n \geq 8$.

Solution Let $P(n)$ be the statement "n cents can be made from 3c and 5c stamps".

Base step. 8 cents can be made from one 3c stamp and one 5c stamp. So $P(8)$ is true.

Induction step. Suppose that $P(k)$ is true for some integer $k \geq 8$. This means that k cents can be made from 3c and 5c stamps.

We want to prove that $P(k+1)$ is true. We must show that $k+1$ cents can be made from 3c and 5c stamps.

If the way to make k cents involves a 5c stamp, then we can replace it with two 3c stamps to make $k+1$ cents.

If the way to make k cents does not involve a 5c stamp, then it is made with all 3c stamps (at least three of them because $k \geq 8$). Then we can replace three 3c stamps with two 5c stamps to make $k+1$ cents.

So $P(k+1)$ is true.

This proves that $P(n)$ is true for each integer $n \geq 8$.

9.3 Sums of series

Induction is commonly used to prove that sum formulas are correct.

Example 5. Prove that $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ for each integer $n \geq 1$.

Solution Let $P(n)$ be the statement "$1+2+3+\cdots+n = \frac{n(n+1)}{2}$".

Base step. The left hand side of $P(1)$ is just 1 and the right hand side is $\frac{1(1+1)}{2} = 1$. So $P(1)$ is true.

Induction step. Suppose that $P(k)$ is true for some integer $k \geq 1$. This means that $1+2+3+\cdots+k = \frac{k(k+1)}{2}$.

We want to prove that $P(k+1)$ is true. We must show that $1+2+3+\cdots+(k+1) = \frac{(k+1)(k+2)}{2}$.

$$\begin{aligned} 1+2+3+\cdots+(k+1) &= (1+2+3+\cdots+k)+(k+1) \\ &= \frac{k(k+1)}{2}+(k+1) && \text{(by } P(k)\text{)} \\ &= (k+1)\left(\frac{k}{2}+1\right) \\ &= (k+1)\left(\frac{k+2}{2}\right) \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

So $P(k+1)$ is true.

This proves that $P(n)$ is true for each integer $n \geq 1$.

Remark. Another proof is to write down

$$\begin{gathered}1+2+3+\cdots+(n-1)+n \\ n+(n-1)+\cdots+3+2+1\end{gathered}$$

and observe that each of the n columns sums to $n+1$. Thus the sum of twice the series is $n(n+1)$, and hence the sum of the series itself is $n(n+1)/2$. This proof uses induction unconsciously, to prove that the sum of each column is the same.

Question 9.3 Is $n^2 + n + 41$ prime for all natural numbers n?

$$
\begin{aligned}
0^2 + 0 + 41 &= 41 \quad \text{prime} \\
1^2 + 1 + 41 &= 43 \quad \text{prime} \\
2^2 + 2 + 41 &= 47 \quad \text{prime} \\
3^2 + 3 + 41 &= 53 \quad \text{prime} \\
4^2 + 4 + 41 &= 61 \quad \text{prime} \\
5^2 + 5 + 41 &= 71 \quad \text{prime} \\
&\vdots \\
39^2 + 39 + 41 &= 1601 \quad \text{prime} \\
40^2 + 40 + 41 &= 1681 = 41 \times 41 \quad \text{not prime}
\end{aligned}
$$

Example: Merge Sort

Merging two already-sorted lists

8	7	5	3
6	4	2	1

8 7 6 5 4 3 2 1

This process needs at most x comparisons where x is the total number of things in the two lists.

The MergeSort algorithm

MergeSort(L)
if L has length 1 **then**
  output L
else
  split L into two "halves" A and B
  set $A' =$ MergeSort(A)
  set $B' =$ MergeSort(B)
  set L' to be the result of merging A' and B' as we did above
  output L'
end if

Example: Merge Sort

Example Show by induction that MergeSort works on lists of length 2^n and needs at most $n2^n$ comparisons.

Base step. Merge sort works on lists of length $2^0 = 1$ and requires $0(2^0) = 0$ comparisons.

Induction step.

- Suppose that MergeSort works on lists of length 2^k and needs at most $k2^k$ comparisons.
- If MergeSort is used on a list of length 2^{k+1} it will split it into two lists A and B of length 2^k.
- It will work on A and B, making sorted lists A' and B', using at most $k2^k$ comparisons for each (this is by our assumption).
- It will then merge A' and B' using at most 2^{k+1} comparisons.
- So it will work on lists of length 2^k and need at most
$$2k(2^k) + 2^{k+1} = k2^{k+1} + 2^{k+1} = (k+1)2^{k+1}$$
comparisons.