Housekeeping

Assignment 1 is due at the beginning of your support class next week (13–17 March).

Mon 13 is not a holiday for Monash. Support classes will run as normal.

Assignment 2, tutorial sheet 2 and tutorial solutions 1 are now available.

MAT1830

Lecture 6: Rules of inference

Last time we saw how to recognise tautologies and logically equivalent sentences by computing their truth tables. Another way is to *infer* new sentences from old by *rules of inference*.

6.1 The rule of replacement

This rule says that any sentence may be replaced by an equivalent sentence. Any series of such replacements therefore leads to a sentence equivalent to the one we started with.

Using replacement is like the usual method of proving identities in algebra – make a series of replacements until the left hand side is found equal to the right hand side.

Why can we say that $(\frac{2x}{2})^2 = x^2$?

Because $\frac{2x}{2} = x$.

And there's a rule of replacement.

It's the same in logic except with $\equiv$ instead of $=$.

So we can say that $p \wedge \neg(q \vee r) \equiv p \wedge (\neg q \wedge \neg r)$ because $\neg(q \vee r) \equiv \neg q \wedge \neg r$.

Examples

1. $x \to y \equiv (\neg y) \to (\neg x)$

Proof.

$$\begin{aligned}
x \to y &\equiv (\neg x) \vee y \\
&\quad \text{by implication law} \\
&\equiv y \vee (\neg x) \\
&\quad \text{by commutative law} \\
&\equiv (\neg\neg y) \vee (\neg x) \\
&\quad \text{by law of double negation} \\
&\equiv (\neg y) \to (\neg x) \\
&\quad \text{by implication law} \qquad \square
\end{aligned}$$

$$x \to y \equiv (\neg y) \to (\neg x)$$

$(\neg y) \to (\neg x)$ is the *contrapositive* of $x \to y$.

Example. The contrapositive of

MCG flooded $\to$ cricket is off

is

Cricket is on $\to$ MCG not flooded.

An implication and its contrapositive are equivalent: they mean the same thing!

Question 6.1 What does "no pain, no gain" mean as an implication?

"no pain" $\rightarrow$ "no gain"

Question 6.2 What is its contrapositive?

$\neg$"no gain" $\rightarrow$ $\neg$"no pain"
OR "gain" $\rightarrow$ "pain"

Contrapositives are not negations!

Don't confuse contrapositives with negations.

We've seen that the contrapositive of $p \to q$ is $\neg q \to \neg p$ and that it is logically equivalent to the original statement.

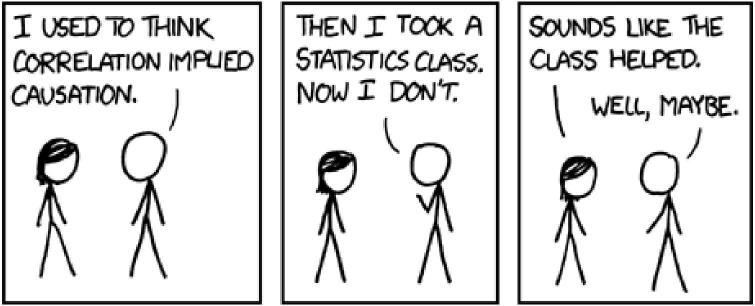
The negation of $p \to q$ is $\neg(p \to q)$. It is not logically equivalent to the original statement.

$$\begin{aligned} \neg(p \to q) &\equiv \neg(\neg p \vee q) \\ &\equiv \neg\neg p \wedge \neg q \\ &\equiv p \wedge \neg q \end{aligned}$$

"If he were the dean then he'd have pants on."

Contrapositive: "If he doesn't have pants on then he isn't the dean."
Logically equivalent to original statement!

Negation: "He's the dean and he doesn't have pants on."
True exactly when the original statement is false!


I USED TO THINK CORRELATION IMPLIED CAUSATION.
THEN I TOOK A STATISTICS CLASS. NOW I DON'T.
SOUNDS LIKE THE CLASS HELPED.
WELL, MAYBE.

Question 6.3 Write down the following sentences as implications and then write their contrapositives.

Sentence: "You can't make an omelette without breaking eggs."
Implication: "you made an omelette" $\rightarrow$ "you broke eggs"
Contrapositive: "you didn't break eggs" $\rightarrow$ "you didn't make an omelette"

Sentence: "If n is even, so is n^2."
Implication: "n is even" $\rightarrow$ "n^2 is even"
Contrapositive: "n^2 is odd" $\rightarrow$ "n is odd"

Sentence: "Haste makes waste."
Implication: "haste" $\rightarrow$ "waste"
Contrapositive: "no waste" $\rightarrow$ "no haste"

2. $p \to (q \to p) \equiv \mathsf{T}$

Proof.

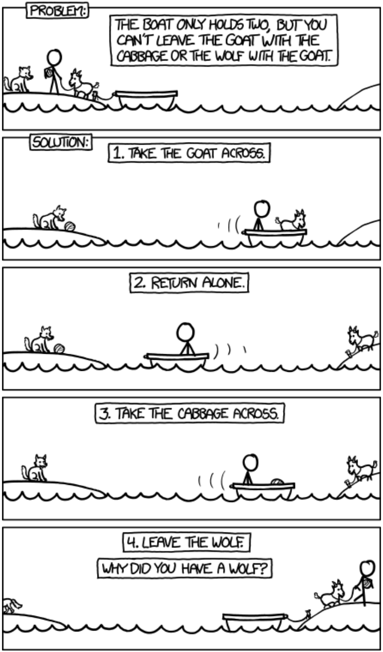
$$
\begin{aligned}
& p \to (q \to p) \\
\equiv\ & (\neg p) \vee (q \to p) \\
& \text{by implication law} \\
\equiv\ & (\neg p) \vee ((\neg q) \vee p) \\
& \text{by implication law} \\
\equiv\ & (\neg p) \vee (p \vee (\neg q)) \\
& \text{by commutative law} \\
\equiv\ & ((\neg p) \vee p) \vee (\neg q) \\
& \text{by associative law} \\
\equiv\ & (p \vee (\neg p)) \vee (\neg q) \\
& \text{by commutatve law} \\
\equiv\ & \mathsf{T} \vee (\neg q) \quad \text{by inverse law} \\
\equiv\ & \mathsf{T} \quad \text{by annihilation law}\square
\end{aligned}
$$

Question 6.4 Show that $p \to (q \to (r \to p))$ is a tautology.

$$
\begin{aligned}
p \to (q \to (r \to p)) &\equiv \neg p \vee (q \to (r \to p)) \\
&\equiv \neg p \vee (\neg q \vee (r \to p)) \\
&\equiv \neg p \vee (\neg q \vee (\neg r \vee p)) \\
&\equiv \neg p \vee \neg q \vee \neg r \vee p \\
&\equiv (\neg p \vee p) \vee \neg q \vee \neg r \\
&\equiv T \vee \neg q \vee \neg r \\
&\equiv T
\end{aligned}
$$

Question 6.5 Find a tautology form with n variables which is $p \to (q \to p)$ for $n = 2$ and $p \to (q \to (r \to p))$ for $n = 3$.

$$p_1 \to (p_2 \to (p_3 \to (p_4 \to \cdots\cdots (p_{n-1} \to (p_n \to p_1)) \cdots)))$$


PROBLEM:
THE BOAT ONLY HOLDS TWO, BUT YOU CAN'T LEAVE THE GOAT WITH THE CABBAGE OR THE WOLF WITH THE GOAT.
SOLUTION:
1. TAKE THE GOAT ACROSS.
2. RETURN ALONE.
3. TAKE THE CABBAGE ACROSS.
4. LEAVE THE WOLF.
WHY DID YOU HAVE A WOLF?

3. $((p \to q) \land p) \to q \equiv \mathsf{T}$

Proof.

$$
\begin{aligned}
& ((p \to q) \land p) \to q \\
\equiv\ & \neg((p \to q) \land p) \lor q \\
& \text{by implication law} \\
\equiv\ & (\neg(p \to q) \lor (\neg p)) \lor q \\
& \text{by de Morgan's law} \\
\equiv\ & \neg(p \to q) \lor ((\neg p) \lor q) \\
& \text{by associative law} \\
\equiv\ & \neg(p \to q) \lor (p \to q) \\
& \text{by implication law} \\
\equiv\ & (p \to q) \lor \neg(p \to q) \\
& \text{by commutative law} \\
\equiv\ & \mathsf{T} \quad \text{by inverse law} \quad \square
\end{aligned}
$$

This tautology says that "if p implies q and p is true then q is true".

6.2 Modus ponens

The tautology $((p \to q) \wedge p) \to q$ also translates into a rule of inference known as *modus ponens:* from sentences $p \to q$ and p we can infer the sentence q.

6.3 Logical consequence

A sentence ψ is a *logical consequence* of a sentence ϕ, if $\psi = \mathsf{T}$ whenever $\phi = \mathsf{T}$. We write this as $\phi \Rightarrow \psi$.

It is the same to say that $\phi \to \psi$ is a tautology, but $\phi \Rightarrow \psi$ makes it clearer that we are discussing a relation between the sentences ϕ and ψ.

Any sentence ψ logically *equivalent* to ϕ is a logical consequence of ϕ, but not all consequences of ψ are equivalent to it.

Example. $p \wedge q \Rightarrow p$

p is a logical consequence of $p \wedge q$, because $p = \mathsf{T}$ whenever $p \wedge q = \mathsf{T}$. However, we can have $p \wedge q = \mathsf{F}$ when $p = \mathsf{T}$ (namely, when $q = \mathsf{F}$). Hence $p \wedge q$ and p are not equivalent.

This example shows that $\Rightarrow$ is not symmetric:

$$(p \wedge q) \Rightarrow p \quad \text{but} \quad p \not\Rightarrow (p \wedge q)$$

This is where $\Rightarrow$ differs from $\equiv$, because if $\phi \equiv \psi$ then $\psi \equiv \phi$.

In fact, we build the relation $\equiv$ from $\Rightarrow$ the same way $\leftrightarrow$ is built from $\rightarrow$:

$$\phi \equiv \psi \ \text{ means } \ (\phi \Rightarrow \psi) \text{ and } (\psi \Rightarrow \phi).$$

Example Show that $p \wedge (q \vee r) \Rightarrow (p \wedge q) \vee r$ using a truth table.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$(p \wedge q) \vee r$	
T	T	T	T	T	T	T	(!)
T	T	F	T	T	T	T	(!)
T	F	T	T	T	F	T	(!)
T	F	F	F	F	F	F	
F	T	T	T	F	F	T	
F	T	F	T	F	F	F	
F	F	T	T	F	F	T	
F	F	F	F	F	F	F	