Housekeeping

Assignment 5 is now available. It is due at the beginning of your support class in week 7 (10–13 April). Submission options for people missing their Fri 14 April tutorial are included on the assignment sheet.

People missing their Fri 14 April tutorial should feel free to attend any other tutorial in week 7 (a list is now up on moodle).

There's a link to the MLC video library under “additional resources” on the moodle page.

MAT1830

Lecture 17: Equivalence Relations

An *equivalence relation* R on a set A is a binary relation with the following three properties.

1. Reflexivity.

$$aRa$$
$$\text{for all } a \in A.$$

2. Symmetry.

$$aRb \Rightarrow bRa$$
$$\text{for all } a, b \in A.$$

3. Transitivity.

$$aRb \text{ and } bRc \Rightarrow aRc$$
$$\text{for all } a, b, c \in A.$$

Equality and congruence mod n (for fixed n) are examples of equivalence relations.

Reflexivity (For a binary relation R on a set A.)

Everywhere I see:

I actually see:

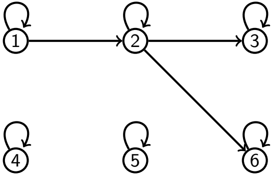
To prove R is reflexive, show that...

For all $x \in A$, xRx.

To prove R is not reflexive, show that...

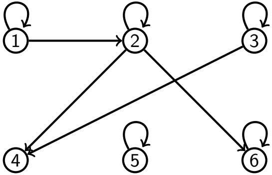
There is an $x \in A$ such that $x \not R x$.

Question Let R be the relation on A pictured below. Is R reflexive?



Yes. xRx for all $x \in A$.

Question Let S be the relation on A pictured below. Is S reflexive?

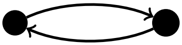


No. $4 \not S 4$.

Symmetry (For a binary relation R on a set A.)

Everywhere I see:

I actually see:



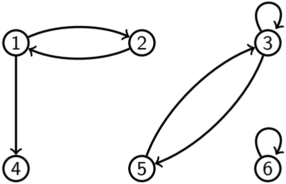
To prove R is symmetric, show that...

For all $x, y \in A$, if xRy then yRx.

To prove R is not symmetric, show that...

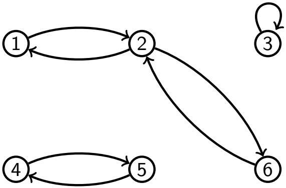
There are some $x, y \in A$ such that xRy but $y \not R x$.

Question Let R be the relation on A pictured below. Is R symmetric?

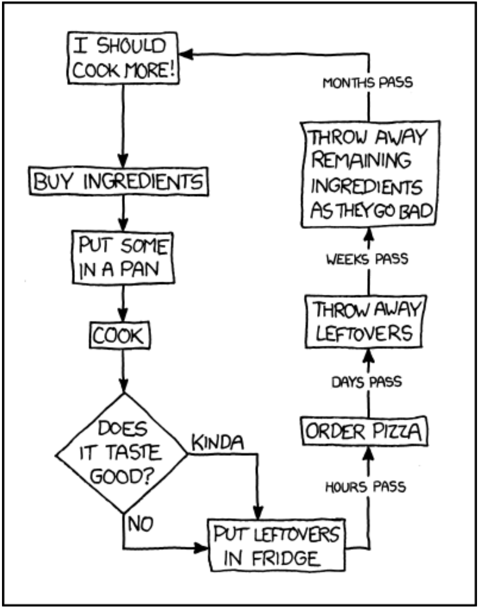


No. $1R4$ but $4 \not R 1$.

Question Let S be the relation on A pictured below. Is S symmetric?

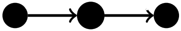


Yes. For all $x, y \in A$ if xSy then ySx.

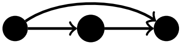

I SHOULD COOK MORE!
BUY INGREDIENTS
PUT SOME IN A PAN
COOK
DOES IT TASTE GOOD?
KINDA
NO
PUT LEFTOVERS IN FRIDGE
HOURS PASS
ORDER PIZZA
DAYS PASS
THROW AWAY LEFTOVERS
WEEKS PASS
THROW AWAY REMAINING INGREDIENTS AS THEY GO BAD
MONTHS PASS

Transitivity (For a binary relation R on a set A.)

Everywhere I see:



I actually see:



Everywhere I see:

I actually see:

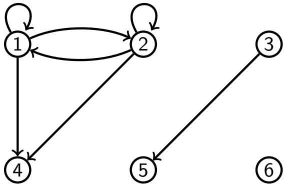
To prove R is transitive, show that...

For all $x, y, z \in A$, if xRy and yRz then xRz.

To prove R is not transitive, show that...

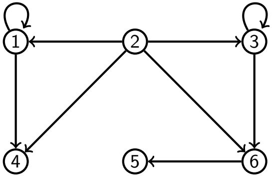
There are some $x, y, z \in A$ such that xRy and yRz but $x \not R z$.

Question Let R be the relation on A pictured below. Is R transitive?



Yes. For all $x, y, z \in A$, if xRy and yRz then xRz.

Question Let S be the relation on A pictured below. Is S transitive?



No, because $3S6$ and $6S5$ but $3 \not S 5$.

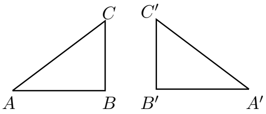
17.1 Other equivalence relations

1. Equivalence of fractions.

Two fractions are equivalent if they reduce to the same fraction when the numerator and denominator of each are divided by their gcd. E.g. $\frac{2}{4}$ and $\frac{3}{6}$ are equivalent because both reduce to $\frac{1}{2}$.

2. Congruence of triangles.

Triangles ABC and $A'B'C'$ are congruent if $AB = A'B'$, $BC = B'C'$ and $CA = C'A'$. E.g. the following triangles are congruent.

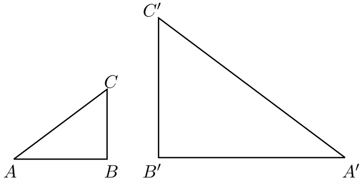


3. Similarity of triangles.

Triangles ABC and $A'B'C'$ are similar if

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}.$$

E.g the following triangles are similar



4. Parallelism of lines.

The relation $L \| M$ (L is parallel to M) is an equivalence relation.

Remark

In all these cases the relation is an equivalence because it says that objects are the *same* in some respect.

1. Equivalent fractions have the same reduced form.

2. Congruent triangles have the same side lengths.

3. Similar triangles have the same shape.

4. Parallel lines have the same direction.

Sameness is always reflexive (a is the same as a), symmetric (if a is the same as b, then b is the same as a) and transitive (if a is the same as b and b is the same as c, then a is the same as c).

Question 17.1 Which of the following relations are equivalence relations on $\mathbb{Z}$?

- $|x| = |y|$

Reflexive: Yes. $|a| = |a|$ for all $a \in \mathbb{Z}$.
Symmetric: Yes. If $|a| = |b|$, then $|b| = |a|$ for all $a, b \in \mathbb{Z}$.
Transitive: Yes. If $|a| = |b|$ and $|b| = |c|$, then $|a| = |c|$ for all $a, b, c \in \mathbb{Z}$.
So it is an equivalence relation.

- $x^3 - y^3 = 0$

Reflexive: Yes. $a^3 - a^3 = 0$ for all $a \in \mathbb{Z}$.
Symmetric: Yes. If $a^3 - b^3 = 0$, then $b^3 - a^3 = 0$ for all $a, b \in \mathbb{Z}$.
Transitive: Yes. If $a^3 - b^3 = 0$ and $b^3 - c^3 = 0$, then $a^3 - c^3 = 0$ for all $a, b, c \in \mathbb{Z}$.
So it is an equivalence relation.

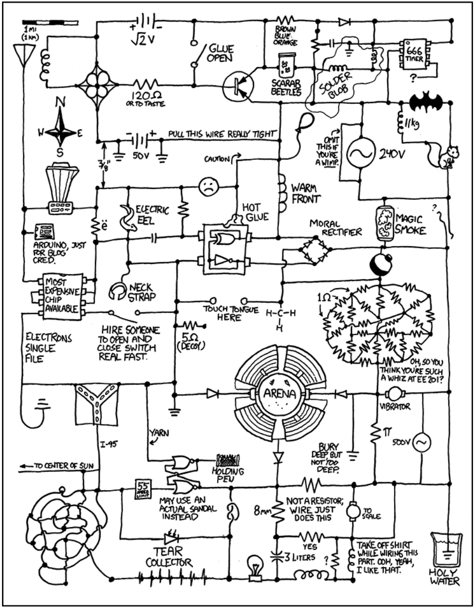
Question 17.1 (cont.) Which of the following relations are equivalence relations on $\mathbb{Z}$?

- $x^3 - y^3 = 1$

Reflexive: No. $1^3 - 1^3 \neq 1$ so $1 \not R 1$.
Symmetric: No. $1^3 - 0^3 = 1$ but $0^3 - 1^3 \neq 1$, so $1R0$ but $0 \not R 1$.
Transitive: No. $1^3 - 0^3 = 1$ and $0^3 - (-1)^3 = 1$ but $1^3 - (-1)^3 \neq 1$, so $1R0$ and $0R(-1)$ but $1 \not R(-1)$.
So it is not an equivalence relation.

- x divides y

Reflexive: Yes. a divides a for all $a \in \mathbb{Z}$.
Symmetric: No. 3 divides 6 but 6 does not divide 3, so $3R6$ but $6 \not R 3$.
Transitive: Yes. If a divides b and b divides c, then a divides c for all $a, b, c \in \mathbb{Z}$.
So it is not an equivalence relation.


1 MI (1 KM)
√2 V
GLUE OPEN
BROWN BLUE ORANGE
666 TIMER
?
120Ω OR TO TASTE
SCARAB BEETLES
SOLDER BLOB
N
W
E
S
PULL THIS WIRE REALLY TIGHT
50V
3/8"
1/kg
OMIT THIS IF YOU'RE A WIMP.
240V
CAUTION
WARM FRONT
ELECTRIC EEL
ë
HOT GLUE
MORAL RECTIFIER
MAGIC SMOKE
?
ARDUINO, JUST FOR BLOG CRED.
MOST EXPENSIVE CHIP AVAILABLE
NECK STRAP
TOUCH TONGUE HERE
H-C-H
H
1Ω
HIRE SOMEONE TO OPEN AND CLOSE SWITCH REAL FAST.
ELECTRONS SINGLE FILE
5Ω (DECOY)
ARENA
OH, SO YOU THINK YOU'RE SUCH A WHIZ AT EE 201?
VIBRATOR
π
500V
BURY DEEP, BUT NOT TOO DEEP.
YARN
I-95
TO CENTER OF SUN
HOLDING PEN
55 SPEED LIMIT
MAY USE AN ACTUAL SANDAL INSTEAD
NOT A RESISTOR; WIRE JUST DOES THIS
8mm
TO SCALE
TEAR COLLECTOR
YES
3 LITERS
?
TAKE OFF SHIRT WHILE WIRING THIS PART. OOH, YEAH, I LIKE THAT.
HOLY WATER

Question 17.1 (cont.) Which of the following relations are equivalence relations on $\mathbb{Z}$?

- 5 divides $x - y$

Yes. This relation is the same as $x \equiv y \pmod{5}$ and we know that's an equivalence relation.

Question 17.3 What is the same about the equivalent objects for these equivalence relations on $\mathbb{Z}$?

- $|x| = |y|$

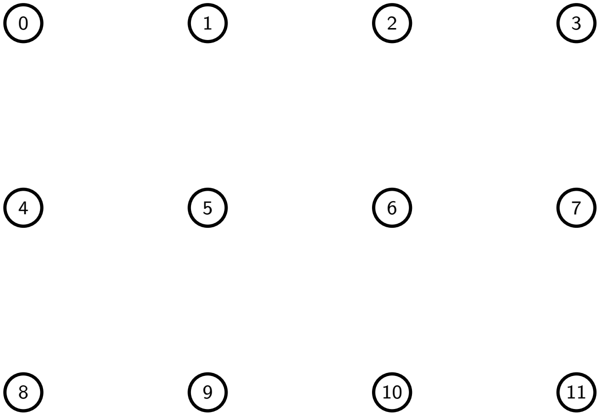
x and y have the same "magnitude".

- $x^3 - y^3 = 0$

x and y are equal!

- 5 divides $x - y$

x and y have the same remainder when divided by 5.


0
1
2
3
4
5
6
7
8
9
10
11

17.2 Equivalence classes

Conversely, we can show that if R is a reflexive, symmetric and transitive relation then aRb says that a and b are the same in some respect: *they have the same R-equivalence class.*

If R is an equivalence relation we define the *R-equivalence class* of a to be

$$[a] = \{s : sRa\}.$$

Thus $[a]$ consists of all the elements related to a. It can also be defined as $\{s : aRs\}$, because sRa if and only if aRs, by symmetry of R.

Examples

- The parallel equivalence class of a line L consists of all lines parallel to L.
- The equivalence class of 1 for congruence mod 2 is the set of all odd numbers.

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Examples

- The parallel equivalence class of a line L consists of all lines parallel to L.
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17.3 Equivalence class properties

Claim. *If two elements are related by an equivalence relation R on a set A, their equivalence classes are equal.*

Proof. Suppose $a, b \in A$ and aRb. Now

$$\begin{aligned} s \in [a] &\Rightarrow sRa \text{ by definition of } [a] \\ &\Rightarrow sRb \text{ by transitivity of } R \\ & \text{ since } sRa \text{ and } aRb \\ &\Rightarrow s \in [b] \text{ by definition of } [b]. \end{aligned}$$

Thus all elements of $[a]$ belong to $[b]$. Similarly, all elements of $[b]$ belong to $[a]$, hence $[a] = [b]$. $\square$

Claim. *If R is an equivalence relation on a set A, each element of A belongs to exactly one equivalence class.*

Proof. Suppose $a, b, c \in A$, and $c \in [a] \cap [b]$.

$$\begin{aligned}
& c \in [a] \text{ and } c \in [b] \\
\Rightarrow\ & cRa \text{ and } cRb \\
& \text{by definition of } [a] \text{ and } [b] \\
\Rightarrow\ & aRc \text{ and } cRb \text{ by symmetry} \\
\Rightarrow\ & aRb \text{ by transitivity} \\
\Rightarrow\ & [a] = [b] \\
& \text{by the previous claim.}
\end{aligned}$$

17.4 Partitions and equivalence classes

A *partition* of a set S is a set of subsets of S such that each element of S is in exactly one of the subsets.

Using what we showed in the last section, we have the following.

If R is an equivalence relation on a set A, then the equivalence classes of R form a partition of A. Two elements of A are related if and only if they are in the same equivalence class.

Example. Let R be the relation on $\mathbb{Z}$ defined by aRb if and only if $a \equiv b \pmod 3$. The three equivalence classes of R are

$$\{x : x \equiv 0 \pmod 3\} = \{3k : k \in \mathbb{Z}\}$$
$$\{x : x \equiv 1 \pmod 3\} = \{3k+1 : k \in \mathbb{Z}\}$$
$$\{x : x \equiv 2 \pmod 3\} = \{3k+2 : k \in \mathbb{Z}\}.$$

These partition the set $\mathbb{Z}$.

Question 17.4 What are the equivalence classes of these equivalence relations on $\mathbb{Z}$?

- $|x| = |y|$

$\{0\}, \{1, -1\}, \{2, -2\}, \{3, -3\} \dots.$

- $x^3 - y^3 = 0$

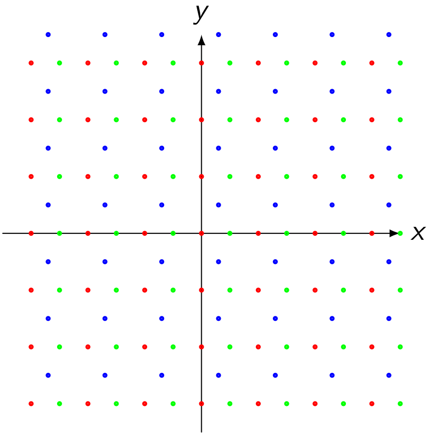
$\dots, \{-3\}, \{-2\}, \{-1\}, \{0\}, \{1\}, \{2\}, \{3\}, \dots.$

- 5 divides $x - y$

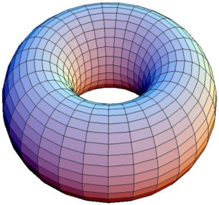
$\{\dots, -10, -5, 0, 5, 10, \dots\},$
$\{\dots, -9, -4, 1, 6, 11, \dots\},$
$\{\dots, -8, -3, 2, 7, 12, \dots\},$
$\{\dots, -7, -2, 3, 8, 13, \dots\},$
$\{\dots, -6, -1, 4, 9, 14, \dots\}$

Example Think of the relation on $\mathbb{R} \times \mathbb{R}$ defined by $(x_1, y_1)R(x_2, y_2)$ if and only if $x_1 - x_2 \in \mathbb{Z}$ and $y_1 - y_2 \in \mathbb{Z}$.

It's not too hard to check this is an equivalence relation.



What geometric space does this relation correspond to?

A torus.