Prepared by Julian García, based on material by
David Albrecht and María García de la Banda

Lecture 10
Complexity

FIT 1008
Introduction to Computer Science

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Running Time and RAM

Insertion sort

Binary Search

Big O

Growth rates

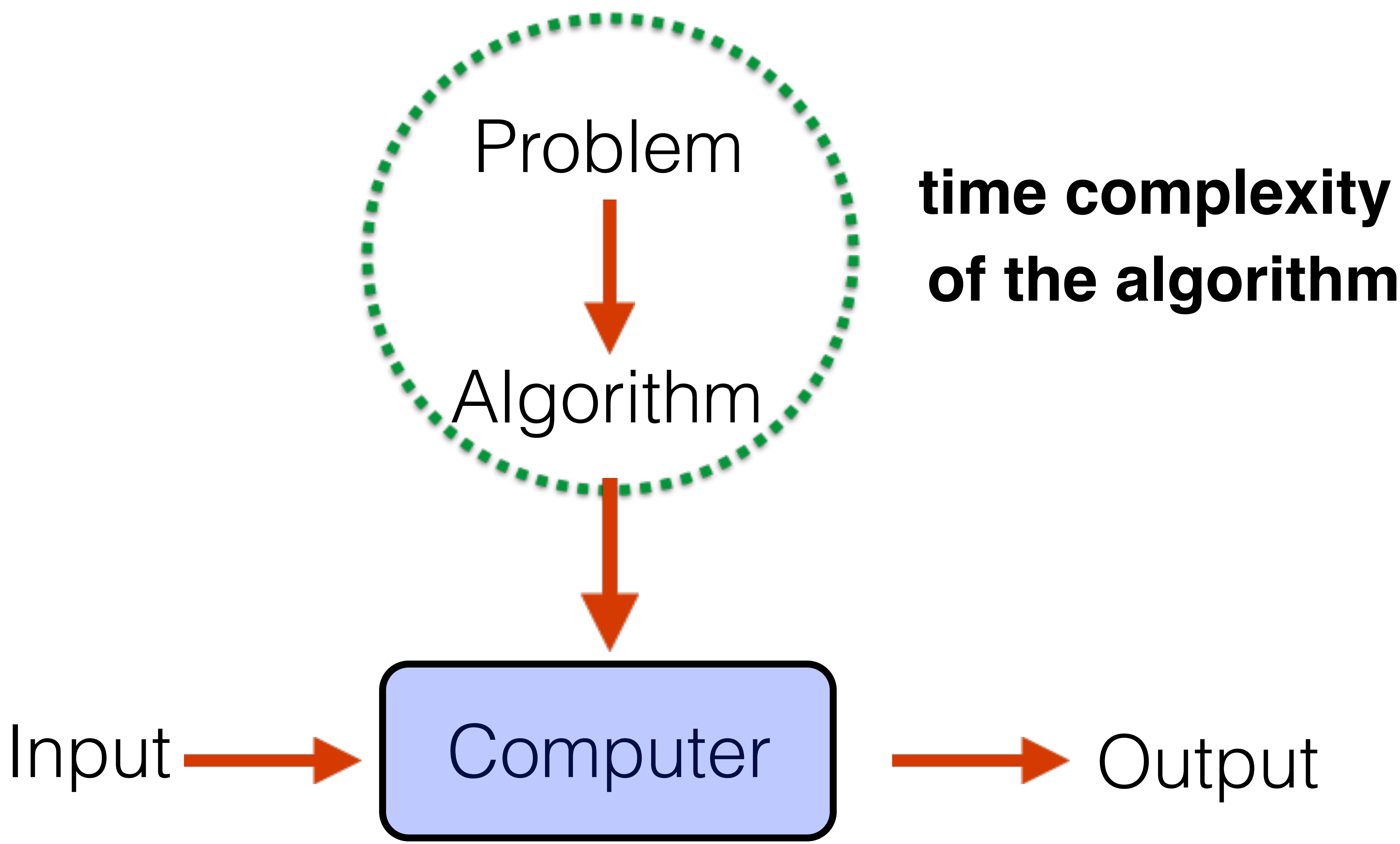
Running Time and RAM

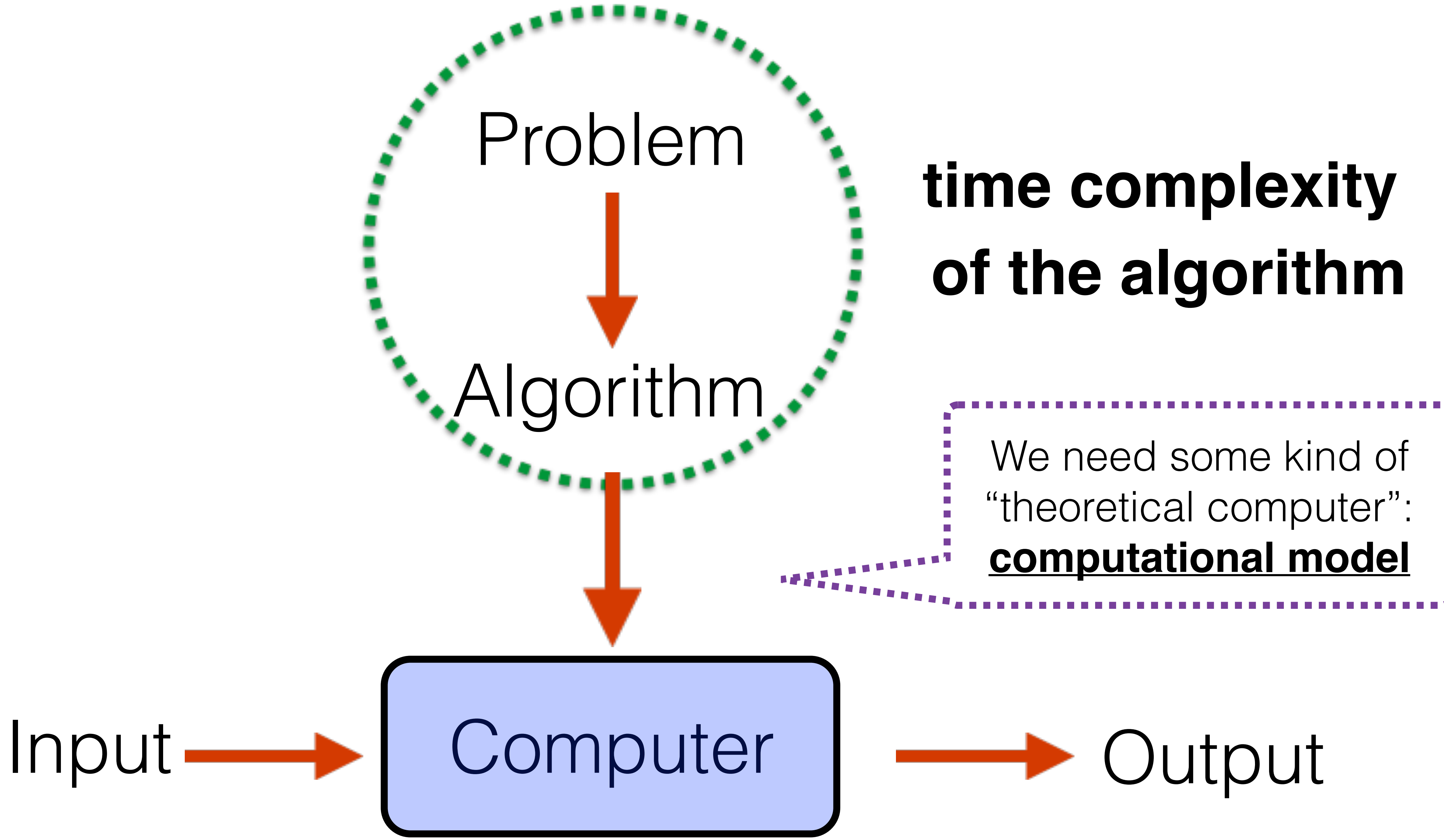
Running Time

Depends on a number of factors including:

- The **input**
- The quality of the code generated by the **compiler**
- The nature and **speed** of the instructions on the **machine** used to execute the program
- The **time complexity of the algorithm**

Jamaica's Usain Bolt celebrating after winning the final of the men's 100 metres athletics event at the 2015 IAAF World Championships in Beijing. AFP PHOTO / PEDRO UGARTE


Problem
Algorithm
time complexity
of the algorithm
Input
Computer
Output


Problem
Algorithm
time complexity
of the algorithm
We need some kind of
"theoretical computer":
computational model
Input
Computer
Output

Simple computation model

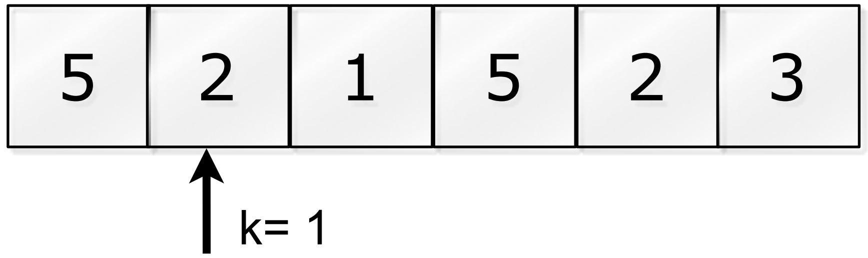
- Each simple operation takes one step
  (e.g., **assignment**, **print** or **return** statement).
- Each **comparison** takes one time step.
- Running time of **a sequence of statements** = **Sum** of the running time of the **statements**.
- **Loops** and **modules**
  - Composition of many simple operations,
    and their running time
  - Depends on how many times each of these simple operations are performed.

RAM model = abstract machine

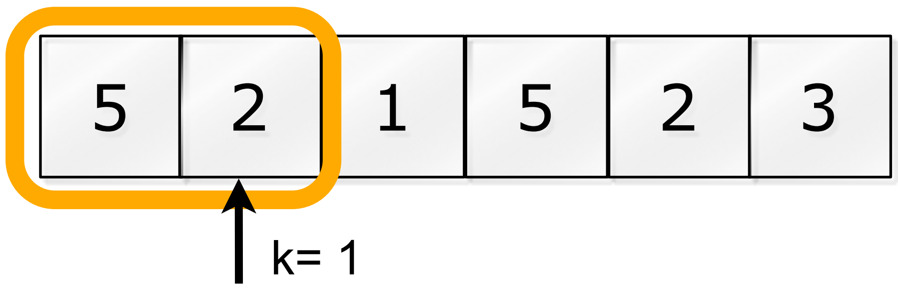
Insertion Sort

(take last,
put it slowly in the right position,
enlarge)

L



L



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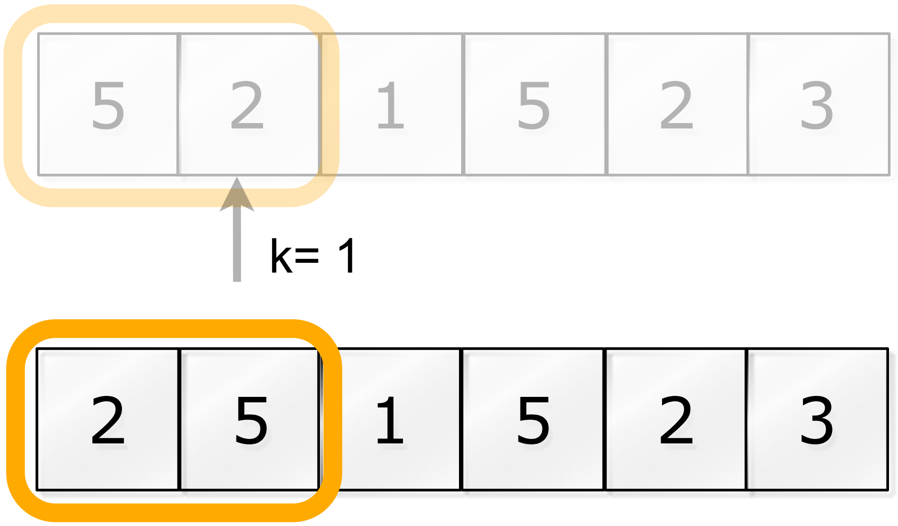
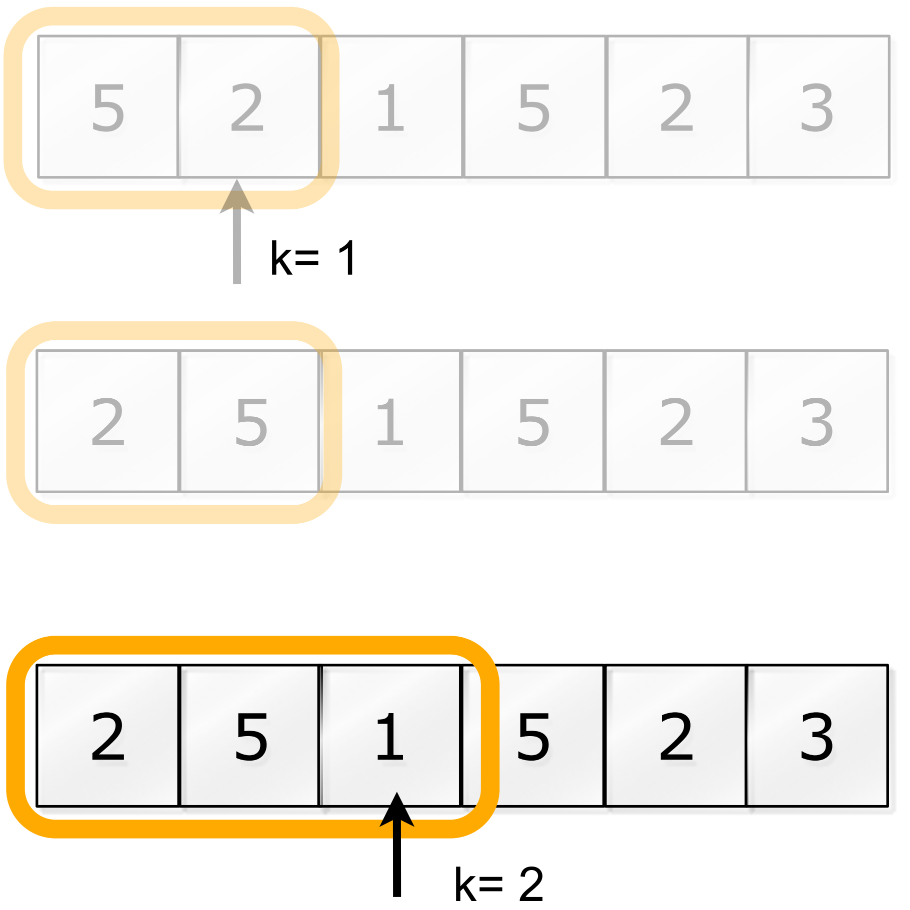


L



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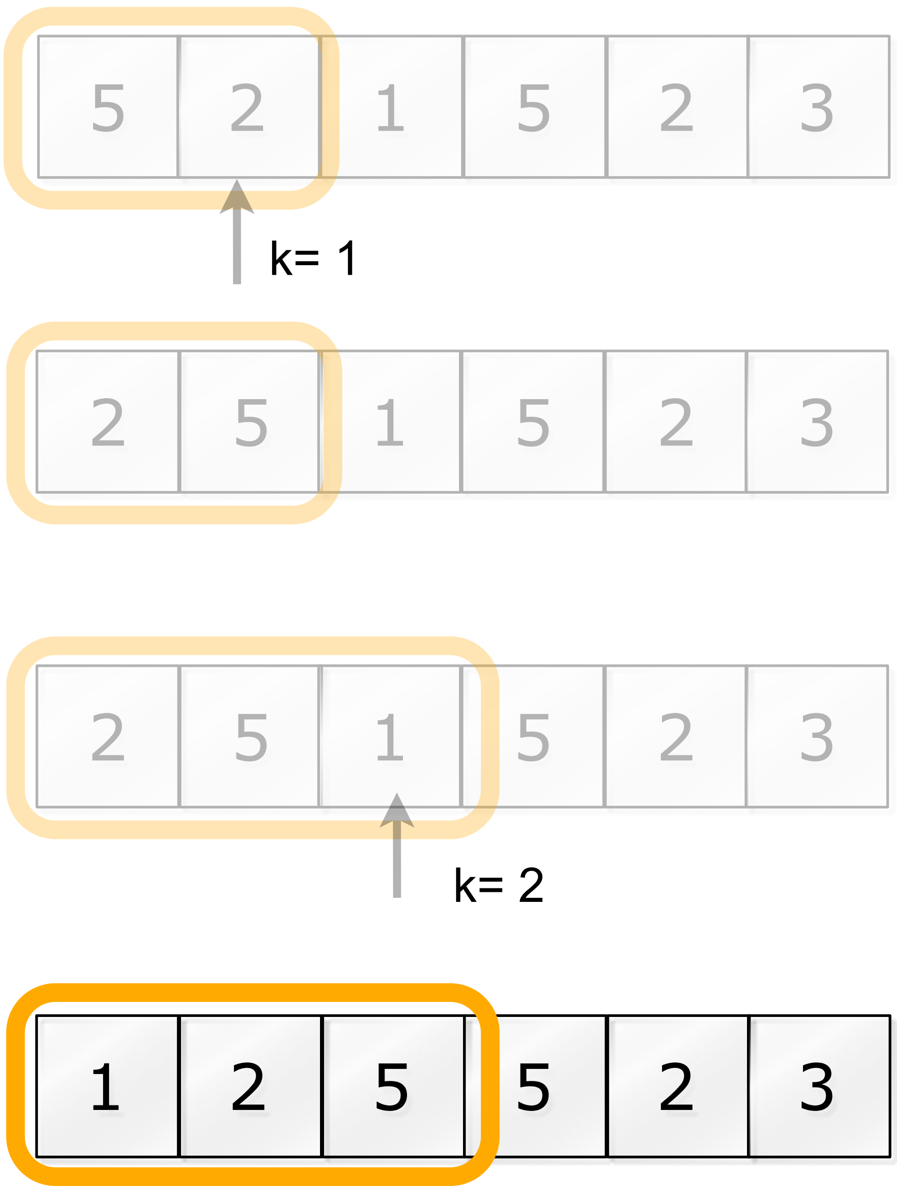

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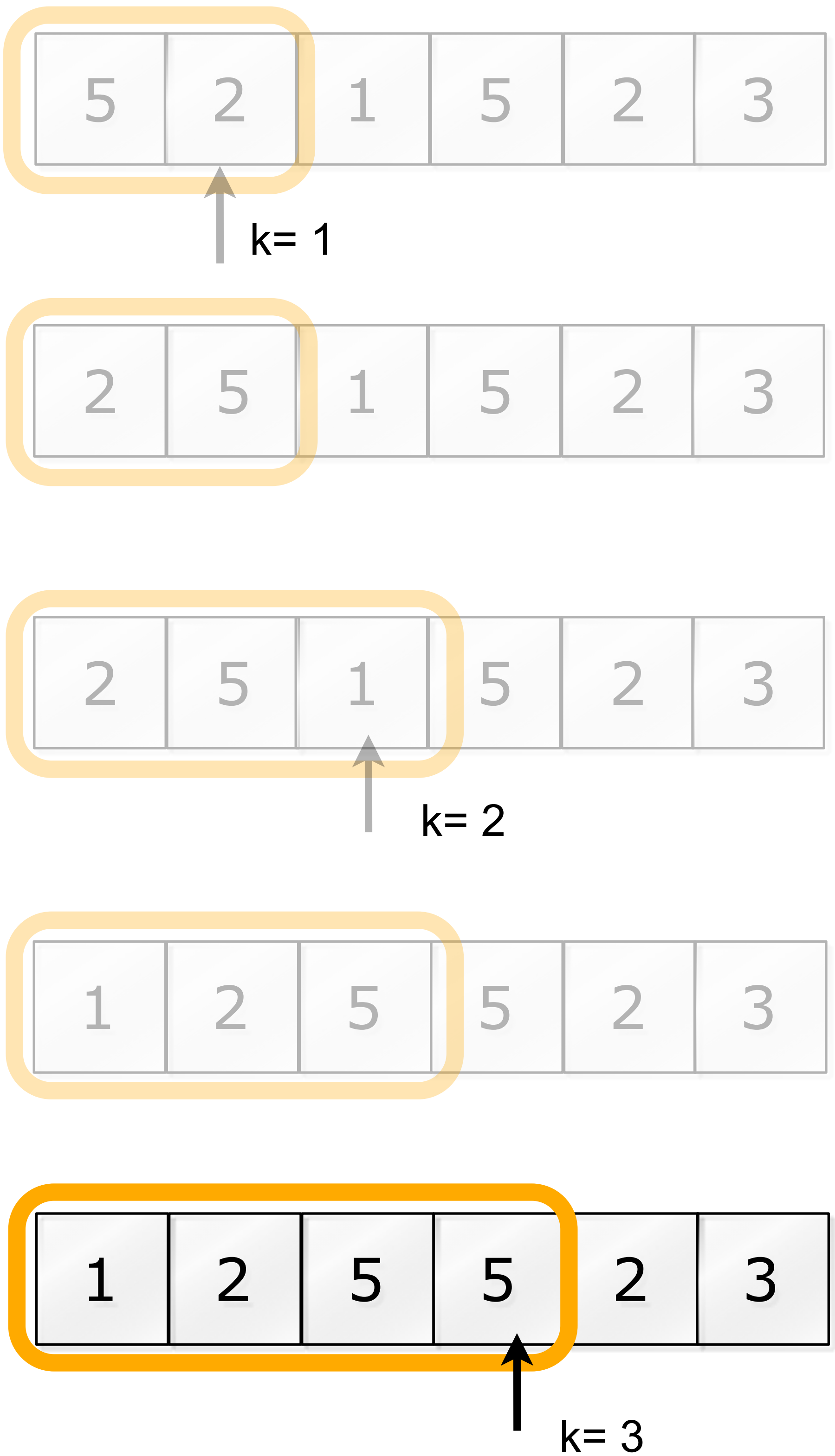

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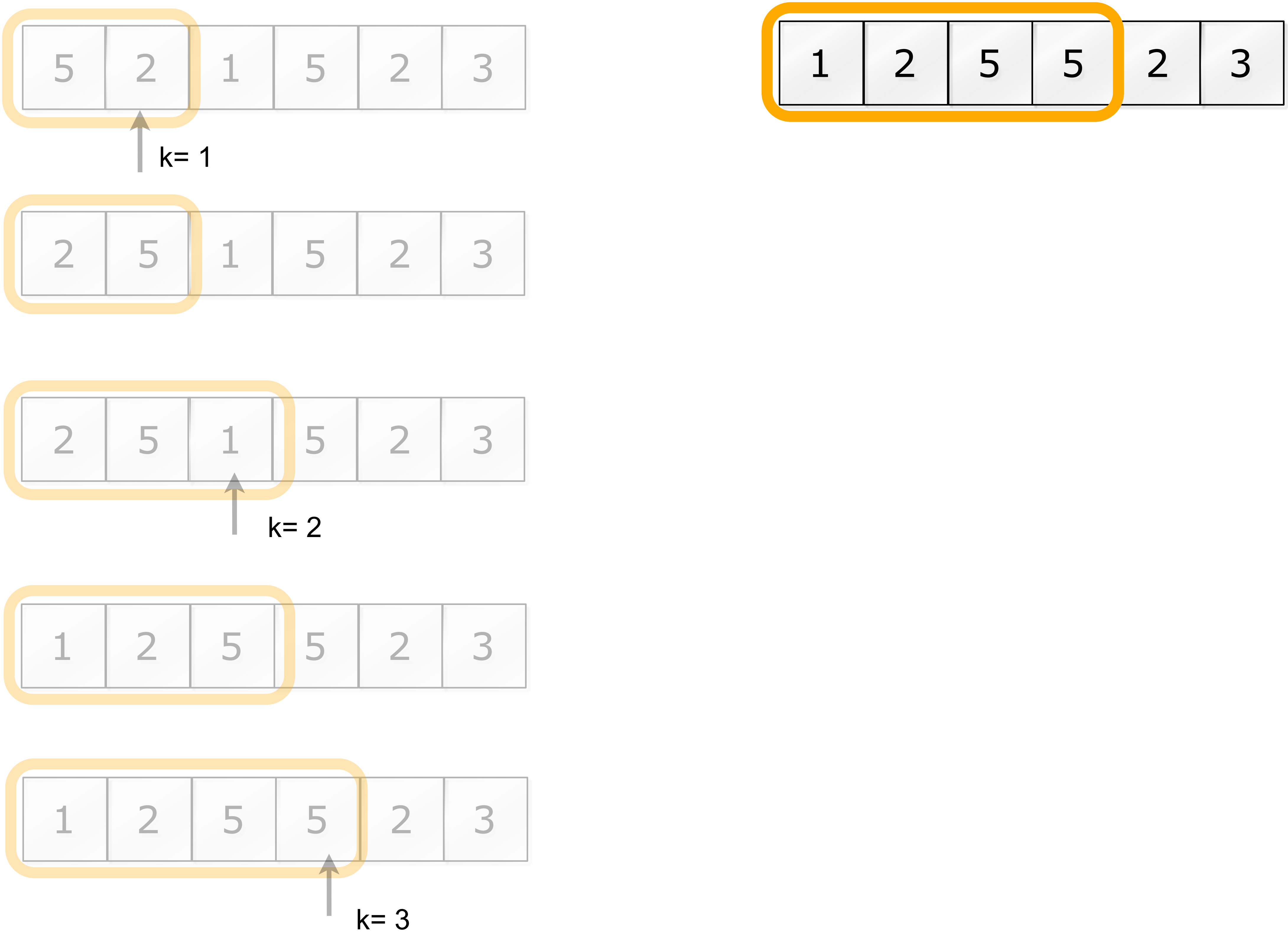

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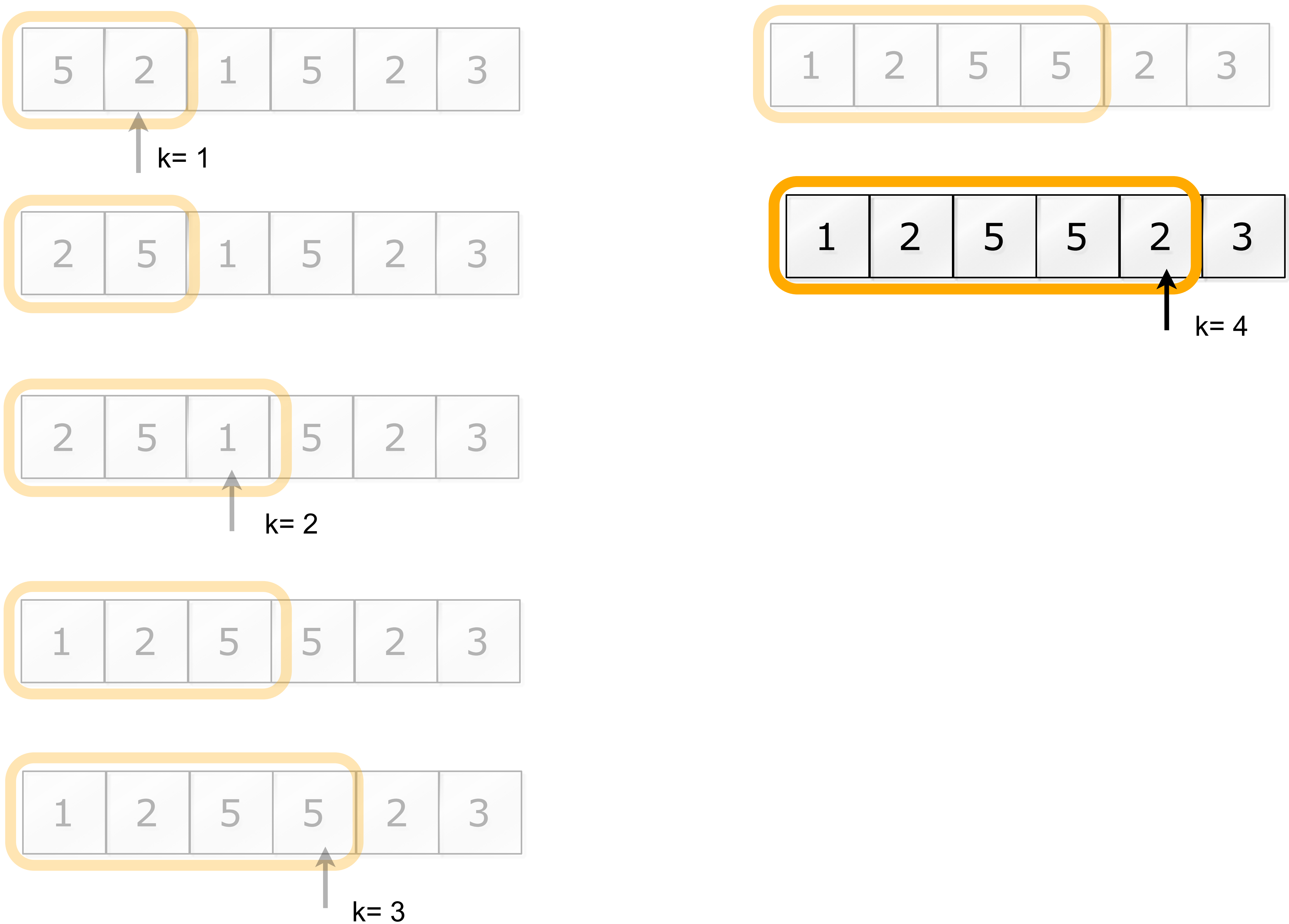
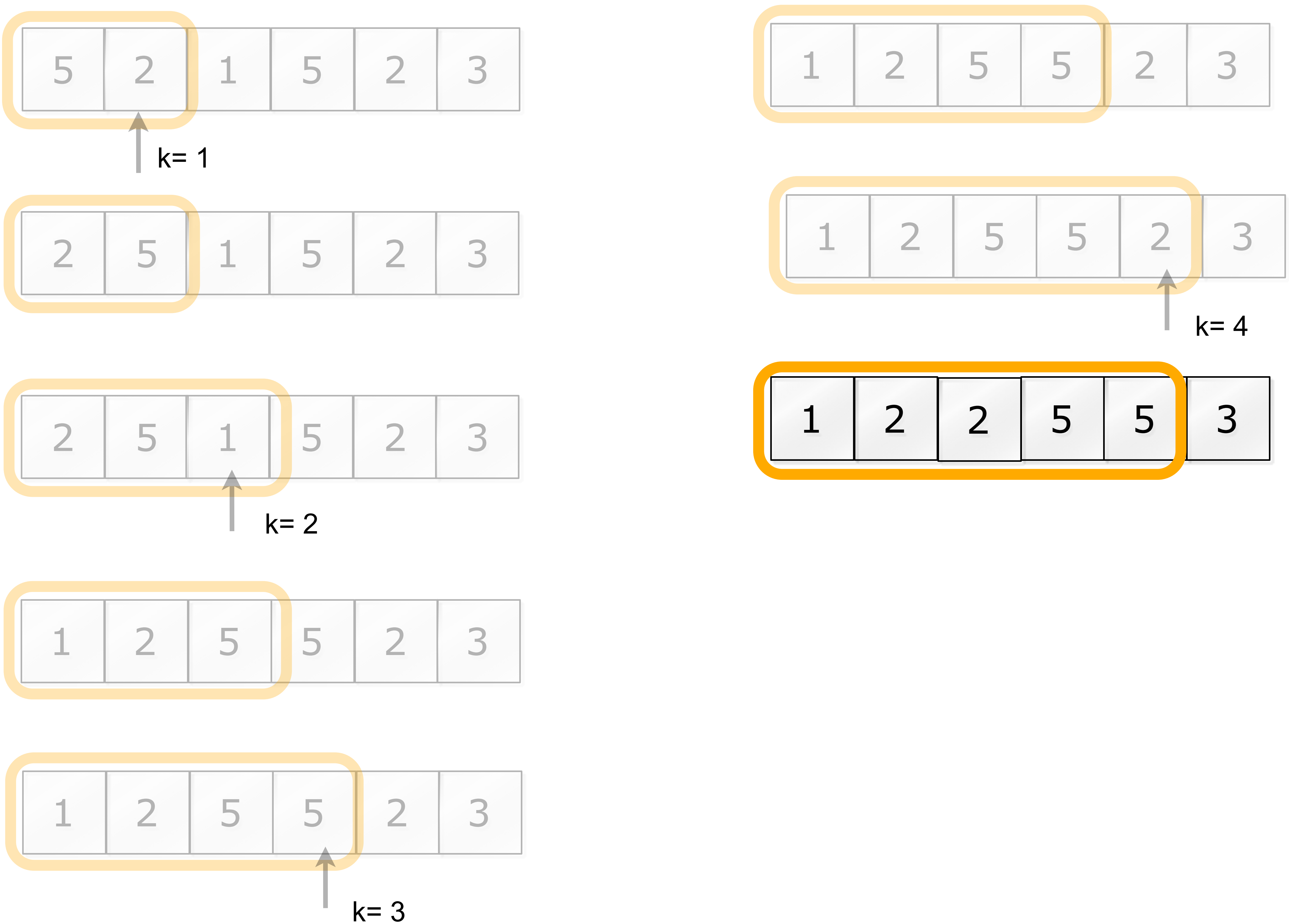

5 2 1 5 2 3
k= 1
2 5 1 5 2 3
2 5 1 5 2 3
k= 2
1 2 5 5 2 3
1 2 5 5 2 3
k= 3
1 2 5 5 2 3
1 2 5 5 2 3
k= 4

L



5 2 1 5 2 3
k= 1
2 5 1 5 2 3
2 5 1 5 2 3
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1 2 5 5 2 3
1 2 5 5 2 3
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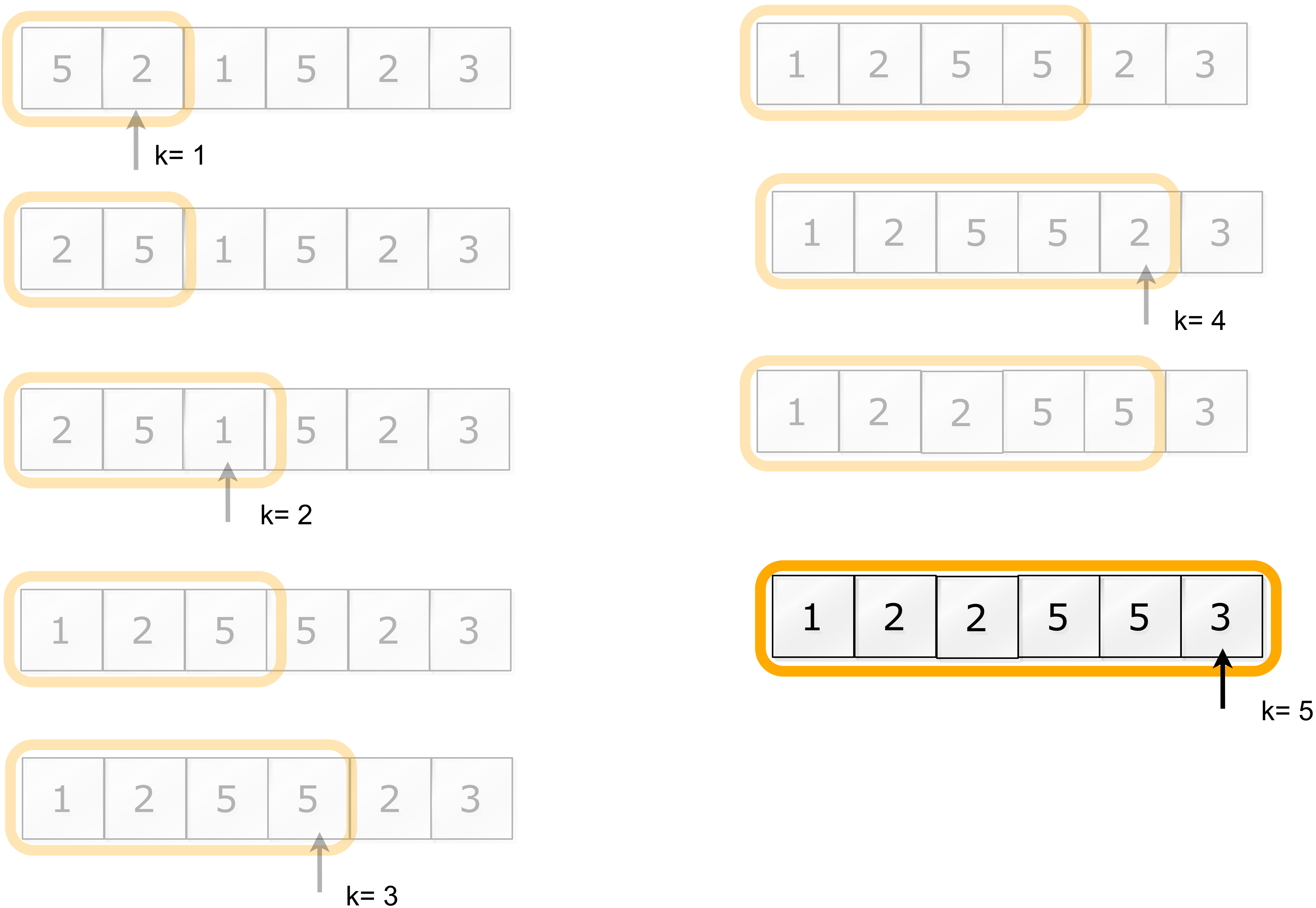
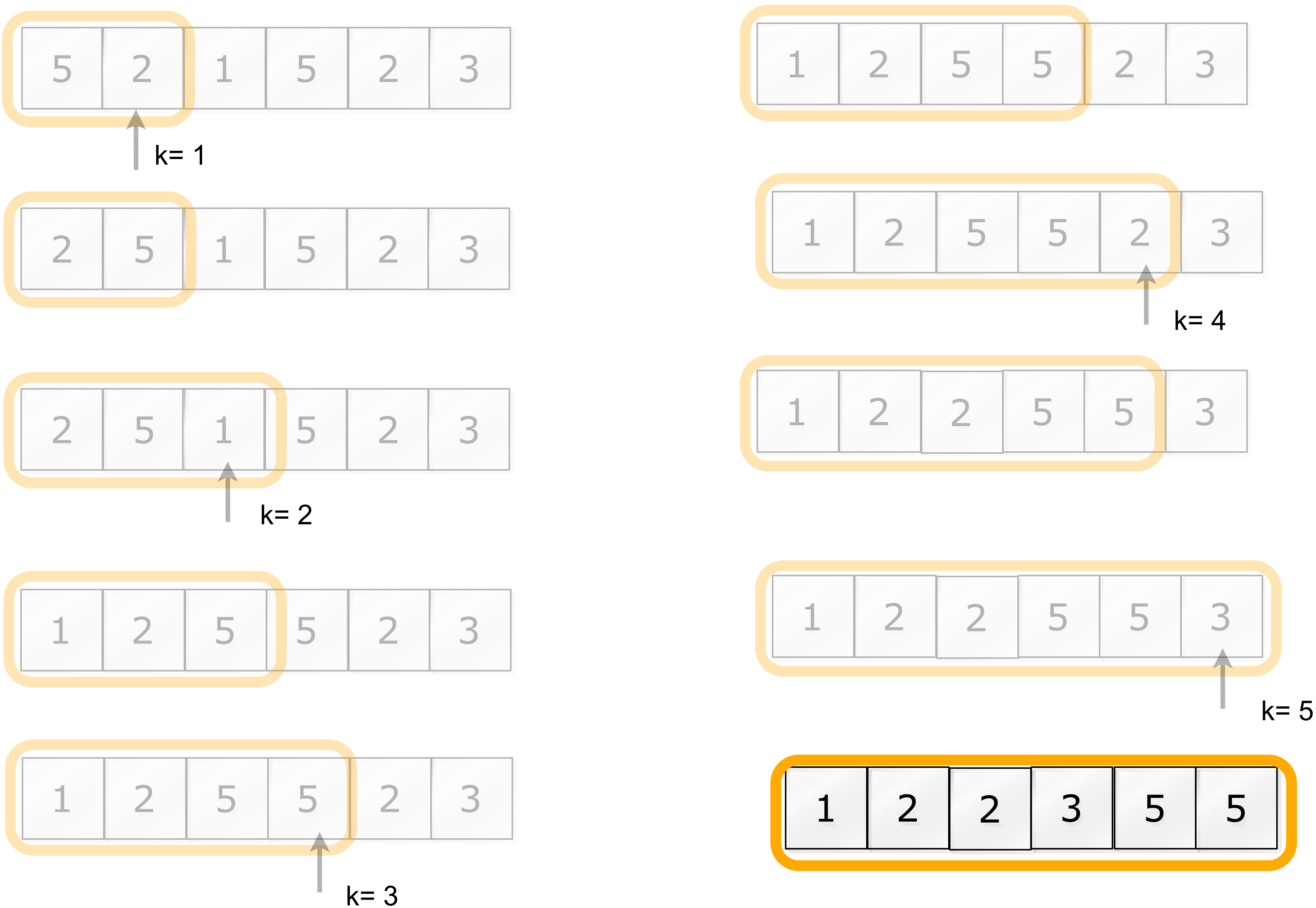
1 2 5 5 2 3
1 2 5 5 2 3
k= 4
1 2 2 5 5 3

L

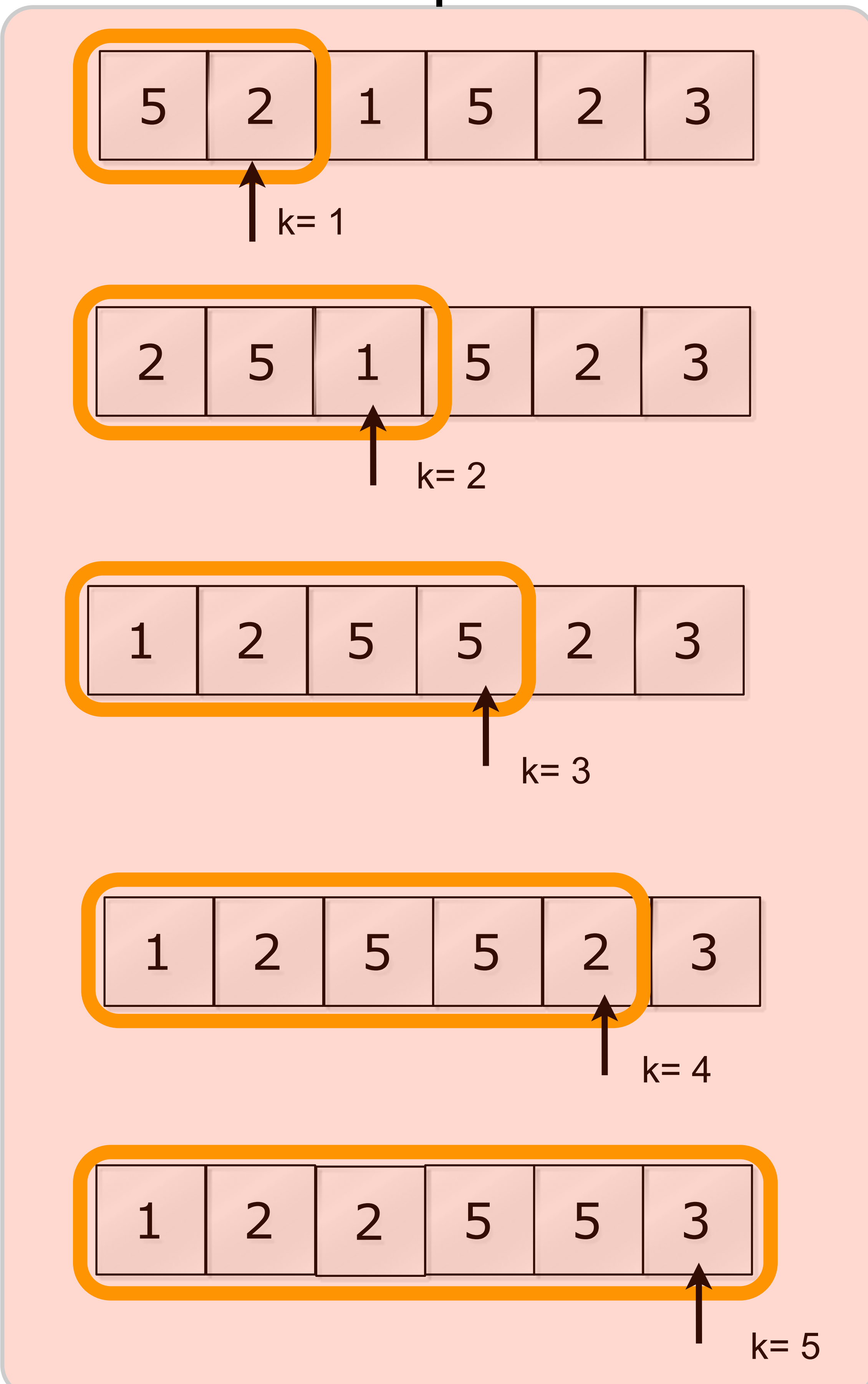


5 2 1 5 2 3
k= 1
2 5 1 5 2 3
2 5 1 5 2 3
k= 2
1 2 5 5 2 3
1 2 5 5 2 3
k= 3
1 2 5 5 2 3
1 2 5 5 2 3
k= 4
1 2 2 5 5 3
1 2 2 5 5 3
k= 5

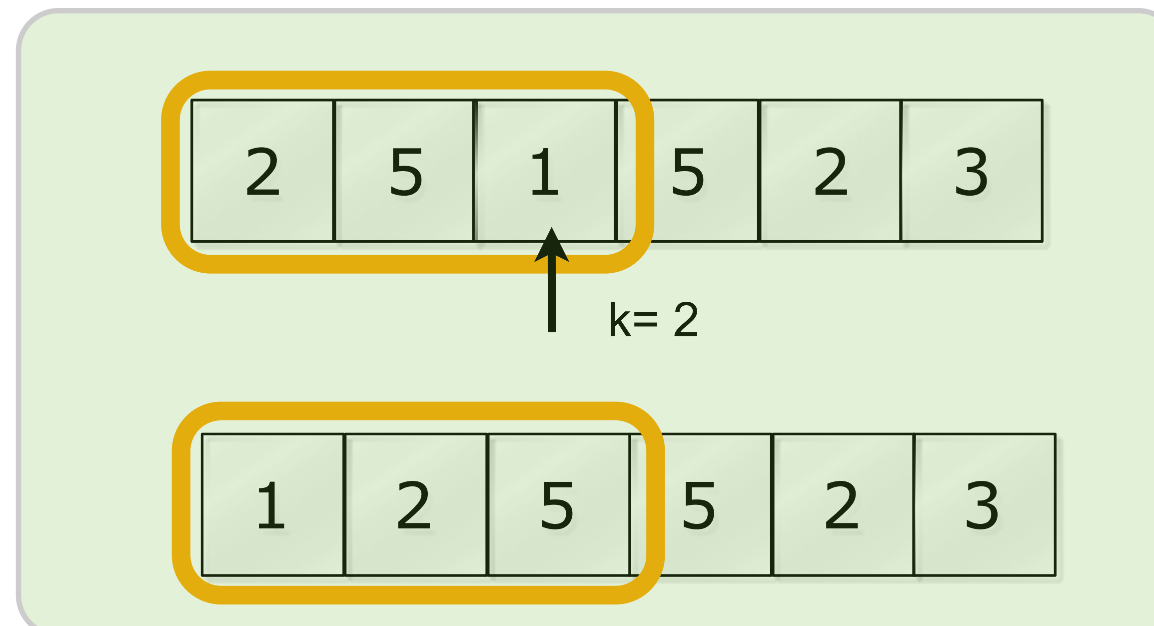
L


5 2 1 5 2 3
k= 1
2 5 1 5 2 3
2 5 1 5 2 3
k= 2
1 2 5 5 2 3
1 2 5 5 2 3
k= 3
1 2 5 5 2 3
1 2 5 5 2 3
k= 4
1 2 2 5 5 3
1 2 2 5 5 3
k= 5
1 2 2 3 5 5

Loop 1


5 2 1 5 2 3
k= 1
2 5 1 5 2 3
k= 2
1 2 5 5 2 3
k= 3
1 2 5 5 2 3
k= 4
1 2 2 5 5 3
k= 5

Loop 2


2 5 1 5 2 3
k= 2
1 2 5 5 2 3

Algorithm InsertionSort(L[0..n-1])

Sorts a list using insertion sort.

Input: A list L[0, n-1] of real numbers

Output: A list sorted in ascending order.

```
k ⟵ 1
while (k < n) do {
        tmp ⟵ list[k]
        j ⟵ k-1
        while (j ≧ 0 and tmp < list[j]) do {
            list[j+1] ⟵ list[j]
            j ⟵ j – 1
        }

        list[j+1] ⟵ tmp
        k ⟵ k + 1
```

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```

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Sorts a list using insertion sort.

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```
k ⟵ 1   1 assignment
while (k < n) do {   1 comparison
      tmp ⟵ list[k]
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      }

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k ⟵ 1                                  1 assignment
while (k < n) do {                     1 comparison
      tmp ⟵ list[k]  \
                      } 2 assignments
      j ⟵ k-1        /
      while (j ≧ 0 and tmp < list[j]) do {
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         j ⟵ j – 1
      }

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      k ⟵ k + 1
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    tmp ⟵ list[k]                       } 2 assignments
    j ⟵ k-1                             }
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        list[j+1] ⟵ list[j]
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    }

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          list[j+1] ⟵ list[j]       }
          j ⟵ j – 1                 } 2 assignments
      }

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      j ⟵ k-1                           }
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         j ⟵ j – 1                      }
      }
                                        } 2 assignments
      list[j+1] ⟵ tmp                   }
      k ⟵ k + 1
```

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Sorts a list using insertion sort.

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Output: A list sorted in ascending order.

```
k ⟵ 1                                        1 assignment
while (k < n) do {                           1 comparison
      tmp ⟵ list[k]                     ⎫
      j ⟵ k-1                           ⎭    2 assignments
      while (j ≧ 0 and tmp < list[j]) do {   2 comparisons      ⎫
          list[j+1] ⟵ list[j]           ⎫                       ⎬ At most k times k < n
          j ⟵ j – 1                     ⎭    2 assignments      ⎭
      }
      list[j+1] ⟵ tmp                   ⎫
      k ⟵ k + 1                         ⎭    2 assignments
```

Algorithm InsertionSort(L[0..n-1])

Sorts a list using insertion sort.

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Output: A list sorted in ascending order.

```
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while (k < n) do {                              1 comparison
      tmp ⟵ list[k]                            2 assignments
      j ⟵ k-1
      while (j ≧ 0 and tmp < list[j]) do {     2 comparisons
           list[j+1] ⟵ list[j]                 2 assignments
           j ⟵ j – 1
      }
      list[j+1] ⟵ tmp                          2 assignments
      k ⟵ k + 1
```

At most
k times
k < n

n-1 times

Algorithm InsertionSort(L[0..n-1])

Sorts a list using insertion sort.

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Output: A list sorted in ascending order.

```
k ⟵ 1                                   1 assignment
while (k < n) do {                       1 comparison
      tmp ⟵ list[k]
      j ⟵ k-1                            2 assignments
      while (j ≧ 0 and tmp < list[j]) do {     2 comparisons
           list[j+1] ⟵ list[j]
           j ⟵ j – 1                     2 assignments
      }
      list[j+1] ⟵ tmp                    2 assignments
      k ⟵ k + 1
```

At most k times $k < n$

n-1 times

Running time does not depend on **n** only

Algorithm InsertionSort(L[0..n-1])

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      j ⟵ k-1
      while (j ≧ 0 and tmp < list[j]) do {
         list[j+1] ⟵ list[j]
         j ⟵ j – 1
      }

      list[j+1] ⟵ tmp
      k ⟵ k + 1
```

1 assignment

1 comparison

2 assignments

2 comparisons

2 assignments

2 assignments

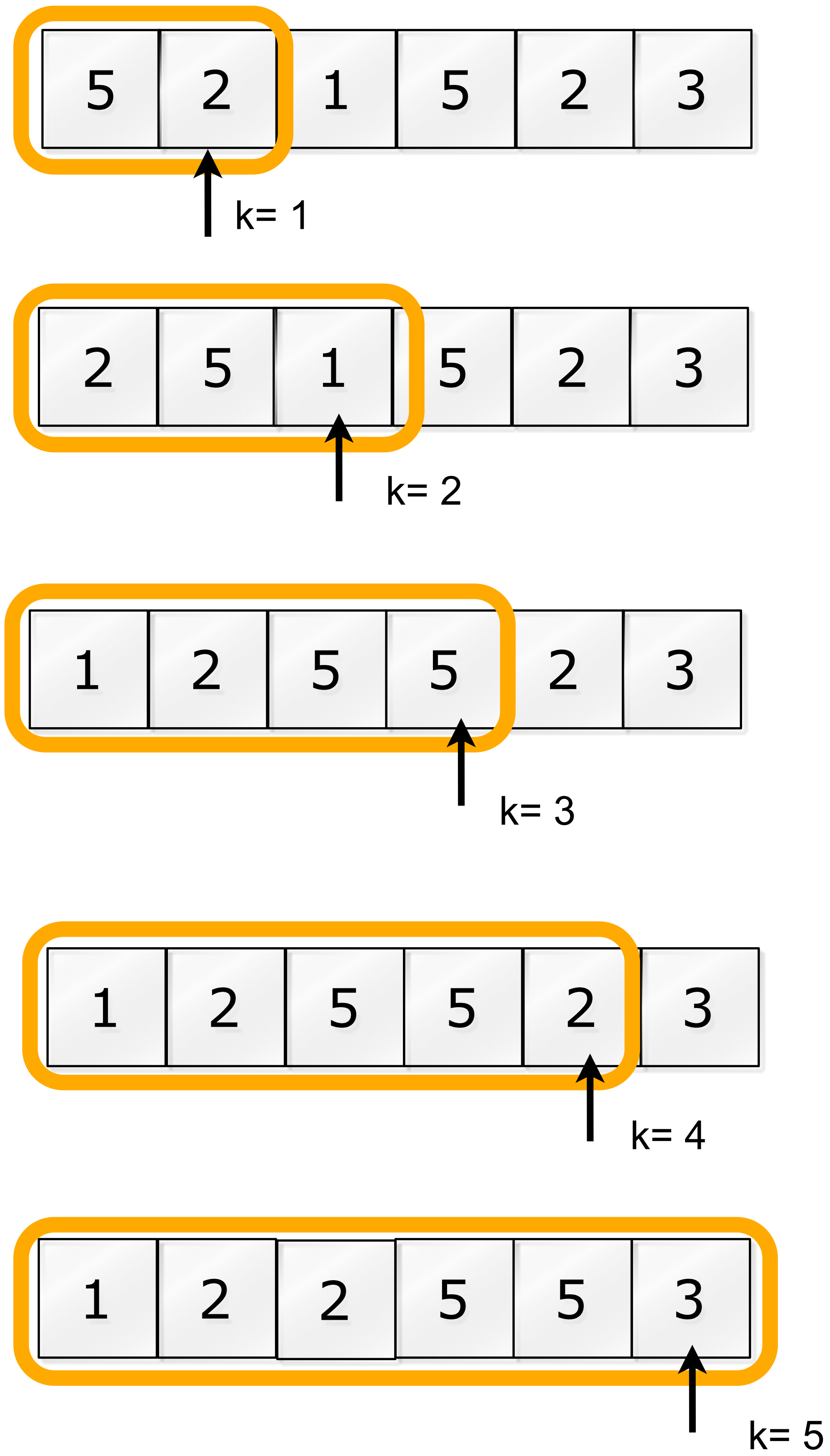
At most k times k < n

n-1 times

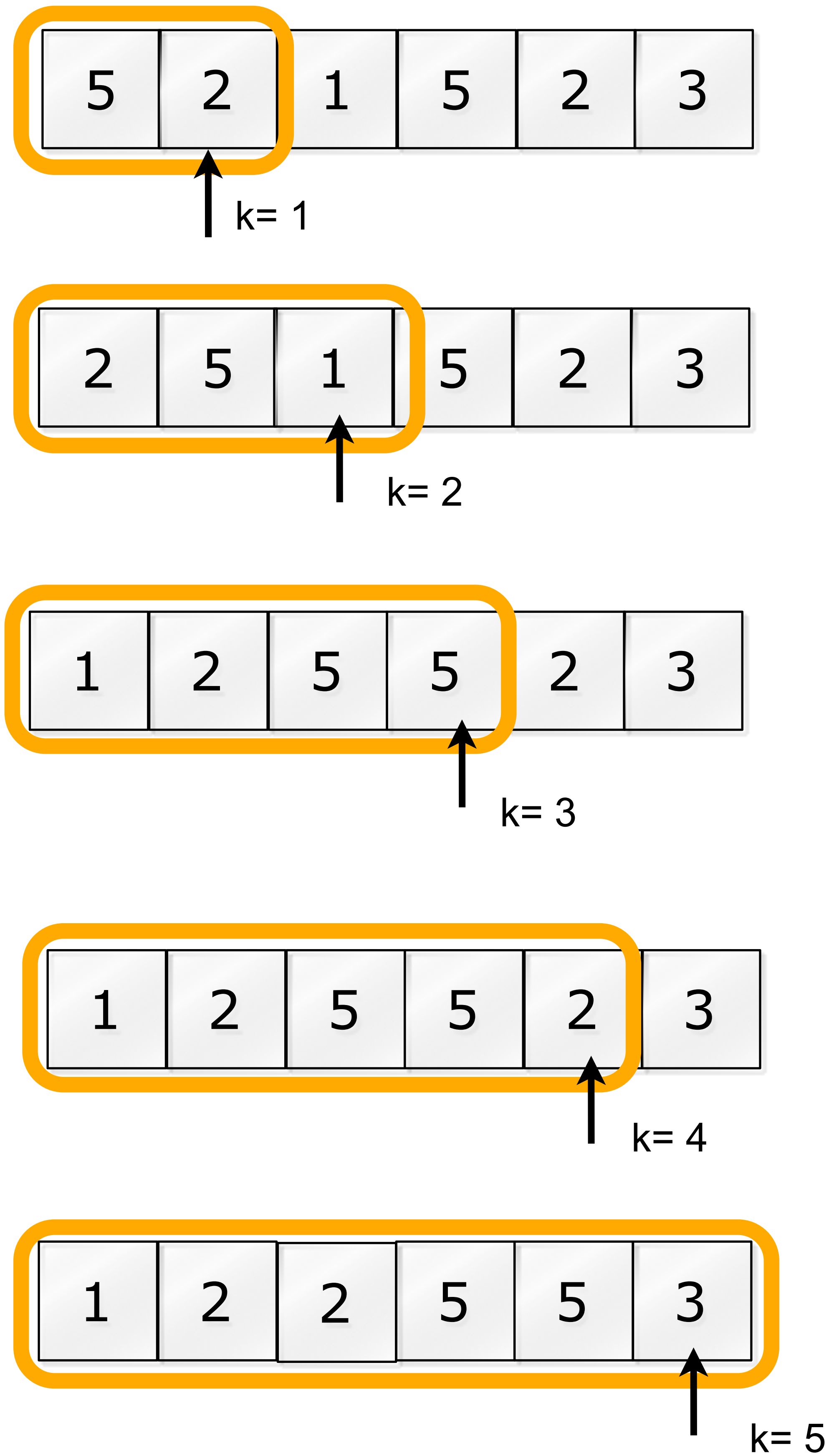
Running time does not depend on **n** only

Best and Worst Case

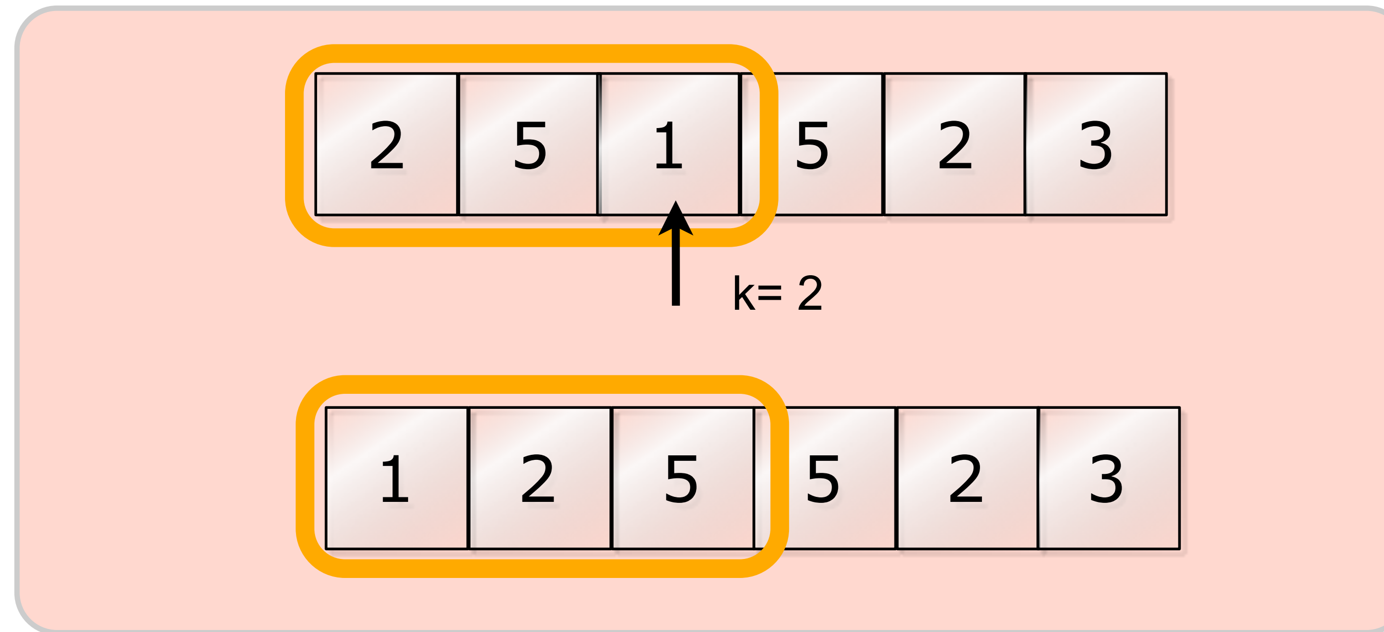
Loop 1



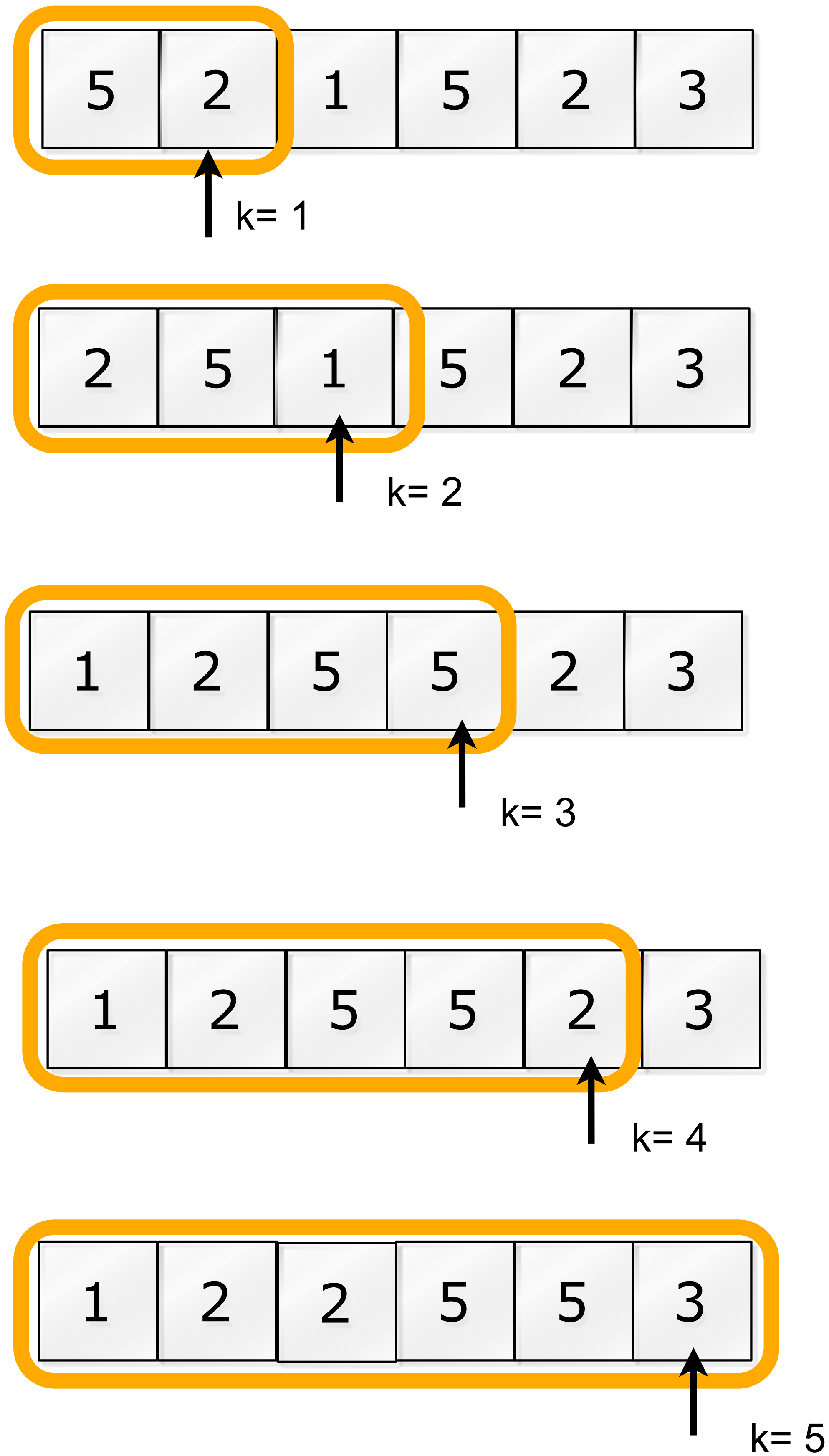
Loop 1


5 2 1 5 2 3
k= 1
2 5 1 5 2 3
k= 2
1 2 5 5 2 3
k= 3
1 2 5 5 2 3
k= 4
1 2 2 5 5 3
k= 5

Loop 2

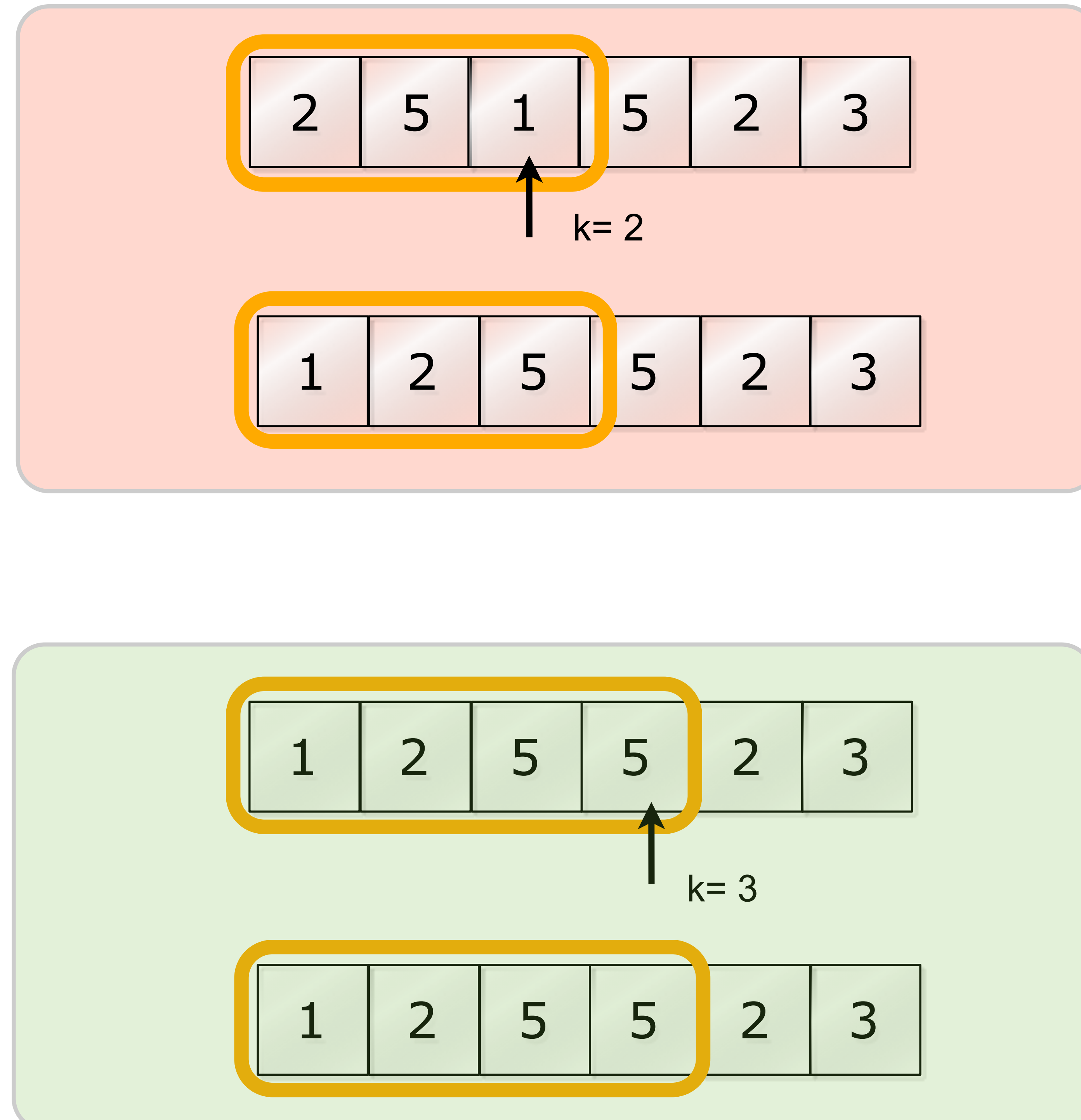

2 5 1 5 2 3
k= 2
1 2 5 5 2 3

Loop 1



5 2 1 5 2 3
k= 1
2 5 1 5 2 3
k= 2
1 2 5 5 2 3
k= 3
1 2 5 5 2 3
k= 4
1 2 2 5 5 3
k= 5

Loop 2



2 5 1 5 2 3
k= 2
1 2 5 5 2 3
1 2 5 5 2 3
k= 3
1 2 5 5 2 3

Insertion Sort: Time complexity

```
k ⟵ 1
while (k < n) do {
        tmp ⟵ list[k]
        j ⟵ k-1
        while (j ≧ 0 and tmp < list[j]) do {
            list[j+1] ⟵ list[j]
            j ⟵ j – 1
        }

        list[j+1] ⟵ tmp
        k ⟵ k + 1
```

Insertion Sort: Time complexity

```
k ⟵ 1
while (k < n) do {
        tmp ⟵ list[k]
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```

- Can we stop any of the two loops early?

Insertion Sort: Time complexity

```
k ⟵ 1
while (k < n) do {
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      while (j ≧ 0 and tmp < list[j]) do {
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      }

      list[j+1] ⟵ tmp
      k ⟵ k + 1
```

- Can we stop any of the two loops early?
  - Yes, the second one, when tmp ≧ list[j]

Insertion Sort: Time complexity

```
k ⟵ 1
while (k < n) do {
      tmp ⟵ list[k]
      j ⟵ k-1
      while (j ≧ 0 and tmp < list[j]) do {
         list[j+1] ⟵ list[j]
         j ⟵ j – 1
      }

      list[j+1] ⟵ tmp
      k ⟵ k + 1
```

- Can we stop any of the two loops early?
  - Yes, the second one, when tmp ≧ list[j]
  - Best and worst cases are going to be different

Insertion Sort: Time complexity

```
k ⟵ 1
while (k < n) do {
    tmp ⟵ list[k]
    j ⟵ k-1
    while (j ≧ 0 and tmp < list[j]) do {
        list[j+1] ⟵ list[j]
        j ⟵ j – 1
    }

    list[j+1] ⟵ tmp
    k ⟵ k + 1
```

- Can we stop any of the two loops early?
  - Yes, the second one, when tmp ≧ list[j]
  - Best and worst cases are going to be different
  - The average case lies in between them.

Insertion Sort: Time complexity

```
k ⟵ 1
while (k < n) do {
        tmp ⟵ list[k]
        j ⟵ k-1
        while (j ≧ 0 and tmp < list[j]) do {
            list[j+1] ⟵ list[j]
            j ⟵ j – 1
        }

        list[j+1] ⟵ tmp
        k ⟵ k + 1
```

- Can we stop any of the two loops early?
    - Yes, the second one, when tmp ≧ list[j]
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Insertion Sort: Time complexity

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      while (j ≧ 0 and tmp < list[j]) do {
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         j ⟵ j – 1
      }

      list[j+1] ⟵ tmp
      k ⟵ k + 1
```

- Can we stop any of the two loops early?
  - Yes, the second one, when tmp ≧ list[j]
  - Best and worst cases are going to be different
  - The average case lies in between them.
- Best case?

Insertion Sort: Time complexity

```
k ⟵ 1
while (k < n) do {
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        j ⟵ k-1
        while (j ≧ 0 and tmp < list[j]) do {
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            j ⟵ j – 1
        }

        list[j+1] ⟵ tmp
        k ⟵ k + 1
```

- Can we stop any of the two loops early?
  - Yes, the second one, when tmp ≧ list[j]
  - Best and worst cases are going to be different
  - The average case lies in between them.
- Best case?
  - [1, 2, 3, 4]

Insertion Sort: Time complexity

```
k ⟵ 1
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    tmp ⟵ list[k]
    j ⟵ k-1
    while (j ≧ 0 and tmp < list[j]) do {
        list[j+1] ⟵ list[j]
        j ⟵ j – 1
    }

    list[j+1] ⟵ tmp
    k ⟵ k + 1
```

- Can we stop any of the two loops early?
  - Yes, the second one, when tmp ≧ list[j]
  - Best and worst cases are going to be different
  - The average case lies in between them.
- Best case?
  - [1, 2, 3, 4]
- Worst case?

Insertion Sort: Time complexity

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```

- Can we stop any of the two loops early?
  - Yes, the second one, when tmp ≧ list[j]
  - Best and worst cases are going to be different
  - The average case lies in between them.
- Best case?
  - [1, 2, 3, 4]
- Worst case?
  - [4, 3, 2, 1]

Best case

```
k ⟵ 1                                        1
while (k < n) do {                           1
      tmp ⟵ list[k]
                                             2
      j ⟵ k-1
      while (j ≧ 0 and tmp < list[j]) do {   2
         list[j+1] ⟵ list[j]                 k times max     n-1 times
         j ⟵ j – 1
      }
                                             2
      list[j+1] ⟵ tmp
      k ⟵ k + 1
```

Best case

```
k ⟵ 1  1
while (k < n) do {    1
    tmp ⟵ list[k]
                        2
    j ⟵ k-1
                                                2
    while (j ≧ 0 and tmp < list[j]) do {
        list[j+1] ⟵ list[j]        2
        j ⟵ j – 1
    }
                            2
    list[j+1] ⟵ tmp
    k ⟵ k + 1
```

k
times
max

n-1
times

Best case

```
k ⟵ 1                                   1
while (k < n) do {                      1
    tmp ⟵ list[k]                       2
    j ⟵ k-1
    while (j ≧ 0 and tmp < list[j]) do {        2
        list[j+1] ⟵ list[j]            2
        j ⟵ j – 1
    }
    list[j+1] ⟵ tmp                     2
    k ⟵ k + 1
```

k times max

n-1 times

Best case

```
k ⟵ 1                                        1
while (k < n) do {                           1
    tmp ⟵ list[k]
    j ⟵ k-1                                  2
    while (j ≧ 0 and tmp < list[j]) do {     2
        list[j+1] ⟵ list[j]                  2
        j ⟵ j – 1
    }

    list[j+1] ⟵ tmp                          2
    k ⟵ k + 1
```

n-1 times

Best case

```
k ⟵ 1   1
while (k < n) do {   1
      tmp ⟵ list[k]
                       2
      j ⟵ k-1
      while (j ≧ 0 and tmp < list[j]) do {   2
         list[j+1] ⟵ list[j]   2
         j ⟵ j – 1
      }
                       2
      list[j+1] ⟵ tmp
      k ⟵ k + 1
```

n-1
times

Best case

```
k ⟵ 1   1 - - - - - - - - - - - - - - - - - - - - -> 1
while (k < n) do {   1
    tmp ⟵ list[k]
                        2
    j ⟵ k-1
    while (j ≧ 0 and tmp < list[j]) do {   2
        list[j+1] ⟵ list[j]   2
        j ⟵ j – 1
    }
                        2
    list[j+1] ⟵ tmp
    k ⟵ k + 1
```

n-1
times

Best case

```
k ⟵ 1                                                  1
while (k < n) do {                                     1
      tmp ⟵ list[k]
      j ⟵ k-1                                          2
      while (j ≧ 0 and tmp < list[j]) do {             2
            list[j+1] ⟵ list[j]                        2
            j ⟵ j – 1
      }
      list[j+1] ⟵ tmp                                  2
      k ⟵ k + 1
```

1

+1

n-1
times

Best case

```
k ⟵ 1                                   1
while (k < n) do {                      1
      tmp ⟵ list[k]
      j ⟵ k-1                           2
      while (j ≧ 0 and tmp < list[j]) do {     2
         list[j+1] ⟵ list[j]            2
         j ⟵ j – 1
      }
      list[j+1] ⟵ tmp                   2
      k ⟵ k + 1
```

1
+1
+2

n-1
times

Best case

```
k ⟵ 1   1
while (k < n) do {   1
      tmp ⟵ list[k]
                        2
      j ⟵ k-1
      while (j ≧ 0 and tmp < list[j]) do {   2
         list[j+1] ⟵ list[j]
                                2
         j ⟵ j – 1
      }
                              2
      list[j+1] ⟵ tmp
      k ⟵ k + 1
```

1

+1

+2

+2

n-1
times

Best case

```
k ⟵ 1                                          1
while (k < n) do {                             1
    tmp ⟵ list[k]
    j ⟵ k-1                                    2
    while (j ≧ 0 and tmp < list[j]) do {       2
        list[j+1] ⟵ list[j]                    2
        j ⟵ j – 1
    }
    list[j+1] ⟵ tmp                            2
    k ⟵ k + 1
```

1

+1

+2

+2

+2

n-1
times

Best case

```
k ⟵ 1                                   1
while (k < n) do {                      1
      tmp ⟵ list[k]                     2
      j ⟵ k-1
      while (j ≧ 0 and tmp < list[j]) do {   2
         list[j+1] ⟵ list[j]            2
         j ⟵ j – 1
      }
      list[j+1] ⟵ tmp                   2
      k ⟵ k + 1
```

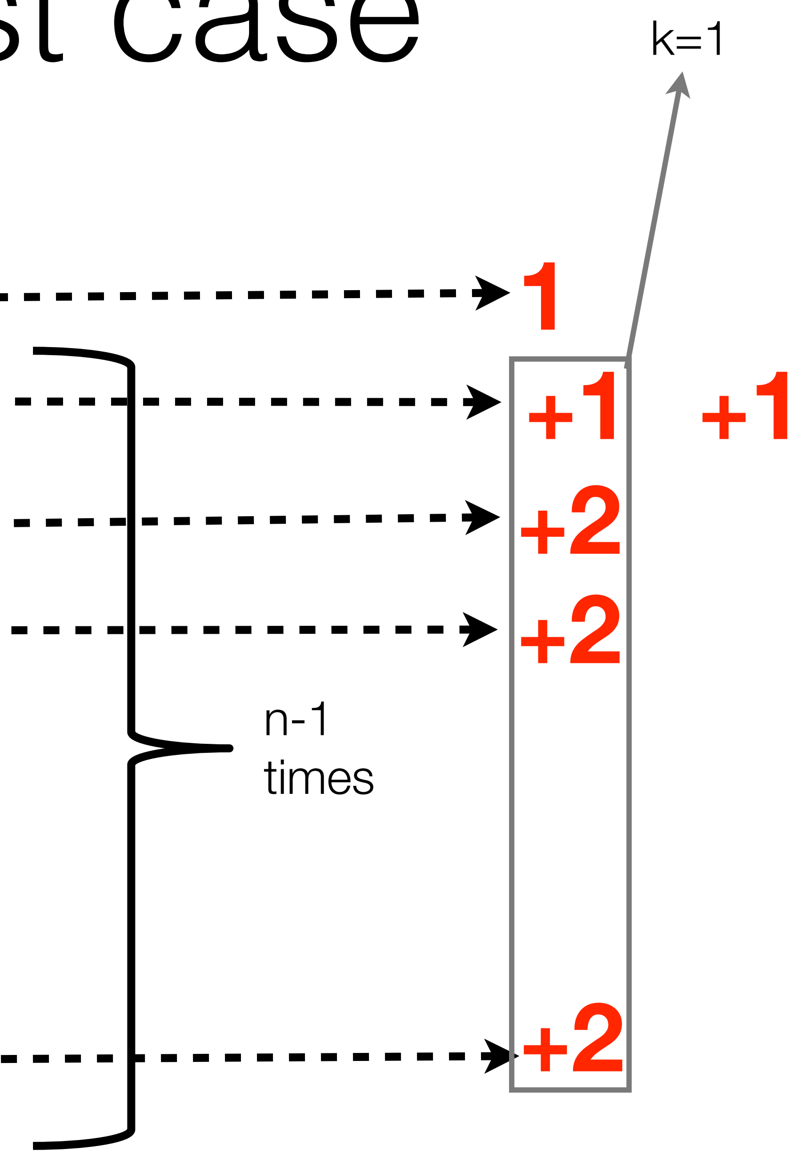
n-1 times

k=1

1
+1
+2
+2
+2

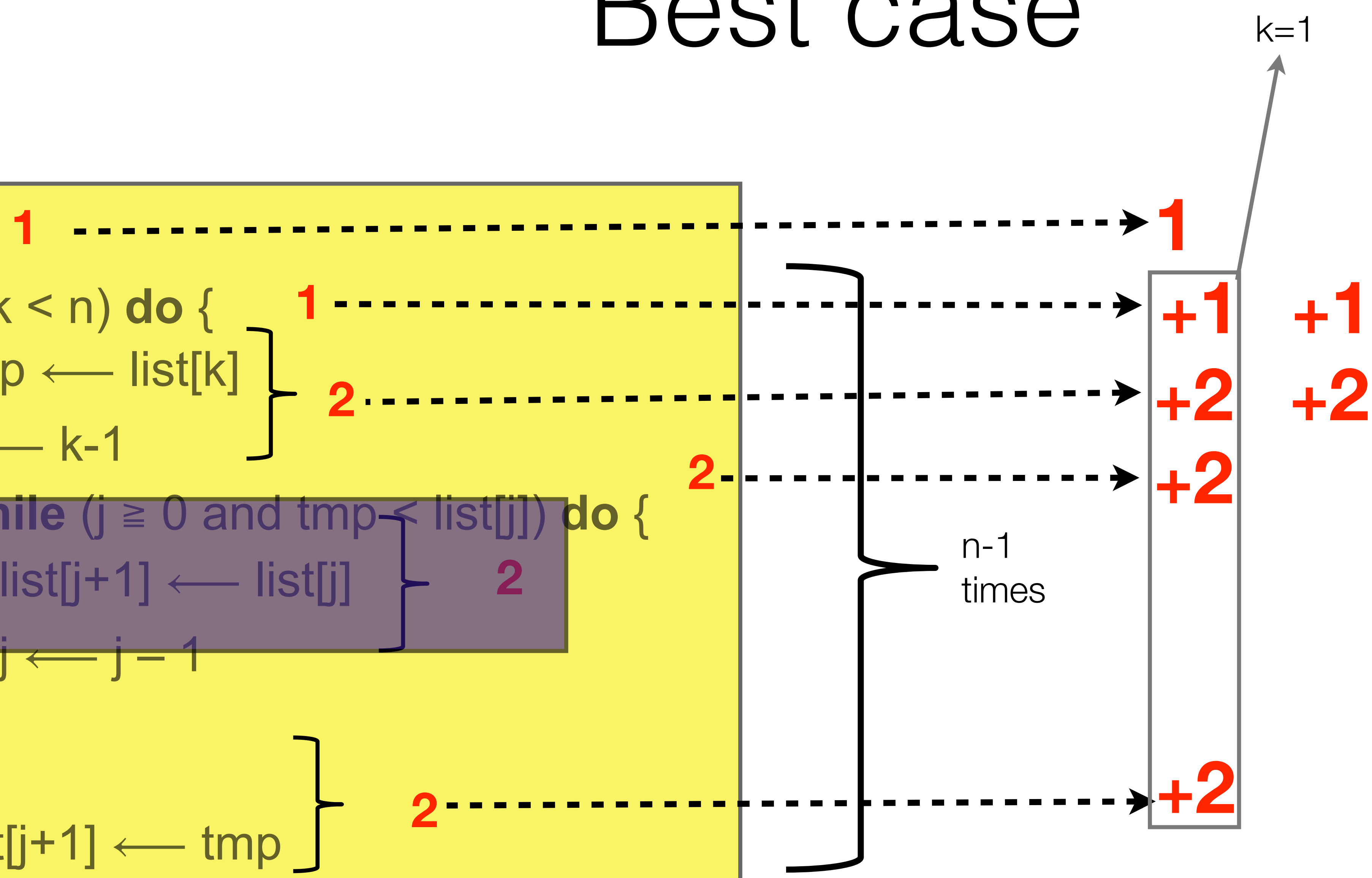
Best case

```
k ⟵ 1  1
while (k < n) do {  1
    tmp ⟵ list[k]
                      2
    j ⟵ k-1
    while (j ≧ 0 and tmp < list[j]) do {  2
        list[j+1] ⟵ list[j]   2
        j ⟵ j – 1
    }

    list[j+1] ⟵ tmp   2
    k ⟵ k + 1
```



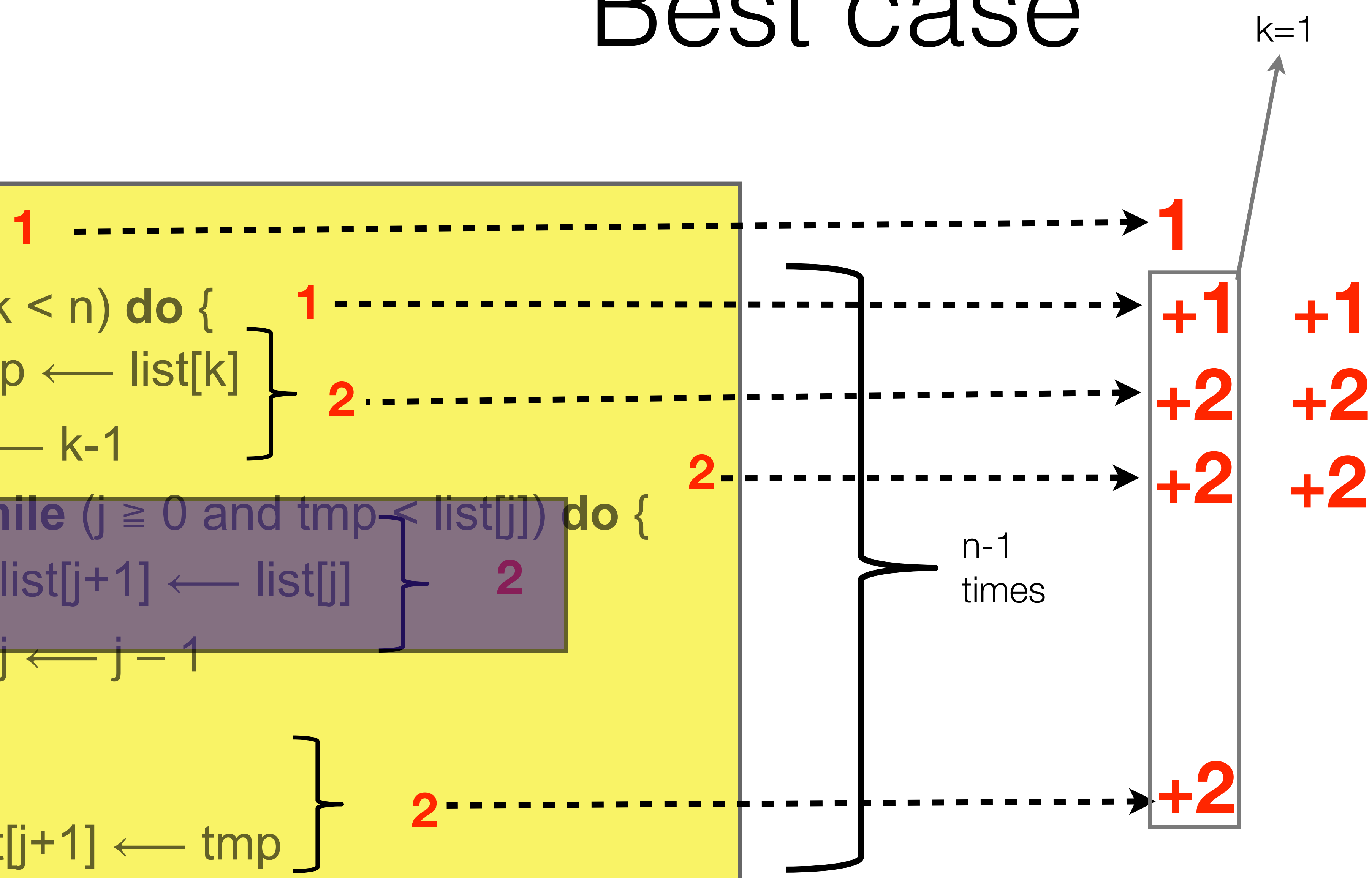
Best case

```
k ⟵ 1
while (k < n) do {
      tmp ⟵ list[k]
      j ⟵ k-1
      while (j ≧ 0 and tmp < list[j]) do {
            list[j+1] ⟵ list[j]
            j ⟵ j – 1
      }

      list[j+1] ⟵ tmp
      k ⟵ k + 1
```



Best case

```
k ⟵ 1
while (k < n) do {
      tmp ⟵ list[k]
      j ⟵ k-1
      while (j ≧ 0 and tmp < list[j]) do {
         list[j+1] ⟵ list[j]
         j ⟵ j – 1
      }

      list[j+1] ⟵ tmp
      k ⟵ k + 1
```



Best case

```
k ⟵ 1                                        1
while (k < n) do {                           1
    tmp ⟵ list[k]
    j ⟵ k-1                                  2
    while (j ≧ 0 and tmp < list[j]) do {     2
        list[j+1] ⟵ list[j]
        j ⟵ j – 1                            2
    }
    list[j+1] ⟵ tmp                          2
    k ⟵ k + 1
```

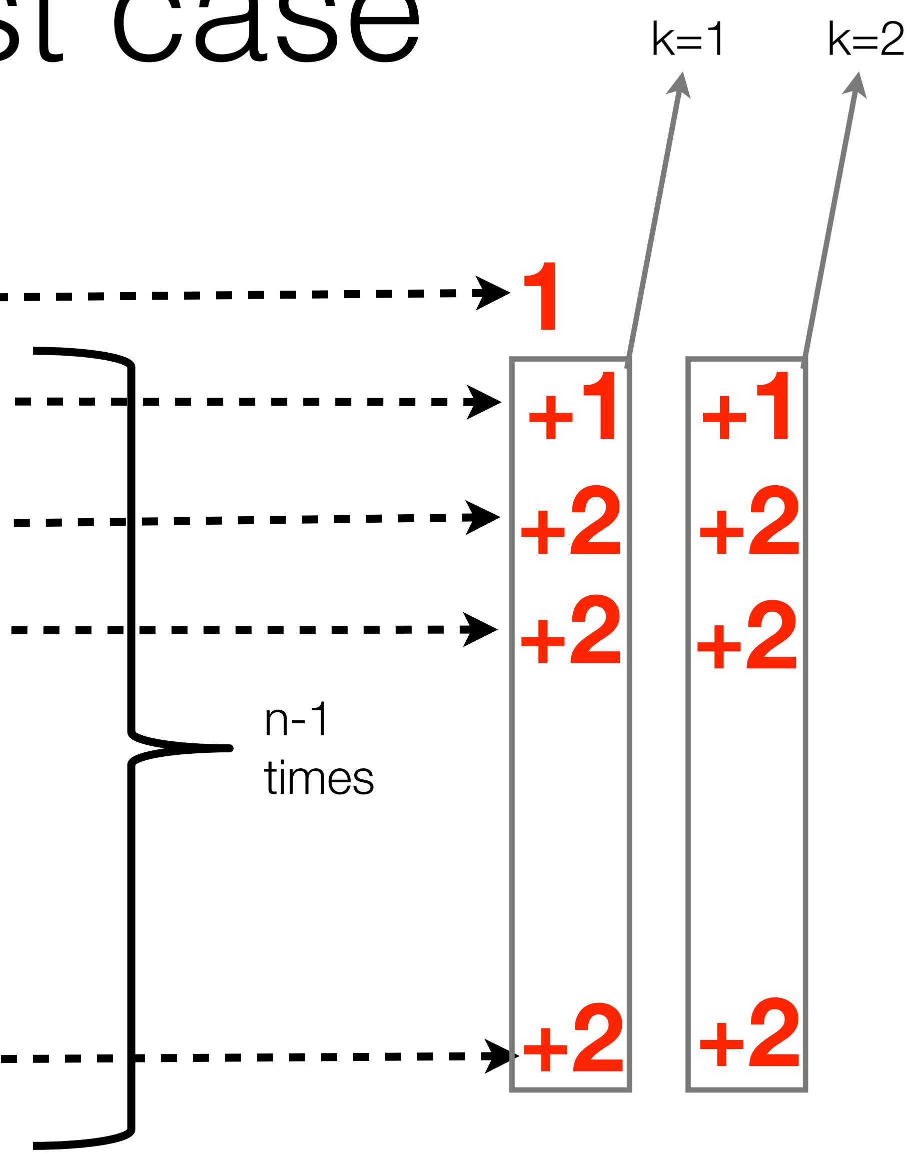
n-1
times

k=1

1	
+1	+1
+2	+2
+2	+2
+2	+2

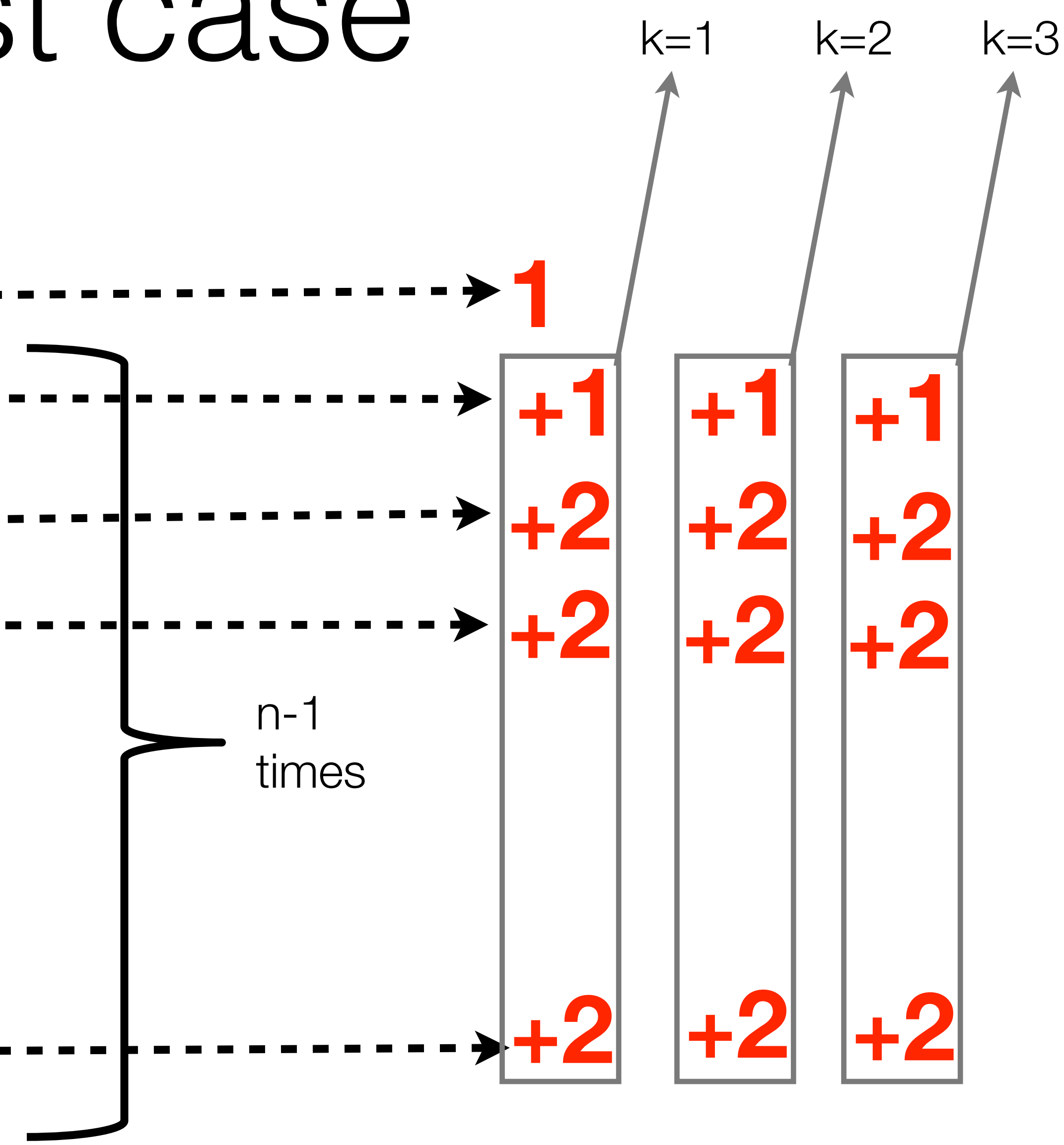
Best case

```
k ⟵ 1   1
while (k < n) do {   1
      tmp ⟵ list[k]
                        2
      j ⟵ k-1
                                2
      while (j ≧ 0 and tmp < list[j]) do {
         list[j+1] ⟵ list[j]       2
         j ⟵ j – 1
      }
                    2
      list[j+1] ⟵ tmp
      k ⟵ k + 1
```



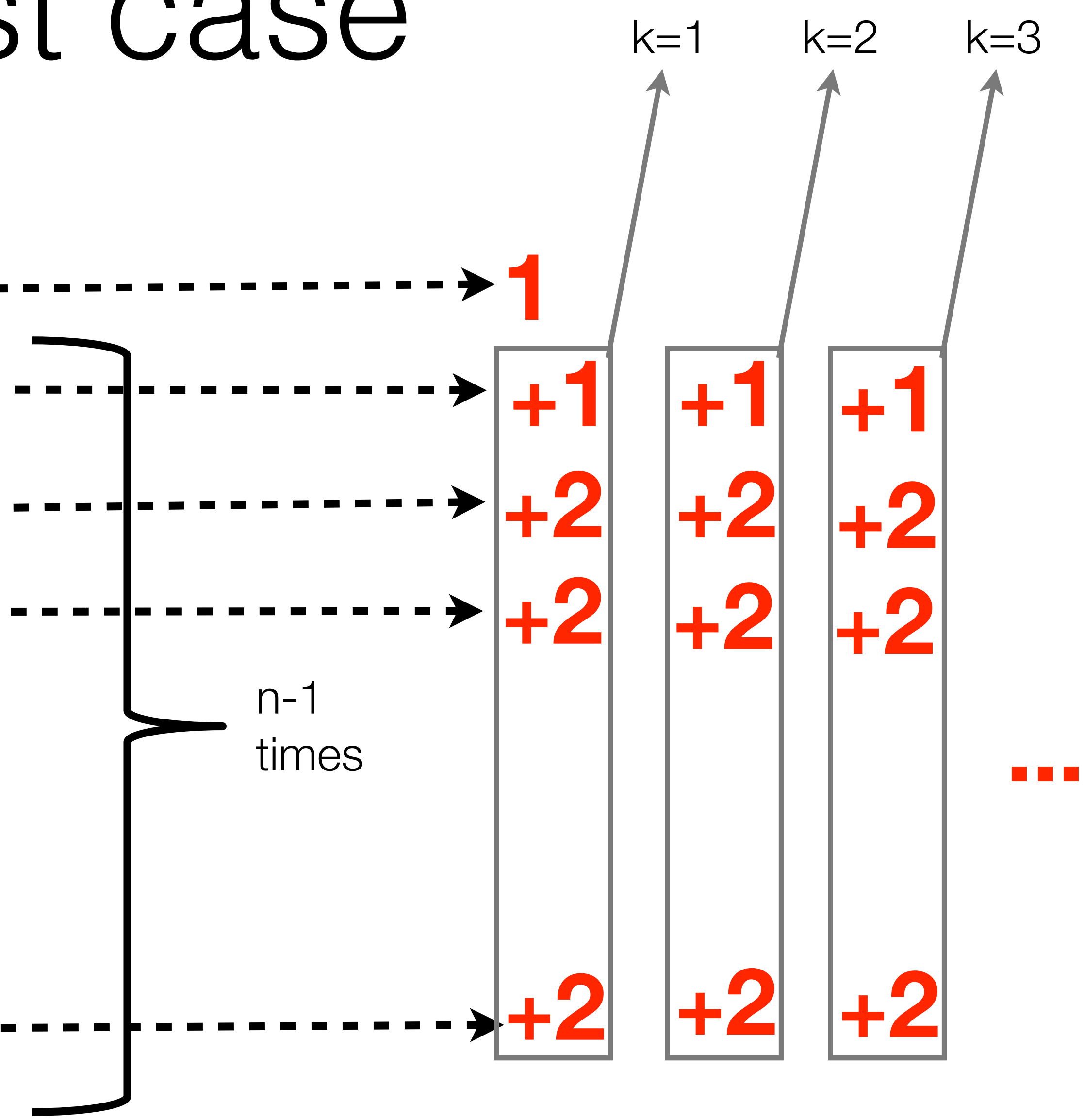
Best case

```
k ⟵ 1                                      1
while (k < n) do {                         1
      tmp ⟵ list[k]
      j ⟵ k-1                              2
      while (j ≧ 0 and tmp < list[j]) do {  2
         list[j+1] ⟵ list[j]
         j ⟵ j – 1                          2
      }
      list[j+1] ⟵ tmp                       2
      k ⟵ k + 1
```



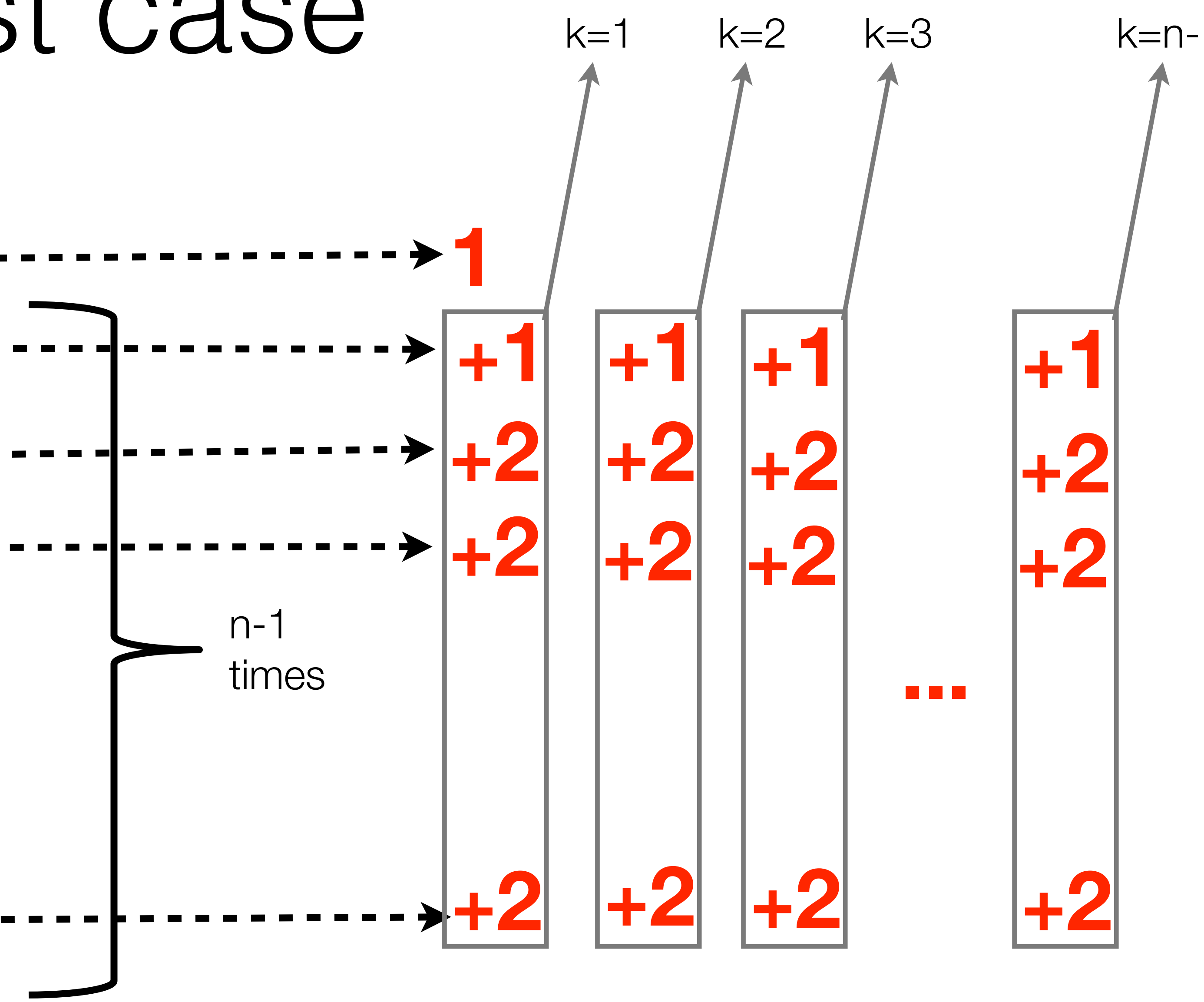
Best case

```
k ⟵ 1   1
while (k < n) do {   1
      tmp ⟵ list[k]
      j ⟵ k-1   2
      while (j ≧ 0 and tmp < list[j]) do {   2
         list[j+1] ⟵ list[j]
         j ⟵ j – 1   2
      }
      list[j+1] ⟵ tmp   2
      k ⟵ k + 1
```



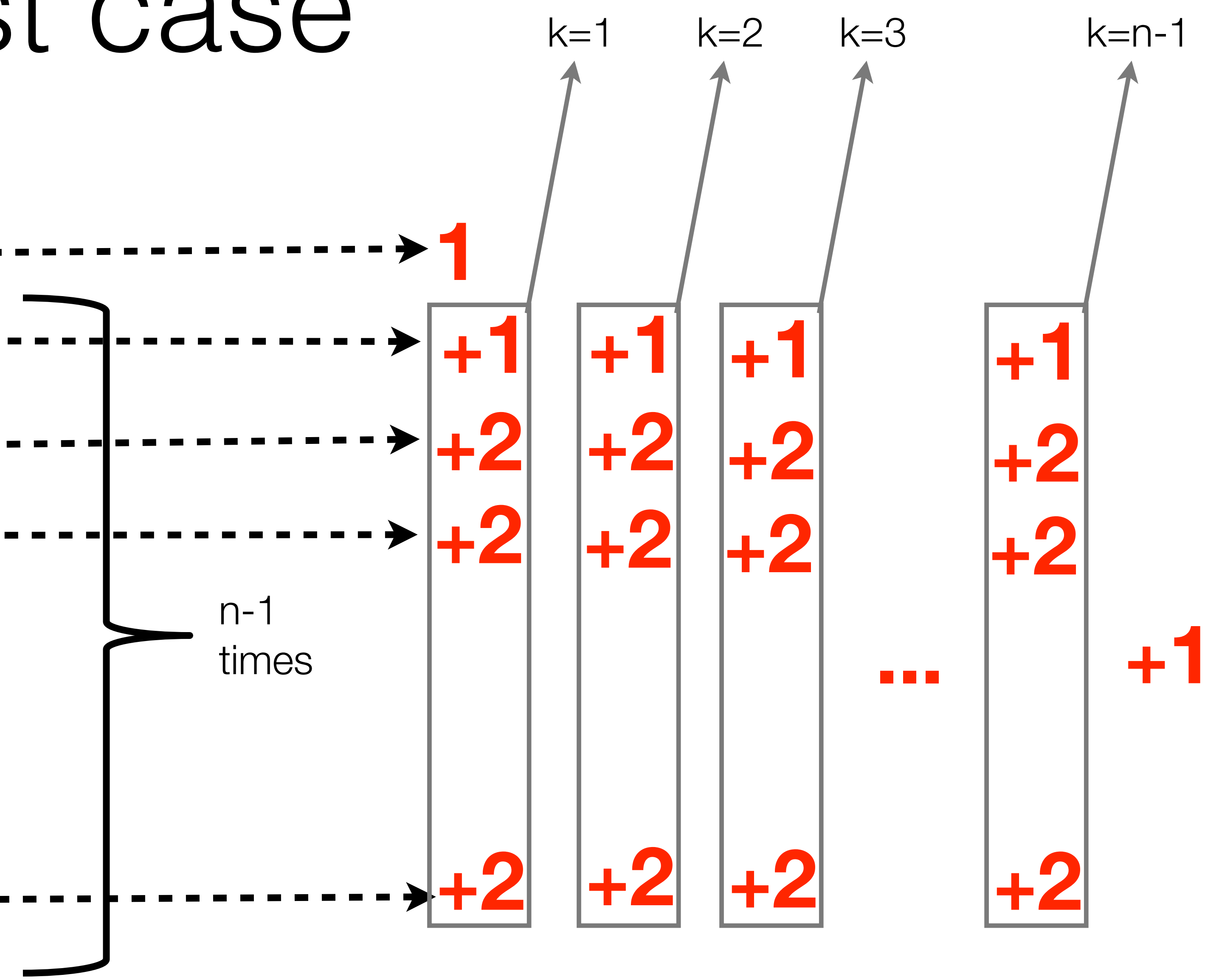
Best case

```
k ⟵ 1                                         1
while (k < n) do {                            1
      tmp ⟵ list[k]                            2
      j ⟵ k-1
      while (j ≧ 0 and tmp < list[j]) do {     2
         list[j+1] ⟵ list[j]                   2
         j ⟵ j – 1
      }
      list[j+1] ⟵ tmp                          2
      k ⟵ k + 1
```

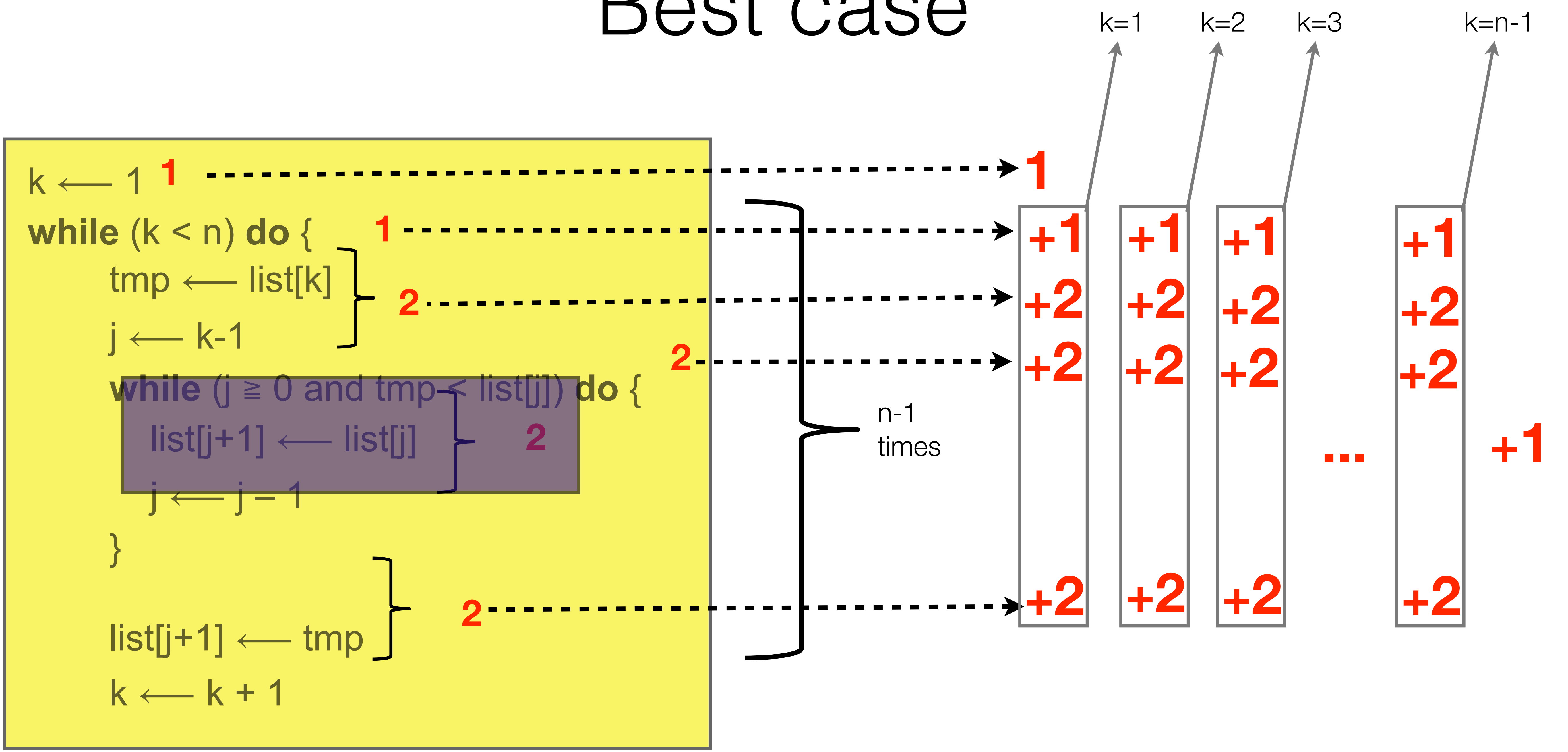


Best case

```
k ⟵ 1                                   1
while (k < n) do {                      1
      tmp ⟵ list[k]                     2
      j ⟵ k-1
      while (j ≧ 0 and tmp < list[j]) do {   2
         list[j+1] ⟵ list[j]            2
         j ⟵ j – 1
      }
      list[j+1] ⟵ tmp                   2
      k ⟵ k + 1
```



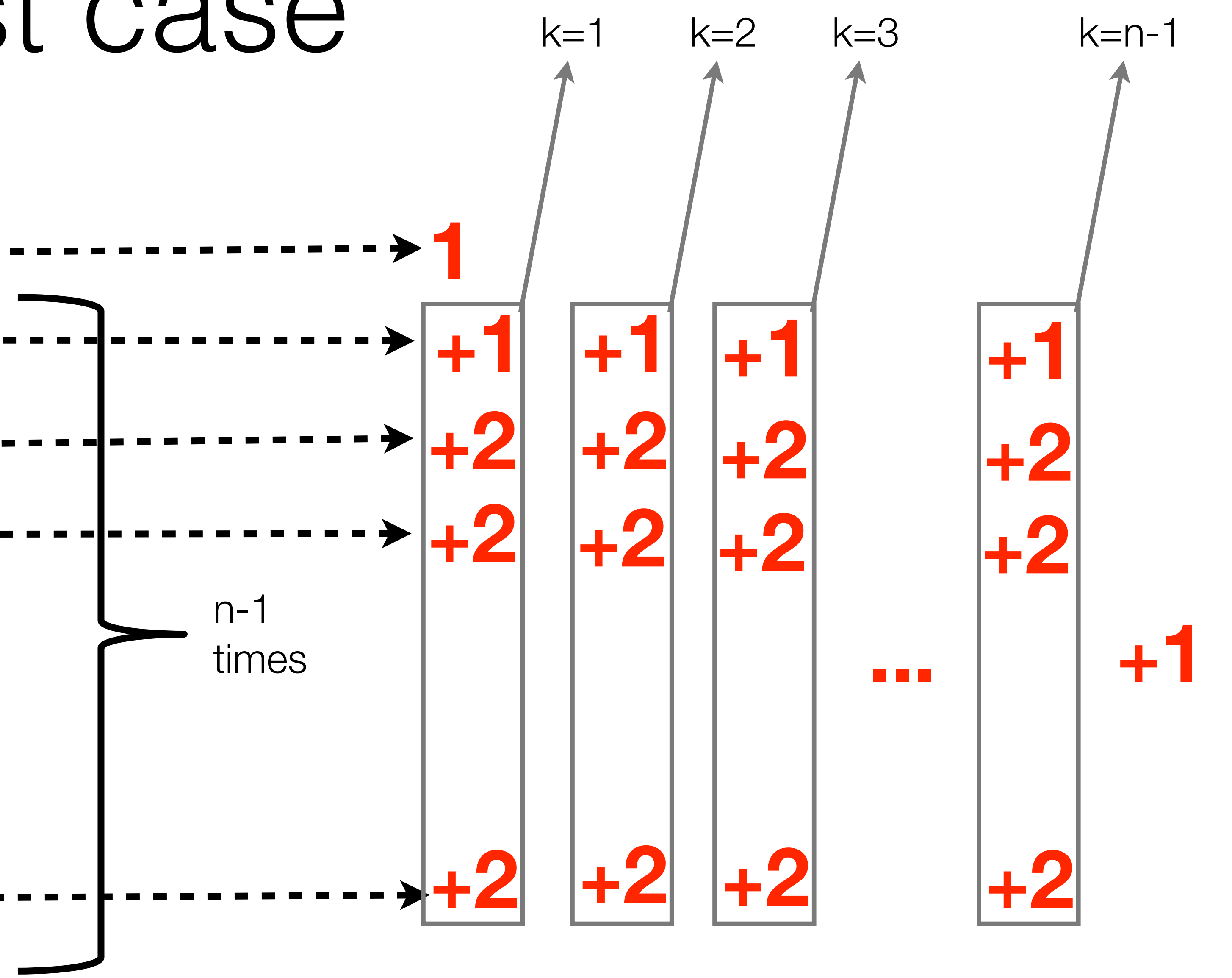
Best case



$1 + 7(n-1) + 1$

Best case

```
k ⟵ 1                                   1
while (k < n) do {                      1
      tmp ⟵ list[k]
      j ⟵ k-1                           2
      while (j ≧ 0 and tmp < list[j]) do {   2
         list[j+1] ⟵ list[j]
         j ⟵ j – 1                      2
      }
      list[j+1] ⟵ tmp                   2
      k ⟵ k + 1
```



$$1 + 7(n-1) + 1$$

$$\boxed{7n - 5}$$

Worst case

```
k ⟵ 1                                          1
while (k < n) do {                             1
        tmp ⟵ list[k]
                                               2
        j ⟵ k-1
        while (j ≧ 0 and tmp < list[j]) do {   4      k         n-1
            list[j+1] ⟵ list[j]                       times     times
            j ⟵ j – 1                                 max
        }
                                               2
        list[j+1] ⟵ tmp
        k ⟵ k + 1
```

Worst case

1

```
k ⟵ 1                                        1
while (k < n) do {                           1
      tmp ⟵ list[k]
      j ⟵ k-1                                2
      while (j ≧ 0 and tmp < list[j]) do {   4
         list[j+1] ⟵ list[j]
         j ⟵ j – 1
      }

      list[j+1] ⟵ tmp                        2
      k ⟵ k + 1
```

k times max

n-1 times

Worst case

```
k ⟵ 1  1
while (k < n) do {  1
    tmp ⟵ list[k]
    j ⟵ k-1  2
    while (j ≧ 0 and tmp < list[j]) do {  4
        list[j+1] ⟵ list[j]
        j ⟵ j – 1
    }

    list[j+1] ⟵ tmp  2
    k ⟵ k + 1
```

1

+1

k times max

n-1 times

Worst case

1

+1+2

```
k ⟵ 1   1
while (k < n) do {   1
        tmp ⟵ list[k]
                        2
        j ⟵ k-1
        while (j ≧ 0 and tmp < list[j]) do {
                                              4
            list[j+1] ⟵ list[j]
            j ⟵ j – 1
        }
                            2
        list[j+1] ⟵ tmp
        k ⟵ k + 1
```

k
times
max

n-1
times

Worst case

1

+1+2 +4

```
k ⟵ 1                                          1
while (k < n) do {                              1
    tmp ⟵ list[k]                               2
    j ⟵ k-1
    while (j ≧ 0 and tmp < list[j]) do {        4
        list[j+1] ⟵ list[j]
        j ⟵ j – 1
    }
                                                2
    list[j+1] ⟵ tmp
    k ⟵ k + 1
```

k times max

n-1 times

Worst case

```
k ⟵ 1                                     1
while (k < n) do {                        1
      tmp ⟵ list[k]
                                          2
      j ⟵ k-1
      while (j ≧ 0 and tmp < list[j]) do {
                                          4
         list[j+1] ⟵ list[j]
         j ⟵ j – 1
      }
                                          2
      list[j+1] ⟵ tmp
      k ⟵ k + 1
```

k times max

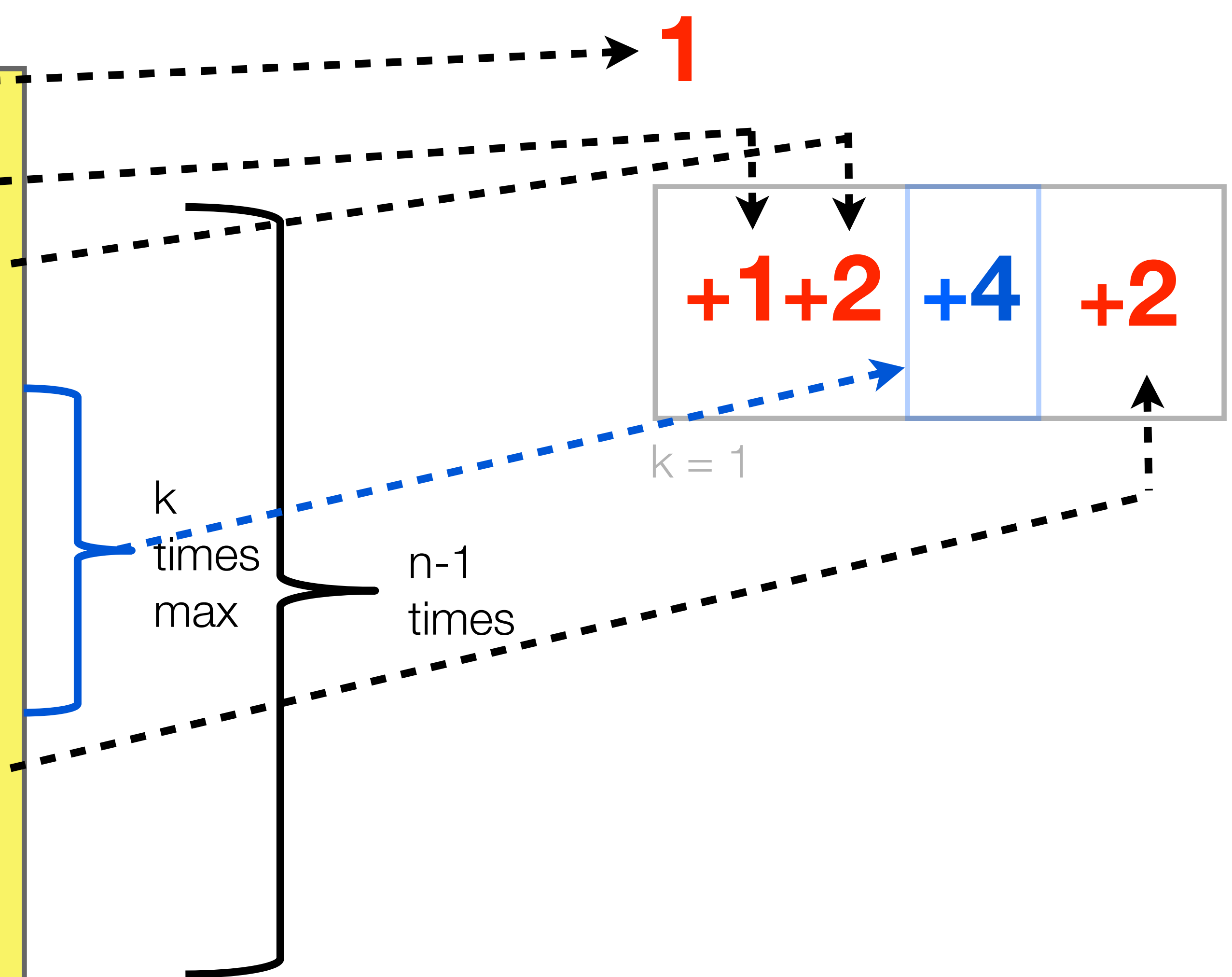
n-1 times

1

+1+2 +4 +2

Worst case

```
k ⟵ 1
while (k < n) do {
    tmp ⟵ list[k]
    j ⟵ k-1
    while (j ≧ 0 and tmp < list[j]) do {
        list[j+1] ⟵ list[j]
        j ⟵ j – 1
    }

    list[j+1] ⟵ tmp
    k ⟵ k + 1
```



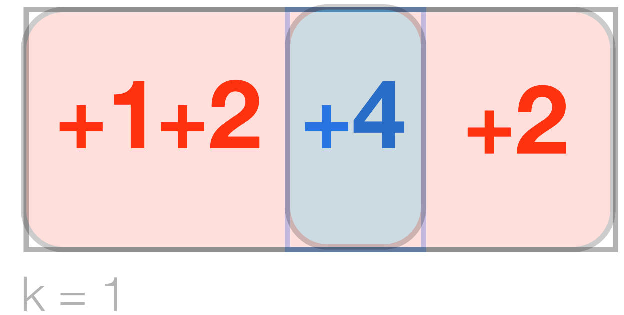
Worst case

1

```
k ⟵ 1                                   1
while (k < n) do {                      1
      tmp ⟵ list[k]                     2
      j ⟵ k-1
      while (j ≧ 0 and tmp < list[j]) do {    4
            list[j+1] ⟵ list[j]
            j ⟵ j – 1
      }
                                          2
      list[j+1] ⟵ tmp
      k ⟵ k + 1
```

k times max

n-1 times



Worst case

1

```
k ⟵ 1                                       1
while (k < n) do {                          1
      tmp ⟵ list[k]                         2
      j ⟵ k-1
      while (j ≧ 0 and tmp < list[j]) do {   4
          list[j+1] ⟵ list[j]
          j ⟵ j – 1
      }
      list[j+1] ⟵ tmp                       2
      k ⟵ k + 1
```

k
times
max

n-1
times

+1+2	+4	+2

k = 1

+1+2	+ 2x4	+2

k = 2

Worst case

```
k ⟵ 1   1
while (k < n) do {   1
      tmp ⟵ list[k]
      j ⟵ k-1        2
      while (j ≧ 0 and tmp < list[j]) do {   4
          list[j+1] ⟵ list[j]
          j ⟵ j – 1
      }
                                2
      list[j+1] ⟵ tmp
      k ⟵ k + 1
```

k times max

n-1 times

1

+1+2	**+4**	**+2**

k = 1

+1+2	**+ 2x4**	**+2**

k = 2

Worst case

```
k ⟵ 1                                           1
while (k < n) do {                              1
      tmp ⟵ list[k]                             
      j ⟵ k-1                                   2
      while (j ≧ 0 and tmp < list[j]) do {      4
         list[j+1] ⟵ list[j]
         j ⟵ j – 1
      }
                                                2
      list[j+1] ⟵ tmp
      k ⟵ k + 1
```

k times max

n-1 times

1

+1+2	+4	+2

k = 1

+1+2	+ 2x4	+2

k = 2

Worst case

```
k ⟵ 1                                          1
while (k < n) do {                             1
      tmp ⟵ list[k]
      j ⟵ k-1                                  2
      while (j ≧ 0 and tmp < list[j]) do {     4
          list[j+1] ⟵ list[j]
          j ⟵ j – 1
      }
      list[j+1] ⟵ tmp                          2
      k ⟵ k + 1
```

k times max

n-1 times

1

+1+2	+4	+2

k = 1

+1+2	+ 2x4	+2

k = 2

+1+2	+ 3x4	+2

k = 3

Worst case

```
k ⟵ 1   1
while (k < n) do {   1
    tmp ⟵ list[k]
    j ⟵ k-1          2
    while (j ≧ 0 and tmp < list[j]) do {   4
        list[j+1] ⟵ list[j]
        j ⟵ j – 1
    }
    list[j+1] ⟵ tmp   2
    k ⟵ k + 1
```

k times max

n-1 times

1

+1+2	+4	+2

k = 1

+1+2	+ 2x4	+2

k = 2

+1+2	+ 3x4	+2

k = 3

⋮

Worst case

```
k ⟵ 1                                   1
while (k < n) do {                      1
    tmp ⟵ list[k]                       2
    j ⟵ k-1
    while (j ≧ 0 and tmp < list[j]) do {    4
        list[j+1] ⟵ list[j]
        j ⟵ j – 1
    }
                                          2
    list[j+1] ⟵ tmp
    k ⟵ k + 1
```

k
times
max

n-1
times

1

+1+2	+4	+2

k = 1

+1+2	+ 2x4	+2

k = 2

+1+2	+ 3x4	+2

k = 3

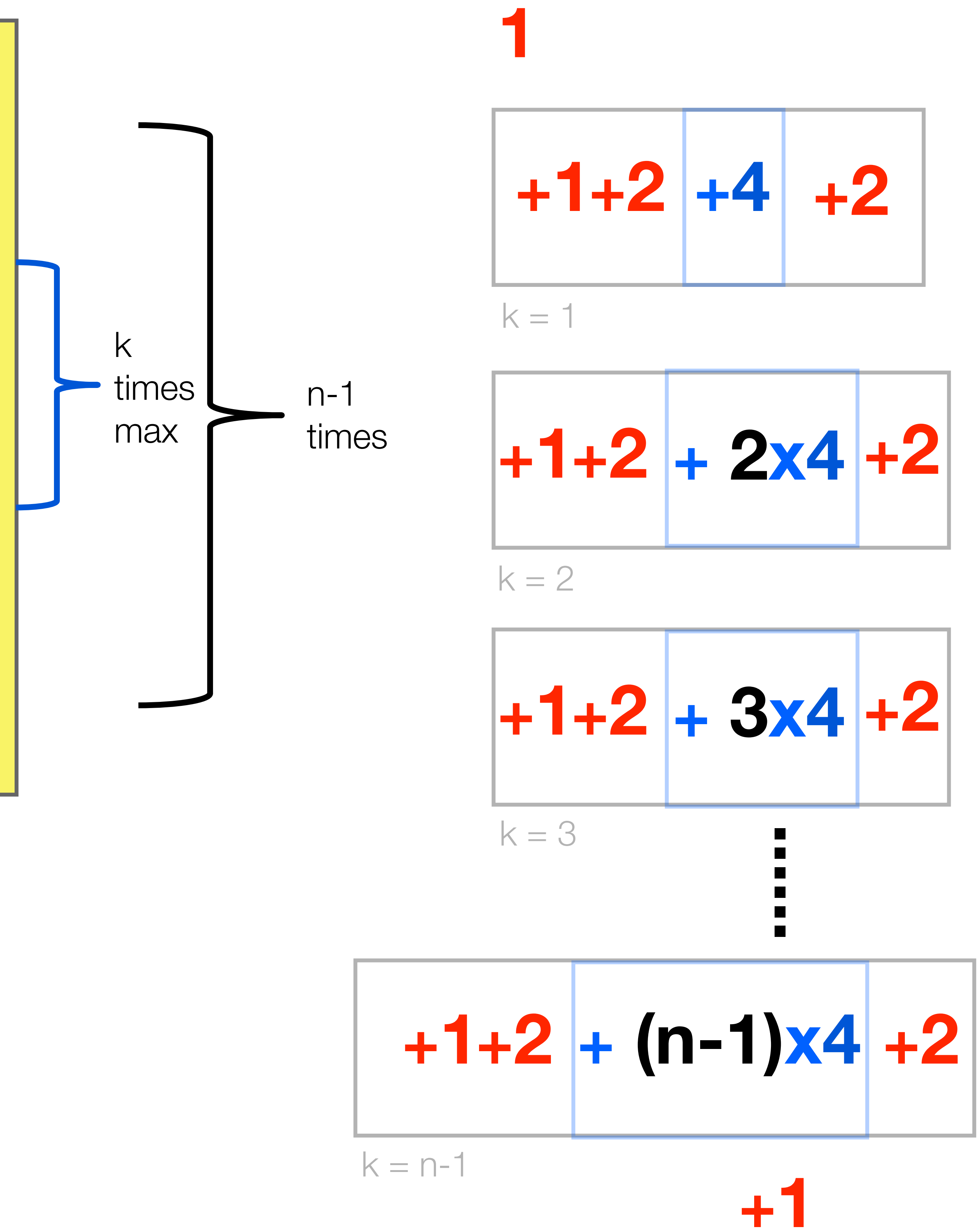
⋮

+1+2	+ (n-1)x4	+2

k = n-1

Worst case

```
k ⟵ 1                                          1
while (k < n) do {                             1
      tmp ⟵ list[k]                            2
      j ⟵ k-1
      while (j ≧ 0 and tmp < list[j]) do {     4
         list[j+1] ⟵ list[j]
         j ⟵ j – 1
      }
                                               2
      list[j+1] ⟵ tmp
      k ⟵ k + 1
```

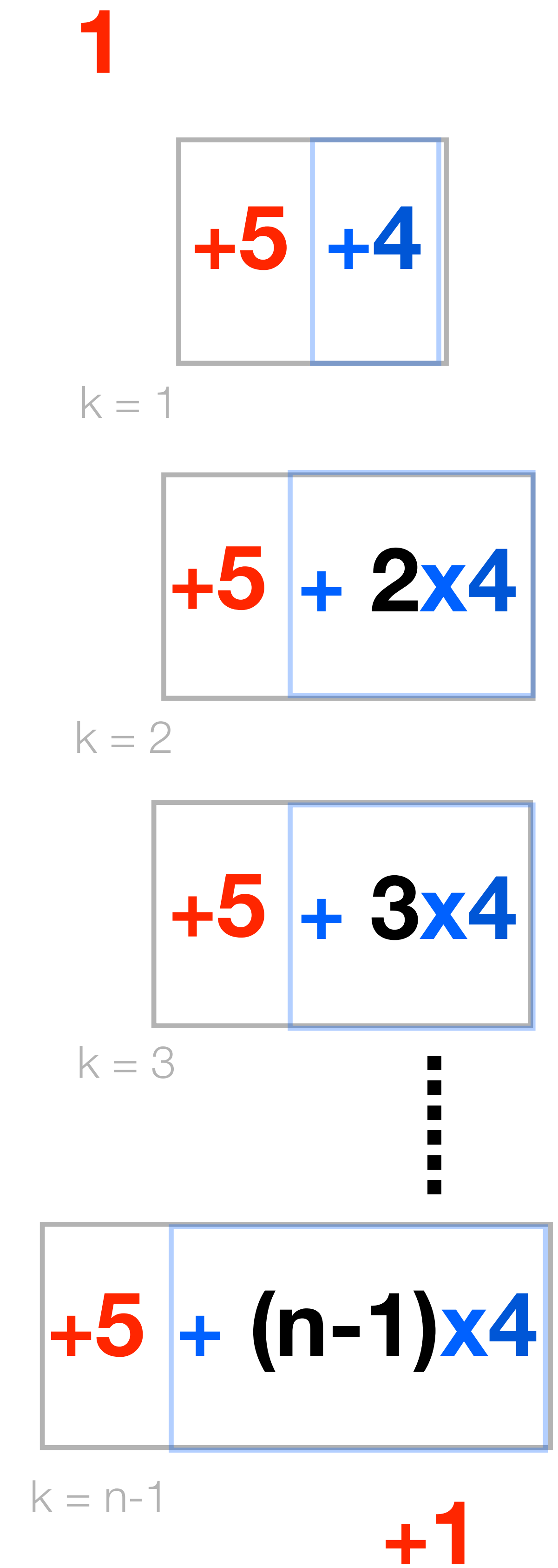


Worst case

```
k ⟵ 1                                        1
while (k < n) do {                           1
      tmp ⟵ list[k]
                                             2
      j ⟵ k-1
      while (j ≧ 0 and tmp < list[j]) do {   4
          list[j+1] ⟵ list[j]
          j ⟵ j – 1
      }
                                             2
      list[j+1] ⟵ tmp
      k ⟵ k + 1
```

k times max

n-1 times



5(n-1) + (1x4 + 2x4 + 3x4+ ... +(n-1)x4) +2

S = 1 + 2 + 3 + 4 + ... + 97 + 98 + 99 + 100

+ S = 100 + 99 + 98 + 97 + ... + 4 + 3 + 2 + 1

2S = 101*100

S = 101*50

S = 5050

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$1 \times 4 + 2 \times 4 + 3 \times 4 + \ldots + (n-1) \times 4 =$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

1x4 + 2x4 + 3x4+ ... +(n-1)x4 =

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

1x4 + 2x4 + 3x4+ ... +(n-1)x4 =

4(1 + 2 + 3 + ... +(n-1))

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

1x4 + 2x4 + 3x4+ ... +(n-1)x4 =

4(1 + 2 + 3 + ... +(n-1))

$$4\,\frac{(n-1(n))}{2}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$1x4 + 2x4 + 3x4 + \ldots + (n-1)x4 =$$

$$4(1 + 2 + 3 + \ldots + (n-1))$$

$$4\,\frac{(n-1(n))}{2}$$

2n(n-1)

Worst case

```
k ⟵ 1                                              1
while (k < n) do {                                 1
        tmp ⟵ list[k]                             2
        j ⟵ k-1
        while (j ≧ 0 and tmp < list[j]) do {      4
            list[j+1] ⟵ list[j]
            j ⟵ j – 1
        }
                                                   2
        list[j+1] ⟵ tmp
        k ⟵ k + 1
```

k
times
max

n-1
times

$$5(n-1) + (1 \times 4 + 2 \times 4 + 3 \times 4 + \ldots + (n-1) \times 4) \quad \mathbf{+2}$$

Worst case

```
k ⟵ 1                                            1
while (k < n) do {                               1
      tmp ⟵ list[k]                              2
      j ⟵ k-1
      while (j ≧ 0 and tmp < list[j]) do {       4
         list[j+1] ⟵ list[j]
         j ⟵ j – 1
      }

      list[j+1] ⟵ tmp                            2
      k ⟵ k + 1
```

k times max

n-1 times

$$5(n-1) + (1x4 + 2x4 + 3x4 + \dots + (n-1)x4) \quad \mathbf{+2}$$

$$5(n-1) + (2n(n-1)) \quad +2$$

Worst case

```
k ⟵ 1                                          1
while (k < n) do {                             1
        tmp ⟵ list[k]
                                                2
        j ⟵ k-1
        while (j ≧ 0 and tmp < list[j]) do {
                                                4
            list[j+1] ⟵ list[j]
            j ⟵ j – 1
        }
                                                2
        list[j+1] ⟵ tmp
        k ⟵ k + 1
```

k times max

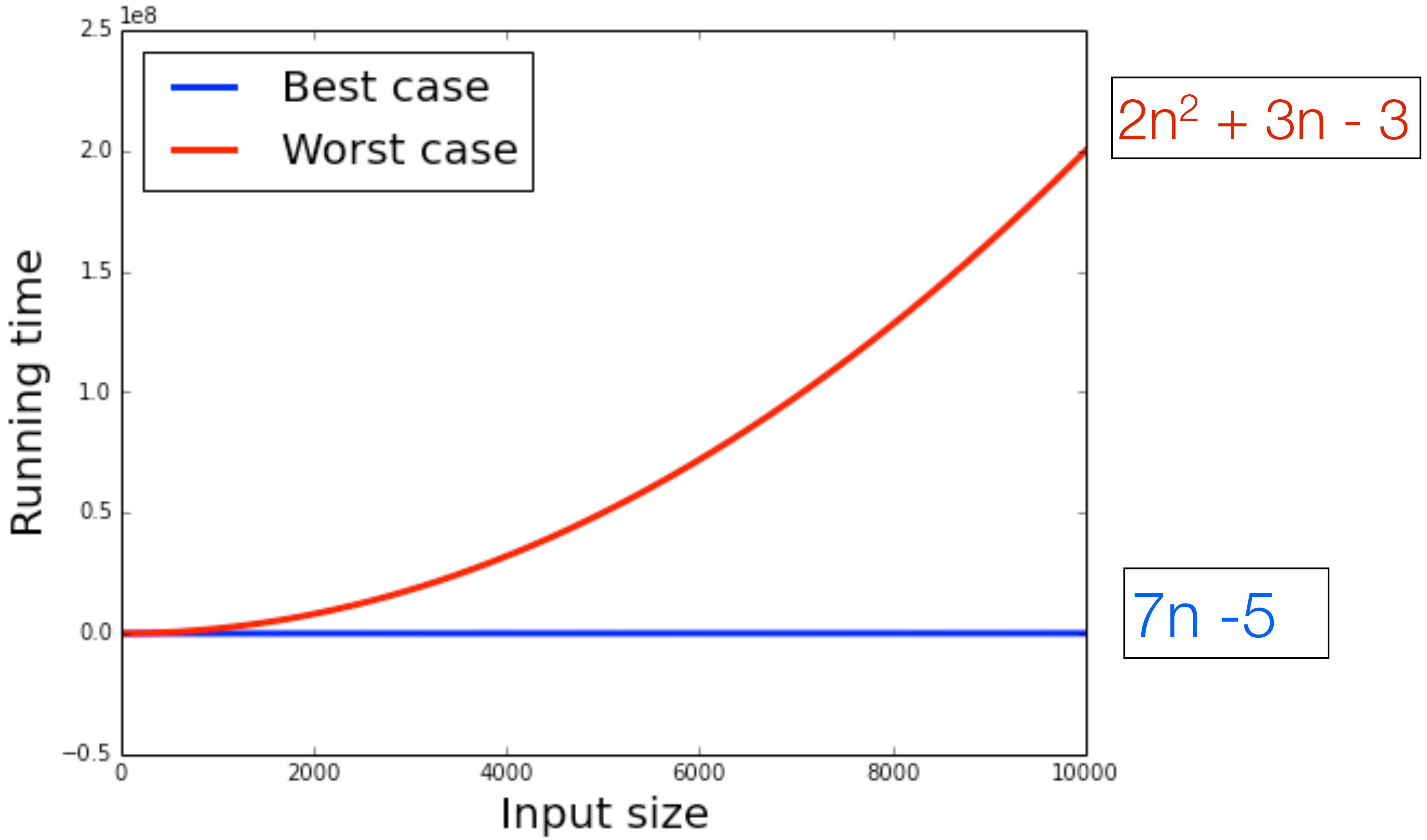
n-1 times

$$5(n-1) + (1\text{x}4 + 2\text{x}4 + 3\text{x}4 + \ldots + (n-1)\text{x}4) \quad \mathbf{+2}$$

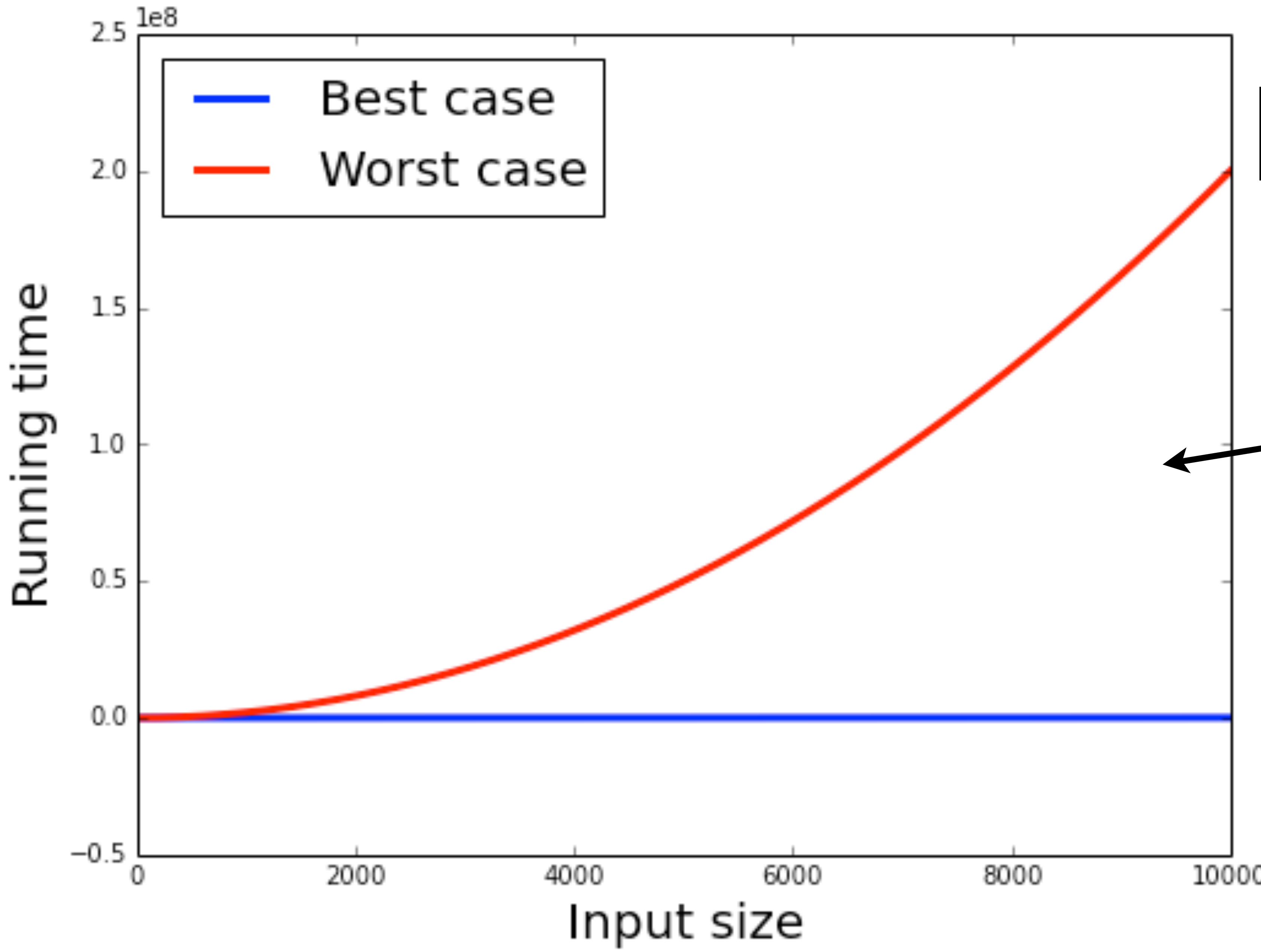
$$5(n-1) + (2n(n-1)) \quad +2$$

$$\boxed{2n^2 + 3n - 3}$$

Insertion Sort running time



Insertion Sort running time



$2n^2 + 3n - 3$

average case must be somewhere in the middle

$7n - 5$

- Select **how to measure the input size**.
- If running time depends only on the input size, then that's great.

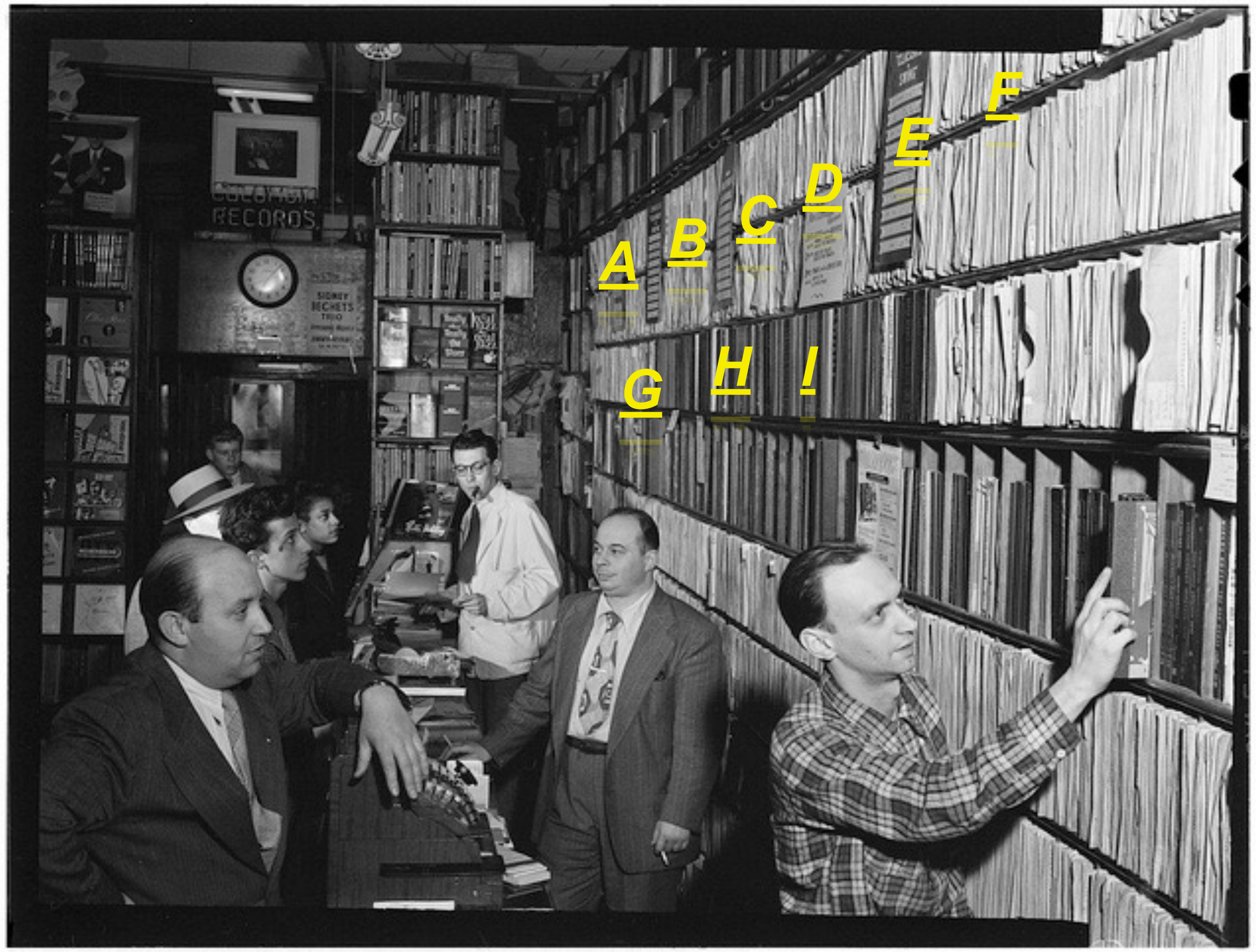
- If running time depends on input size **and other characteristics** of the input:
    - Analyse best case separately *(can I leave any loops early).*
    - Analyse worst case separately.
    - Together best and worst case are informative.

Insertion Sort: Code

```
k ⟵ 1
while (k < n) do {
    tmp ⟵ list[k]
    j ⟵ k-1
    while (j ≥ 0 and tmp < list[j]) do {
        list[j+1] ⟵ list[j]
        j ⟵ j – 1
    }

    list[j+1] ⟵ tmp
    k ⟵ k + 1
```

```python
def insertion_sort(the_list):
    n = len(the_list)
    for k in range(1, n):
        temp = the_list[k]
        i = k - 1
        while i >= 0 and the_list[i] > temp:
            the_list[i + 1] = the_list[i]
            i -= 1
        the_list[i + 1] = temp
```

Binary Search Assumptions

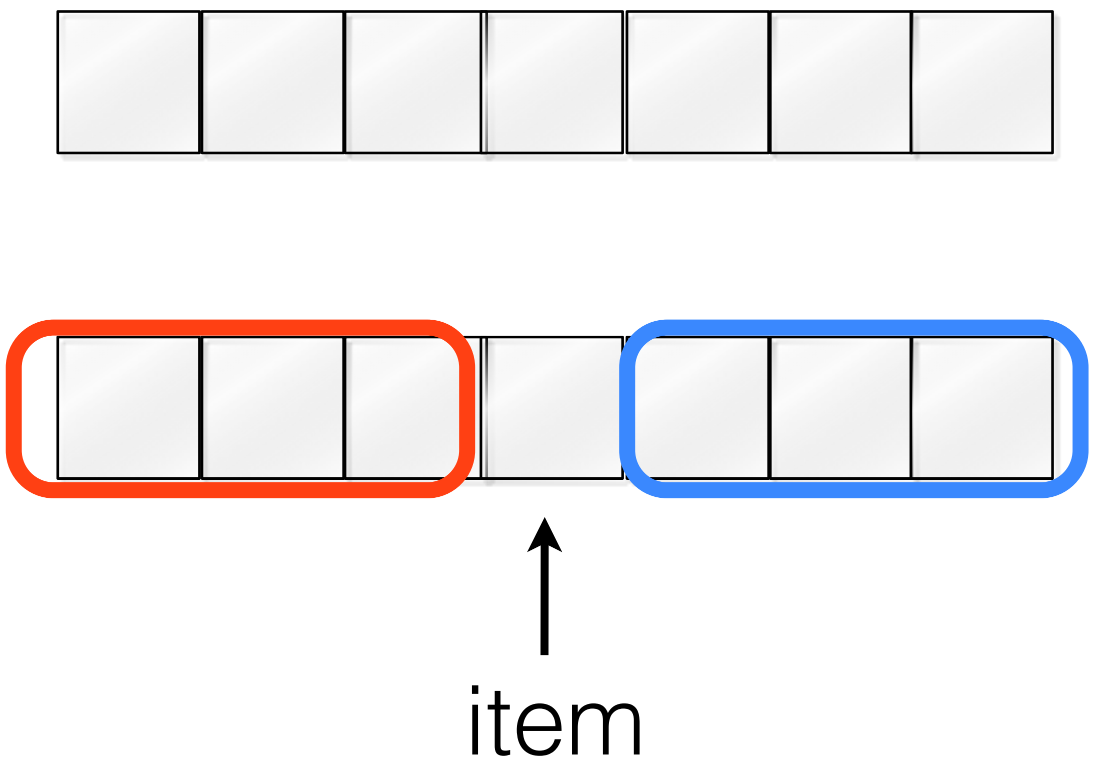


- The list is sorted

- We can random access the list
  (you can get the value of any position in the list)

Binary Search

```
item ⟵ the item in the middle of the list

if (item = target)
{
        return index of item
}

if (target < item)
{
        search the first part of the list
}

if (target > item)
{
        search the second part of the list
}
```



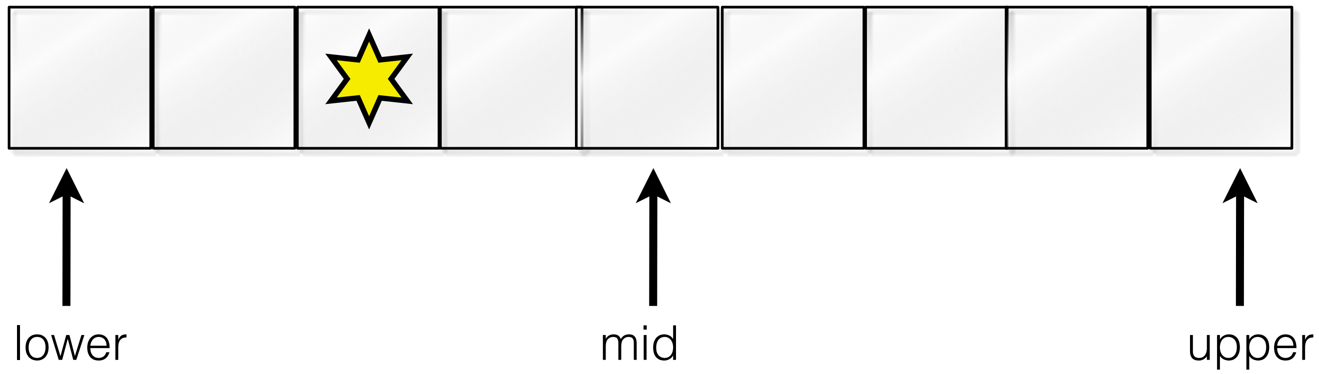
item < target

item = target

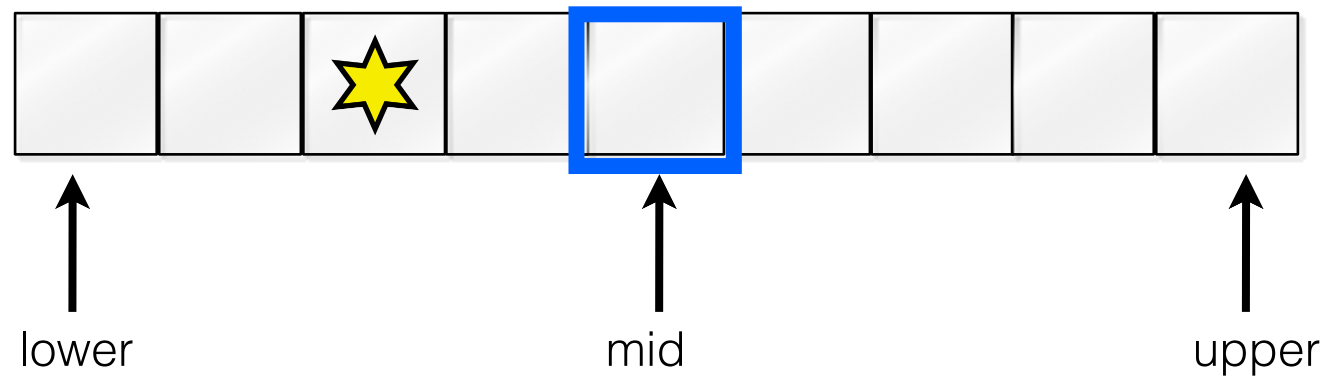
item > target

More concrete…

target



target



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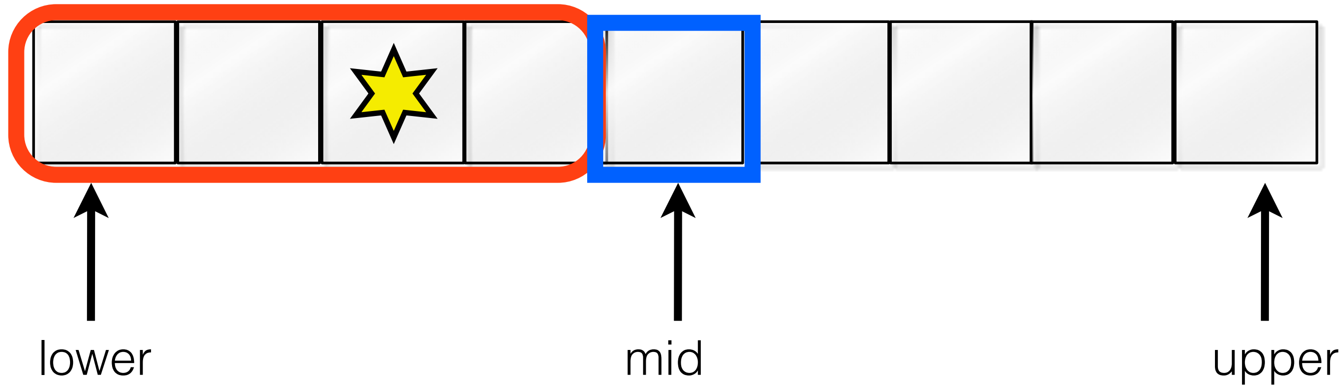

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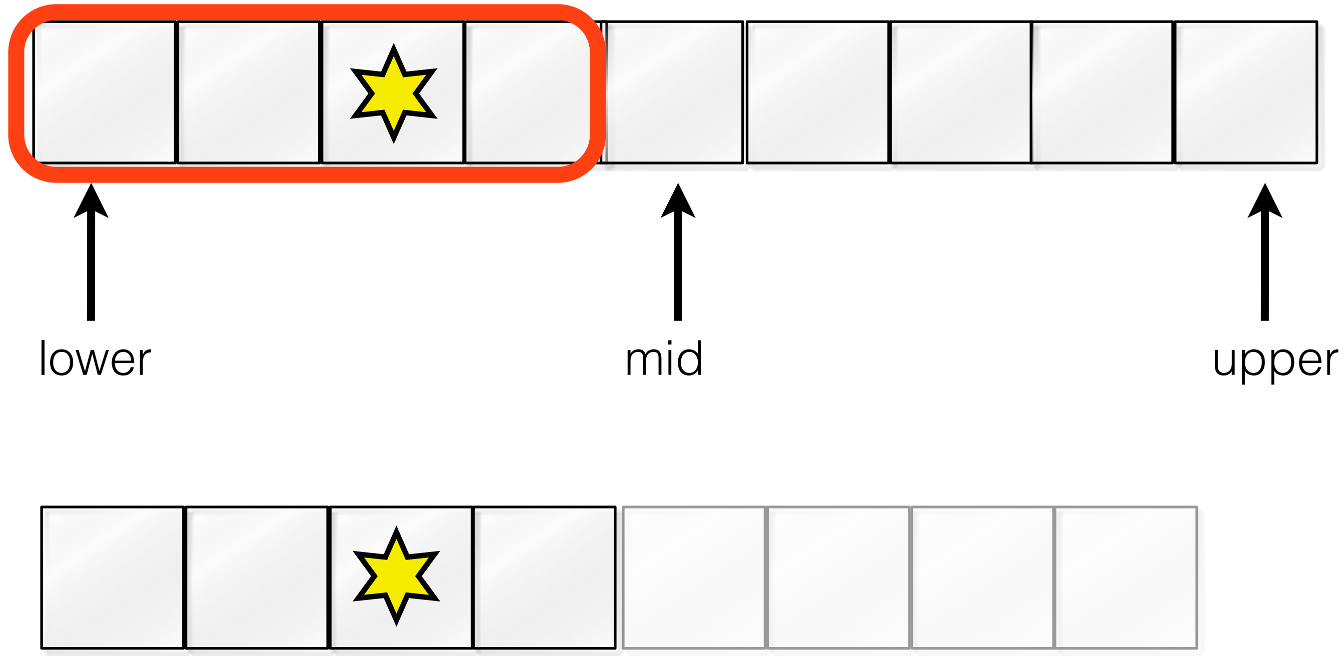
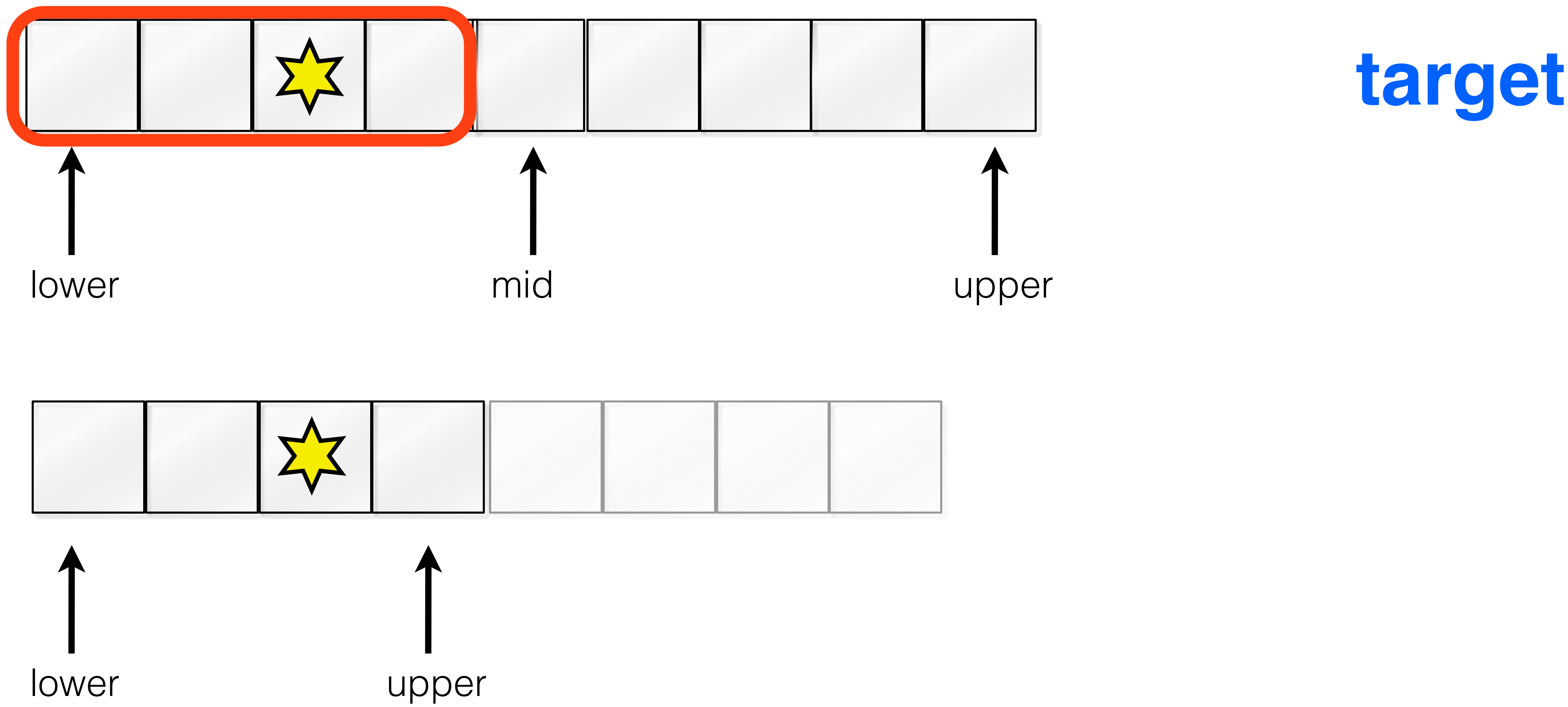
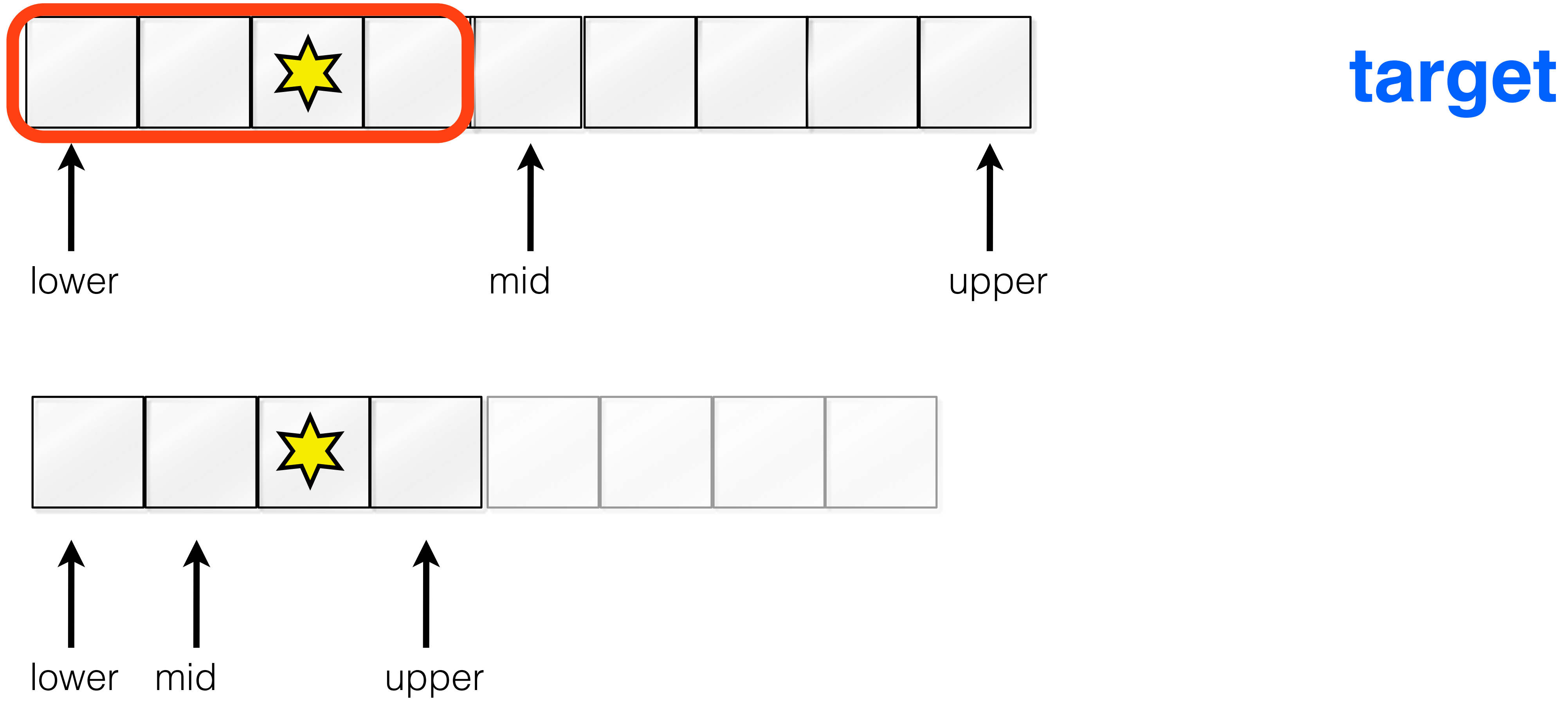
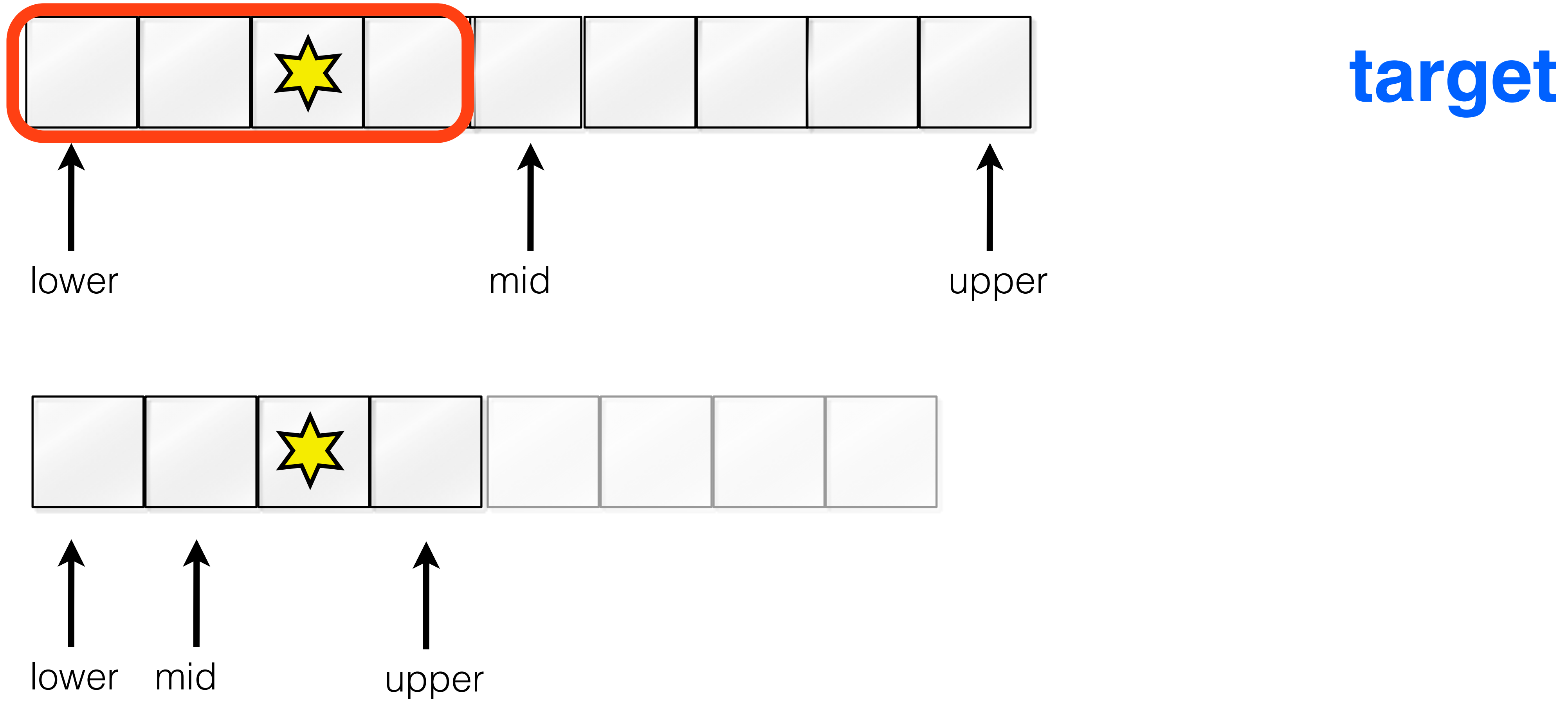


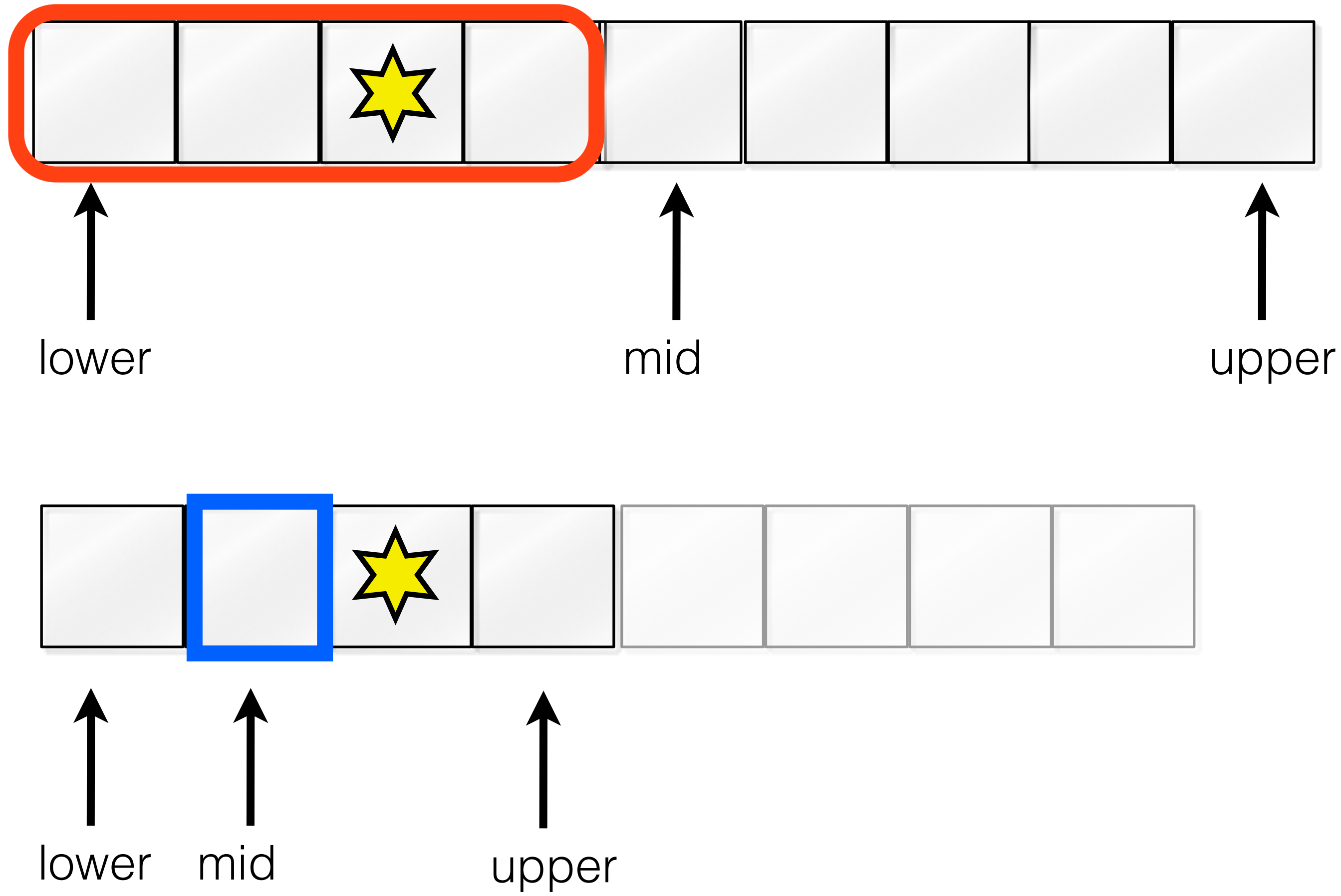
target


target
lower
mid
upper
lower
upper

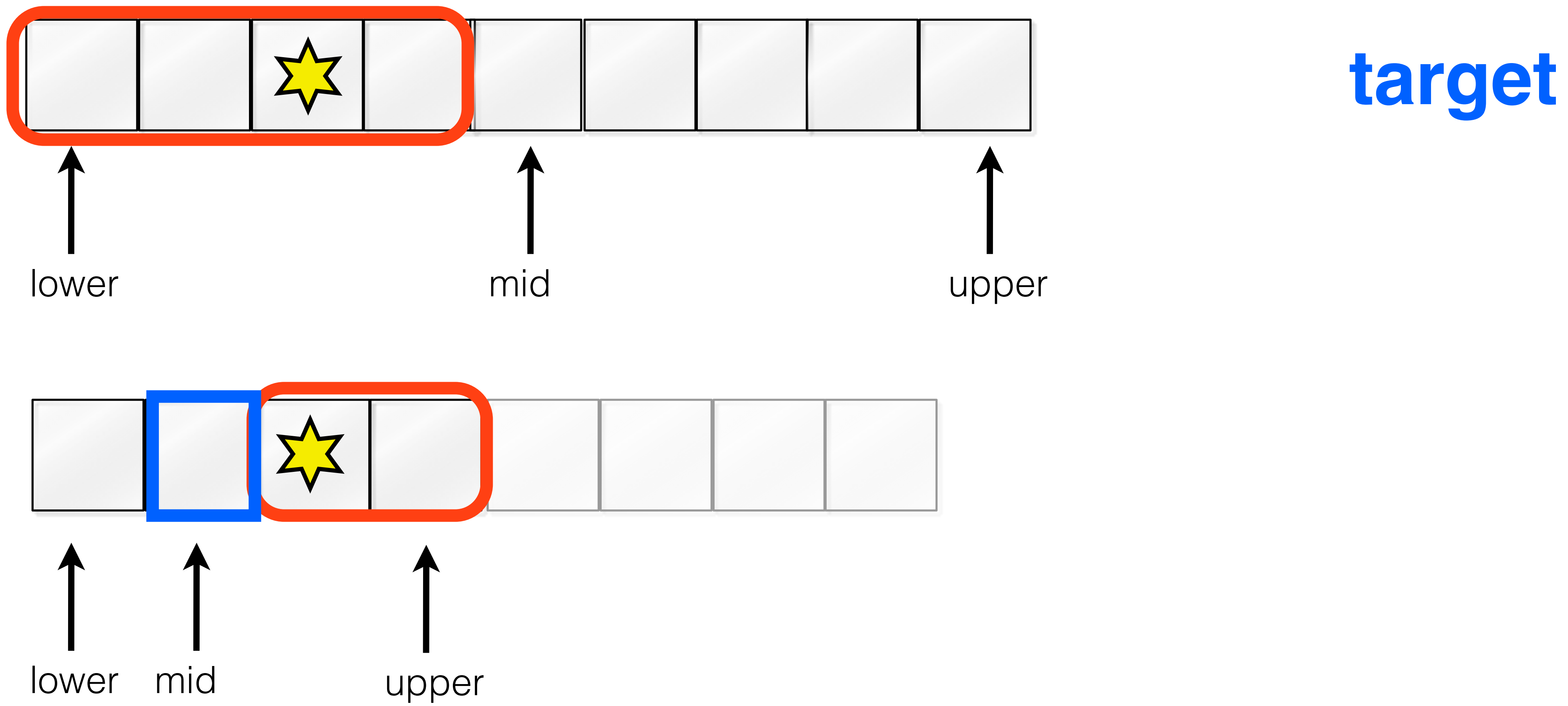

target
lower
mid
upper
lower
mid
upper



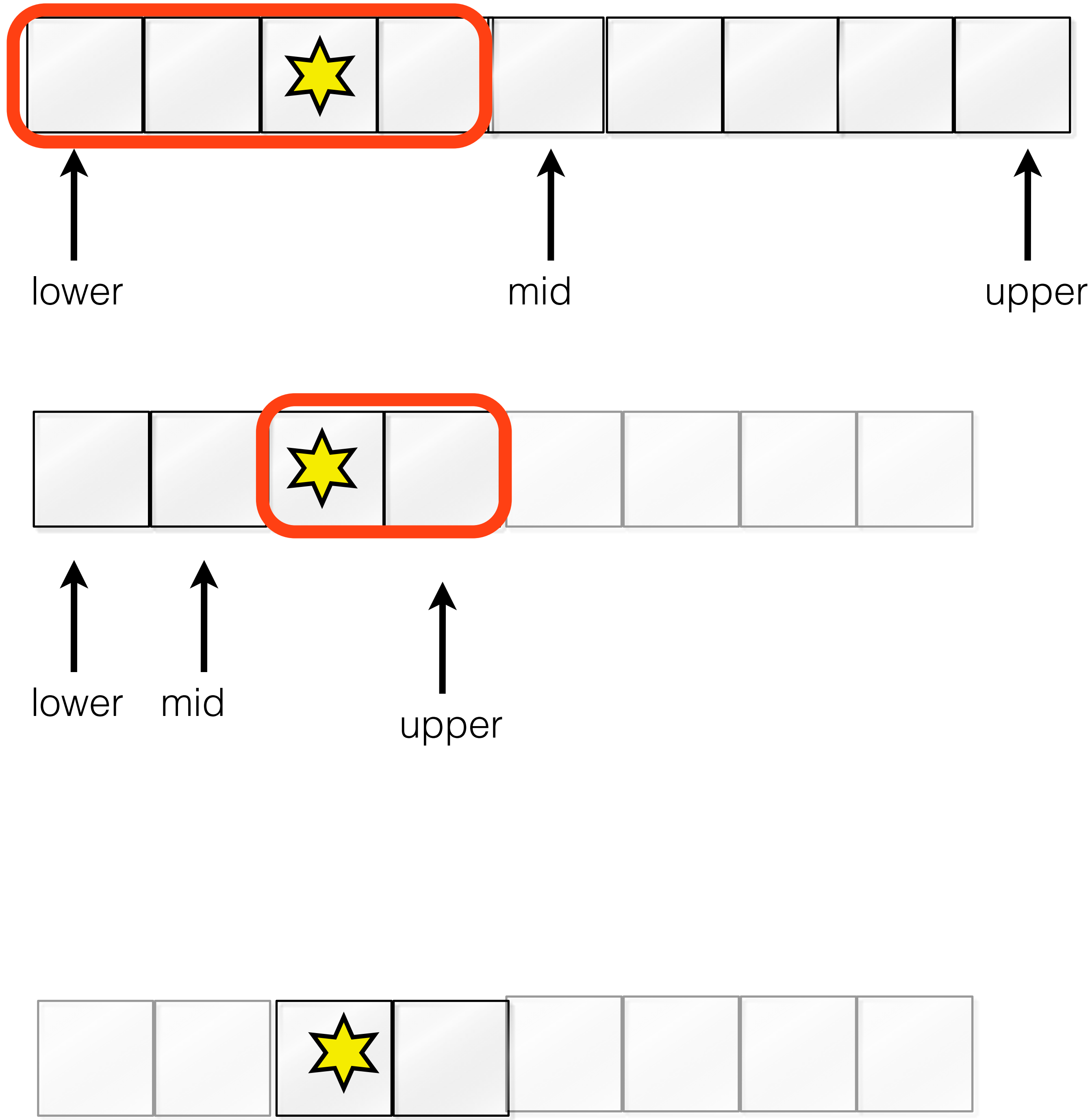
target
lower
mid
upper
lower
mid
upper

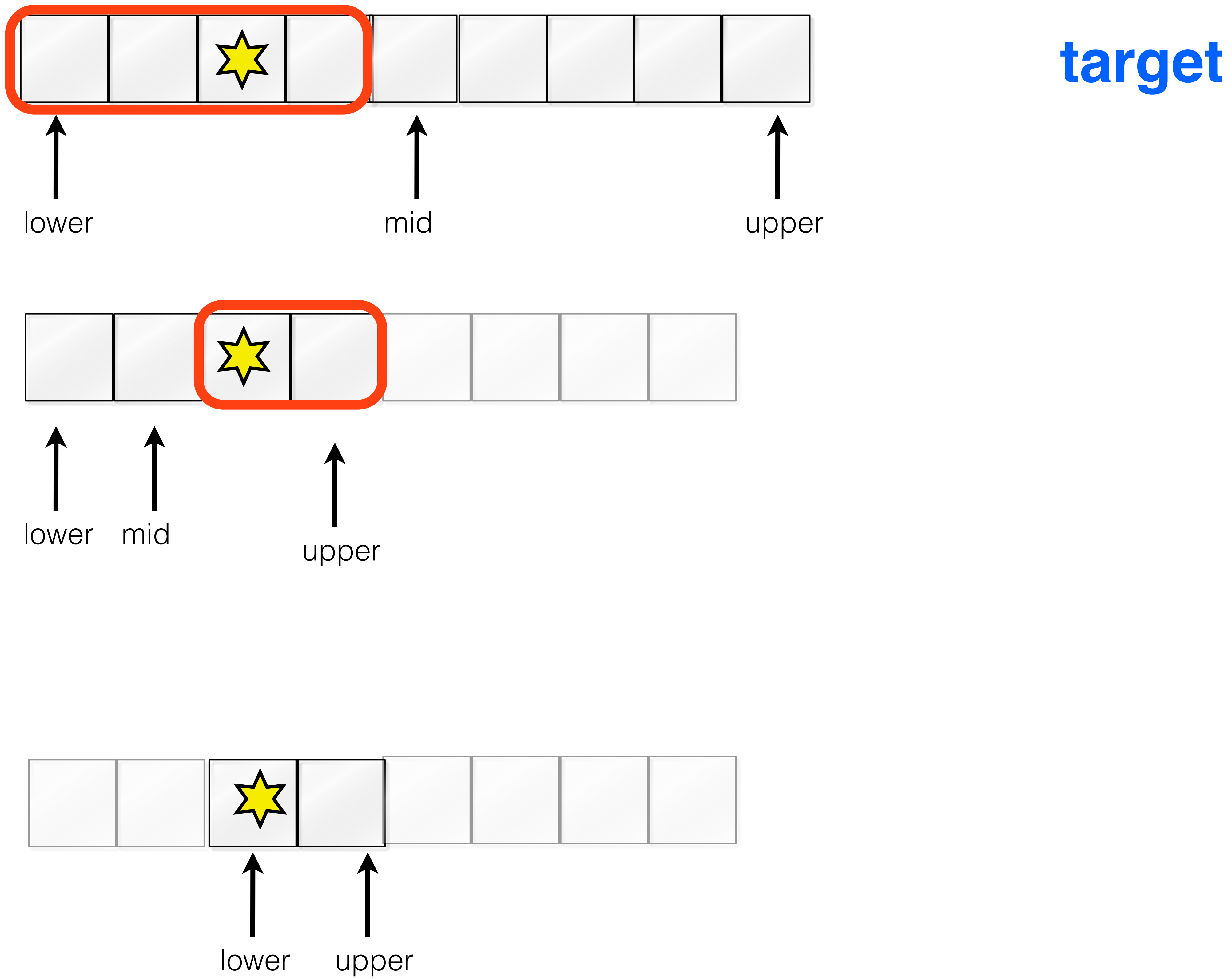


target

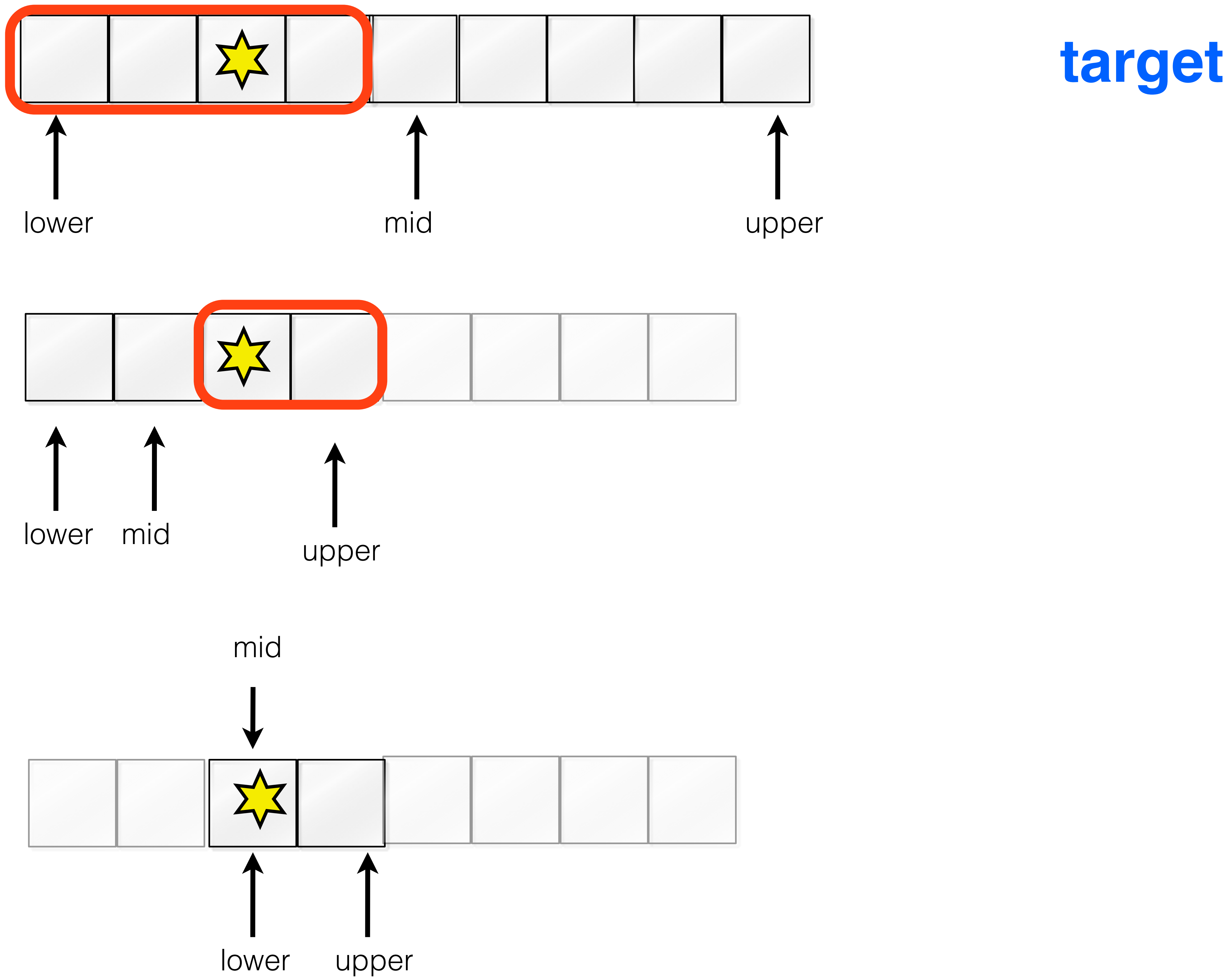

target
lower
mid
upper
lower
mid
upper

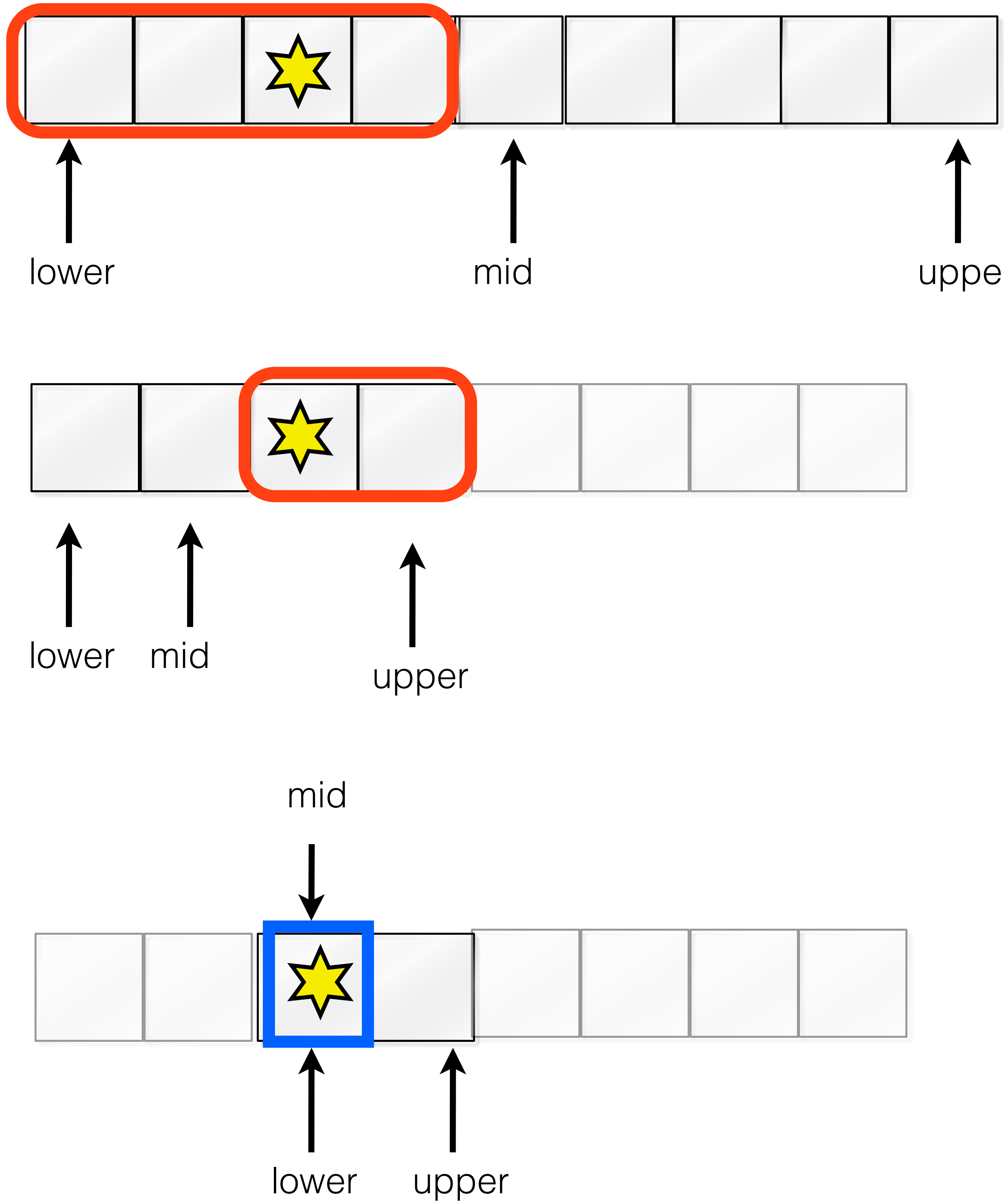
target





target
lower
mid
upper
lower
mid
upper
lower
upper


target
lower
mid
upper
lower
mid
upper
mid
lower
upper



target

Binary Search

```
Algorithm BinarySearch(target,  L[0..n-1])
// Find the index such that L[index] = target
// Input:  target and list L[0..n-1]
// Output:  If target is in L,  return the index of the first
// item with that value. Otherwise return -1.

lower ⟵ 0
upper ⟵ n-1
while (lower ≤ upper) do {
      mid =  ⌊ (lower + upper)/2 ⌋
      if (target == L[mid])
           return  mid
      if (target < L[mid])
            upper = mid – 1
      if (target > L[mid])
            lower = mid + 1
}
return -1
```

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}
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```

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      if (target > L[mid])
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}
return -1
```
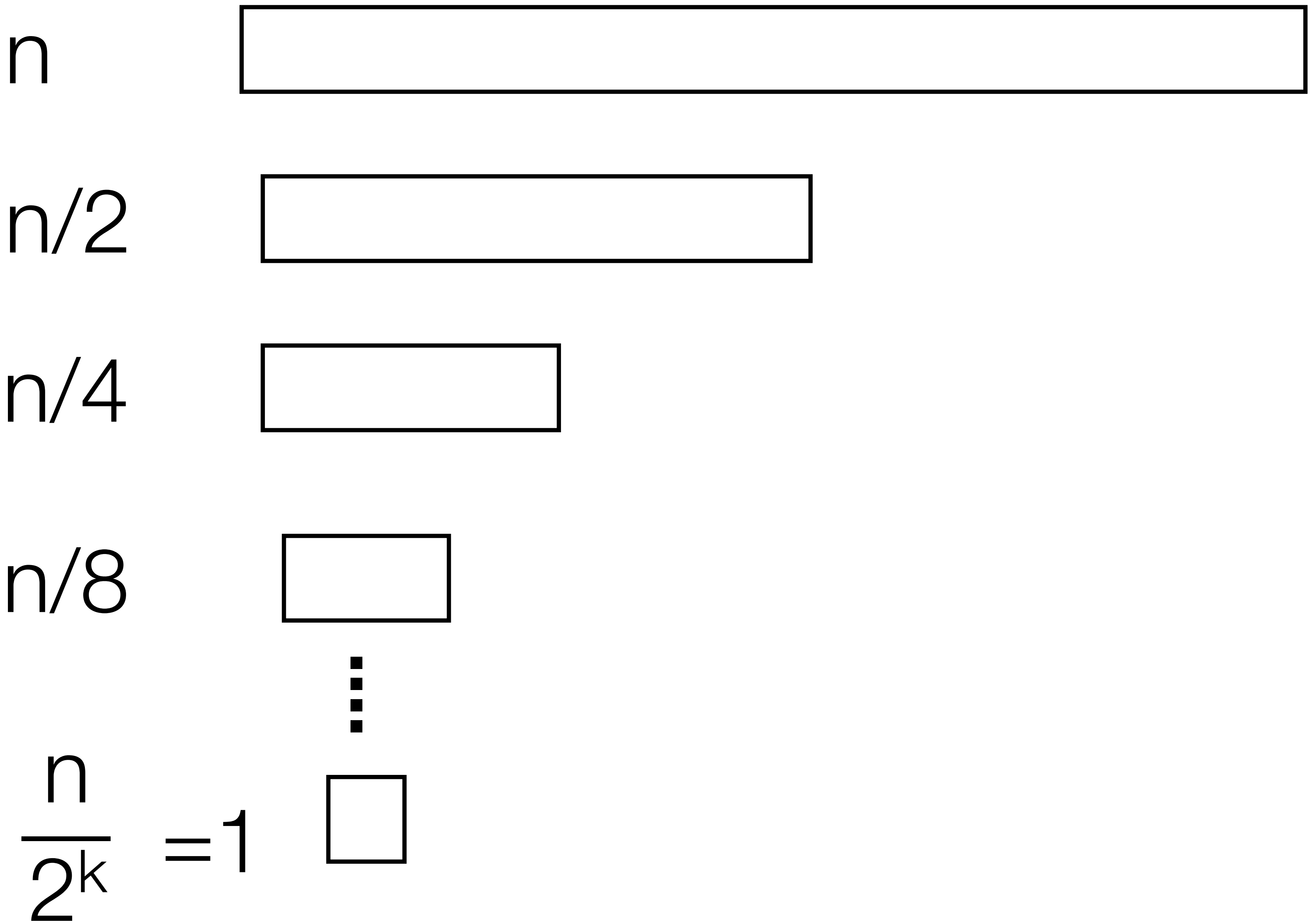

At most $\log_2(n)$ times.



$$\frac{n}{2^k} = 1$$

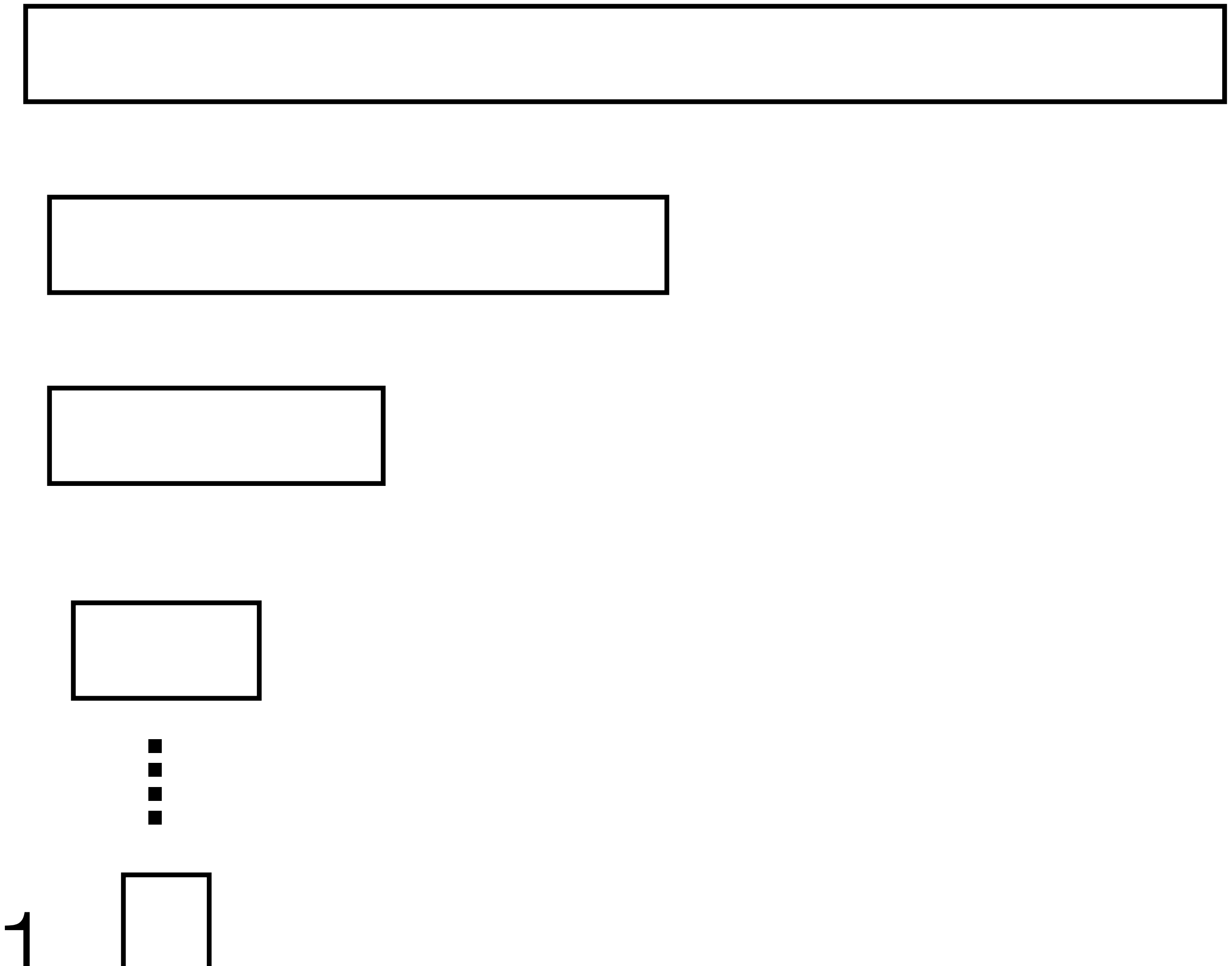
n

$n/2$

$n/4$

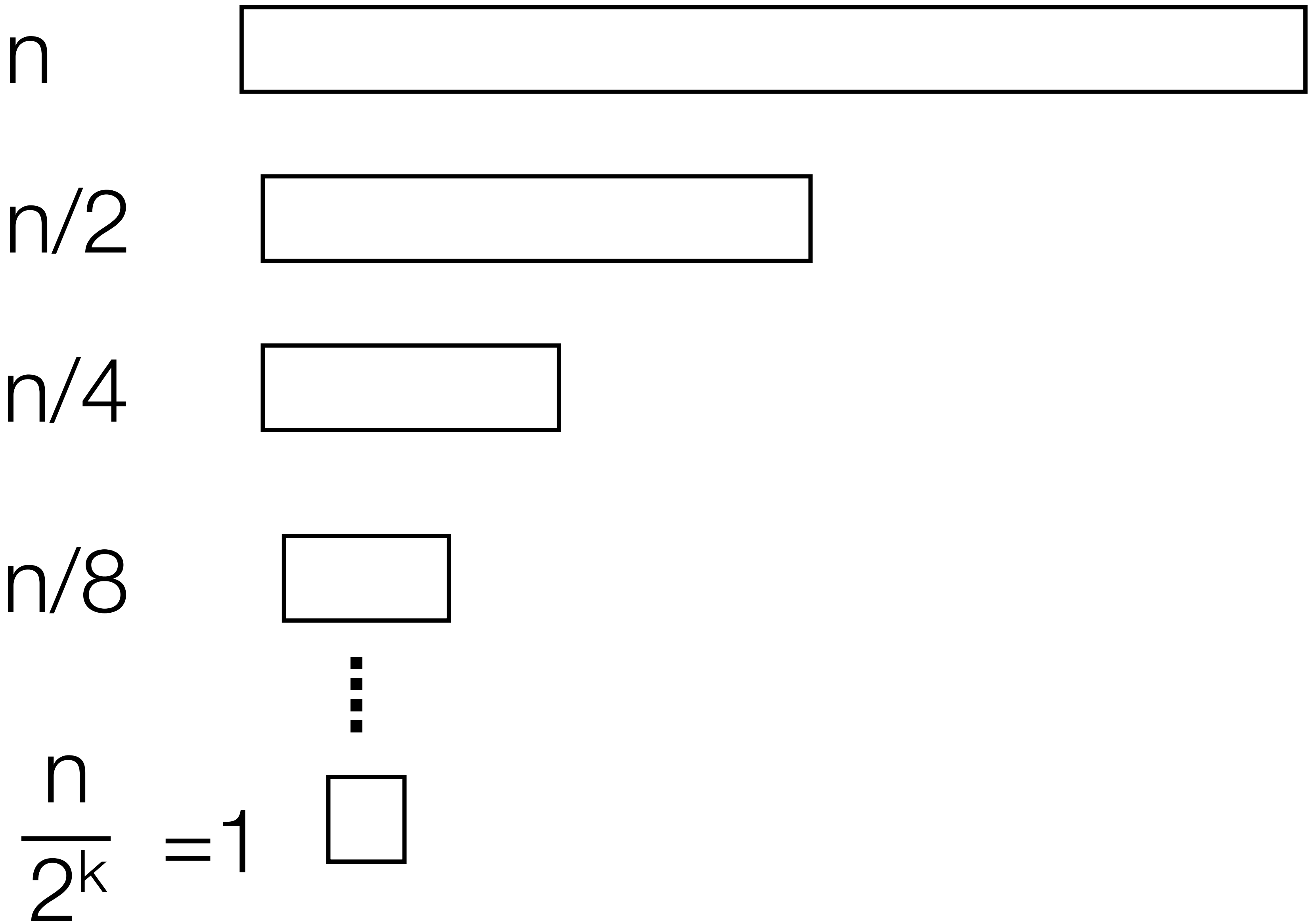
$n/8$

$$\frac{n}{2^k} = 1$$



$$\frac{n}{2^k} = 1$$

$$n = 2^k$$



$$\frac{n}{2^k} = 1$$

$$n = 2^k$$

$$\log_2(n) = k$$

Worst case

$2 + \log_2(n)\ (1 + 1 + 1 + 3) + 1 + 1$

```
lower ⟵ 0
upper ⟵ n-1
while (lower ≤ upper) {
      mid ⟵ ⌊(lower + upper)/2⌋
      if (target = L[mid])
            return mid
      if (target < L[mid])
            upper ⟵ mid – 1
      if (target > L[mid])
            lower ⟵ mid + 1
```

2 assignments

1 comparison

1 assignment

1 comparison

1 return

3 operations

1 return

Target not in List

at most $\log_2(n)$ times

Worst case

$\boxed{6 \log_2(n) + 4}$

$2 + \log_2(n)\ (1 + 1 + 1 + 3) + 1 + 1$

```
lower ⟵ 0
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```

2 assignments

1 comparison

1 assignment

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1 return

3 operations

1 return

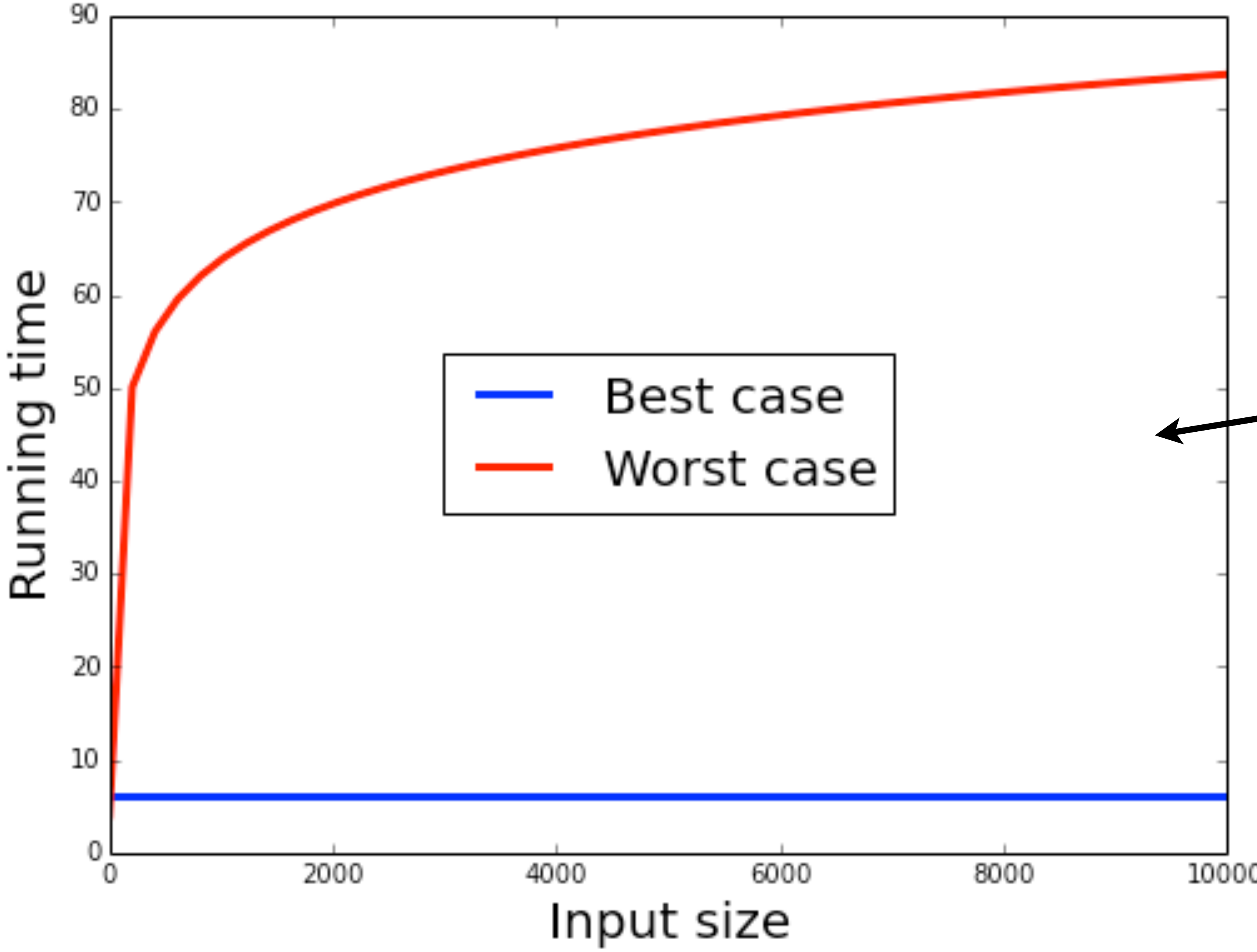
Target not in List

at most $\log_2(n)$ times

Big O

Focus on the big picture

Binary Search running time



$6 \log_2(n) + 4$

average case must be somewhere in the middle

6

$6 \log_2(n) + 4$

```
n = 1, 4.0 = 0.0 + 4.0
n = 2, 10.0 = 6.0 + 4.0
n = 3, 13.5 = 9.5 + 4.0
n = 5, 17.9 = 13.9 + 4.0
n = 10, 23.9 = 19.9 + 4.0
n = 100, 43.9 = 39.9 + 4.0
n = 1000, 63.8 = 59.8 + 4.0
n = 10000, 83.7 = 79.7 + 4.0
n = 100000, 103.7 = 99.7 + 4.0
n = 1000000, 123.6 = 119.6 + 4.0
```

Ignore parts that do not contribute significantly, when the input is large

Worst case

$$d + k \log_2(n) + 1$$

```
lower ⟵ 0
upper ⟵ n-1
while (lower ≤ upper) {
    mid ⟵ ⌊(lower + upper)/2⌋

    if (target = L[mid])
        return mid
    if (target < L[mid])
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    if (target > L[mid])
        lower ⟵ mid + 1
```

d

k

1

Target not in List

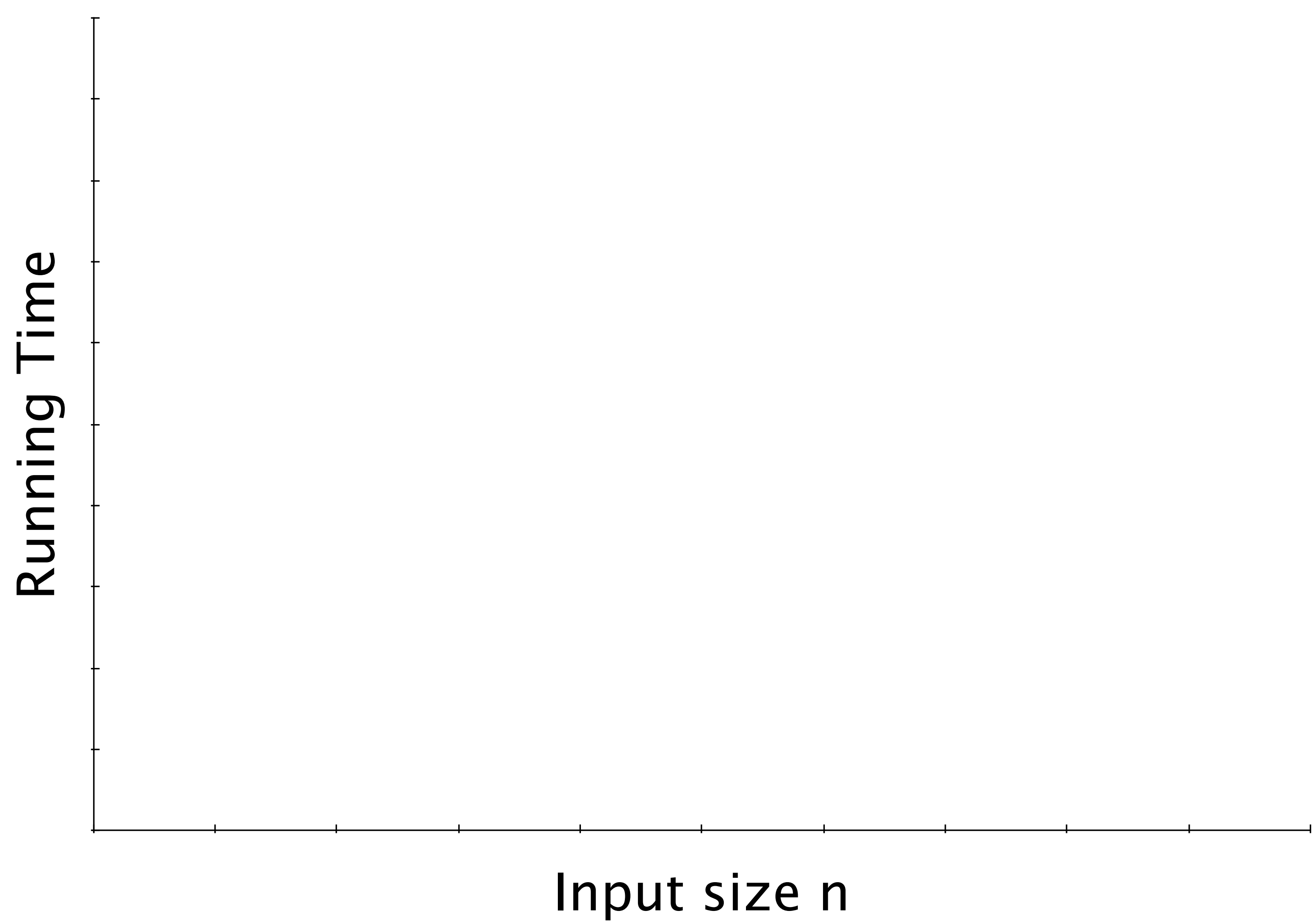
at most
$\log_2(n)$
times

$$d + k \log_2(n) + 1$$

Big O notation

Function **g(n)** is said to be **O(f(n))** if there exist constants **k** and **L** such that:

$$g(n) < k*f(n) \text{ for all } n > L$$



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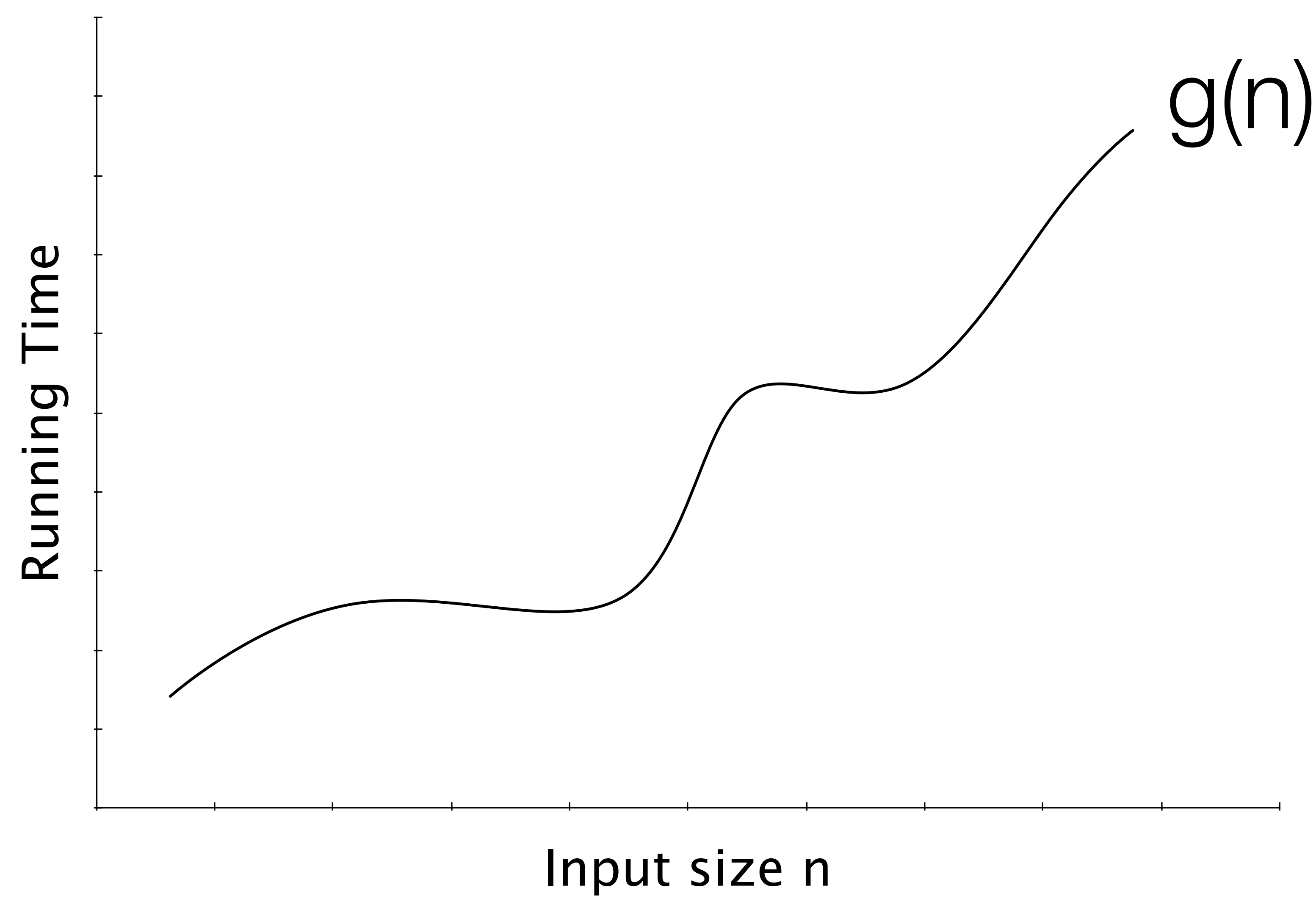

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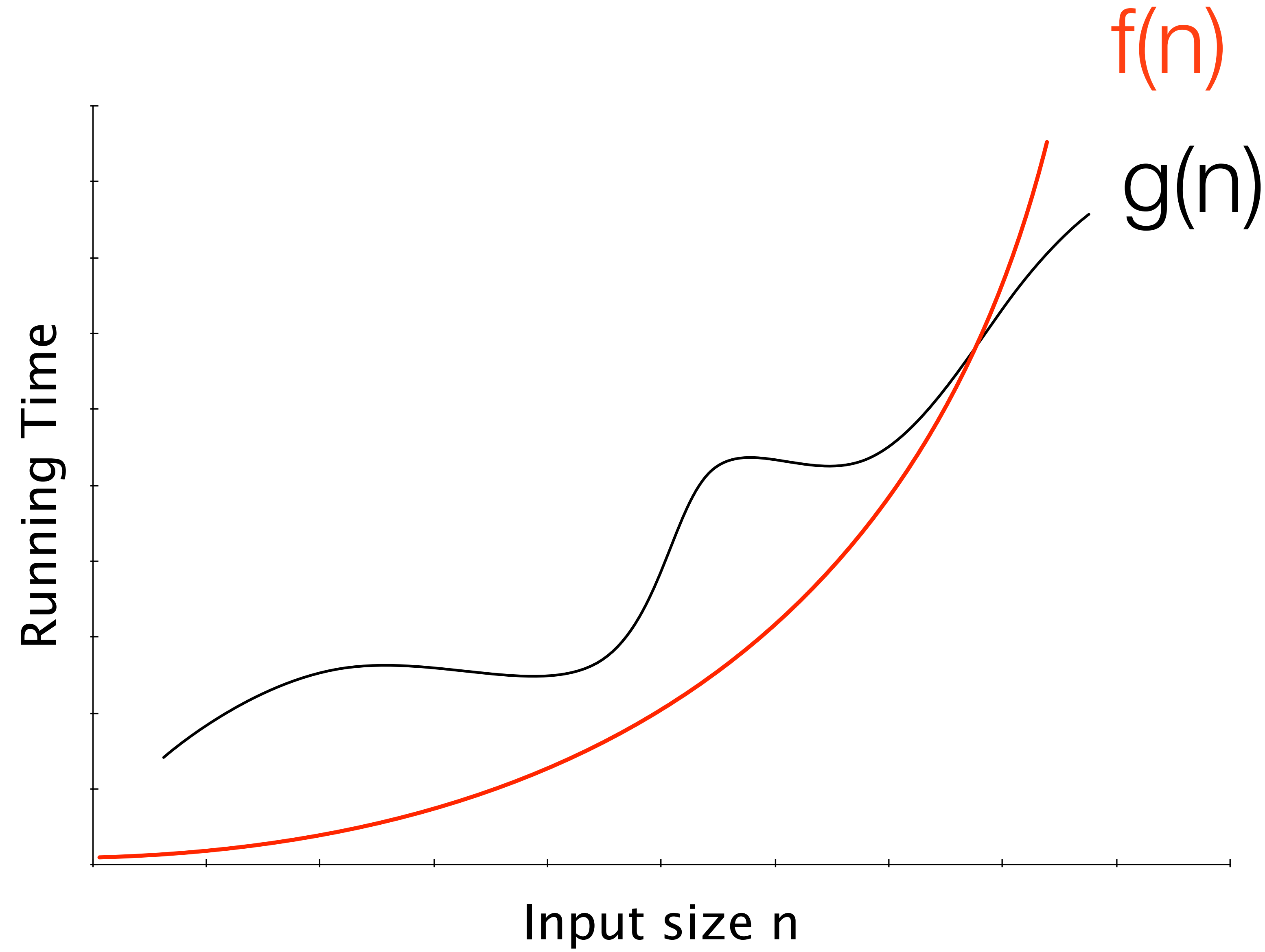

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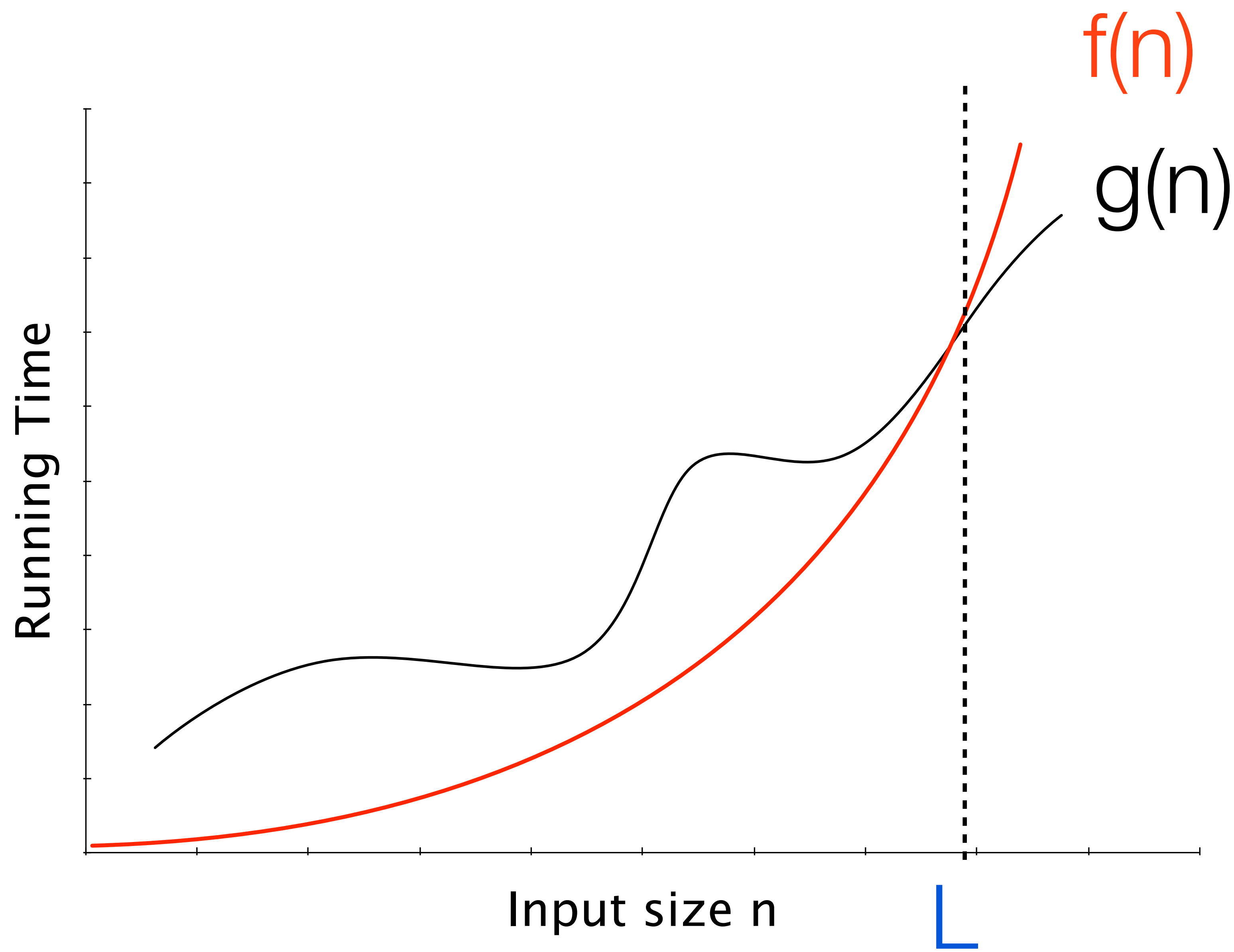

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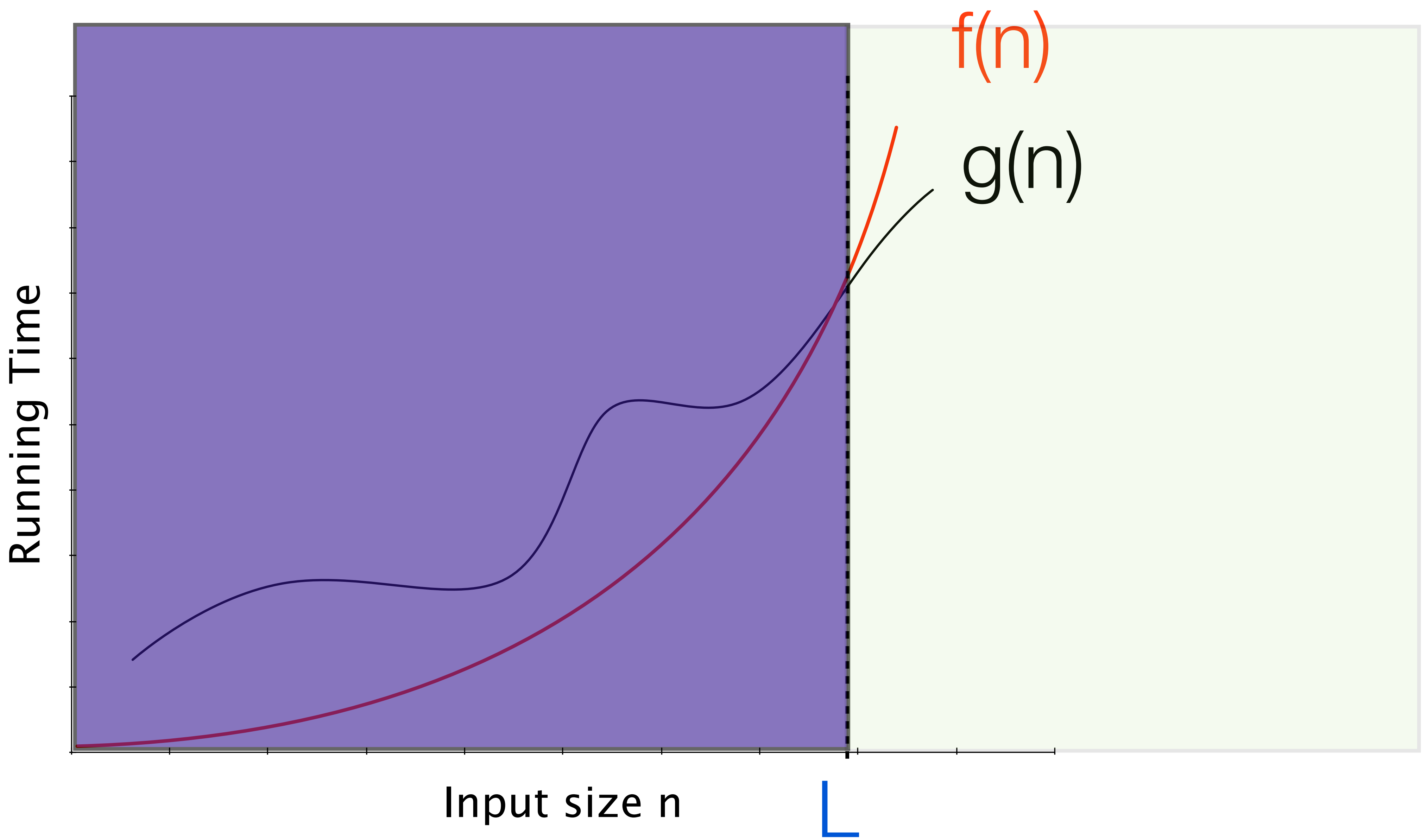

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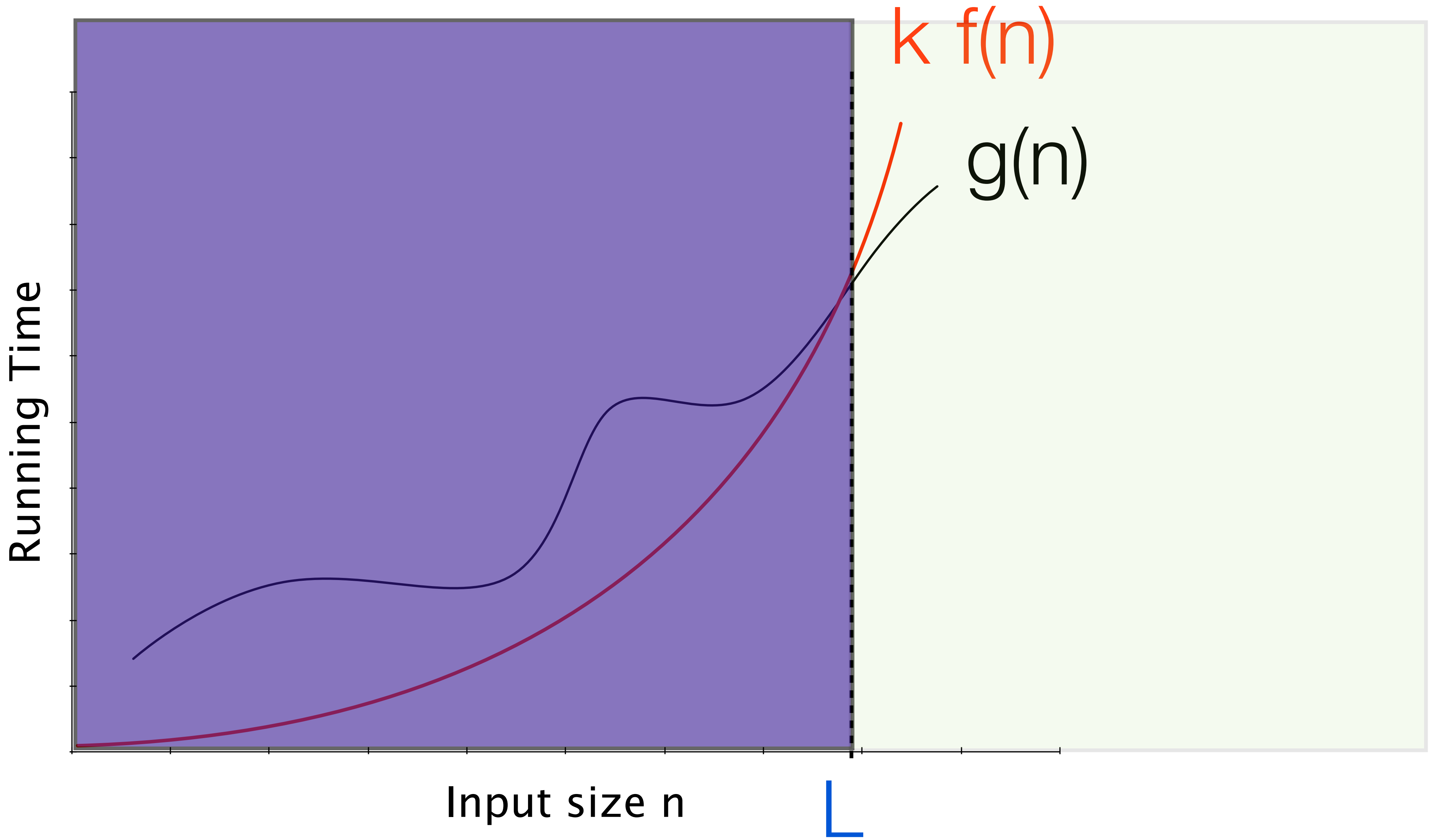

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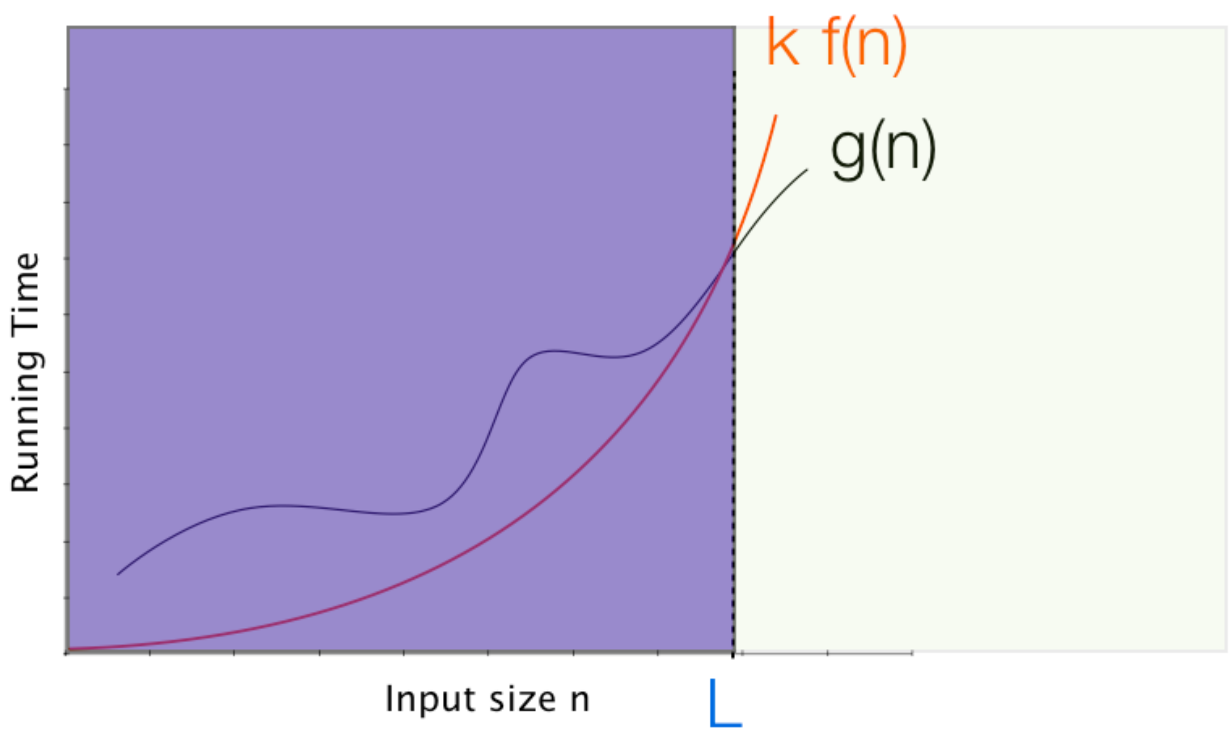
g(n) is **O(f(n))**



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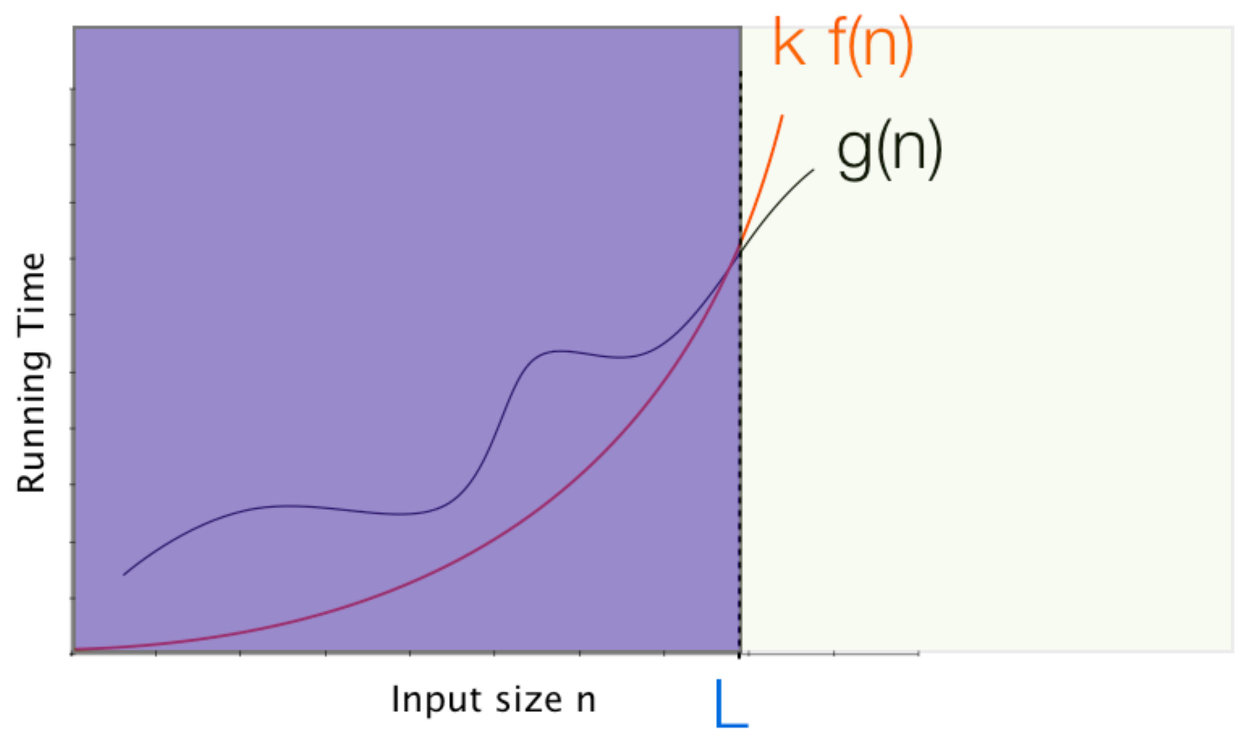
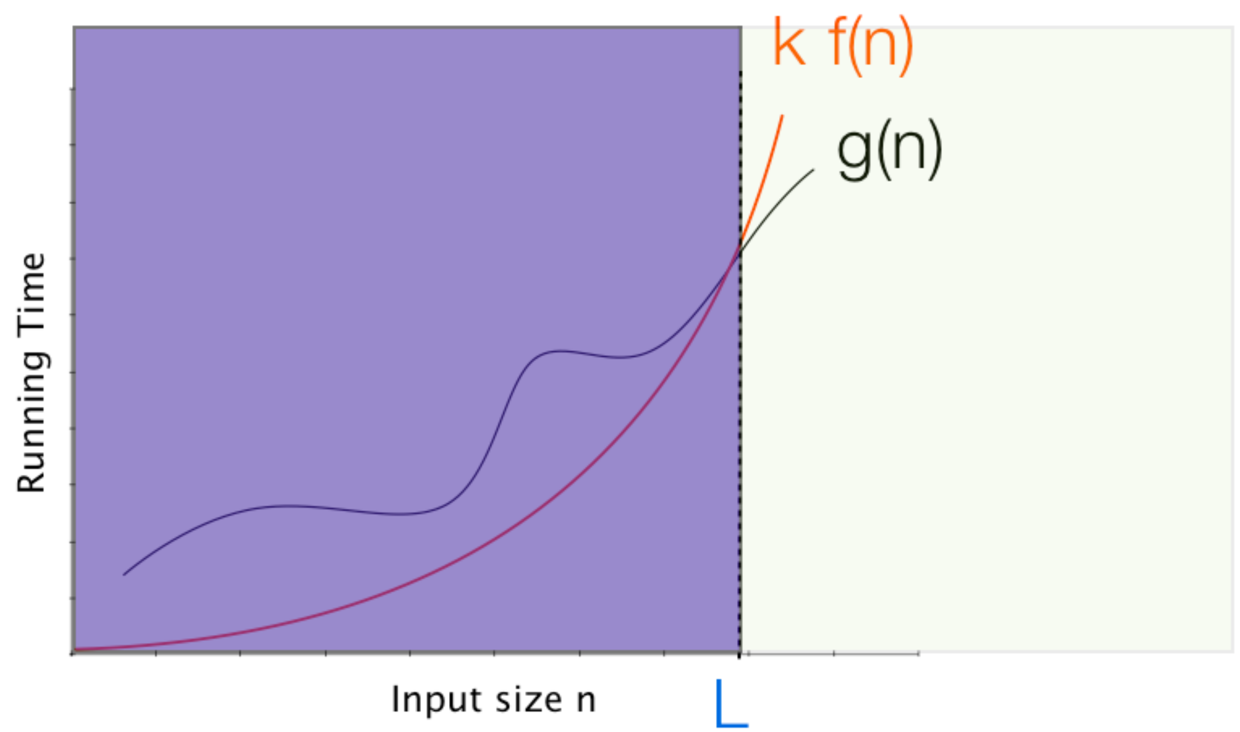


- Intuitively:
  f(n) gives an **upper bound** to running time g(n), which:

Big O notation

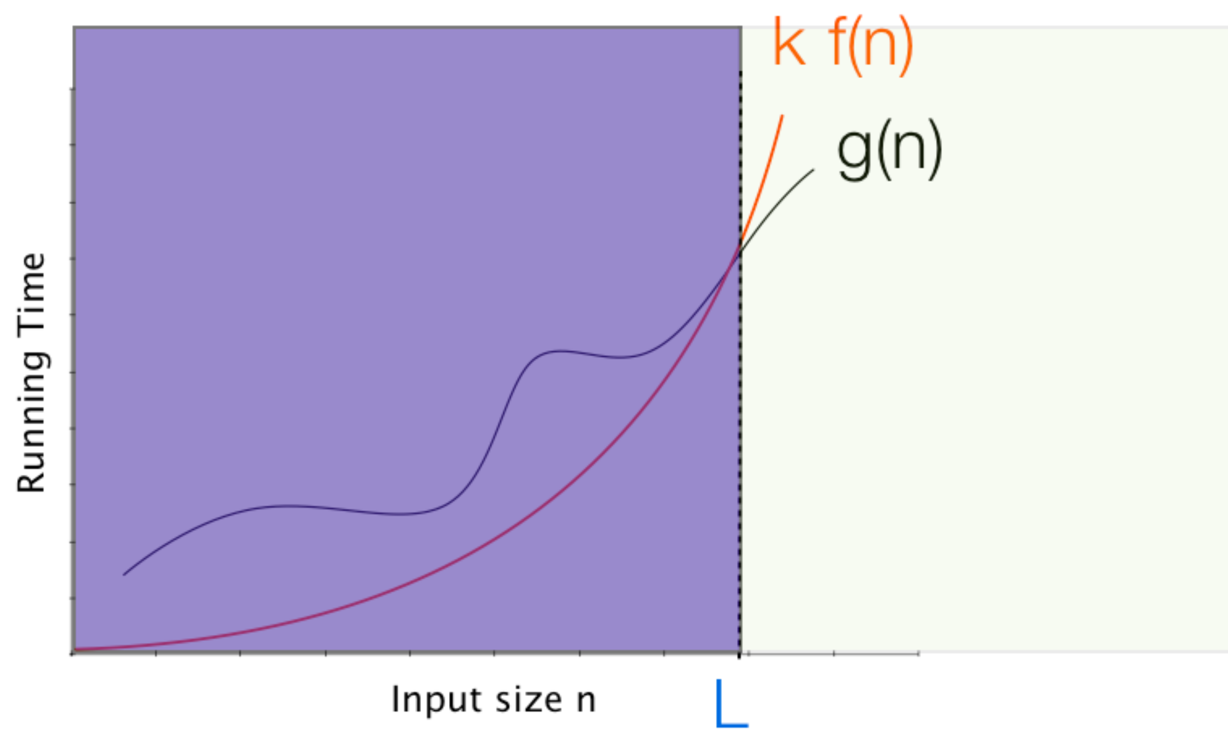
g(n) is **O(f(n))**



- Intuitively:
  f(n) gives an **upper bound** to running time g(n), which:
- ignores parts of the algorithm that do not contribute significantly to the total running time

Big O notation

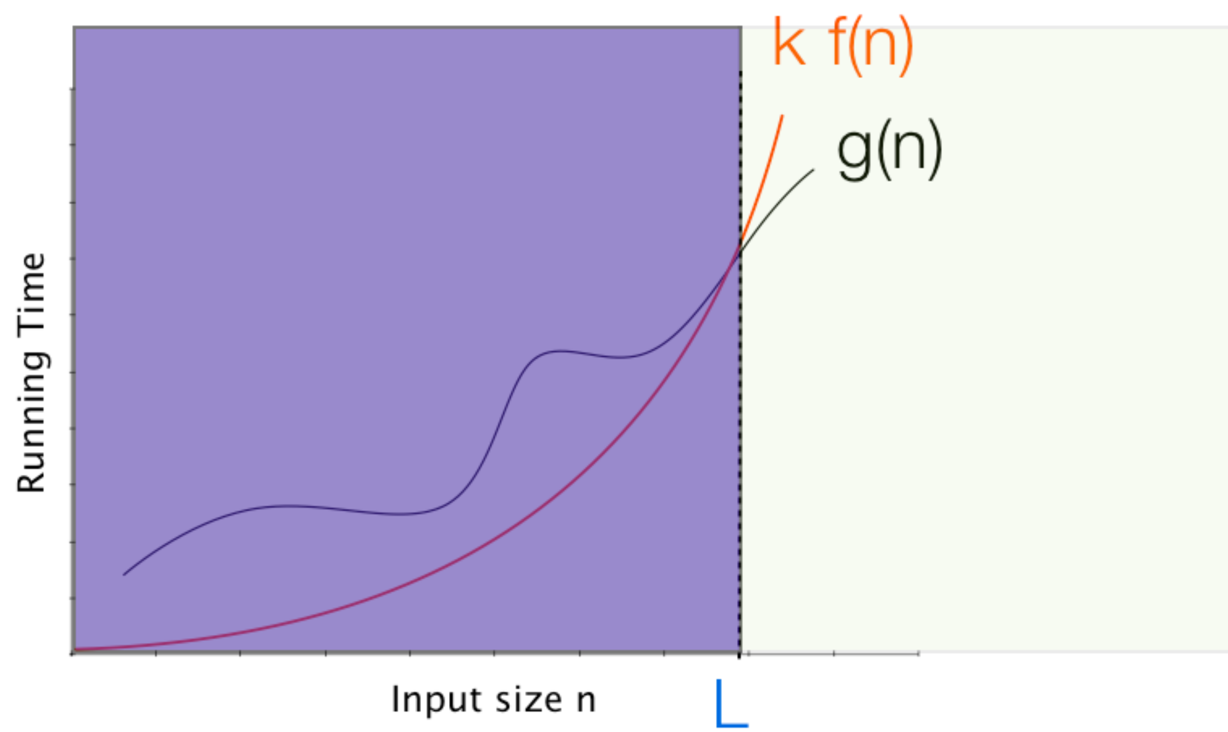
g(n) is **O(f(n))**



- Intuitively:
  f(n) gives an **upper bound** to running time g(n), which:
- ignores parts of the algorithm that do not contribute significantly to the total running time
- bounds the error made when ignoring small terms in g

Big O notation

g(n) is **O(f(n))**



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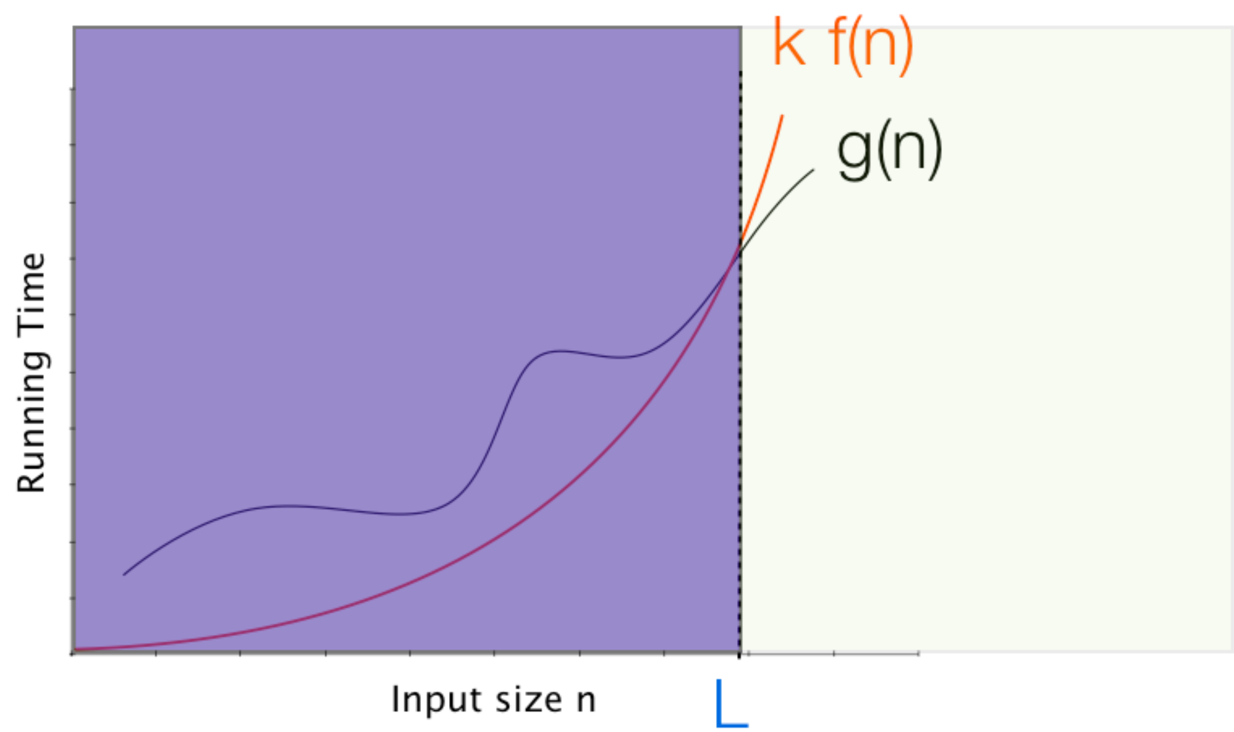

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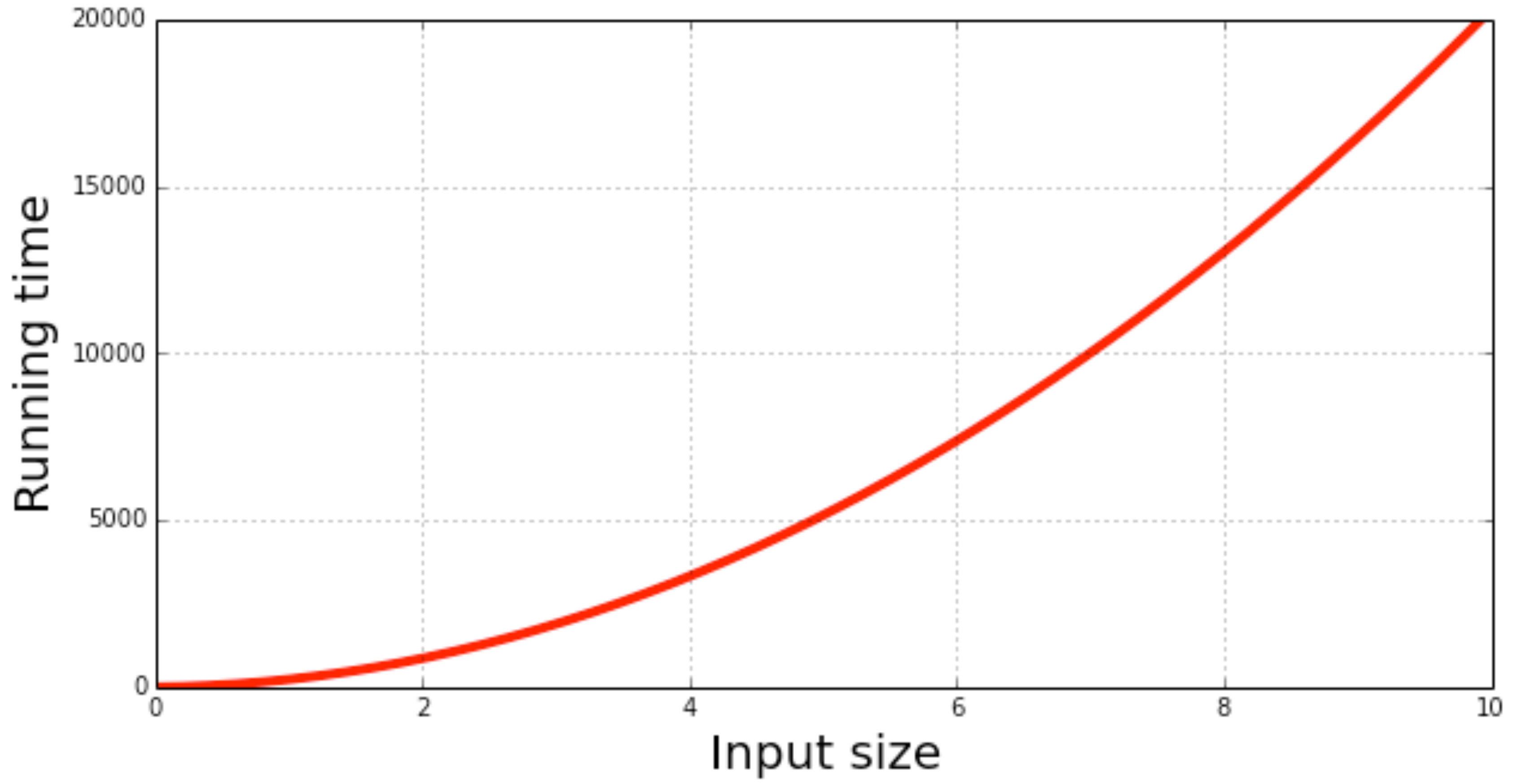
Big O gives us an idea of g(n)'s behaviour for **large inputs**.
Simple but formal.

Insertion sort worst case

$2n^2 + 3n - 3$



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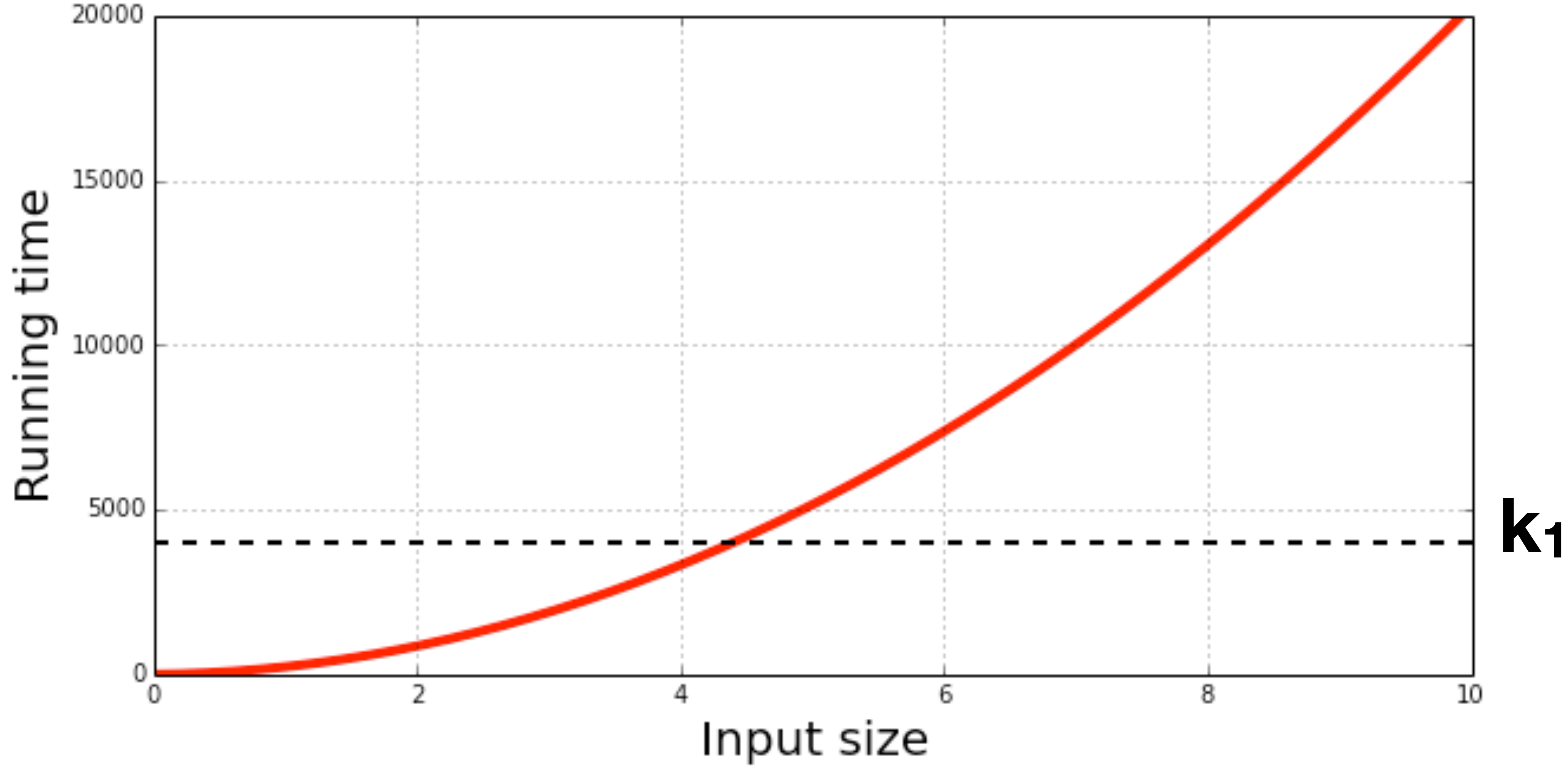

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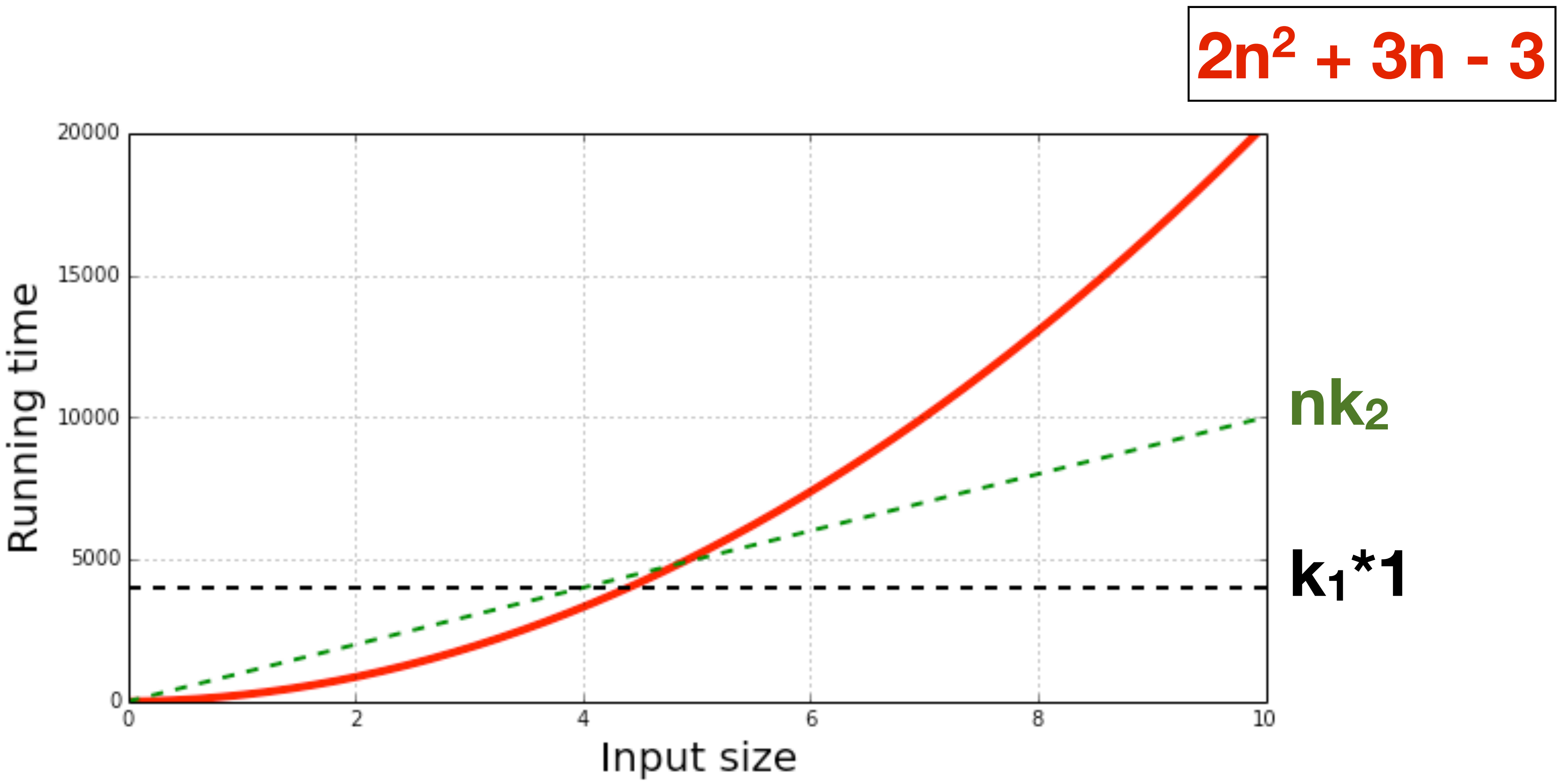

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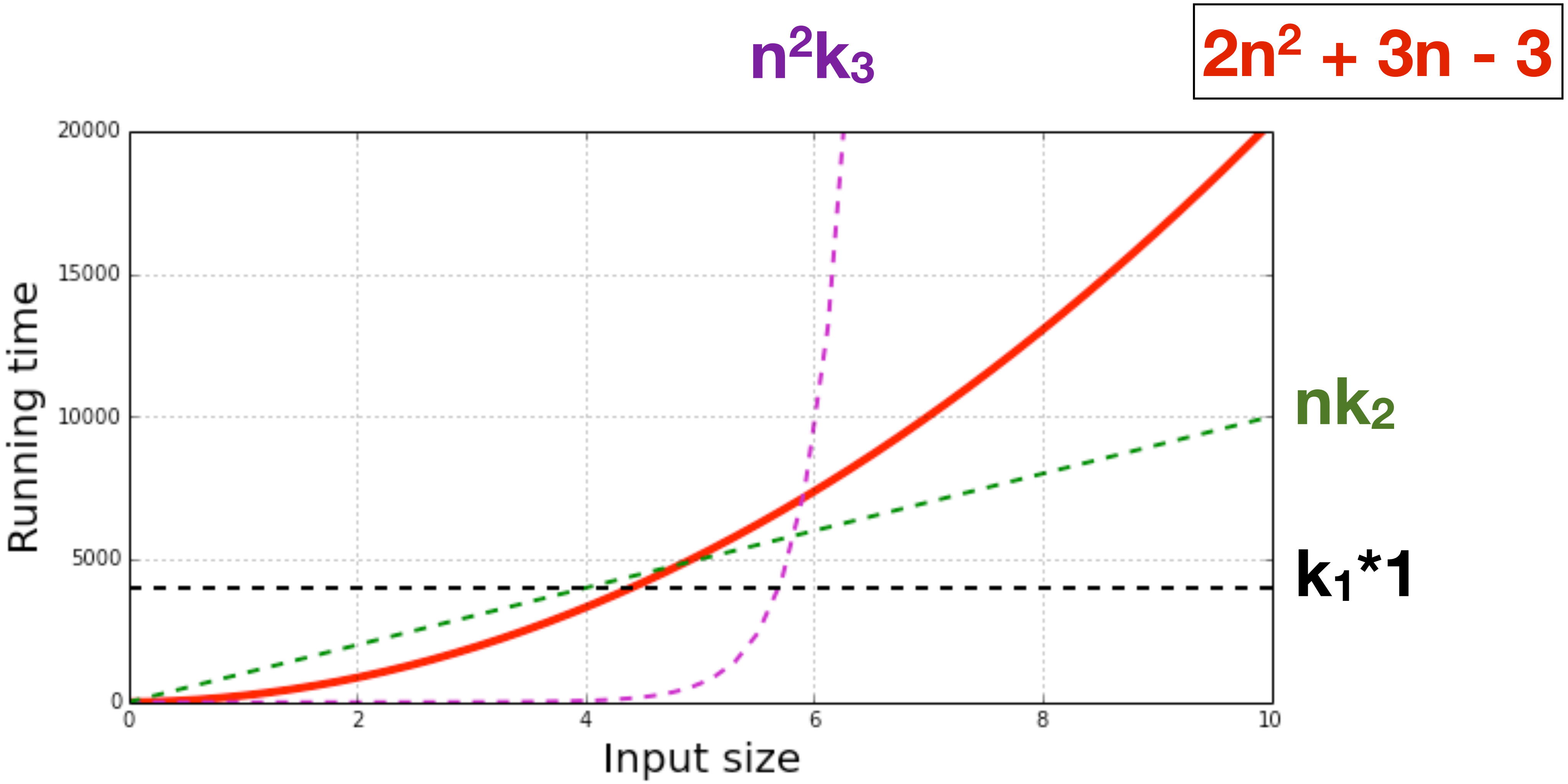

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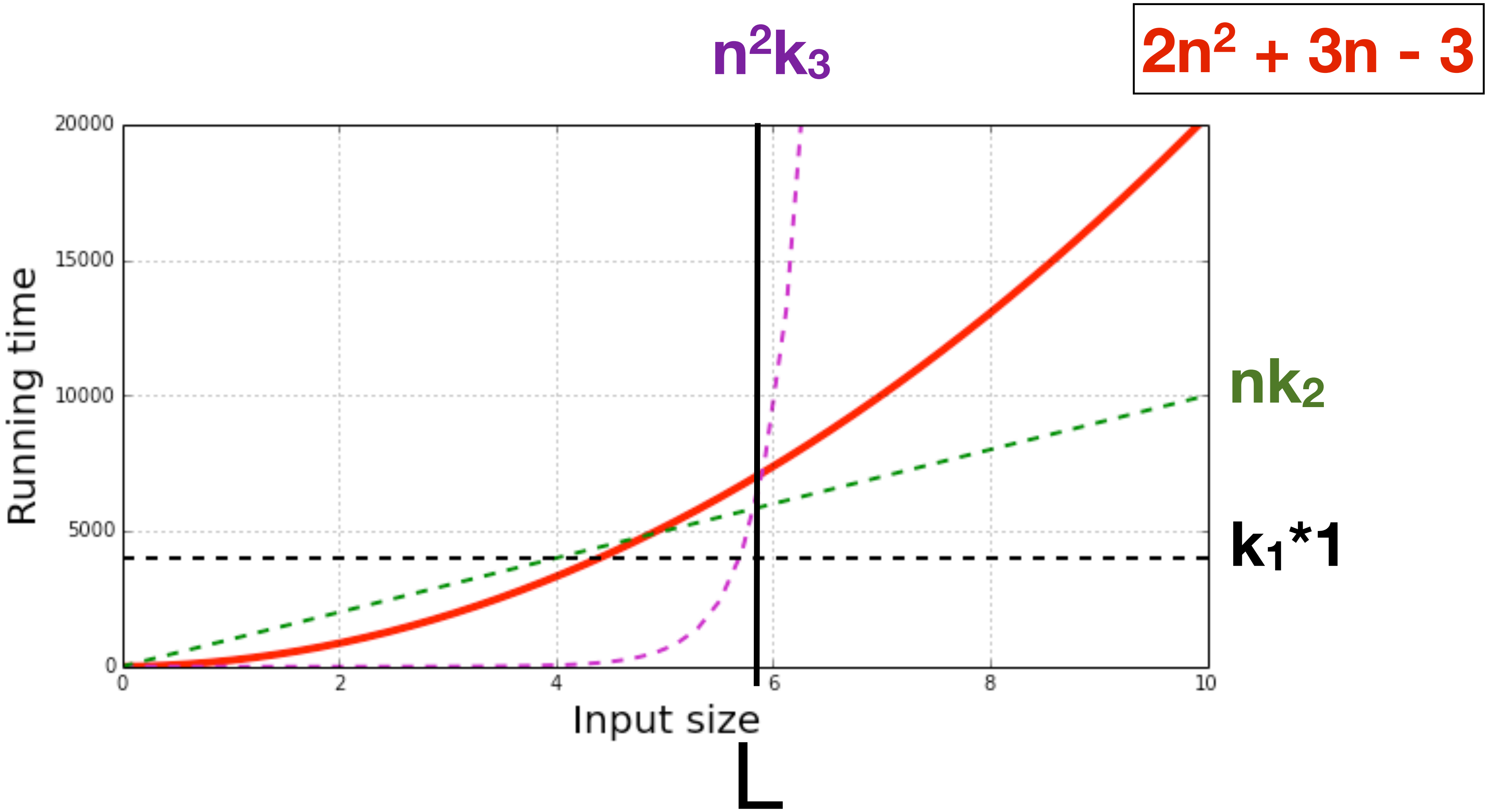
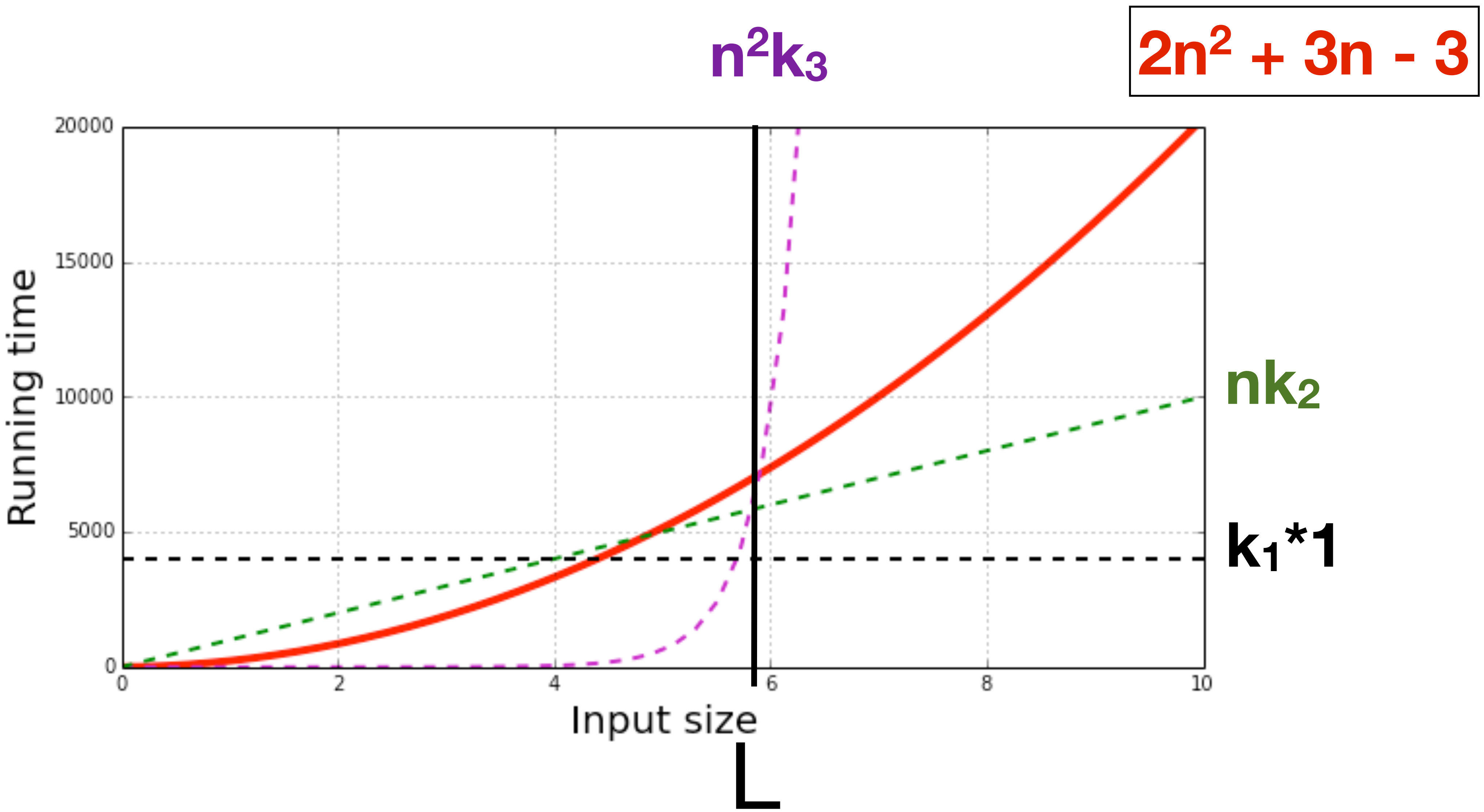


Insertion sort worst case



$2n^2 + 3n - 3$ is $O(n^2)$

Big O notation

Ignore constants

Ignore parts that do not contribute significantly

Basic efficiency classes

In order of increasing time complexity:

- Constant $O(1)$
- Logarithmic $O(\log N)$
- Linear $O(N)$
- Superlinear $O(N \log N)$
- Quadratic $O(N^2)$
- Exponential $O(2^N)$
- Factorial $O(N!)$

Basic efficiency classes

In order of increasing time complexity:

- Constant — $O(1)$ 😎
- Logarithmic — $O(\log N)$ 😆
- Linear — $O(N)$ 😃
- Superlinear — $O(N \log N)$ 🙂
- Quadratic — $O(N^2)$ 😒
- Exponential — $O(2^N)$ 😰
- Factorial — $O(N!)$ 😱

💩

Constant	$O(1)$	Running time does not depend on N	N doubles, T remains constant
Logarithmic	$O(\log N)$	Problem is broken up into smaller problems and solved independently. Each step cuts the size by a constant factor.	If N doubles, running time T gets slightly slower
Linear	$O(N)$	Each element requires a certain (fixed) amount of processing	If N doubles, running time T doubles (2*T)
Superlinear	$O(N \log N)$	Problem is broken up in sub-problems. Each step cuts the size by a constant factor and the final solution is obtained by combining the solutions.	If N doubles, running time T gets slightly bigger than double (2*T and a bit)
Quadratic	$O(N^2)$	Processes pairs of data items. Often occurs when you have double nested loop	If N doubles, running time T increases four times (4*T)
Exponential	$O(2^N)$	Combinatorial explosion (think about a family tree)	If N doubles, running time T squares (T*T)
Factorial	$O(N!)$	Finding all the permutations of N items	

Growth Rates

N	log(N)	N	Nlog(N)	N^2	2^N	N!
10	0.003 μs	0.01 μs	0.033 μs	0.1 μs	1 μs	3.63 ms
20	0.004 μs	0.02 μs	0.086 μs	0.4 μs	1 ms	77.1 years
30	0.005 μs	0.03 μs	0.147 μs	0.9 μs	1 sec	8.4×10^{15} years
40	0.005 μs	0.04 μs	0.213 μs	1.6 μs	18.3 min	
50	0.006 μs	0.05 μs	0.282 μs	2.5 μs	13 days	
100	0.007 μs	0.1 μs	0.644 μs	10 μs	4×10^{13} years	
1,000	0.010 μs	1 μs	9.966 μs	1 ms		
10,000	0.013 μs	10 μs	130 μs	100 ms		
100,000	0.017 μs	100 μs	1.67 ms	10 sec		
1,000,000	0.020 μs	1 ms	19.93 ms	16.7 min		
10,000,000	0.023 μs	10 ms	0.23 sec	1.16 days		
100,000,000	0.027 μs	0.1 sec	2.66 sec	115.7 days		
1,000,000,000	0.030 μs	1 sec	29.90 sec	31.7 years		

Measured in nanoseconds (10^{-9} secs)