MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #7 Solutions

Unless you're told otherwise, it's always OK to leave the answers to these kinds of questions as mathematical expressions rather than evaluating them as (sometimes huge) numbers. I give the numbers below as well as the expressions just to give you an idea of the sizes involved.

1. (a) $4! = 4 \times 3 \times 2 \times 1 = 24$
   (b) $\frac{10!}{8!} = 10 \times 9 = 90$
   (c) $\binom{10}{8} = \frac{10!}{8!2!} = \frac{10\times 9}{2} = 45$
   (d) $\binom{7}{3} = \frac{7!}{3!4!} = \frac{7\times6\times5}{3!} = 35$

2. (a) This is the number of ordered selections without repetition of 3 elements chosen from a set of 10 elements. So it is $\frac{10!}{(10-3)!} = \frac{10!}{7!} = 10 \times 9 \times 8 = 720$.
   (b) This is the number of unordered selections without repetition of 3 elements chosen from a set of 10 elements. So it is $\binom{10}{3} = \frac{10!}{7!\times 3!} = \frac{10\times9\times8}{3\times2} = 120$.
   (c) The selection in (a) is ordered (because president, treasurer and secretary are different roles), while the selection in (b) is unordered.

       Noticing this means you know that the answer to (a) will be bigger than the answer to (b) without calculation: there are always more ways to take an ordered selection (of at least two things) than an unordered selection. In this case for every 3 person team there are $3! = 6$ ways to appoint them as president, treasurer and secretary and so the answer for (a) is 6 times the answer for (b).
   (d) This is the number of unordered selections with repetition of 3 items from a set of 10 items. So it is $\binom{10+3-1}{10-1} = \binom{12}{9} = \frac{12!}{9!\times3!} = \frac{12\times11\times10}{3\times2} = 220$.
   (e) Each possible way to divide the prizes corresponds to a sequence of length 5 with each term in the set $\{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ with 3 elements (for example giving the first three prizes to Anastasia and the last two to Cadel corresponds to AAACC). So there are 3^5 possible ways.
   (f) This is the number of ordered selections without repetition of 6 elements chosen from a set of 6 elements (or permutations of length 6). So it is $6! = 6 \times 5 \times 4 \times 3 \times 2 = 720$.

3. If it fires 80 (or fewer) missiles, then each A-wing could have 5 (or fewer) missiles locked on to it. But if it fires 81, then the pigeonhole principle guarantees that at least one A-wing will have at least $\lceil \frac{81}{16} \rceil = 6$ missiles locked on.

4. (a) By the binomial theorem, the terms of this expansion will be $\binom{20}{i}x^i 2^{20-i}$ for $i = 0, 1, \ldots, 20$. So the relevant term will be for $i = 9$. This term is $\binom{20}{9}x^9 2^{11} = \binom{20}{9}2^{11}x^9$. So the coefficient is $\binom{20}{9}2^{11}$.
   (b) By the binomial theorem, the terms of this expansion will be $\binom{20}{i}(3x)^i 2^{20-i}$ for $i = 0, 1, \ldots, 20$. Because $(3x)^9 = 3^9x^9$, the relevant term will be for $i = 9$. This term is $\binom{20}{9}(3x)^9 2^{11} = \binom{20}{9}2^{11}3^9x^9$. So the coefficient is $\binom{20}{9}2^{11}3^9$.
   (c) By the binomial theorem, the terms of this expansion will be $\binom{20}{i}x^i(2)^{20-i}$ for $i = 0, 1, \ldots, 20$. Because $(3x^3)^i = 3^i x^{3i}$, the relevant term will be for $3i = 9$, so for $i = 3$. This term is $\binom{20}{3}(3x^3)^3 2^{17} = \binom{20}{3}2^{17}3^3x^9$. So the coefficient is $\binom{20}{3}2^{17}3^3$.

5. (a) There are $\binom{9}{4}$ ways to choose A and then, once it is chosen, there are $\binom{5}{3}$ ways to choose B so that $A \cap B = \emptyset$. So there are $\binom{9}{4}\binom{5}{3}$ ways to do this.

   (b) There are $\binom{9}{3}$ ways to choose C and then, once it is chosen, there are $\binom{6}{4}$ ways to choose D so that $C \cap D = \emptyset$. So there are $\binom{9}{3}\binom{6}{4}$ ways to do this.

   (c) There are $\binom{9}{7}$ ways to choose E and then, once it is chosen, there are $\binom{7}{3}$ ways to choose F so that $F \subseteq E$. So there are $\binom{9}{7}\binom{7}{3}$ ways to do this.

   (d) All of them equal 1260. This is because all of the questions are effectively asking how many ways there are to partition S into a set of size 4, a set of size 3, and a set of size 2.