MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #8 and Additional Practice Questions

1. The sample space is $\{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{HTT}, \mathrm{THH}, \mathrm{THT}, \mathrm{TTH}, \mathrm{TTT}\}$ (where HTH means heads on the first flip, tails on the second, heads on the third, and so on). Each of these outcomes occurs with probability $(\frac{1}{2})^3$ because the three flips are independent.

   $X = 0$ if the outcome is TTT.

   $X = 1$ if the outcome is in $\{\mathrm{HHT}, \mathrm{HTH}, \mathrm{THH}\}$.

   $X = 2$ if the outcome is in $\{\mathrm{HTT}, \mathrm{THT}, \mathrm{TTH}\}$.

   $X = 3$ if the outcome is HHH.

   Thus the probability distribution of X is given by

x	0	1	2	3
$\Pr(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

   .

2. (a) Without any further information, the best the doctor can answer is to say that about one in every three pairs of twins worldwide is a pair of identical twins and hence the probability is about $\frac{1}{3}$.

   (b) Let I be the event the twins are identical and M be the event they're both male.
   $\Pr(I) = \frac{1}{3}$ from the question.
   $\Pr(M|I) = \frac{1}{2}$ from the question.
   $\Pr(M|\overline{I}) = \frac{1}{4}$ from the question.
   By Bayes' theorem,

   $$\begin{aligned}\Pr(I|M) &= \frac{\Pr(M|I)\Pr(I)}{\Pr(M|I)\Pr(I) + \Pr(M|\overline{I})\Pr(\overline{I})} \\ &= \frac{\frac{1}{2}\times\frac{1}{3}}{(\frac{1}{2}\times\frac{1}{3}) + (\frac{1}{4}\times(1-\frac{1}{3}))} \\ &= \frac{1}{2}.\end{aligned}$$

   So the doctor can say that the probability is about $\frac{1}{2}$.

3. For all $x \in \{1,2,3,4,5,6\}$, $\Pr(X = x) = \frac{1}{6}$. For all $y \in \{1,2,3,4,5,6\}$, $\Pr(Y = y) = \frac{1}{6}$. Note that X and Y are independent.

   (a) $\Pr(Z = 7 \wedge Y = 2) = \Pr(X = 5 \wedge Y = 2) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.
   $\Pr(Y = 2) = \frac{1}{6}$.
   So $\Pr(Z = 7 \mid Y = 2) = \frac{1}{36} / \frac{1}{6} = \frac{1}{6}$.

   (b) Again $\Pr(Y = 2 \wedge Z = 7) = \Pr(X = 5 \wedge Y = 2) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.
   $Z = 7$ exactly when $(X,Y) \in \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$ and each event in this set has probability $\frac{1}{36}$. So $\Pr(Z = 7) = 6 \times \frac{1}{36} = \frac{1}{6}$.
   So $\Pr(Y = 2 \mid Z = 7) = \frac{1}{36} / \frac{1}{6} = \frac{1}{6}$.

   (c) Yes. There are at least three ways to show this:
   - by noting that $\Pr(Z = 7 \mid Y = 2) = \Pr(Z = 7)$ from (a);
   - by noting that $\Pr(Y = 2 \mid Z = 7) = \Pr(Y = 2)$ from (b);
   - by noting that $\Pr(Y = 2 \wedge Z = 7) = \Pr(Y = 2)\Pr(Z = 7)$.

   (d) $\Pr(Z = 6 \wedge Y = 2) = \Pr(X = 4 \wedge Y = 2) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.
   $\Pr(Y = 2) = \frac{1}{6}$.
   So $\Pr(Z = 6 \mid Y = 2) = \frac{1}{36} / \frac{1}{6} = \frac{1}{6}$.

   (e) What is $\Pr(Y = 2 \mid Z = 6)$?
   Again $\Pr(Y = 2 \wedge Z = 6) = \Pr(X = 4 \wedge Y = 2) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.
   $Z = 6$ exactly when $(X,Y) \in \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$ and each event in this set has probability $\frac{1}{36}$. So $\Pr(Z = 6) = 5 \times \frac{1}{36} = \frac{5}{36}$.
   So $\Pr(Y = 2 \mid Z = 6) = \frac{1}{36} / \frac{5}{36} = \frac{1}{5}$.

   (f) No. There are at least three ways to show this:
   - by noting that $\Pr(Z = 6 \mid Y = 2) \neq \Pr(Z = 6)$ from (d);
   - by noting that $\Pr(Y = 2 \mid Z = 6) \neq \Pr(Y = 2)$ from (e);
   - by noting that $\Pr(Y = 2 \wedge Z = 6) \neq \Pr(Y = 2)\Pr(Z = 6)$.

   (g) No, this follows from our answer to (f). To show two random variables Y and Z are not independent it is enough to find some y and z such that

   $$\Pr(Y = y \wedge Z = z) \neq \Pr(Y = y)\Pr(Z = z).$$

4. (a) $\Pr(X = 8) = \frac{1}{256}$ because exactly when the string is 11111111.
   $\Pr(Y = 8) = \frac{1}{256}$ because exactly when the string is 00000000.
   $\Pr(X = 8 \wedge Y = 8) = 0$ because there is no binary string of length 8 with 8 0s and 8 1s.
   Thus $\Pr(X = 8 \wedge Y = 8) \neq \Pr(X = 8)\Pr(Y = 8)$ and so X and Y are not independent.
   (There are many other examples that will show this, as well).

   (b) Because the string has length 8, $Z = X + Y$ is always 8. So the probability distribution of Z is given by

z	8
$\Pr(Z = z)$	1

   .