Monash University
Faculty of Information Technology

Lecture 14
Turing Machines and Computability

Slides by David Albrecht (2011),
some additions and modifications by Graham Farr (2013, 2016).

FIT2014 Theory of Computation

Overview

- Turing Machines
- Converting Finite Automaton to a Turing Machine
- Building Turing Machines
- Turing machines for computing functions
- Church's thesis

Effective Process

- Can be done with pencil and paper.
- Is a finite set of instructions.
- Demands neither insight or ingenuity.
- Will definitely work without error.
- Produces in a finite number of steps either:
  - A final result, or
  - If the result is a sequence, each symbol in the sequence.

Alan Turing (1912-1954)
http://www.npg.org.uk/collections/search/portrait/mw165875

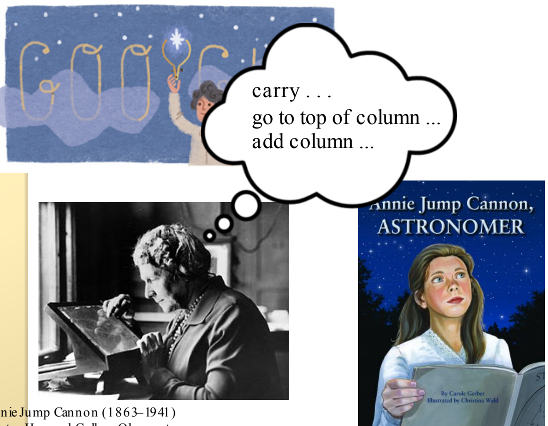
How to model computation?

Consider a person doing a computation (pencil & paper).

At any given time, the person is …
- focused on some particular **position** on the paper;
- reading the **symbol** at the current position;
- in some particular mental **state**, i.e., is doing some particular part of the computation.

Depending on the state and symbol, the person then …
- writes a **symbol** there
  - (possibly overwriting what is already there);
- may change their **state**;
- **moves** their attention nearby.

www.google.com 11 Dec 2014



Annie Jump Cannon (1863–1941)
Photo: Harvard College Observatory
http://www.skyandtelescope.com/astronomy-news/digitizing-harvards-century-of-sky/

http://www.pelicanpub.com/proddetail.php?prod=9781589809116

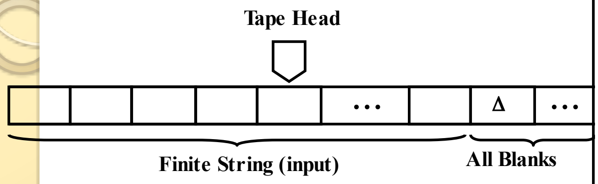
Turing machine

Set-up:
- infinitely long **tape** divided into cells
- each cell may contain a **symbol** from a finite alphabet
- **head** scans one tape cell at a time
- at any time, the machine is in some **state**

Program:
- For each *state* and *symbol*, specify the *next state*, *next symbol*, and *direction* (one step left, or one step right).

Computation step:
- At each step, apply the appropriate instruction.

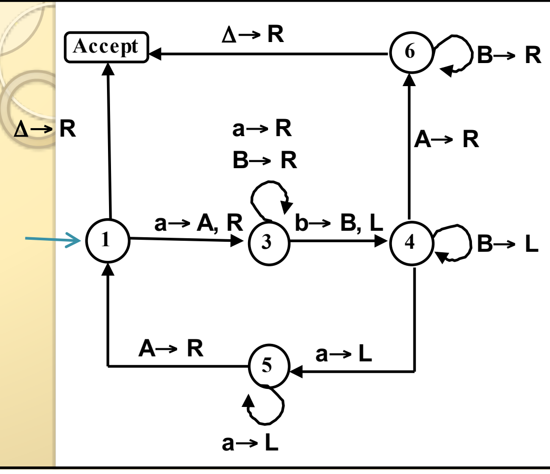
Turing machine



Tape Head can:

- Move **left** and **right**.
- **Read letters** from the tape.
- **Write letters** onto the tape.

Computation is ***deterministic***.



TM Components

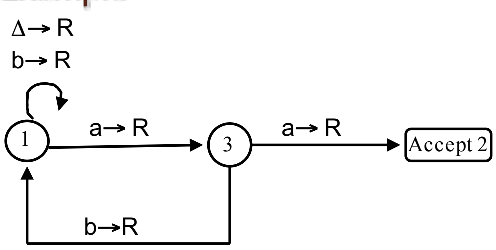
- A Tape and Tape Head
- An Input Alphabet
- A Tape Alphabet
- A finite set of states
  - Each state is numbered by an integer ≥ 1.
  - **Start State** (1)
  - **Accept State** (2)
  - Note: a *Reject state* is optional (if crash = reject).
- A finite set of rules **letter → letter, direction** between states.

Definitions

For a Turing Machine T

- **Accept(T)**
  - The set of strings leading to the **Accept** state.
  - Called the **language accepted by T.**
- **Reject(T)**
  - The set of strings that **crash**, or lead to a **Reject** state (if there is one), during execution.
- **Loop(T)**
  - The set of strings that cause T to loop forever.

Example



- **Accept(T)** = strings with a double **aa**
- **Reject(T)** = strings without a double **aa** that end in **a**
- **Loop(T)** = ε or strings without a double **aa** that end in **b**

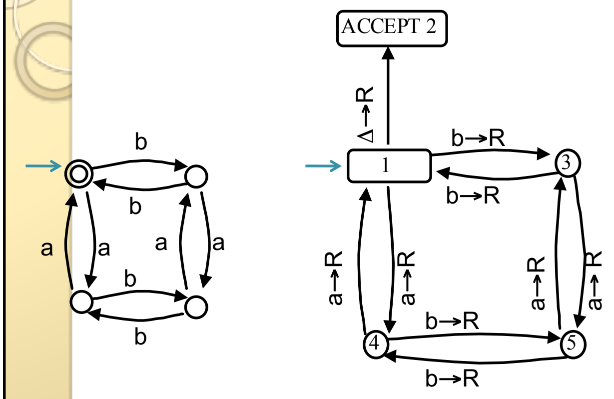
Regular Languages

Every **Regular Language** can be accepted by a **Turing Machine**.

Convert Finite Automaton into a Turing Machine

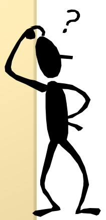
- Label start state with **1**.
- Label all other states with a integer ≥ 3.
- Change the edge labels.
  - **a** to **a→R**
  - **b** to **b→R**
- Delete the second circle from all the Final states, and add an edge from each Final state to **Accept 2**, labelled with **Δ →R**.

EVEN-EVEN



Problem

Build a Turing Machine that accepts the language **{a^nb^n : n ≥ 0}**.



If the current letter is blank, then **Accept** string.

Loop {

If current letter is **a**, then change **a** to **A** & move **right**.

Move **right** over any **a**'s and **B**'s.

If current letter is **b**, then change **b** to **B** & move **left.**

Move **left** over any **B**'s.

If current letter is **A**, then move **right** & **exit** the loop.

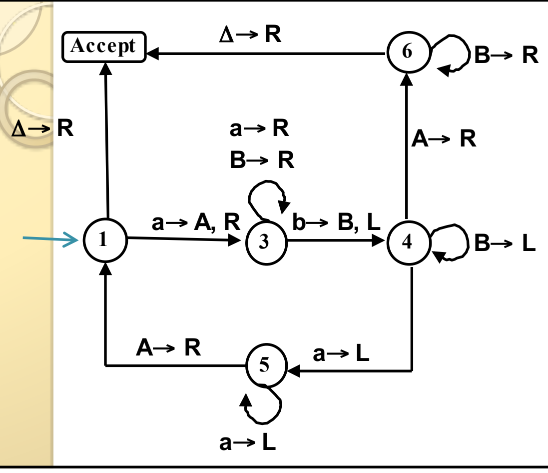
Else if current letter is **a** move **left** over any **a**'s.

If current letter is **A**, then move **right**.

}

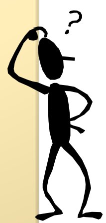
Move **right** over any **B**'s.

If current letter is **blank**, then **Accept** string.



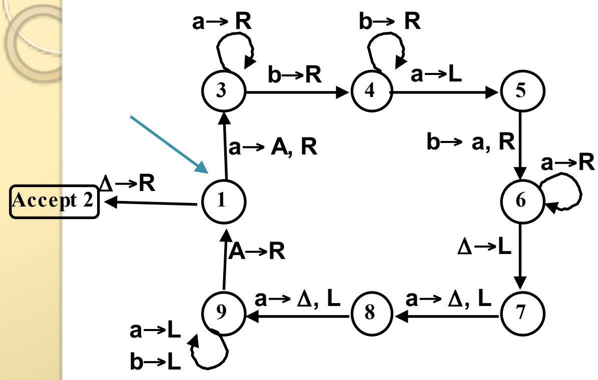
Problem

Build a Turing Machine that accepts the language **{$a^nb^na^n$: n ≥ 0}**.



Loop {
  If current letter is **blank**, then **Accept** string.
  If current letter is **a**, then change **a** to **A** & move **right**.
  Move **right** over **a*bb***.
  If current letter is **a**, then move **left**.
  If current letter is **b**, then change **b** to **a** & move **right**.
  Move **right** over any **a**' s.
  If current letter is **blank**, then **delete** 2 **a**' s on the left.
  Move **left** over any **a**' s and **b**' s.
  If current letter is **A**, then move **right**.
}

TM for $a^n b^n a^n$



Other Machines

- **Queue automaton**
  - Like a deterministic PDA, but uses a Queue.
- **2PDA**
  - Like a deterministic PDA, but with 2 Stacks.
- **NTM**
  - A Nondeterministic Turing Machine.
- **kTM**
  - A Turing Machine with k Tapes.

Theorems

- Any language which a Turing machine can accept can also be defined by any of these machines, and visa-versa.
- There are algorithms to convert all these machines (including Turing Machines) into each other.

Turing machines for computing functions

So far, our Turing machines just accept/reject.
TMs can also compute functions.

What kinds of objects can Turing machines work with?

- any objects that can be encoded as strings …

ASCII Code

0	bbaaaa	5	bbabab
1	bbaaab	6	bbabba
2	bbaaba	7	bbabbb
3	bbaabb	8	bbbaaa
4	bbabaa	9	bbbaab

Binary Code

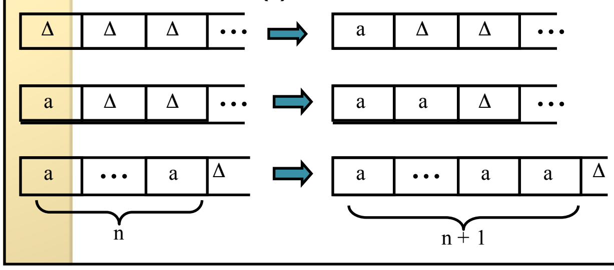
0	a
1	b
2	ba
3	bb
4	baa
5	bab
6	bba
7	bbb
:	:

Unary Code

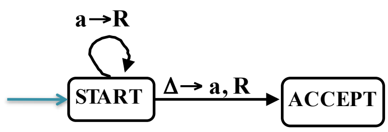
0	Δ	Δ
1	a	a
2	aa	a^2
3	aaa	a^3
4	aaaa	a^4
5	aaaaa	a^5
6	aaaaaa	a^6
7	aaaaaaa	a^7
:	:	:

Successor

Using the unary code for natural numbers build a Turing Machine that represents the function
f(n) = n + 1.

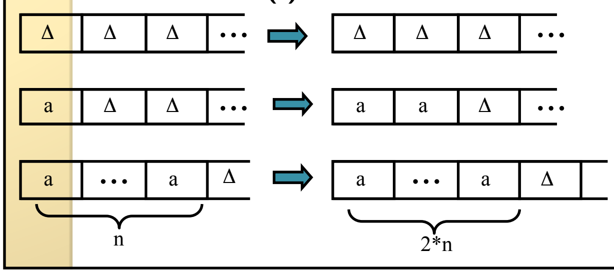


Successor

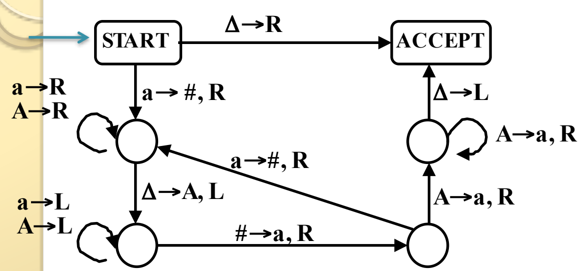


Double

Using the unary code for natural numbers build a Turing Machine that represents the function
f(n) = 2*n.



Double



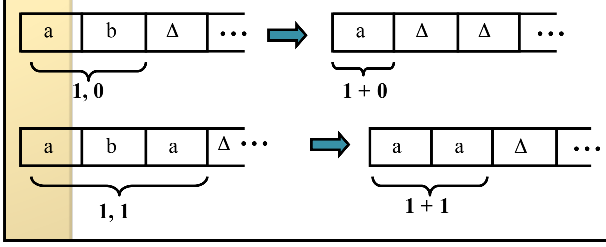
Unary Code for Tuples of Integers

- Tuples of natural numbers
- Example: **1,0,2,3**
- Encoding:
  - Each integer is coded using the unary code as a string of **a**'s
  - Integers are separated by a **b**.
- Example: **abbaabaaa**

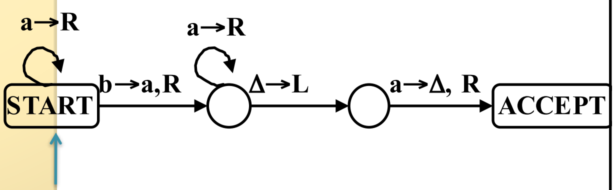
Addition

Using the unary code of natural numbers build a Turing Machine that represents the function

f(n, m) = n+m.



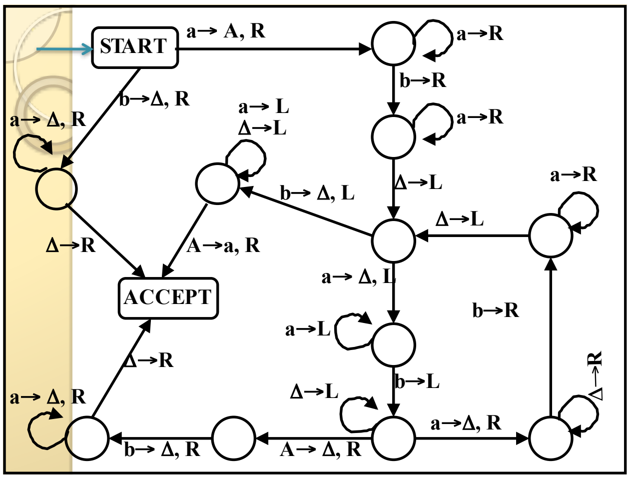
Addition



Monus

Using the unary code of tuples of natural numbers build a Turing Machine that represents the function:

$$f(n,m) = \begin{cases} n - m, & n \geq m \\ 0, & \text{Otherwise} \end{cases}$$



Definition

A function is **computable** if it can be represented as:

- A Turing Machine
- Input
  - sequences of natural numbers
- Output
  - one natural number

Variations on Turing machines

- Direction: stay still, as well as Left/Right
- Tapes:
  - two-way infinite
  - multiple tapes
  - separate input, output, work tapes
  - "tapes" of 2 or more dimensions
- ……

Same class of computable functions

Other approaches to computability

recursive function theory
- starting with Kurt Gödel, 1931

lambda calculus
- Alonzo Church, 1936

Turing machines
- Alan Turing, 1936-37

Same class of computable functions

Kurt Gödel (1906-1978)

http://www-history.mcs.st-andrews.ac.uk/Biographies/Godel.html

Alonzo Church (1903-1995)

http://www-history.mcsst-andrews.ac.uk/Biographies/Church.html

Church-Turing Thesis

Any function which can defined by an algorithm can be represented by a Turing Machine.

Note: not a Theorem! But widely accepted.

Evidence:
- different approaches to computability end up in agreement
- long experience, that algorithms can be implemented as programs, and therefore on Turing machines
- no known counterexamples, i.e., no algorithms which seem to be unimplementable

Alan Turing

Alan Turing Centenary Year (2012) website: http://www.turingcentenary.eu/

B. Jack Copeland, *Turing: Pioneer of the Information Age*, OUP, 2013.
Andrew Hodges, *Alan Turing: The Enigma*, Vintage, London, 1983.
Andrew Hodges, *Turing*, Phoenix, London, 1997.

Turing bibliography: http://www.turing.org.uk/sources/biblio.html

G. Farr, Calls for a posthumous pardon … but who was Alan Turing?, *The Conversation*, 22 Dec 2011, https://theconversation.com/calls-for-a-posthumous-pardon-but-who-was-alan-turing-4773

G. Farr, The Imitation Game: is it history, drama or myth?, *The Conversation*, 9 Jan 2015, https://theconversation.com/the-imitation-game-is-it-history-drama-or-myth-35849

Revision

- Know what a Turing Machine is, and how to use one.
- Be able to convert a Finite Automaton into a TM.
- Be able to build a Turing Machine to define a language.
- Know the unary code for natural numbers, and tuples
- Know what a computable function is, and how to define one using a TM.
- Know and understand the Church-Turing Thesis.

Turing machine software (see Moodle)

- Tuatara (graphical environment)

Reading:

- Sipser, Ch 3: Section 3.1, pp. 165-176, 181-190.

Preparation:

- Sipser, Ch 3, start & end of Section 3.2; Section 3.3