Housekeeping

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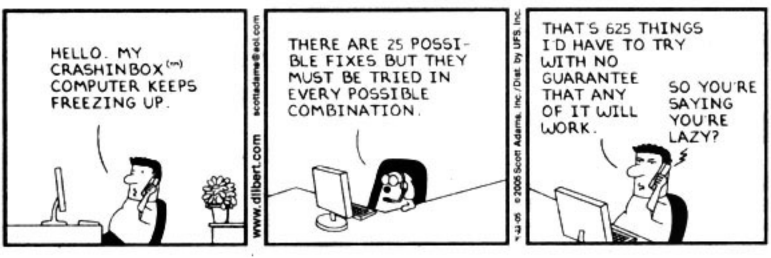
Office hours are Tues 12-1pm; Wed 1-2pm.

Other times by appointment.

There will be no catch-up lecture for the class we miss on Friday.
If your tutorial is on Friday then consult moodle for instructions.

Lecture 19: Selections and Arrangements



Ordered selections without repetition

A reviewer is going to compare ten phones and list, in order, a top three. In how many ways can she do this? More generally, how many ways are there to arrange r objects chosen from a set of n objects?

In our example, the reviewer has 10 options for her favourite, but then only 9 for her second-favourite, and 8 for third-favourite. So there are $10 \times 9 \times 8$ ways she could make her list.

For an ordered selection without repetition of r elements from a set of n elements there are

$$\begin{array}{rl}
n & \text{options for the 1st element} \\
n-1 & \text{options for the 2nd element} \\
n-2 & \text{options for the 3rd element} \\
\vdots & \qquad\qquad\vdots \\
n-r+1 & \text{options for the } r\text{th element.}
\end{array}$$

So we have the following formula.

The number of ordered selections without repetition of r elements from a set of n elements ($0 \leqslant r \leqslant n$) is

$$n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

When $r = n$ and all the elements of a set S are ordered, we just say that this is a *permutation of S*. Our formula tells us there are $n!$ such permutations. For example, there are $3! = 6$ permutations of the set $\{a, b, c\}$:

$$(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a).$$

Unordered selections without repetition

What if our reviewer instead chose an unordered top three? In how many ways could she do that? More generally, how many ways are there to choose (without order) r objects from a set of n objects?

> A *combination* of r elements from a set S is a subset of S with r elements.

For every unordered list our reviewer could make there are $3! = 6$ corresponding possible ordered lists. And we've seen that she could make $10 \times 9 \times 8$ ordered lists. So the number of unordered lists she could make is $\frac{10 \times 9 \times 8}{6}$.

For every combination of r elements from a set of n elements there are $r!$ corresponding permutations. So, using our formula for the number of permutations we have the following.

The number of combinations of r elements from a set of n elements $(0 \leqslant r \leqslant n)$ is

$$\frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}.$$

Notice that the notation $\binom{n}{r}$ is used for $\frac{n!}{r!(n-r)!}$. Expressions like this are called *binomial coefficients*. We'll see why they are called this in the next lecture.

Ordered selections with repetition

An ordered selection of r elements from a set X is really just a sequence of length r with each term in X. If X has n elements, then there are n possibilities for each term and so:

> The number of sequences of r terms, each from some set of n elements, $(0 \leqslant r \leqslant n)$ is
>
> $$\underbrace{n \times n \times \cdots \times n}_{r} = n^r.$$

Questions

19.1 A bank requires a PIN that is a string of four decimal digits. How many such PINs are there? How many are made of four different digits?

A PIN is an ordered selection with repetition of four elements from the set $\{0, \ldots, 9\}$. So there are $10^4 = 10000$ possible PINS.

A PIN with four *different* digits is a permutation of four elements from the set $\{0, \ldots, 9\}$. So there are $\frac{10!}{6!} = 5040$ possible PINS with four different digits.

19.2 How many binary strings of length 5 are there? How many of these contain exactly two 1s?

There are $2^5 = 32$ binary strings of length 5.
There are $\binom{5}{2} = 10$ that contain exactly two 1s.

Unordered selections with repetition

A shop has a special deal on any four cans of soft drink. Cola, lemonade and sarsaparilla flavours are available. In how many ways can you select four cans?

We can write a selection in a table, for example,

C	L	S
•	••	•

and

C	L	S
	•	•••

.

We can change a table like this into a string of 0s and 1s, by moving from left to right reading a "•" as a 0 and a column separator as a 1. The tables above would be converted into

$$0\,1\,0\,0\,1\,0 \quad \text{and} \quad 1\,0\,1\,0\,0\,0$$

Notice that each string has four 0s (one for each can selected) and two 1s (one fewer than the number of flavours). We can choose a string like this by beginning with a string of six 0s and then choosing two 0s to change to 1s. There are $\binom{6}{2}$ ways to do this and so there are $\binom{6}{2}$ possible can selections.

An unordered selection of r elements, with repetition allowed, from a set X of n elements can be thought of as a multiset with r elements, each in X. As in the example, we can represent each such multiset with a string of r 0s and $n-1$ 1s. We can choose a string like this by beginning with a string of $n+r-1$ 0s and then choosing $n-1$ 0s to change to 1s.

> The number of multisets of r elements, each from a set of n elements, is
> $$\binom{n+r-1}{n-1} = \frac{(n+r-1)!}{(n-1)!r!}.$$

The pigeonhole principle

The pigeonhole principle is a reasonably obvious statement, but can still be very useful.

> If n items are placed in m containers with $n > m$, then at least one container has at least two items.

Example

If a drawer contains only blue, black and white socks and you take out four socks without looking at them, then you are guaranteed to have two of the same colour.

We can generalise the pigeonhole principle as follows.

> If n items are placed in m containers, then at least one container has at least $\lceil \frac{n}{m} \rceil$ items.

In the above $\lceil \frac{n}{m} \rceil$ means the smallest integer greater than $\frac{n}{m}$ (or $\frac{n}{m}$ "rounded up").

Example

If 21 tasks have been distributed between four processor cores, the busiest core must have been assigned at least 6 tasks.

Questions

19.3 In a game, each of ten players holds red, blue and green marbles, and places one marble in a bag. How many possibilities are there for the colours of marbles in the bag? If each player chooses their colour at random are all of these possibilities equally likely?

There are $\binom{n+r-1}{n-1} = \binom{3+10-1}{3-1} = \binom{12}{2} = \frac{12!}{10!2!} = 66$ possibilities. There are *not* equally likely. For example, the "all red" case is less likely than having 3 red, 3 blue and 4 green.

19.4 How many ways are there to partition (that is, split up) a set with 10 elements into one class of 5 elements, one class of 3 elements and one class of 2 elements? How many ways are there to partition a set with 10 elements into one class of 6 elements and two classes of 2 elements each?

In the first case, there are $\binom{10}{5}\binom{5}{3} = 252 \times 10 = 2520$ ways.
In the second case, there are $\binom{10}{6}\binom{4}{2}/2 = 210 \times 3 = 630$ ways.