MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #9 and Additional Practice Questions

Tutorial Questions

1. (a) Find the expected value and variance of a random variable X with probability distribution given by the table below.

x	0	1	2
$\Pr(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

   (b) Let p and q be real numbers. A random variable Y has expected value 2 and its probability distribution is given by the table below.

y	0	1	2	3
$\Pr(Y = y)$	p	$\frac{1}{12}$	$\frac{1}{3}$	q

   Find p and q.

2. Write down the next four values of each of the following recursive sequences.

   (a) $r_0 = 3, \qquad r_n = 2r_{n-1} - 1$ for all integers $n \geq 1$.

   (b) $t_0 = 1, t_1 = 1, t_2 = -2, \qquad t_n = t_{n-1}t_{n-3}$ for all integers $n \geq 3$.

3. Captain Kirk fires photon torpedoes, one at a time, at a Romulan ship until it is destroyed. Each torpedo has a 10% chance of destroying the ship (independent of any other torpedoes). What is the probability the ship is destroyed by one of his first three torpedoes?

4. Two children, Mary and Tran, are inventing a game using a spinner that spins green, blue or red so that Pr(green) $= \frac{1}{4}$, Pr(blue) $= \frac{5}{8}$, Pr(red) $= \frac{1}{8}$.

   - Mary wants a spin of green to mean that Tran has to give her 6 of his jellybeans.
   - Tran wants a spin of blue to mean that Mary has to give him 3 of her jellybeans.
   - They both want the game to be "fair".

   What should happen on a spin of red? What definition of "fair" are you using?

5. (a) Invent an example of a random variable X such that $\Pr(X = E[X]) = 0$.

   (b) Invent an example of a random variable Y such that $\mathrm{E}[Y]$ is negative but Y is extremely likely to be positive.

   (c) Invent an example of a random variable Z that is never negative and has a high probability of being at least $3\mathrm{E}[Z]$ (where "high" means as high as you can make it).

6. Markov's inequality is an important result in probability. It says that, for any random variable X which only takes non-negative values and any positive real number a, we have $\Pr(X \geq a) \leq \frac{E[X]}{a}$.

   (a) How good was your answer to 5(c)?

   (b) Verify this inequality for the random variables in 1(a) and 1(b) using $a = 2$ and $a = 3$.

   (c) The mean income of a certain country is \$10000 per year, what is the greatest fraction of the country's population that can earn at least \$100000 per year? When does this happen?

Practice Questions

1. (a) A number X is chosen uniformly at random from the set $\{100, 101, \ldots, 199\}$. What is $\mathrm{E}[X]$? What is $\mathrm{Var}[X]$?

   (b) A number Y is chosen uniformly at random from the set $\{101, 102, \ldots, 200\}$. What is $\mathrm{E}[Y]$? What is $\mathrm{Var}[Y]$?

2. (a) Chebyshev's inequality says, for any random variable Y and positive integer b, that

$$\Pr\big(|Y - \mathrm{E}[Y]| \geq b\big) \leq \frac{\mathrm{Var}[Y]}{b^2}.$$

   Use Markov's inequality to prove Chebyshev's inequality.

   Hint: Define a random variable $X = (Y - \mathrm{E}[Y])^2$. What is $E[X]$?

   (b) Prove Markov's inequality.

   Hint: Define a random variable I such that $I = 0$ if $X < a$ and $I = 1$ if $X \geq a$. Show that aI is always less than or equal to X. Now what can you say about $\mathrm{E}[aI]$ and $\mathrm{E}[X]$?

   (c) Who has a cooler name, Markov or Chebyshev?

3. The sides of three fair six-sided dice are numbered as follows.

   - Die A: 3, 3, 3, 3, 3, 6
   - Die B: 2, 2, 2, 5, 5, 5
   - Die C: 1, 4, 4, 4, 4, 4

   (a) For each die, what is the expected value of a roll of that die?

   (b) If die A and die B are rolled together, what is the probability that A rolls a greater number than B?

   (c) If die B and die C are rolled together, what is the probability that B rolls a greater number than C?

   (d) If die C and die A are rolled together, what is the probability that C rolls a greater number than A?

   (e) A hare and a tortoise have these three dice in front of them. When the starter's gun fires they will each take one of the dice and roll it. Whoever rolls the higher number wins. The hare has lightning reflexes and poor impulse control. The tortoise, like all tortoises, is an expert mathematician. Who's your money on?