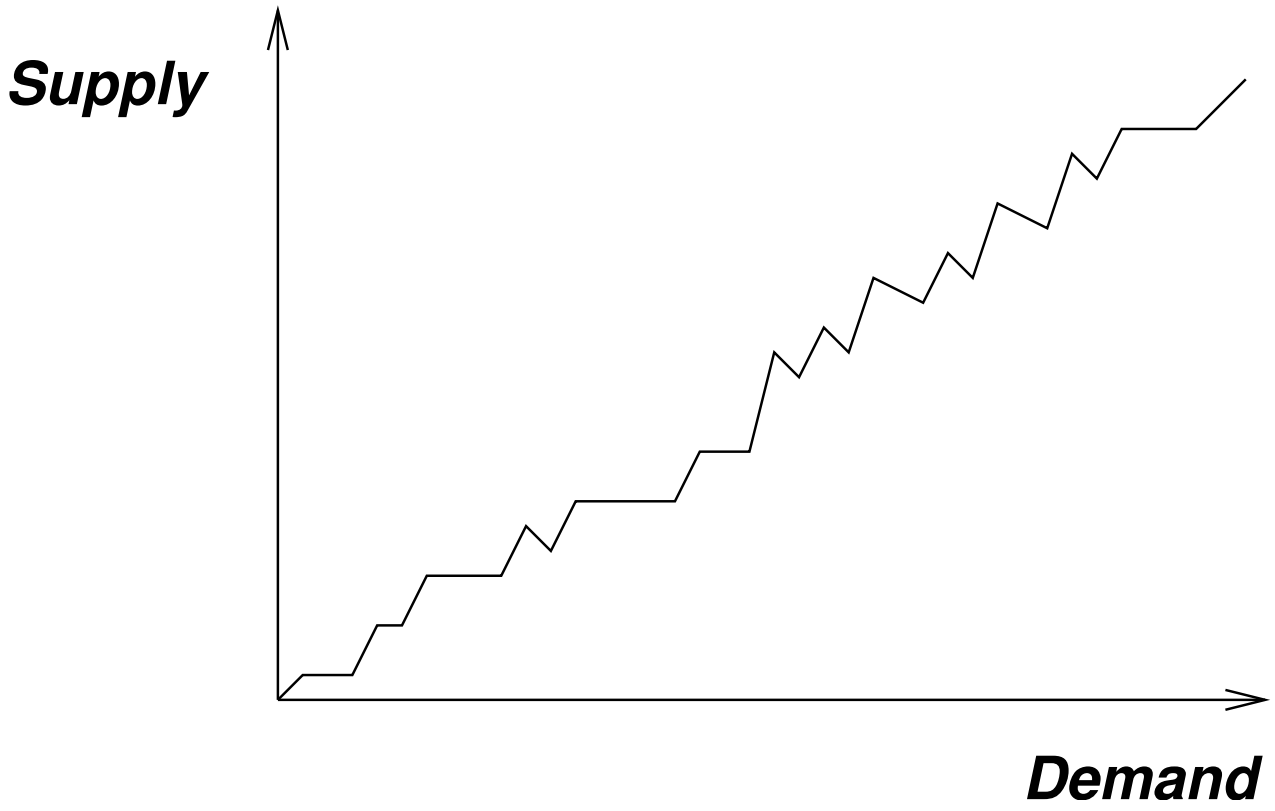
Lecture 29: Graph Theory

What graph theory is NOT about



What graph theory IS about

A graph consists of a set of objects called vertices and a list of pairs of vertices, called edges.

Graphs are normally represented by pictures, with vertex A represented by a dot labelled A and each edge AB represented by a curve joining A and B.

Such pictures are helpful for displaying data or relationships, and they make it easy to recognise properties which might otherwise not be noticed.

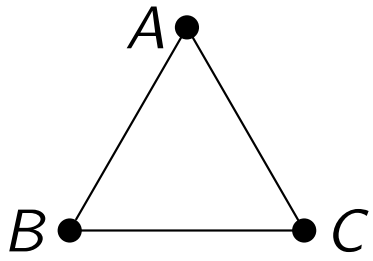
The description by lists of vertices and edges is useful when graphs have to be manipulated by computer. It is also a useful starting point for precise definitions of graph concepts.

Examples of graphs

Description

Vertices: A, B, C
Edges: AB, BC, CA

Picture

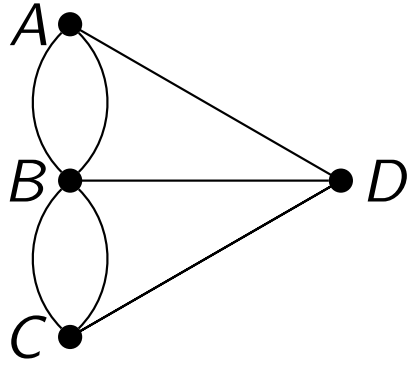


Such a graph, with at most one edge between each pair of vertices, and no vertex joined to itself, is called a simple graph.

Description

Vertices: A, B, C, D
Edges: $AB, AB, BC, BC,$
AD, BD, CD

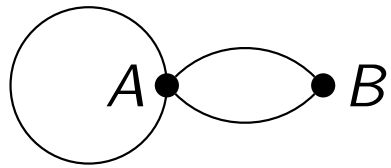
Picture



The two edges AB which join the same pair of vertices are called parallel edges.

Description **Picture**

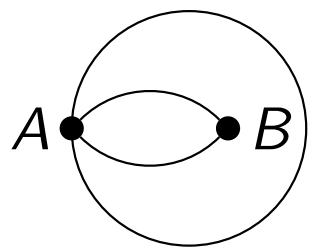
Vertices: A, B
Edges: AA, AB, AB



The edge joining A to A is called a loop.

The name multigraph is used when loops and/or parallel edges are *allowed*.

Warning: A graph can be represented by pictures that look very different. This last example could be redrawn as:



Problems given by graphs

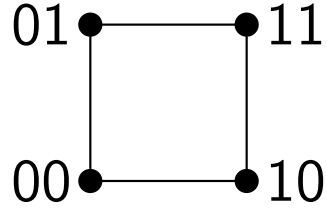
Many problems require vertices to be connected by a "path" of successive edges. We shall define paths (and related concepts) next lecture, but the following examples illustrate the idea and show how often it comes up.

They also show how helpful it is to have graph pictures when searching for paths.

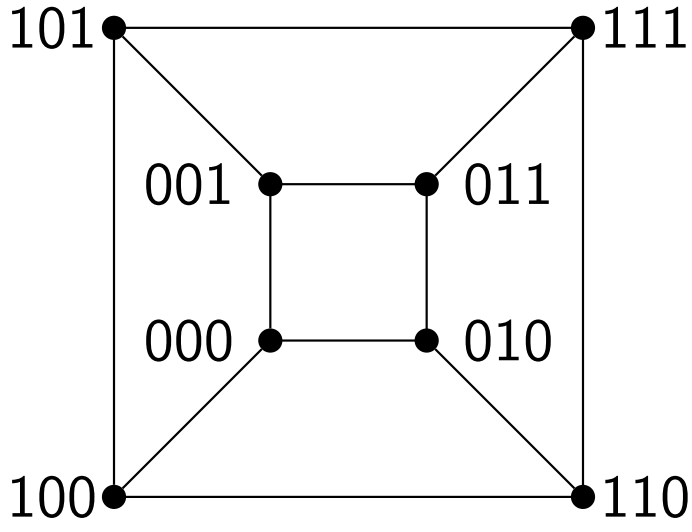
1. Gray Codes

The binary strings of length n are taken as the vertices, with an edge joining any two vertices that differ in only one digit. This graph is called the n-cube.

E.g. the 2-digit binary strings form a square (a "2-cube").

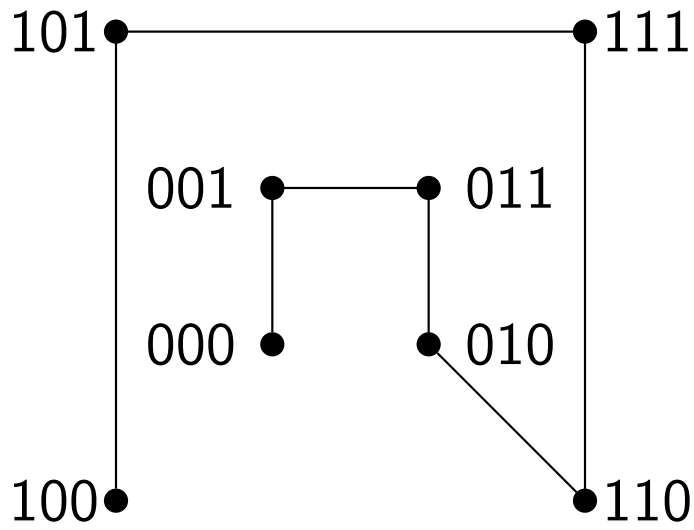


and the 3-digit binary strings form an ordinary cube (a "3-cube").



A Gray code of length n is a path which includes each vertex of the n-cube exactly once. E.g. here is a path in the 3-cube which gives the Gray code

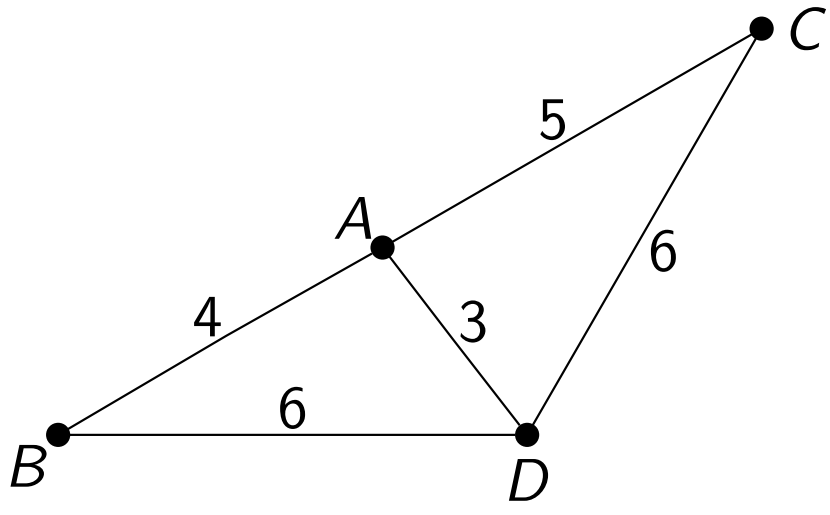
$$000, 001, 011, 010, 110, 111, 101, 100$$



The n-cube has been popular as a *computer architecture* in recent years. Processors are placed at the vertices of an n-cube (for $n = 15$ or so) and connected along its edges.

2. Travelling salesman problem

Vertices are towns. Two towns are joined by an edge labelled ℓ if there is a road of length ℓ between them. E.g.



The problem is to find a path of minimal length which includes all towns, in this case $BADC$.

3. Jug problems

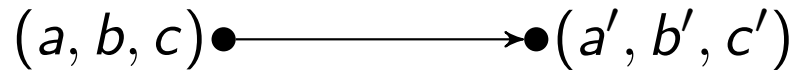
Suppose we have three jugs, which hold exactly 3, 4 and 6 litres respectively. We fill the 6-litre jug, and then pour from one jug to another, always stopping when the jug being poured to becomes full or when the jug being poured from becomes empty.
Is it possible to reach a state where one jug contains 1 litre and another contains 5 litres?
We represent each possible state by a triple (a, b, c), where

$$a = \text{number of litres in the 3-litre jug}$$

$$b = \text{number of litres in the 4-litre jug}$$

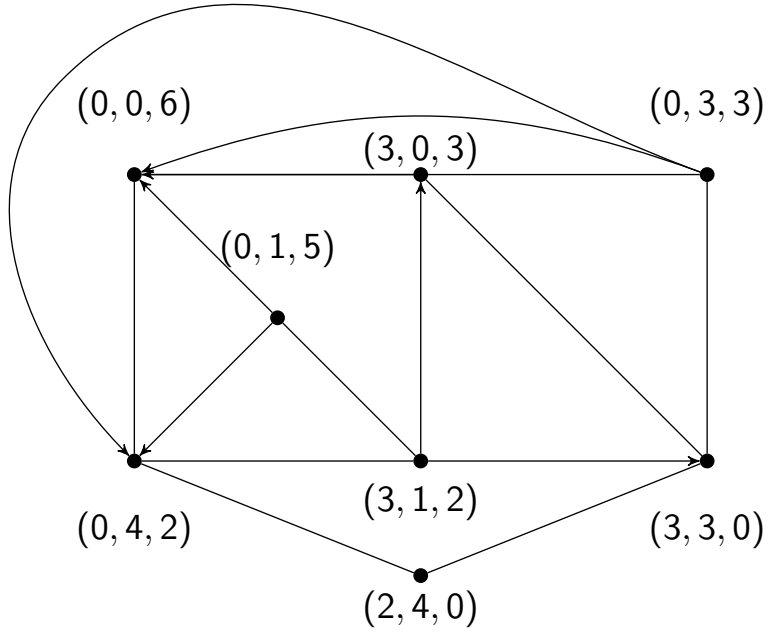
$$c = \text{number of litres in the 6-litre jug}$$

Each state is a vertex, and if state (a', b', c') can be reached from (a, b, c) by pouring as described above, we put a directed edge in the graph:



If (a, b, c) can also be reached from (a', b', c'), we join them by an ordinary edge.

Then, listing the states that can be reached from $(0, 0, 6)$, we find the graph on the right.



A graph like this, with some directed edges, is called a directed graph or digraph.

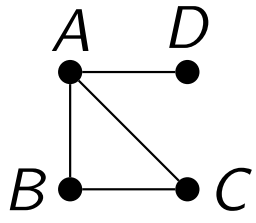
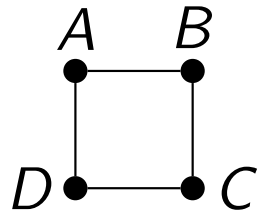
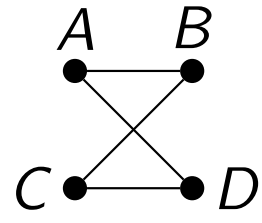
This particular digraph shows that

$$(0, 0, 6) \to (0, 4, 2) \to (3, 1, 2) \to (0, 1, 5)$$

hence we can start with a full 6-litre jug and in three pourings get 1 litre in the 4-litre jug and 5 litres in the 6-litre jug.

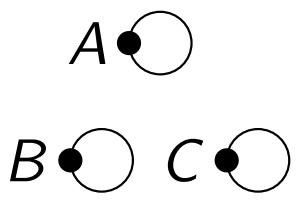
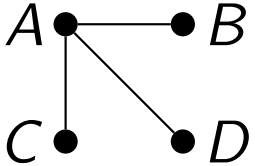
Questions

29.1 Write down descriptions of the following graphs

picture			
description	Vertices A,B,C,D Edges AB,AC,AD,BC	Vertices A,B,C,D Edges AB,AD,BC,CD	Vertices A,B,C,D Edges AB,AD,BC,CD

Questions

29.2 Draw pictures of graphs with the following descriptions.

	A, B, C	A, B, C, D
description	AA, BB, CC	AB, AC, AD
picture		

Questions

29.3 Use the following picture of the 4-cube to find a Gray code of length 4.