Lecture 30: Walks, paths and trails

There are several ways of “travelling” around the edges of a graph.

A *walk* is a sequence

$$V_1, e_1, V_2, e_2, V_3, e_3, \ldots, e_{n-1}, V_n,$$

where each e_i is an edge joining vertex V_i to vertex V_{i+1}. (In a simple graph, where at most one edge joins V_i and V_{i+1}, it is sufficient to list the vertices alone.)

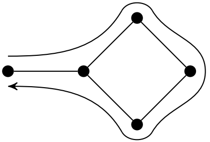
If $V_n = V_1$ the walk is said to be *closed*.

A *path* is a walk with no repeated vertices.

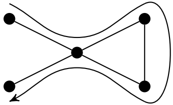
A *trail* is a walk with no repeated edges.

Examples

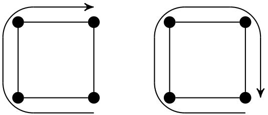
In these pictures, a walk is indicated by a directed curve running alongside the actual edges in the walk.



A walk which is not a trail or a path. (Repeated edge, repeated vertex.)



A trail which is not a path. (Repeated vertex.)



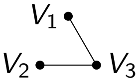
A nonclosed walk and a closed walk.

Adjacency matrix

If two vertices are joined by an edge we say that they are *adjacent*.

A simple graph G with vertices $V_1, V_2, \ldots, V_n$ is described by an *adjacency matrix* which has (i, j) entry (i^{th} row and j^{th} column) a_{ij} given by

$$a_{ij} = \begin{cases} 1 & \text{if } V_i \text{ is adjacent to } V_j \text{ in } G, \\ 0 & \text{otherwise.} \end{cases}$$

For example, the graph 

has adjacency matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Adjacency matrix powers

The *product* of matrices

$$\begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & \cdots \\ b_{21} & b_{22} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

is the matrix whose (i,j) entry is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots ,$$

the "dot product" of the i^{th} row

$$a_{i1} \quad a_{i2} \quad a_{i3} \quad \cdots$$

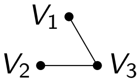
of the matrix on the left with the j^{th} column

$$\begin{matrix} b_{1j} \\ b_{2j} \\ b_{3j} \\ \vdots \end{matrix}$$

of the matrix on the right.

The (i, j) entry in the k^{th} power of the adjacency matrix gives the number of walks of length k between V_i and V_j.

For example, suppose we want the number of walks of length 2 from V_3 to V_3 in the graph



The adjacency matrix M tells us that the following edges exist.

$$\begin{array}{ll}
\begin{pmatrix} \cdots & \cdots & 1 \\ \cdots & \cdots & 1 \\ 1 & 1 & 0 \end{pmatrix} & \begin{array}{l} \leftarrow V_1 \text{ to} V_3 \\ \leftarrow V_2 \text{ to} V_3 \\ \end{array} \\
\begin{array}{cc} \uparrow & \uparrow \\ V_3 & V_3 \\ \text{to} & \text{to} \\ V_1 & V_2 \end{array} &
\end{array}$$

So when we square this matrix, the $(3,3)$ entry in M^2

$$\begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 \times 1 + 1 \times 1 = 2$$

counts the walks from V_3 to V_3, namely

$$V_3 \to V_1 \to V_3 \text{ and } V_3 \to V_2 \to V_3.$$

Similarly, the (i,j) entry in M^2 is the number of walks of length 2 from V_i to V_j. The (i,j) entry in M^3 is the number of walks of length 3 from to V_i to V_j, and so on.

In fact,

$$M^2 \times M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

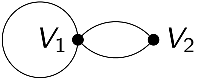
has $(3, 2)$ entry

$$\begin{pmatrix} 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2.$$

Hence the number of walks of length 3 from V_3 to V_2 is 2.

General adjacency matrix

The adjacency matrix can be generalised to multigraphs by making the (i,j) entry the *number* of edges from V_i to V_j (special case: count each loop twice).

For example, the graph  has adjacency matrix

$$N = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}$$

and so

$$N^2 = \begin{pmatrix} 8 & 4 \\ 4 & 4 \end{pmatrix}.$$

The $(1,1)$ entry $2 \times 2 + 2 \times 2$ in N^2, for example, indicates that there are 8 walks of length 2 from V_1 to V_1: 4 walks twice around the loop, and 4 walks from V_1 to V_2 (2 ways) then V_2 to V_1 (2 ways).

This count distinguishes between different directions around the loop. It may help to regard the loop as a pair of opposite *directed* loops.

We can generalise the adjacency matrix M to directed graphs by letting the (i, j) entry be the number of directed edges (which include ordinary edges) from V_i to V_j.

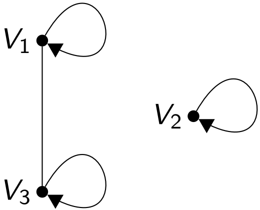
With this definition, the (i, j) entry of M^k gives the number of directed walks from V_i to V_j (i.e. walks that obey the directions of edges).

Questions

30.1 Draw the graph/digraph/multigraph with adjacency matrix

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

using V_1, V_2 and V_3 as names for the vertices corresponding to columns 1, 2 and 3 respectively.



Questions

30.2 Calculate M^2, and use it to find the number of walks of length 2 from V_1 to V_3. Does this calculation give the number you would expect from the graph?

$$M^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix}.$$

Hence the number of walks of length 2 from V_1 to V_3 is 2.

They are
(i) Walk around the loop at V_1, then along the edge from V_1 to V_3
and
(i) Walk along the edge from V_1 to V_3, then around the loop at V_3.

Questions

30.3 Without any calculation, show that the middle row of *any* power M^n is $(0\ 1\ 0)$.

Vertex V_2 is not connected to any other vertices; it just has a loop connecting it to itself.

Row 2 of M^n counts walks of length n from V_2 to the other vertices.

The only walk of length n from V_2 is to go around the loop n times ending up back at V_2. This shows that the middle entry in the middle row is 1, recording the fact that there is only one way to reach V_2 from V_2 after n steps. The other two entries are zero, since V_1 and V_3 cannot be reached from V_2 (in any number of steps).