MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #4 Solutions

1. (a) "Dwayne, the first two terms of the sequence are positive and each term past there is obtained by adding the previous two. It's obvious that we'll never get a negative term. Why not? Well suppose we've just gotten our first negative term - then the two previous terms were positive and they added to give a negative – that can't happen." (A more formal argument could use strong induction.)

   (b) Let $P(n)$ be the statement "3 divides $n^3 - 7n + 6$".

   Base step. $0^3 - 7(0) + 6 = 6$ and 3 divides 6, so $P(0)$ is true.

   Induction step. Assume $P(k)$ is true for some integer $k \geq 0$. So 3 divides $k^3 - 7k + 6$. Equivalently, $k^3 - 7k + 6 = 3a$ for some integer a.

   We need to prove that $P(k+1)$ is true, that is, that 3 divides $(k+1)^3 - 7(k+1) + 6$. Now,

$$\begin{aligned}(k+1)^3 - 7(k+1) + 6 &= k^3 + 3k^2 + 3k + 1 - 7k - 7 + 6 \\ &= (k^3 - 7k + 6) + (3k^2 + 3k - 6) \\ &= 3a + 3(k^2 + k - 2) \qquad \text{(by } P(k)\text{)} \\ &= 3(a + k^2 + k - 2).\end{aligned}$$

   So 3 divides $(k+1)^3 - 7(k+1) + 6$ (note that $a + k^2 + k - 2$ is an integer). So $P(k+1)$ is true.

   So we have proved by induction that $P(n)$ is true for all integers $n \geq 0$.

2. (a) Are the following true or false?

      i. true
      ii. false
      iii. true
      iv. true
      v. false
      vi. true
      vii. true (the empty set is a subset of everything)

   (b) $\{\{\}, \{-1\}, \{0\}, \{1\}, \{-1, 0\}, \{-1, 1\}, \{0, 1\}, \{-1, 0, 1\}\}$

   (c) $2^{10} = 1024$ elements

      i. yes
      ii. yes
      iii. no
      iv. no
      v. yes

3. (a) $\{-1, 0, 1, 2\}$
   (b) $\{1\}$
   (c) $\{(1,-1),(1,0),(1,1),(2,-1),(2,0),(2,1)\}$
   (d) No. For example, when $X = \{1,2\}$, $Y = \{1,2\}$ and $Z = \{1\}$, we have $(X \cup Y) \cap Z = \{1\}$ and $X \cup (Y \cap Z) = \{1,2\}$.
   (e) Yes. Let Y and Z be any sets. Now

$$\begin{aligned} X \in (\mathcal{P}(Y) \cap \mathcal{P}(Z)) &\equiv (X \in \mathcal{P}(Y)) \wedge (X \in \mathcal{P}(Z)) \\ &\equiv (X \subseteq Y) \wedge (X \subseteq Z) \\ &\equiv X \subseteq (Y \cap Z) && \text{(see } * \text{ below)} \\ &\equiv X \in \mathcal{P}(Y \cap Z). \end{aligned}$$

   So $\mathcal{P}(Y) \cap \mathcal{P}(Z) = \mathcal{P}(Y \cap Z)$ is true for any sets Y and Z.

   *To see that $(X \subseteq Y) \wedge (X \subseteq Z) \equiv X \subseteq (Y \cap Z)$, notice that

$$\begin{aligned} (X \subseteq Y) \wedge (X \subseteq Z) &\equiv (\text{every element of } X \text{ is in } Y) \wedge (\text{every element of } X \text{ is in } Z) \\ &\equiv \text{every element of } X \text{ is in } Y \cap Z \\ &\equiv X \subseteq (Y \cap Z). \end{aligned}$$

4. (a) Here's an argument by strong induction. You could also make an argument by regular induction similar to the stamp example in Lecture 9.

   Let $P(n)$ be the statement "\$$n$ can be made from \$7 notes and \$4 notes".

   Base steps. \$18 can be made from two \$7 notes and one \$4 note. So $P(18)$ is true.
   \$19 can be made from one \$7 note and three \$4 notes. So $P(19)$ is true.
   \$20 can be made from five \$4 notes. So $P(20)$ is true.
   \$21 can be made from three \$7 notes. So $P(21)$ is true.

   Induction step. For some integer $k \geq 21$, assume that $P(18), P(19), \ldots, P(k)$ are true. We need to show that $P(k+1)$ is true, that is, that $k+1$ duckbucks can be made from \$4 and \$7 notes.

   We know that $P(k-3)$ is true and so $k-3$ duckbucks can be made from \$4 and \$7 notes (note that $k - 3 \geq 18$ because $k \geq 21$). Simply adding a \$4 note to this makes $k+1$ duckbucks. So $P(k+1)$ is true.

   So we have proved by strong induction that $P(n)$ is true for each integer $n \geq 18$.

   (b) "Dwayne, just keep adding \$4 notes until the amount left to pay is \$18 or \$19 or \$20 or \$21. Then use this cheat sheet." (The cheat sheet is made from the base steps for (b).)