MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #9 Solutions

1. (a) Using the definition of expected value,

$$\mathrm{E}[X] = \tfrac{1}{2} \times 0 + \tfrac{1}{3} \times 1 + \tfrac{1}{6} \times 2 = \tfrac{2}{3}.$$

   Now, using $\mathrm{E}[X] = \frac{2}{3}$,

$$\mathrm{Var}[X] = \tfrac{1}{2} \times (0 - \tfrac{2}{3})^2 + \tfrac{1}{3} \times (1 - \tfrac{2}{3})^2 + \tfrac{1}{6} \times (2 - \tfrac{2}{3})^2 = \tfrac{2}{9} + \tfrac{1}{27} + \tfrac{8}{27} = \tfrac{5}{9}.$$

   (b) Because $E[Y] = 2$ we have

$$2 = E[Y] = p \times 0 + \tfrac{1}{12} \times 1 + \tfrac{1}{3} \times 2 + q \times 3 = \tfrac{3}{4} + 3q.$$

   Solving $2 = \frac{3}{4} + 3q$, we see $q = \frac{5}{12}$.
   Then, because $p + \frac{1}{12} + \frac{1}{3} + q = 1$, we have that $p = \frac{1}{6}$.

2. (a)
$$\begin{aligned}
r_1 &= 2r_0 - 1 &&= 2(3) - 1 &&= 5\\
r_2 &= 2r_1 - 1 &&= 2(5) - 1 &&= 9\\
r_3 &= 2r_2 - 1 &&= 2(9) - 1 &&= 17\\
r_4 &= 2r_3 - 1 &&= 2(17) - 1 &&= 33
\end{aligned}$$

   (b)
$$\begin{aligned}
t_3 &= t_2 t_0 &&= (-2)(1) &&= -2\\
t_4 &= t_3 t_1 &&= (-2)(1) &&= -2\\
t_5 &= t_4 t_2 &&= (-2)(-2) &&= 4\\
t_6 &= t_5 t_3 &&= (4)(-2) &&= -8
\end{aligned}$$

3. The Romulan ship is destroyed by one of Kirk's first three torpedoes if and only if one of the following (mutually exclusive) events occurs.

   - The first torpedo destroys the ship. The probability of this is $\frac{1}{10}$.
   - The first torpedo doesn't destroy the ship, but the second does. The probability of this is $\frac{9}{10} \times \frac{1}{10} = \frac{9}{100}$.
   - The first two torpedoes don't destroy the ship, but the third does. The probability of this is $\frac{9}{10} \times \frac{9}{10} \times \frac{1}{10} = \frac{81}{1000}$.

   So the probability the ship is destroyed by one of Kirk's first three torpedoes is $\frac{1}{10} + \frac{9}{100} + \frac{81}{1000} = \frac{271}{1000}$ or 27.1%.

4. Suppose that on a spin of red Tran must give Mary k jellybeans (if Mary must give Tran jellybeans then k is negative).

   Let X be the number of jellybeans that Tran gives to Mary on a given spin (if Mary gives Tran jellybeans then X is negative). The probability distribution of X is given by

x	6	-3	k
$\Pr(X = x)$	$\frac{1}{4}$	$\frac{5}{8}$	$\frac{1}{8}$

   So $\mathrm{E}[X] = \frac{1}{4} \times 6 + \frac{5}{8} \times (-3) + \frac{1}{8} \times k = \frac{k-3}{8}$.
   One reasonable definition of "fair" is that $\mathrm{E}[X] = 0$ (that is, the expected number of jellybeans going from Tran to Mary on a given spin is 0). So, using the equation above, to make the game fair we need that $k = 3$. This means that on a spin of red Tran gives 3 jellybeans to Mary.

5. (a) One example is a random variable X with probability distribution given by

x	0	1
$\Pr(X = x)$	$\frac{1}{2}$	$\frac{1}{2}$

   Then $E[X] = \frac{1}{2}$, but $\Pr(X = \frac{1}{2}) = 0$.

   (b) One example is a random variable Y with probability distribution given by

y	-1000000	1
$\Pr(Y = y)$	$\frac{1}{1000}$	$\frac{999}{1000}$

   Then $E[Y] = \frac{1}{1000} \times -1000000 + \frac{999}{1000} \times 1 = -999.001$, but $\Pr(Y > 0) = \frac{999}{1000}$.

   (c) The best you'll manage is $\frac{1}{3}$ (see Question 6 for why). One example would be

z	0	3
$\Pr(Z = z)$	$\frac{2}{3}$	$\frac{1}{3}$

   Then $E[Z] = \frac{1}{3} \times 3 = 1$, and $\Pr(Z \geq 3\mathrm{E}[Z]) = \Pr(Z \geq 3) = \frac{1}{3}$.

6. (a) Mine was awesome. I think yours was too.

   (b) Using the definitions of X and Y from 1(a) and 1(b) we have the following.
   - $\Pr(X \geq 2) = \frac{1}{6}$ and $\frac{\mathrm{E}[X]}{2} = (\frac{2}{3})/2 = \frac{1}{3}$.
   - $\Pr(X \geq 3) = 0$ and $\frac{\mathrm{E}[X]}{3} = (\frac{2}{3})/3 = \frac{2}{9}$.
   - $\Pr(Y \geq 2) = \frac{1}{3} + \frac{5}{12} = \frac{3}{4}$ and $\frac{\mathrm{E}[Y]}{2} = \frac{2}{2} = 1$.
   - $\Pr(Y \geq 3) = \frac{5}{12}$ and $\frac{\mathrm{E}[Y]}{3} = \frac{2}{3}$.

   In each case the first number is less than or equal to the second and so the inequality holds.

   (c) Let X be the income (in dollars) of a person from the country selected uniformly at random. Then $E[X] = 10000$. By Markov's inequality $\Pr(X \geq 100000) \leq \frac{10000}{100000} = \frac{1}{10}$. This means that at most $\frac{1}{10}$ of the country's population can earn at least \$100000 per year.
   This will happen when $\frac{1}{10}$ of the population earns exactly \$100000 per year and the remaining $\frac{9}{10}$ earn nothing.