Learning Outcomes for MAT1830

At the completion of this unit, students should be able to:

- identify basic methods of proof, particularly induction, and apply them to solve problems in mathematics and computer science;
- manipulate sets, relations, functions and their associated concepts, and apply these to solve problems in mathematics and computer science;
- use and analyse simple first and second order recurrence relations;
- use trees and graphs to solve problems in computer science;
- apply counting principles in combinatorics;
- describe the principles of elementary probability theory, evaluate conditional probabilities and use Bayes' Theorem.

Please fill out SETU (Student Evaluation of Teaching and Units)

Lecture 24: Expectation and variance

A standard die is rolled some number of times and the average of the rolls is calculated. If the die is rolled only once, this average is just the value rolled and is equally likely to be 1, 2, 3, 4, 5 or 6. If the die is rolled ten times, then the average might be between 1 and 2 but this is pretty unlikely – it's much more likely to be between 3 and 4. If the die is rolled ten thousand times, then we can be almost certain that the average will be very close to 3.5. We will see that 3.5 is the *expected value* of a random variable representing the die roll.

Expected value

When we said "average" above, we really meant "mean".

Remember that the *mean* of a collection of numbers is the sum of the numbers divided by how many of them there are. So the mean of $x_1, \ldots, x_t$ is $\frac{x_1 + \cdots + x_t}{t}$.

The mean of 2, 2, 3 and 11 is $\frac{2+2+3+11}{4} = 4.5$, for example.

The expected value of a random variable is calculated as a weighted average of its possible values.

If X is a random variable with distribution

$$\begin{array}{r||c|c|c|c} x & x_1 & x_2 & \cdots & x_t \\ \hline \Pr(X = x) & p_1 & p_2 & \cdots & p_t \end{array},$$

then the *expected value* of X is

$$\mathrm{E}[X] = p_1 x_1 + p_2 x_2 + \cdots + p_t x_t.$$

Example. If X is a random variable representing a die roll, then

$$\mathrm{E}[X] = \frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \cdots + \frac{1}{6} \times 6 = 3.5.$$

Example. Someone estimates that each year the share price of Acme Corporation has a 10% chance of increasing by \$10, a 50% chance of increasing by \$4, and a 40% chance of falling by \$10. Assuming that this estimate is good, are Acme shares likely to increase in value over the long term?

We can represent the change in the Acme share price by a random variable X with distribution

$$\begin{array}{r||c|c|c} x & -10 & 4 & 10 \\ \hline \Pr(X = x) & \frac{2}{5} & \frac{1}{2} & \frac{1}{10} \end{array}.$$

Then

$$\mathrm{E}[X] = \tfrac{2}{5} \times -10 + \tfrac{1}{2} \times 4 + \tfrac{1}{10} \times 10 = -1.$$

Because this value is negative, Acme shares will almost certainly decrease in value over the long term.

Notice that it was important that we weighted our average using the probabilities here. If we had just taken the average of -10, 4 and 10 we would have gotten the wrong answer by ignoring the fact that some values were more likely than others.

Law of large numbers

Our initial die-rolling example hinted that the average of a large number of independent trials will get very close to the expected value. This is mathematically guaranteed by a famous theorem called the *law of large numbers*.

Let $X_1, X_2, \ldots$ be independent random variables, all with the same distribution and expected value μ. Then

$$\lim_{n \to \infty} \frac{1}{n}(X_1 + \cdots + X_n) = \mu.$$

Questions

24.1 Do you agree or disagree with the following statement? "The expected value of a random variable is the value it is most likely to take."

ANS: You should disagree! In fact it may not even be possible for the value of a random variable to equal its expected value! For example, we saw that the expected value of one die roll is 3.5, but that value never occurs as the outcome of a die roll!

The expected value is a weighted average of its possible values. It is what we would expect to average over a large number of trials.

Linearity of expectation

We saw in the last lecture that adding random variables can be difficult. Finding the expected value of a sum of random variables is easy if we know the expected values of the variables.

If X and Y are random variables, then

$$\mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y].$$

This works even if X and Y are not independent.

Similarly, finding the expected value of a scalar multiple of a random variable is easy if we know the expected value of the variable.

If X is a random variable and $s \in \mathbb{R}$, then

$$\mathrm{E}[sX] = s\mathrm{E}[X].$$

Example. Two standard dice are rolled. What is the expected total?
Let X_1 and X_2 be random variables representing the first and second die rolls. From the earlier example $\mathrm{E}[X_1] = \mathrm{E}[X_2] = 3.5$ and so

$$\mathrm{E}[X_1 + X_2] = \mathrm{E}[X_1] + \mathrm{E}[X_2] = 3.5 + 3.5 = 7.$$

Example. What is the expected number of '11' substrings in a binary string of length 5 chosen uniformly at random?
For $i = 1, \ldots, 4$, let X_i be a random variable that is equal to 1 if the ith and $(i+1)$th bits of the string are both 1. Then $X_1 + \cdots + X_4$ is the number of '11' substrings in the string.
Because the bits are independent, $\Pr(X_i = 1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ and $\mathrm{E}[X_i] = \frac{1}{4}$ for $i = 1, \ldots, 4$. So,

$$\mathrm{E}[X_1 + \cdots + X_4] = \mathrm{E}[X_1] + \cdots + \mathrm{E}[X_4] = \frac{4}{4} = 1.$$

Note that the variables $X_1, \ldots, X_4$ in the above example were not independent, but we were still allowed to use linearity of expectation.

Variance

Think of the random variables X, Y and Z whose distributions are given below.

x	-1	99
$\Pr(X = x)$	$\frac{99}{100}$	$\frac{1}{100}$

y	-1	1
$\Pr(Y = y)$	$\frac{1}{2}$	$\frac{1}{2}$

z	-50	50
$\Pr(Z = z)$	$\frac{1}{2}$	$\frac{1}{2}$

These variables are very different. Perhaps X corresponds to buying a raffle ticket, Y to making a small bet on a coin flip, and Z to making a large bet on a coin flip. However, if you only consider expected value, all of these variables look the same – they each have expected value 0.

To give a bit more information about a random variable we can define its variance, which measures how "spread out" its distribution is.

If X is a random variable with $\mathrm{E}[X] = \mu$,

$$\mathrm{Var}[X] = \mathrm{E}[(X - \mu)^2].$$

So the variance is a measure of how much we expect the variable to differ from its expected value.

Example. The variable X above will be 1 smaller than its expected value with probability $\frac{99}{100}$ and will be 99 larger than its expected value with probability $\frac{1}{100}$. So

$$\mathrm{Var}[X] = \frac{99}{100} \times (-1)^2 + \frac{1}{100} \times 99^2 = 99$$

Similarly,

$$\mathrm{Var}[Y] = \frac{1}{2} \times (-1)^2 + \frac{1}{2} \times 1^2 = 1$$
$$\mathrm{Var}[Z] = \frac{1}{2} \times (-50)^2 + \frac{1}{2} \times 50^2 = 2500.$$

Notice that the variance of X is much smaller than the variance of Z because X is very likely to be close to its expected value whereas Z will certainly be far from its expected value.

Example. Let X be a random variable with distribution given by

x	0	2	6
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

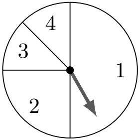
Then the expected value of X is

$$\mathrm{E}[X] = \tfrac{1}{6} \times 0 + \tfrac{1}{2} \times 2 + \tfrac{1}{3} \times 6 = 3.$$

So, the variance of X is

$$\mathrm{Var}[X] = \tfrac{1}{6} \times (0-3)^2 + \tfrac{1}{2} \times (2-3)^2 + \tfrac{1}{3} \times (6-3)^2 = 5.$$

Questions



24.2 Let X be the sum of 1000 spins of the spinner from Lecture 21, and let Y be 1000 times the result of a single spin. Find $\mathrm{E}[X]$ and $\mathrm{E}[Y]$.

The distribution of Y is

y	1000	2000	3000	4000
$\Pr(Y = y)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

So $\mathrm{E}[Y] = \frac{1}{2} \times 1000 + \frac{1}{4} \times 2000 + \frac{1}{8} \times 3000 + \frac{1}{8} \times 4000 = 1875$. We could also have got this from calculating the expectation of one spin S (which is $\mathrm{E}[S] = 1.875$) then using the scalar multiple law ($\mathrm{E}[sS] = s\mathrm{E}[S]$, with $s = 1000$).

I won't write down the distribution of X (why not?) but I don't need to. Since $X = X_1 + X_2 + \cdots + X_{1000}$ where each X has the same distribution as S we can use linearity of expectation to find that $\mathrm{E}[X] = \mathrm{E}[X_1] + \mathrm{E}[X_2] + \cdots + \mathrm{E}[X_{1000}] = 1000\mathrm{E}[S] = 1875$.

Which of X and Y do you think would have greater variance?
ANS: Y will have much larger variance than X, since it has a high probability of being far from its expected value.

Questions

24.3 Let X be the number of heads occurring when three fair coins are flipped. Find $\mathrm{E}[X]$ and $\mathrm{Var}[X]$.

The distribution of X is

x	0	1	2	3
$\Pr(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

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So $\mathrm{E}[X] = \frac{1}{8} \times 0 + \frac{3}{8} \times 1 + \frac{3}{8} \times 2 + \frac{1}{8} \times 3 = \frac{3}{2}$.

$$\begin{aligned}\mathrm{Var}[X] &= \tfrac{1}{8} \times (0 - \tfrac{3}{2})^2 + \tfrac{3}{8} \times (1 - \tfrac{3}{2})^2 + \tfrac{3}{8} \times (2 - \tfrac{3}{2})^2 + \tfrac{1}{8} \times (3 - \tfrac{3}{2})^2 \\ &= \tfrac{1}{8} \times \tfrac{9}{4} + \tfrac{3}{8} \times \tfrac{1}{4} + \tfrac{3}{8} \times \tfrac{1}{4} + \tfrac{1}{8} \times \tfrac{9}{4} = \tfrac{3}{4}.\end{aligned}$$