Monash University
Faculty of Information Technology

Lecture 9
(A) Closure properties.
(B) Pumping Lemma for Regular Languages.

Slides by David Albrecht (2011), revisions and additions by Graham Farr (2013).

FIT2014 Theory of Computation

Overview

- Closure properties of regular languages
- Circuits in FAs
- Pumping Lemma
- Non-regular Languages

Closure properties of regular languages

If doing some operation on regular languages *always* produces another regular language, then we say that regular languages are ***closed*** under that operation.

We will see that regular languages are closed under:

complement, union, intersection, concatenation

Theorem

The complement of a regular language is regular.

We prove this using Kleene's Theorem.

Closure properties of regular languages

Theorem

The complement of a regular language is regular.

Proof (outline):

Suppose we have a Regular Language.

There must be a regular expression that defines it.

So, by Kleene's Theorem, there is a Finite Automaton (FA) that defines this language.

We can convert this FA into one that defines the complement of the language.

So, by Kleene's Theorem, there is a regular expression that defines the complement. Q.E.D.

Closure properties of regular languages

Theorem

The union of two regular languages is regular.

Proof

- Suppose $\mathbf{L_1}$ and $\mathbf{L_2}$ are regular.
- By definition of "regular language", there exist regular expressions $\mathbf{R_1}$ and $\mathbf{R_2}$ that describe $\mathbf{L_1}$ and $\mathbf{L_2}$, respectively.
- Then $\mathbf{R_1} \cup \mathbf{R_2}$ is a regular expression that describes $\mathbf{L_1} \cup \mathbf{L_2}$.

  This uses part 3(iii) of the inductive definition of regular expressions in Lecture 5.
- So $\mathbf{L_1} \cup \mathbf{L_2}$ is regular. Q.E.D.

Closure properties of regular languages

Theorem

The intersection of two regular languages is regular.

We can't just mimic the proof that regular languages are closed under union, since there is no $\cap$ operation on regular expressions.

Closure properties of regular languages

Theorem

The intersection of two regular languages is regular.

Proof

- Suppose L_1 and L_2 are regular.
- We know that their complements $\overline{L_1}$ and $\overline{L_2}$ are regular, and the union of these, $\overline{L_1} \cup \overline{L_2}$, is therefore regular, by the previous Theorem.
- Its complement, $\overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2$, must also be regular.

Q.E.D.

Closure properties of regular languages

Exercises

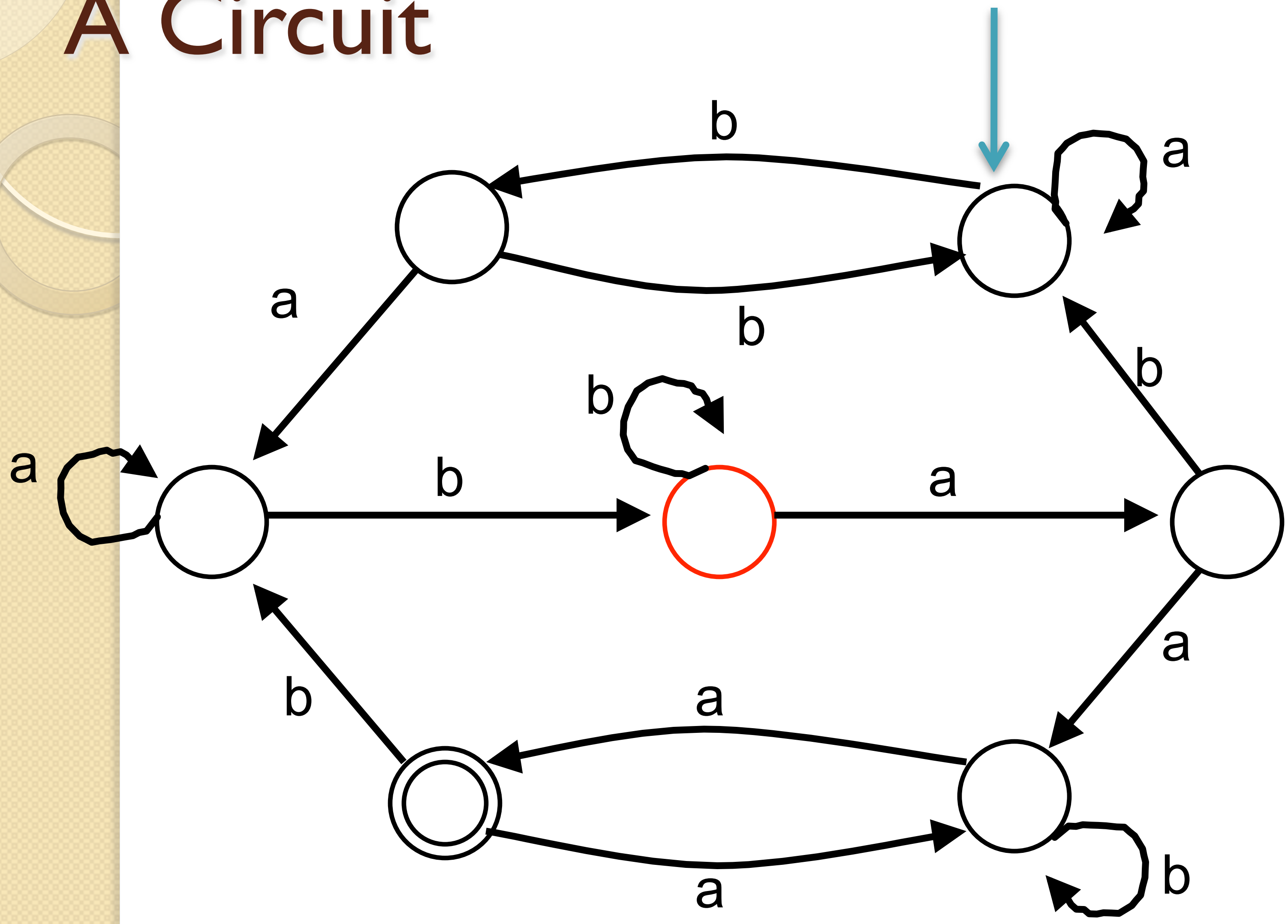
Prove that regular languages are closed under concatenation.

$$L_1 L_2 = \{ xy : x \text{ is in } L_1,\ y \text{ is in } L_2 \}$$

Prove that regular languages are closed under *symmetric difference.* (You can use the closure results we've already proved.)

$$L_1 \Delta L_2 = \{ \text{strings in } L_1 \text{ but not in } L_2, \text{ or in } L_2 \text{ but not in } L_1 \}$$

A Circuit



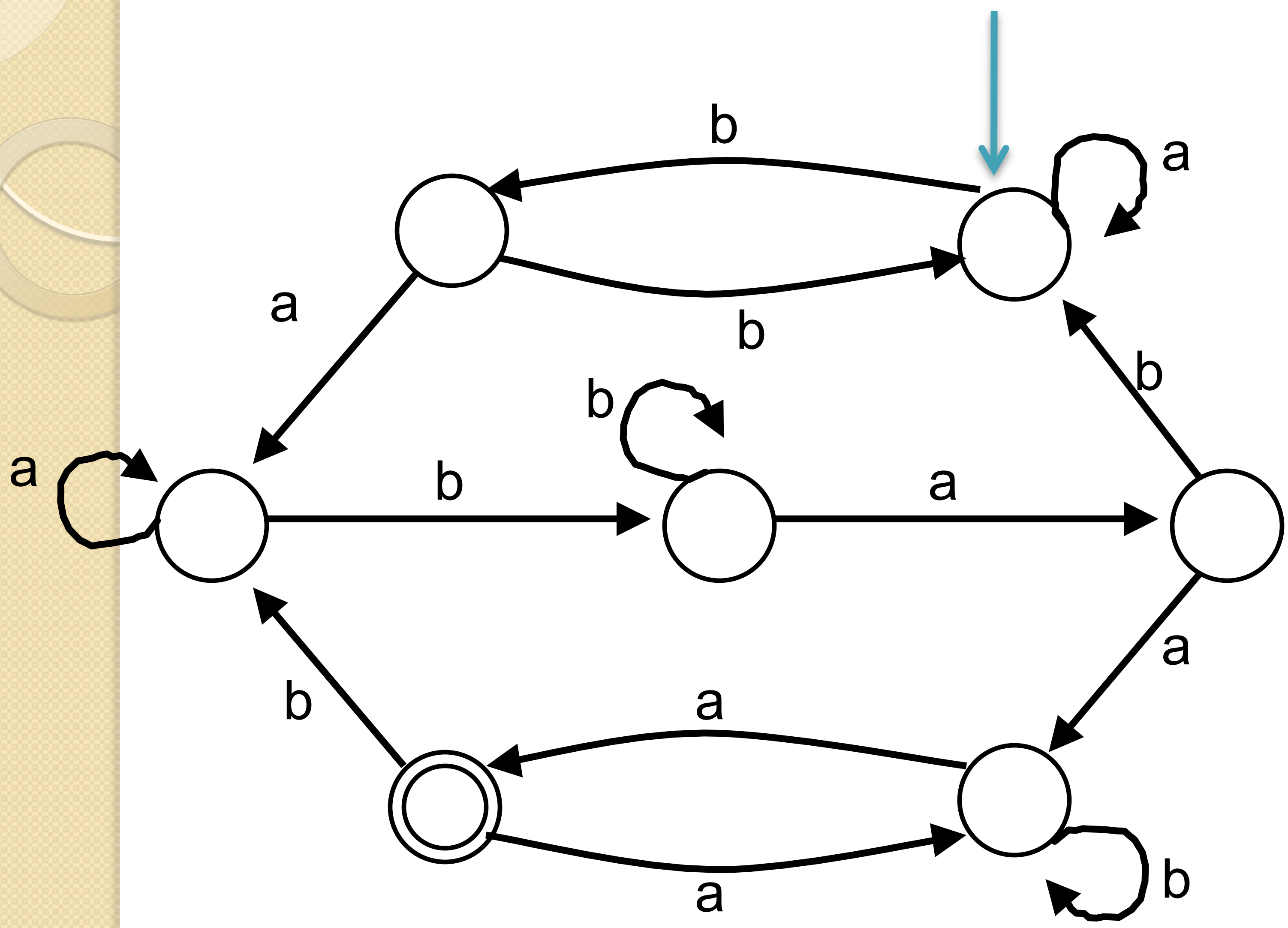
… abbab ...

Definitions

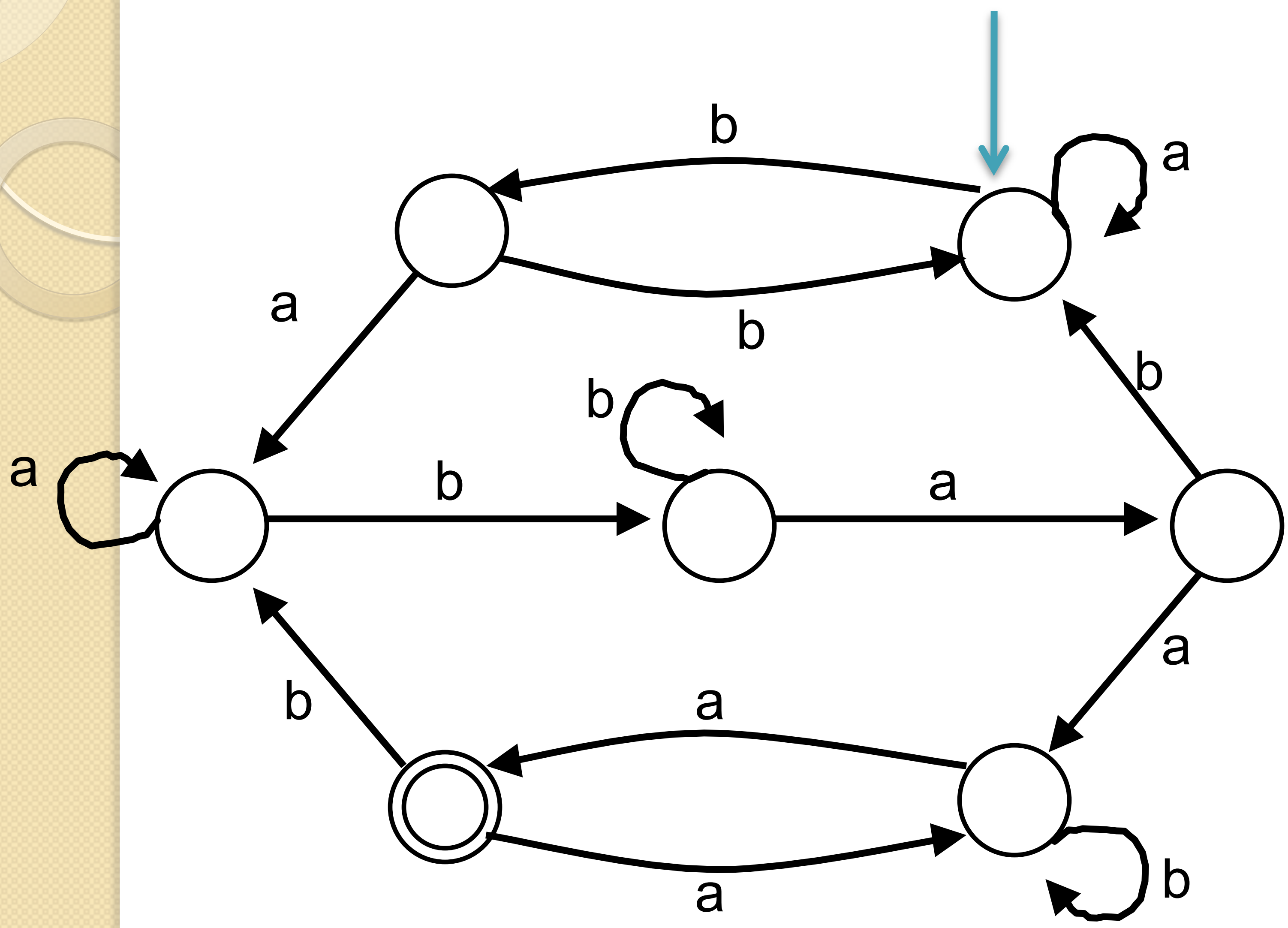
- A **circuit** is a path which starts and ends at the same state.
- The **length** of a circuit is the number of edges in the path.

Observation

- Take any Finite Automaton.
- Take any string with has more letters than there are states in that Finite Automaton.
- Then the path taken when this string is used as input must contain a circuit.

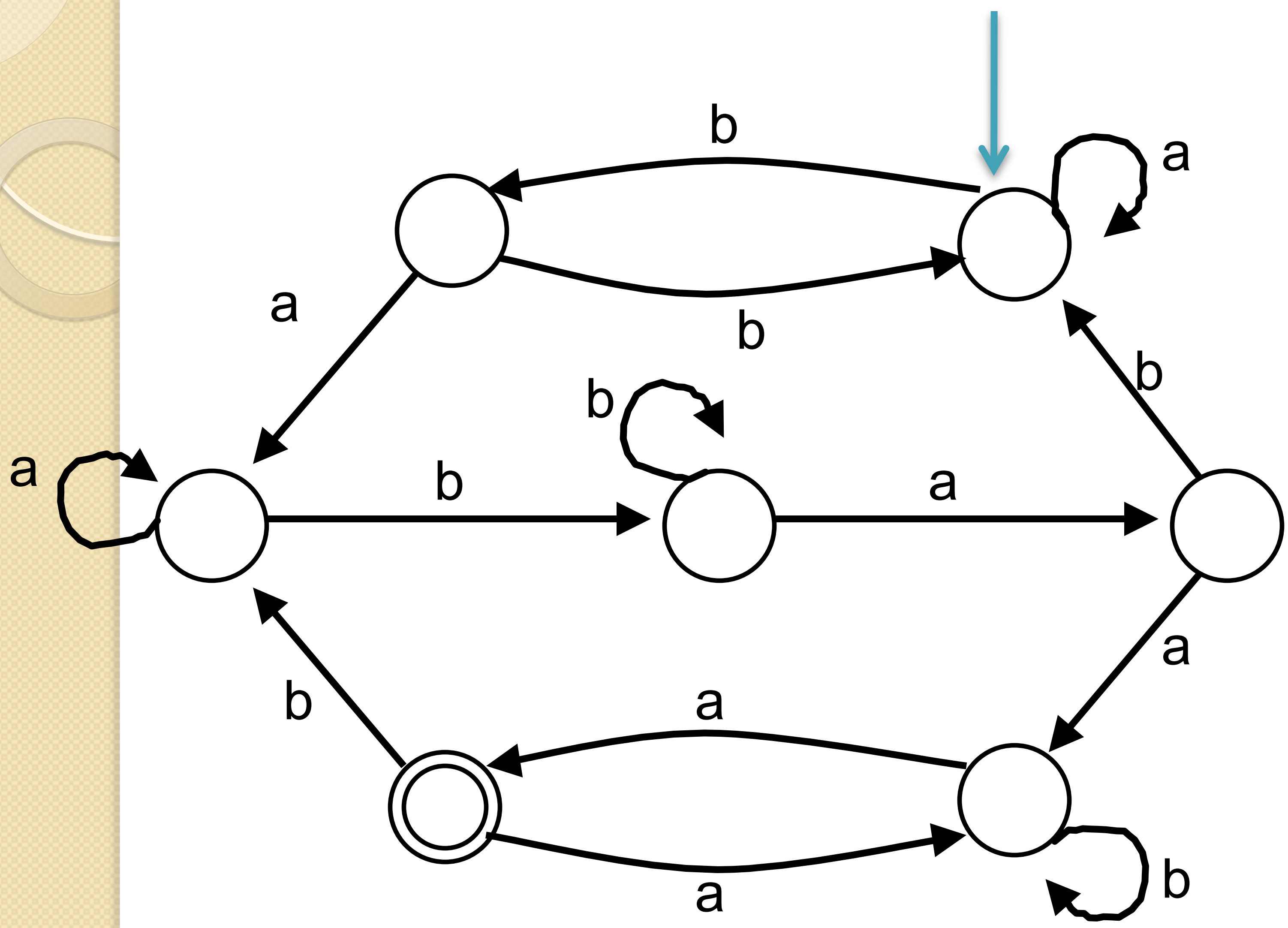


babaaabbab



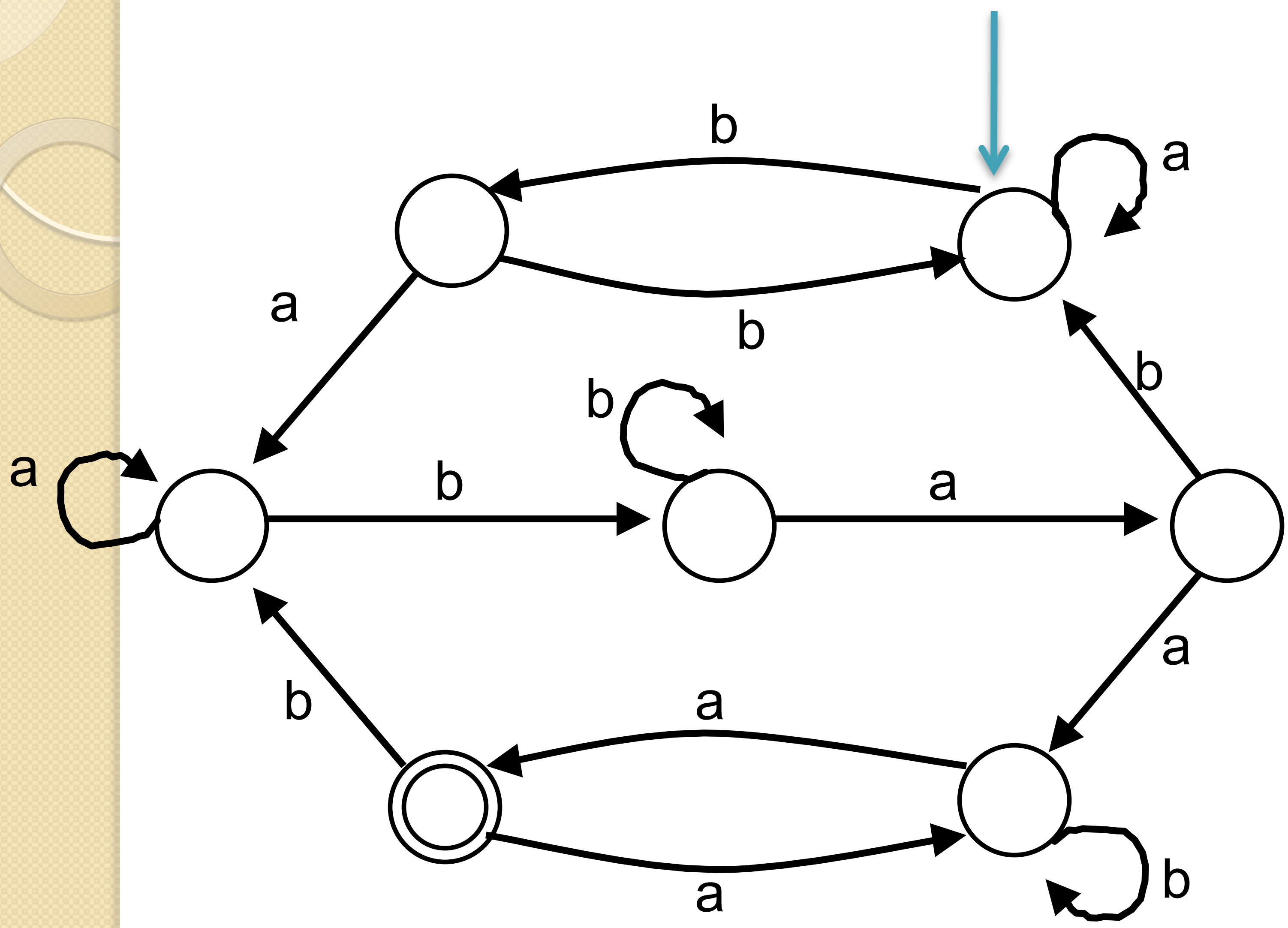
ba baaab bab

x y z

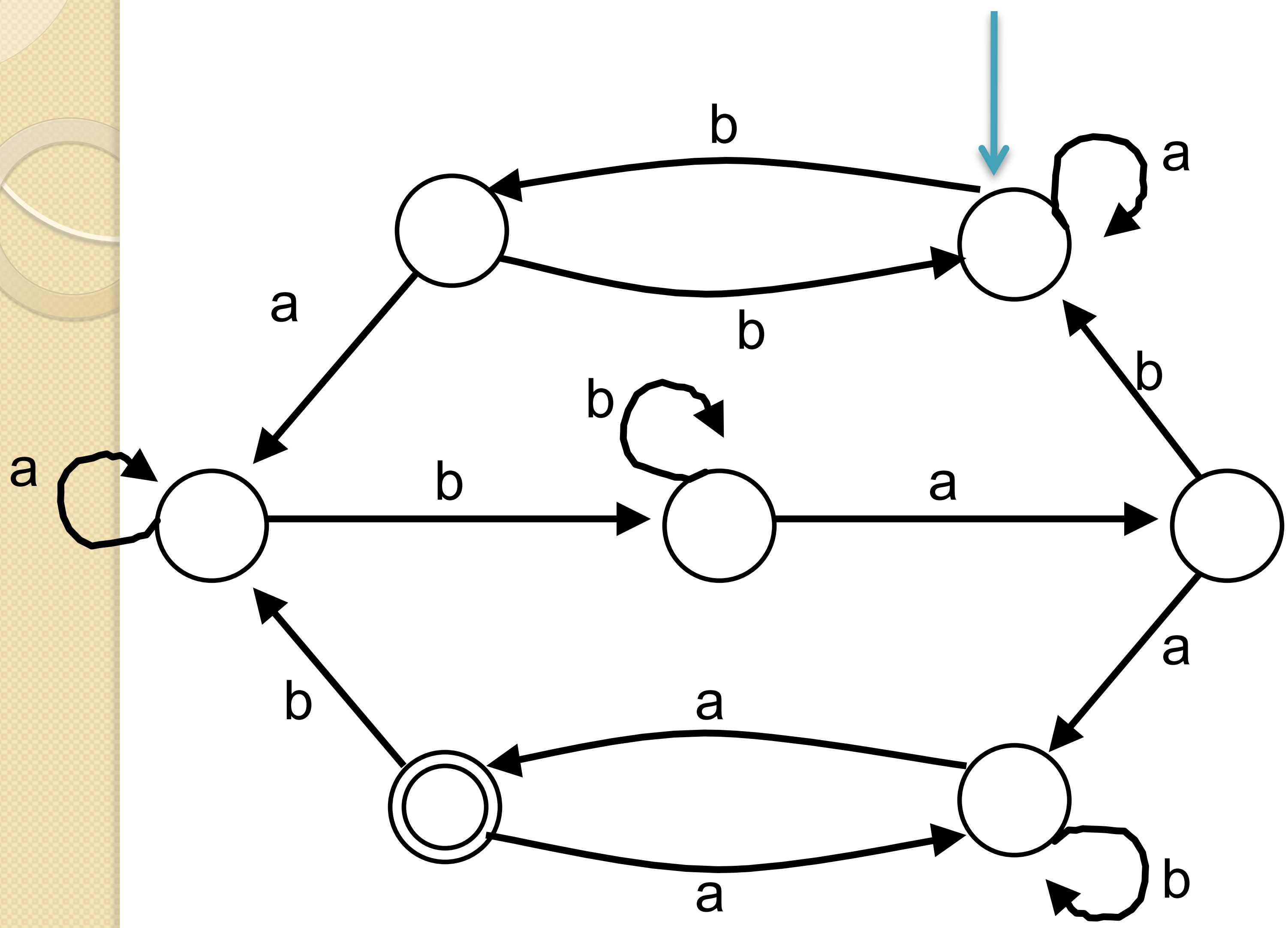


ba baaab baaab bab

x y y z

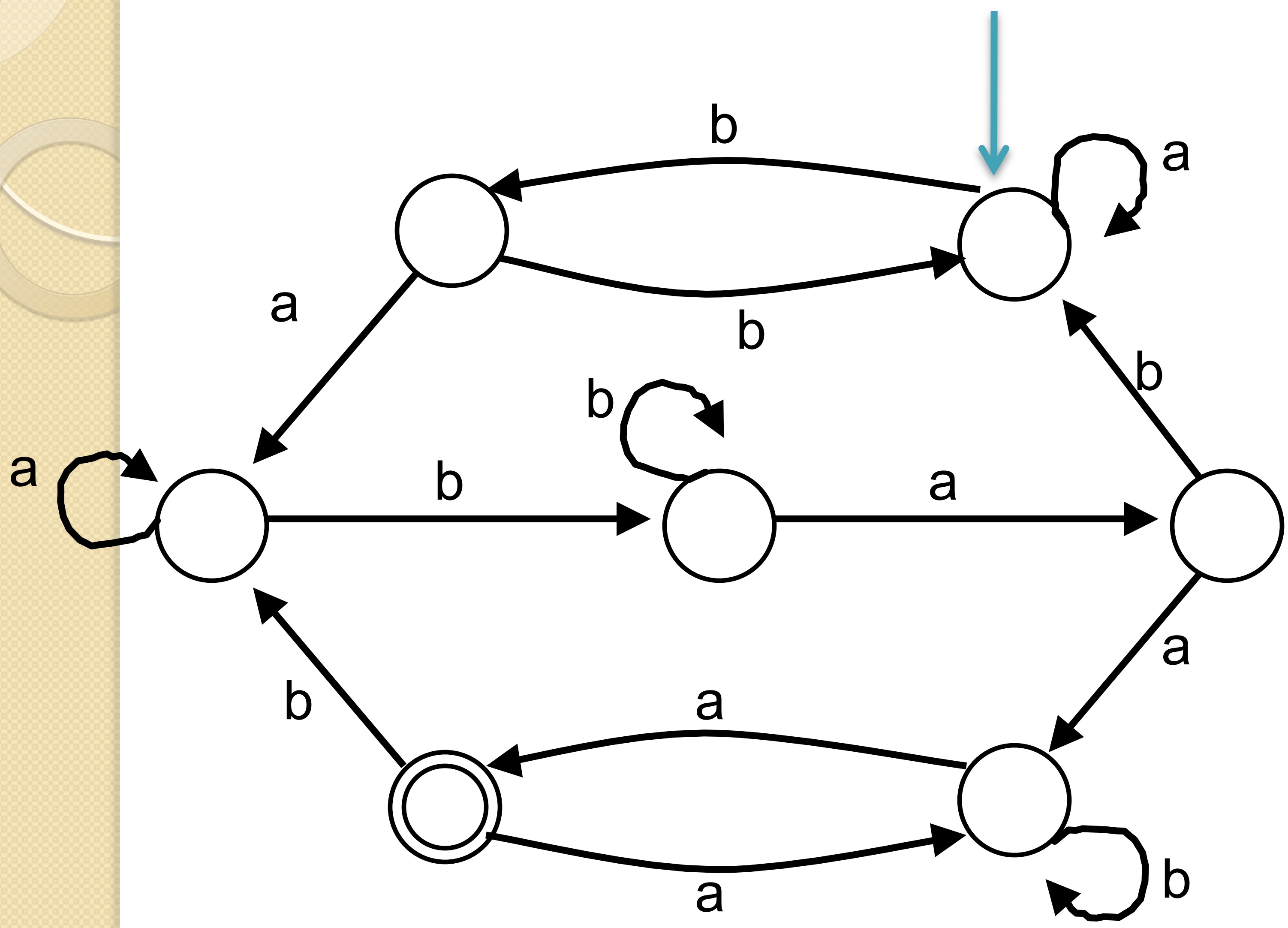


babbabbabaaba



ba bbabba baaba

x y z



ba bbabba bbabba baaba

x y y z

Theorem (Pumping Lemma)

- Let **L** be a regular language with an infinite number of words, accepted by a FA with **N** states.
- Then for all words **w** in **L** with more than **N** letters,

there exist strings **x**, **y**, **z**, with **y** ≠ **ε**, such that

- **w = xyz**
- **length(x) + length(y) ≤ N**
- for all $i \geq 0$, $\mathbf{xy}^i\mathbf{z}$ is in **L**.

i.e.,

- **xyz, xyyz, .. , $\mathbf{xy}^n\mathbf{z}$** are words in L

Proof

- Take any word **w** in **L** with > **N** letters.
- Let
  - **x** be the letters up to the first circuit.
  - **y** be the letters corresponding to the circuit.
  - **z** be the remaining letters.
- **w = xyz** by construction.
- **length(x) + length(y) ≤ N**, since the FA reads **xy** without repeating any state.

…

Proof (continued)

- Since **w = xyz** is in **L**, and **y** starts and finishes at endState(**x**), and **z** goes from endState(**x**) to a Final State, we can repeat y any number of times (or none) and still we end up at the same Final State.

Q.E.D.

Consequence

- Using the Pumping Lemma we can show there are non-regular languages.
- Method
  - Assume **L** is regular
  - Then there exist **x**, **y** $\neq \varepsilon$, and **z**

    s.t. all $\mathbf{xy^nz}$ are words in **L**.
  - Show for some **n > 1**, $\mathbf{xy^nz}$ is not a word in **L**.
  - Contradiction.

$L = \{a^n b^n\}$

$L = \{\varepsilon,\ ab,\ aabb,\ aaabbb,\ \ldots\}$

Claim: **L** is not regular.

Proof (by contradiction):

- Assume **L** is regular
- Then there exists **x**, **y** $\neq \varepsilon$, and **z**

  s.t. **xyz, xyyz, …, xy^nz,…** are words in L.
- Case 1: **y** is all **a**’s
- Case 2: **y** is all **b**’s
- Case 3: **y** contains an **ab**
- Now consider **xyyz**
- Contradiction. Q.E.D.

EQUAL

- All words which have an equal number of **a**'s as **b**'s.

{ε, ab, ba, aabb, abab, abba, baba, … }

- $\{\mathbf{a}^n\,\mathbf{b}^n\} = \mathbf{EQUAL} \cap \mathbf{a^*b^*}$
- $\{\mathbf{a}^n\,\mathbf{b}^n\}$ is non-regular (as just shown)
- **a*b*** is regular
- Therefore, by closure of regular expressions under intersection,
- **EQUAL** is non-regular

PALINDROME

- All the strings which are the same if they are spelt backwards

- E.g.

  ε , a, b, aa, bb, aaa, aba, bab, bbb

PALINDROME is non-regular

Proof (by contradiction):

- Assume **PALINDROME** is regular.
- Then exists a FA with **N** states which accepts **PALINDROME**.
- Let $\mathbf{w = a^N b a^N}$
- There exists strings **x**, **y** $\neq \varepsilon$, and **z** s.t.
  - **w = xyz**
  - **length(x) + length(y) ≤ N**
  - **xyz, xyyz, .. , xy^{n}z** are words in **PALINDROME**
- Consider **xyyz.**
- Contradiction Q.E.D.

Revision

- Know the closure properties of regular languages.
- Know what the Pumping Lemma is used to show.
- Know some examples of non-regular languages.

- Sipser, Section 1.4, pp 77-82.

Preparation

- Read

M. Sipser, , "***Introduction to the Theory of Computation***", Chapter 2.