MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #10 and Additional Practice Questions

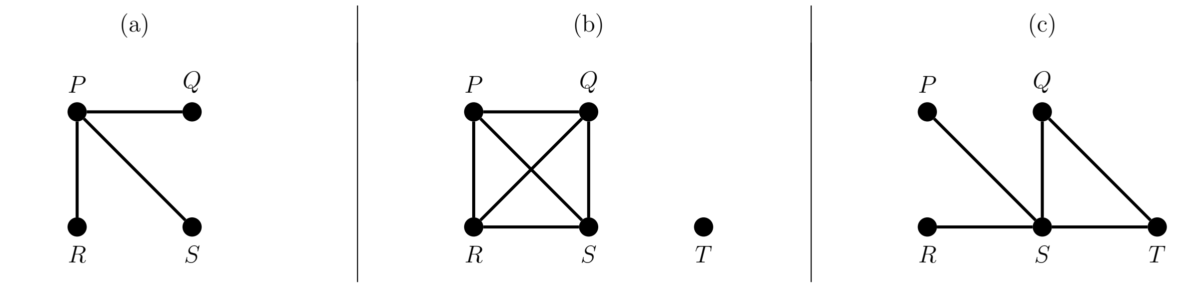
Tutorial Questions

1. Find recursive definitions for the following.

   (a) The sequence $a_0, a_1, a_2, \ldots$ where $a_n = 2^n$ for $n \geq 0$.

   (b) The sequence $b_0, b_1, b_2, \ldots$ where $b_n = n^2$ for $n \geq 0$. (Your recurrence may involve n, but not n^2.)

2. Write lists of vertices and edges for the following graphs.



3. Rewrite the following expressions without using $\sum$ or $\prod$.

   (a) $\displaystyle\sum_{i=6}^{10} \frac{1}{2i+1}$

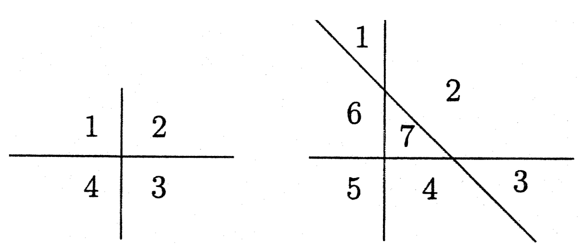
   (b) $\displaystyle\prod_{i=4}^{6} \left(\frac{x^i}{2i} + i\right)$

4. Let s_n be the number of ways (order being important) of writing n as a sum of 1s and 2s. For example $s_4 = 5$ because 4 can be written in five ways:

$$1+1+1+1, \quad 1+1+2, \quad 1+2+1, \quad 2+1+1, \quad 2+2.$$

   Find a recurrence for s_n.

5. Let r_n be the number of regions created when the plane is divided by n straight lines, with no two lines parallel and no three meeting in a single point. For example,



$r_2 = 4$ $\qquad$ $r_3 = 7$

   Find a recurrence for r_n.

(See over for practice questions.)

Practice Questions

1. Write down the next four values of each of the following recursive sequences.

   (a) $s_0 = 0$, $s_1 = 2$, $\quad s_n = 2s_{n-1} - 3s_{n-2}$ for $n \geq 2$.

   (b) $u_0 = 2$, $\quad u_n = 3u_{n-1} + n$ for $n \geq 1$.

2. Consider the following pseudo code of a function "foo".

   function foo(x) (input: a positive integer)
       if $x = 0$ **then**
           return 1
       else
           return $x \times \text{foo}(x-1)$
       end if
   end function

   (a) What will foo return when given input 4?

   (b) What is the recurrence relation corresponding to foo?

   (c) What function of x does foo calculate?

3. Suppose you want to network some computers together in such a way that

   - each computer is directly connected to at most three others; and
   - any two computers are either directly connected or are both directly connected to some third computer.

   Can you find a way to network 7 computers like this? 8? 9? 10?

   (For a way to do this for 10 computers, google "Petersen graph")

4. Let $S(n,k)$ be the number of equivalence relations on the set $\{1, 2, \ldots, n\}$ with exactly k (non-empty) equivalence classes. Prove that $S(n,k) = kS(n-1,k) + S(n-1,k-1)$ for all integers n and k such that $n > k > 1$.

   ($S(n,k)$ are sometimes called *Stirling numbers of the second kind.*)

5. The Fibonacci sequence is defined recursively by

   $$t_0 = 0,\ t_1 = 1, \quad t_k = t_{k-1} + t_{k-2} \text{ for } t \geq 1.$$

   Use strong induction to prove that, for all $n \geq 0$,

   $$t_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$