

**FIT2014**  
**Solutions for Tutorial 1**  
**Logic and Proofs**

1. We must show that, if  $w \in \text{PALINDROMES}$  then  $w \in \overline{\text{ODD-ODD}}$ .

Suppose  $w \in \text{PALINDROMES}$ . Then there is a string  $x$  such that *either*  $w = x\overleftarrow{x}$  or  $w = xy\overleftarrow{x}$ , where  $\overleftarrow{x}$  denotes the reverse of  $x$  and  $y \in \{a, b\}$  is a single letter.

Suppose  $w = x\overleftarrow{x}$ . The numbers of a's and b's in  $x\overleftarrow{x}$  are both even, since each is twice the number in  $x$ . So  $w \in \overline{\text{ODD-ODD}}$ .

Now suppose  $w = xy\overleftarrow{x}$ . Whichever letter in  $\{a, b\}$  is *not*  $y$  must appear an even number of times in  $w$ , by the same argument we have used previously. So that letter does not appear an odd number of times in  $w$ . So  $w \in \overline{\text{ODD-ODD}}$ .

*Alternative argument for the second case*, pointed out by an FIT2014 student in 2013:

Now suppose  $w = xy\overleftarrow{x}$ . Since the length of  $w$  is odd, then it cannot have an both an odd number of a's and an odd number of b's (else its length would be even). So  $w \in \overline{\text{ODD-ODD}}$ .

2.

(a)  $k$ -th odd number  $= 2k - 1$

(b) Inductive basis: when  $k = 1$ , the sum of the first  $k$  odd numbers is just the first odd number, 1, which equals  $1^2$ , so it equals  $k^2$ .

(c)

Sum of the first  $k + 1$  odd numbers

$$\begin{aligned} &= 1 + 3 + \cdots + ((k + 1)\text{-th odd number}) \\ &= (1 + 3 + \cdots + (k\text{-th odd number})) + ((k + 1)\text{-th odd number}) \\ &= (\text{sum of the first } k \text{ odd numbers}) + ((k + 1)\text{-th odd number}) \\ &= (\text{sum of the first } k \text{ odd numbers}) + 2(k + 1) - 1 \\ &\quad (\text{using our formula from part (a)}) \end{aligned}$$

(d) Continuing from above,

$$\begin{aligned} &\dots \\ &= k^2 + 2(k + 1) - 1 \quad (\text{by the Inductive Hypothesis}) \end{aligned}$$

(e) Continuing from above,

$$\begin{aligned} &\dots \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2. \end{aligned}$$

This completes the Inductive Step (i.e., going from  $k$  to  $k + 1$ ).

(f) So, by the Principle of Mathematical Induction, it is true for all  $k$  that the sum of the first  $k$  odd numbers is  $k^2$ .

3.

Base case:  $n = 3$ :

$3! = 6$ , while  $(3 - 1)^3 = 2^3 = 8$ , so the inequality is true for  $n = 3$ .

Inductive Step:

Suppose that  $n! \leq (n - 1)^n$  is true for a particular number  $n$ , where  $n \geq 3$ .

Let's look at what happens at  $n + 1$ .

$$\begin{aligned}
 (n + 1)! &= (n + 1) \cdot n! && \text{(to express it in terms of a smaller case)} \\
 &\leq (n + 1) \cdot (n - 1)^n && \text{(by the Inductive Hypothesis, i.e., } n! \leq (n - 1)^n) \\
 &= (n + 1)(n - 1)(n - 1)^{n-1} && \text{(a slight rearrangement ...)} \\
 &\quad \dots \text{we are hoping to get } n + 1 \text{ factors, all } \leq n \dots \\
 &= (n^2 - 1)(n - 1)^{n-1} && \text{(a little high-school algebra ...)} \\
 &< n^2(n - 1)^{n-1} && \text{(and now we } \textit{do} \text{ have } n + 1 \text{ factors, all } \leq n) \\
 &< n^{n+1} \\
 &= ((n + 1) - 1)^{n+1}.
 \end{aligned}$$

This last line is just to make it clear that the expression is of the required form.

So, by Mathematical Induction, it is true for all  $n$  that  $n! \leq (n - 1)^n$ .

4. (a)

(b)

Inductive basis:

Suppose  $n \leq 100$ . Applying `wc` to standard output of this length gives a one-line standard output stating the numbers of lines, words and characters, with the number of characters being  $n$ . This output has 1 line, 3 (or 4?) words and some small number of characters consisting of the digit 1, the digit 3, a couple of digits (at most) for  $n$ , and some number of spaces (say, 21 altogether, but the analysis is much the same if this number is different). Applying `wc` again gives one line with these numbers in it: 1, 3, 25, again alongside 21 spaces. Another application of `wc` gives the same result. So the claim is true for  $n \leq 99$ .

Inductive step:

Now suppose the claim is true when the file/string has  $\leq k$  characters, where  $k \geq 100$ . Suppose we are given a file/string of  $k + 1$  characters. Applying `wc` gives a one-line standard output, giving numbers of lines, words and characters as  $l$ ,  $w$  and  $k + 1$ , respectively, set out something like

$$l \quad w \quad k + 1$$

For any number  $x$ , write  $\text{digits}(x)$  for the number of digits in  $x$ . Our one-line output has some number  $s$  of spaces, say  $s = 21$ ; the number of non-space characters is

$$\text{digits}(l) + \text{digits}(w) + \text{digits}(k + 1).$$

Now,  $l \leq k + 1$  and  $w \leq k + 1$ .<sup>1</sup> Therefore the number of characters in the one-line output above is  $\leq s + 3 \cdot \text{digits}(k + 1)$ . We claim this is  $\leq k$  if  $k$  is large enough.

To see this, first try  $k = 100$ . Then

$$\begin{aligned} s + 3 \cdot \text{digits}(k + 1) &= 21 + 3 \cdot \text{digits}(100 + 1) \\ &= 21 + 3 \cdot \text{digits}(101) \\ &= 21 + 3 \cdot 3 \\ &= 30, \end{aligned}$$

which is indeed  $\leq k$ . (Only minor changes are needed here if the actual value of  $s$  is not 21; it will not be *much* different from 21.) Now, whenever  $k$  increases by 1,  $\text{digits}(k + 1)$  either stays the same or increases by 1. But it only increases rarely, when  $k + 1$  becomes 100, then when it becomes 1000, and so on. So, given that  $s + 3 \cdot \text{digits}(k + 1) \leq k$  for  $k = 100$  (and by a good margin), this inequality continues to hold for all higher  $k$ .<sup>2</sup>

Since this is now  $\leq k$ , the Inductive Hypothesis tells us that some further applications of **wc** will eventually give constant output.

Therefore the result follows for all  $n$ , by the Principle of Mathematical Induction.

## 5.

1.  $\mathbf{S_A} = \mathbf{B} \wedge \neg \mathbf{C}$
2.  $\mathbf{S_B} = \mathbf{A} \rightarrow \mathbf{C}$
3.  $\mathbf{S_C} = \neg \mathbf{C} \wedge (\mathbf{A} \vee \mathbf{B})$
4. Yes, since

$$(\mathbf{S_A} \wedge \mathbf{S_B} \wedge \mathbf{S_C}) \rightarrow (\mathbf{B} \wedge \neg \mathbf{A} \wedge \neg \mathbf{C})$$

is a tautology.

## 6.

(a) We prove it by constructing the truth table of each. It can be convenient to do this in stages.

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<sup>1</sup>In fact,  $w \leq (k + 1)/2$ , since every consecutive pair of words must have at least one space between them. But we don't need this better upper bound on  $w$ .

<sup>2</sup>Thanks to FIT2014 tutor Nathan Companez for part of this argument.

$P$	$Q$	$R$	$Q \wedge R$	$P \vee (Q \wedge R)$	$P$	$Q$	$R$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
F	F	F	F	F	F	F	F	F	F	F
F	F	T	F	F	F	F	T	F	T	F
F	T	F	F	F	F	T	F	T	F	F
F	T	T	T	T	F	T	T	T	T	T
T	F	F	F	T	T	F	F	T	T	T
T	F	T	F	T	T	F	T	T	T	T
T	T	F	F	T	T	T	F	T	T	T
T	T	T	T	T	T	T	T	T	T	T

The right-hand columns of each table are identical, so the two expressions (at the tops of those columns) are logically equivalent.

(b)

$$\begin{aligned}
P \wedge (Q \vee R) &= \neg\neg P \wedge (\neg\neg Q \vee \neg\neg R) \\
&= \neg\neg P \wedge \neg(\neg Q \wedge \neg R) && \text{(by one of De Morgan's Laws)} \\
&= \neg(\neg P \vee (\neg Q \wedge \neg R)) && \text{(by the other of De Morgan's Laws)} \\
&= \neg((\neg P \vee \neg Q) \wedge (\neg P \vee \neg R)) && \text{(by part (a) of this question)} \\
&= \neg(\neg(P \wedge Q) \wedge \neg(P \wedge R)) && \text{(by De Morgan, twice)} \\
&= \neg\neg(P \wedge Q) \vee \neg\neg(P \wedge R) && \text{(by De Morgan, one last time)} \\
&= (P \wedge Q) \vee (P \wedge R)
\end{aligned}$$

Here we have used equality to stand for logical equivalence, which is normal.

7.

$$\begin{aligned}
(P_1 \wedge \cdots \wedge P_n) \Rightarrow C &= \neg(P_1 \wedge \cdots \wedge P_n) \vee C \\
&= (\neg P_1 \vee \cdots \vee \neg P_n) \vee C,
\end{aligned}$$

using De Morgan's Law.

Remark:

A disjunction of the form  $\neg P_1 \vee \cdots \vee \neg P_n \vee C$ , where  $P_1, \dots, P_n, C$  are each variables that can be True or False, is called a *Horn clause*. These play a big role in the theory of logic programming.

8.

(a)  $L_{B,n} \vee L_{W,n} \vee L_{U,n}$

(b) If  $L_{B,n}$  is true, then it's ok for vertex  $n+1$  to be Black (since it then joins the black chain that includes vertex  $n$ , which must be ok as the position up to vertex  $n$  is legal). It could also be Uncoloured, since adding a new uncoloured vertex next to an existing vertex can never make a legal position illegal. But vertex  $n+1$  cannot be White, as it is then in a chain of its own which has no Uncoloured neighbour.

Similarly, if  $L_{W,n}$  is true, then vertex  $n+1$  can be White or Uncoloured, but it cannot be Black.

Lastly, if  $L_{U,n}$  is true, then vertex  $n+1$  can be in any of the three states, since if it is coloured then it forms a chain of one vertex that already has an uncoloured neighbour, namely vertex  $n$ .

(c) If  $A_{B,n}$  is true, then vertex  $n+1$  can be Uncoloured, since that never hurts legality. But it cannot be Black or White. If it were Black, then it would join the Black chain that contains vertex  $n$  but does not yet have an uncoloured neighbour, so the position would remain almost legal but it wouldn't be legal. If vertex  $n$  were White, it would become a single-vertex chain with no uncoloured neighbour, so the position would be illegal. (Furthermore, the Black chain containing vertex  $n$  would not have an uncoloured neighbour, giving another reason for illegality, so the position is now not even *almost* legal.)

The same holds true for  $A_{W,n}$ : vertex  $n+1$  can be Uncoloured, but not Black or White.

$A_{U,n}$  is impossible, since an almost legal position must have its final vertex coloured.

(d)

$L_{B,n+1}$  can be expressed as

$$(L_{B,n} \vee L_{U,n}) \wedge V_{B,n+1}.$$

$L_{W,n+1}$  can be expressed as

$$(L_{W,n} \vee L_{U,n}) \wedge V_{W,n+1}.$$

$L_{U,n+1}$  can be expressed as

$$(L_{B,n} \vee L_{W,n} \vee L_{U,n} \vee A_{B,n} \vee A_{W,n}) \wedge V_{U,n+1}.$$

$A_{B,n+1}$  can be expressed as

$$(L_{W,n} \vee A_{B,n}) \wedge V_{B,n+1}.$$

$A_{W,n+1}$  can be expressed as

$$(L_{B,n} \vee A_{W,n}) \wedge V_{W,n+1}.$$

9.

i.  $\text{taller}(\text{father}(\text{max}), \text{max}) \wedge \neg \text{taller}(\text{father}(\text{max}), \text{father}(\text{claire}))$

- ii.  $\exists X \text{ taller}(X, \text{father}(\text{claire}))$
- iii.  $\forall X \exists Y \text{ taller}(X, Y)$
- iv.  $\forall X (\text{taller}(X, \text{claire}) \rightarrow \text{taller}(X, \text{max}))$

## Supplementary exercises

10.

1.  $S_I = (K \wedge \neg A) \rightarrow I$
2.  $S_A = (I \wedge K) \rightarrow A$
3.  $S_K = K \leftrightarrow ((\neg I \wedge \neg A \wedge \neg K) \vee (\neg I \wedge A \wedge K) \vee (I \wedge \neg A \wedge K) \vee (I \wedge A \wedge \neg K))$
4. No, since
 
$$(S_I \wedge S_A \wedge S_K) \rightarrow (A \wedge K \wedge \neg I)$$
 is not a tautology.
5. Yes, since
 
$$(S_I \wedge \neg S_A \wedge S_K) \rightarrow (I \wedge K \wedge \neg A)$$
 is a tautology.