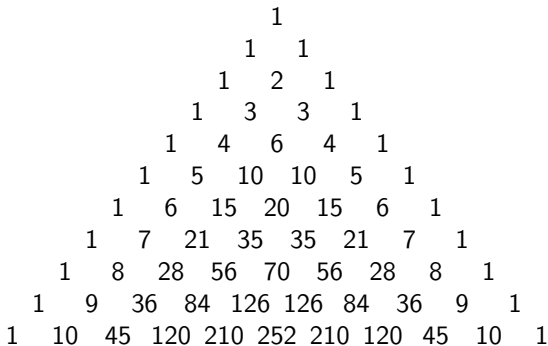


Lecture 20: Pascal's triangle and the binomial theorem

We can write the binomial coefficients in an (infinite) triangular array as follows:

$$\begin{array}{ccccccc} & & & \binom{0}{0} & & & \\ & & \binom{1}{0} & & \binom{1}{1} & & \\ & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\ & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\ & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\ \binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6} \\ \vdots & & & & \vdots & & & & \vdots & & & & \end{array}$$

Here are the first ten rows with the entries as integers:



This triangular array is often called *Pascal's triangle* (although Pascal was nowhere near the first to discover it).

Patterns

Writing the binomial coefficients this way reveals a lot of different patterns in them. Perhaps the most obvious is that every row reads the same left-to-right and right-to-left. Choosing r elements from a set of n elements to be in a combination is equivalent to choosing $n - r$ elements from the same set to not be in the combination. So:

$$\binom{n}{r} = \binom{n}{n-r} \quad \text{for } 0 \leq r \leq n.$$

This shows that every row reads the same left-to-right and right-to-left.

Another pattern is that every “internal” entry in the triangle is the sum of the two entries above it. To see why this is, we’ll begin with an example.

Example. Why is $\binom{6}{2} = \binom{5}{2} + \binom{5}{1}$?

There are $\binom{6}{2}$ combinations of 2 elements of $\{1, 2, 3, 4, 5, 6\}$. Every such combination either

- ▶ does not contain a 6, in which case it is one of the $\binom{5}{2}$ combinations of 2 elements of $\{1, 2, 3, 4, 5\}$; or
- ▶ does contain a 6, in which case the rest of the combination is one of the $\binom{5}{1}$ combinations of 1 element of $\{1, 2, 3, 4, 5\}$.

So $\binom{6}{2} = \binom{5}{2} + \binom{5}{1}$.

We can make a similar argument in general. Let X be a set of n elements and x is a fixed element of X . For any $r \in \{1, \dots, n\}$, there are $\binom{n-1}{r}$ combinations of r elements of X that do not contain x and there are $\binom{n-1}{r-1}$ combinations of r elements of X that do contain x . So:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \quad \text{for } 1 \leq r \leq n.$$

This shows that every internal entry in Pascal's triangle is the sum of the two above it.

The binomial theorem

$$\begin{aligned}(x+y)^0 &= 1 \\(x+y)^1 &= x+y \\(x+y)^2 &= x^2+2xy+y^2 \\(x+y)^3 &= x^3+3x^2y+3xy^2+y^3 \\(x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4 \\(x+y)^5 &= x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5\end{aligned}$$

Notice that the coefficients on the right are exactly the same as the entries in Pascal's triangle. Why does this happen? Think about expanding $(x+y)^3$ and finding the coefficient of xy^2 , for example.

$$\begin{aligned}(x+y)(x+y)(x+y) &= xxx + xxy + xyx + \underline{xyy} \\&\quad + yxx + \underline{yxy} + \underline{yyx} + yyy \\&= x^3 + 3x^2y + \underline{3xy^2} + y^3\end{aligned}$$

The coefficient of xy^2 is 3 because we have three terms in the sum above that contain two y 's (those underlined). This is because there are $\binom{3}{2}$ ways to choose two of the three factors in a term to be y 's.

The same logic holds in general. The coefficient of $x^{n-r}y^r$ in $(x+y)^n$ will be $\binom{n}{r}$ because there will be $\binom{n}{r}$ ways to choose r of the n factors in a term to be y 's. This fact is called the *binomial theorem*.

Binomial theorem: For any $n \in \mathbb{N}$,

$$(x+y)^n = \binom{n}{0}x^ny^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n-1}x^1y^{n-1} + \binom{n}{n}x^0y^n.$$

Questions

20.1 Substitute $x = 1$ and $y = -1$ into the statement of the binomial theorem. What does this tell you about the rows of Pascal's triangle?

$$(1 - 1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots \pm \binom{n}{n}.$$

The alternating sum of the terms in each row is zero except the first row!

20.2 Find a pattern in the sums of the rows in Pascal's triangle. Prove your pattern holds using the binomial theorem. Also prove it holds by considering the powerset of a set.

Substitute $x = 1, y = 1$ in the binomial theorem to get that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = (1 + 1)^n = 2^n.$$

If $|X| = n$ then $|\mathcal{P}(X)| = 2^n$. We count all subsets by counting those with 0 elements, then those with 1 element, then those with 2 elements and so on, up to those with n elements.

Inclusion-exclusion

A school gives out prizes to its best ten students in music and its best eight students in art. If three students receive prizes in both, how many students get a prize? If we try to calculate this as $10 + 8$ then we have counted the three overachievers twice. To compensate we need to subtract three and calculate $10 + 8 - 3 = 15$.

In general, if A and B are finite sets then we have

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

With a bit more care we can see that if A , B and C are sets then we have

$$\begin{aligned} |A \cup B \cup C| = & |A| + |B| + |C| - |A \cap B| \\ & - |A \cap C| - |B \cap C| + |A \cap B \cap C|. \end{aligned}$$

This is part of a more general law called the *inclusion-exclusion principle*.

Let X_1, X_2, \dots, X_t be finite sets. To calculate $|X_1 \cup X_2 \cup \dots \cup X_t|$:

- ▶ add the sizes of the sets;
- ▶ subtract the sizes of the 2-way intersections;
- ▶ add the sizes of the 3-way intersections;
- ▶ subtract the sizes of the 4-way intersections;
- ▶ \vdots
- ▶ add/subtract the size of the t -way intersection.

To see why this works, think of an element x that is in n of the sets X_1, X_2, \dots, X_t . It is counted

$$\binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \binom{n}{4} + \dots \pm \binom{n}{n}$$

times. By the Binomial theorem with $x = 1$ and $y = -1$ (see Question 20.1), this is equal to 1. So each element is counted exactly once.

Question

20.3 Use inclusion-exclusion to work out how many numbers in the set $\{1, \dots, 100\}$ are divisible by 2 or 3 or 5.

Let X_2 be the set of numbers in $\{1, \dots, 100\}$ that are divisible by 2.

Let X_3 be the set of numbers in $\{1, \dots, 100\}$ that are divisible by 3.

Let X_5 be the set of numbers in $\{1, \dots, 100\}$ that are divisible by 5.

We want $|X_2 \cup X_3 \cup X_5|$, which P.I.E. tells us is

$$|X_2| + |X_3| + |X_5| - |X_2 \cap X_3| - |X_2 \cap X_5| - |X_3 \cap X_5| + |X_2 \cap X_3 \cap X_5|.$$

These are much easier sets to count. For example, $|X_2 \cap X_5|$ counts the numbers that are divisible by 10 (of which there are 10).

$$\text{So } |X_2 \cup X_3 \cup X_5| = 50 + 33 + 20 - 16 - 10 - 6 + 3 = 74.$$