Housekeeping

Assignment 5 is now available. It is due at the beginning of your support class in week 7 (10–13 April). Submission options for people missing their Fri 14 April tutorial are included on the assignment sheet.

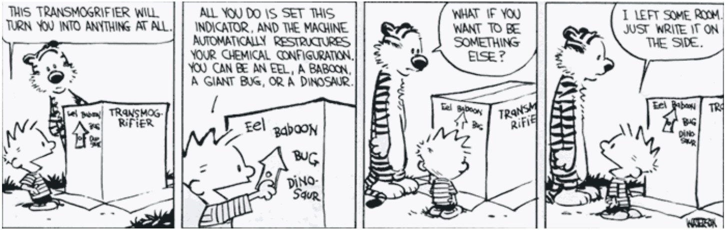
People missing their Fri 14 April tutorial should feel free to attend any other tutorial in week 7 (a list will be posted on moodle).

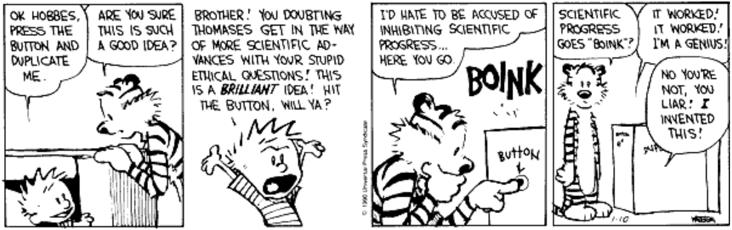
Tutorial sheet 5 and tutorial solutions 4 are also now available.

Assignment 4 is due at your support class next week.

MAT1830

Lecture 15: Composition and Inversion


THIS TRANSMOGRIFIER WILL TURN YOU INTO ANYTHING AT ALL.
Eel Baboon Bug
TRANSMOG-RIFIER
ALL YOU DO IS SET THIS INDICATOR, AND THE MACHINE AUTOMATICALLY RESTRUCTURES YOUR CHEMICAL CONFIGURATION. YOU CAN BE AN EEL, A BABOON, A GIANT BUG, OR A DINOSAUR.
Eel
BABOON
BUG
DINO-SAUR
WHAT IF YOU WANT TO BE SOMETHING ELSE?
I LEFT SOME ROOM. JUST WRITE IT ON THE SIDE.
Eel Baboon
BUG
DINO-SAUR
WATTERSON


OK HOBBES, PRESS THE BUTTON AND DUPLICATE ME.
ARE YOU SURE THIS IS SUCH A GOOD IDEA?
BROTHER! YOU DOUBTING THOMASES GET IN THE WAY OF MORE SCIENTIFIC AD-VANCES WITH YOUR STUPID ETHICAL QUESTIONS! THIS IS A BRILLIANT IDEA! HIT THE BUTTON, WILL YA?
I'D HATE TO BE ACCUSED OF INHIBITING SCIENTIFIC PROGRESS... HERE YOU GO.
BOINK
BUTTON
SCIENTIFIC PROGRESS GOES "BOINK"?
IT WORKED! IT WORKED! I'M A GENIUS!
NO YOU'RE NOT, YOU LIAR! I INVENTED THIS!
1-10
WATTERSON

Complicated functions are often built from simple parts. For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x^2 + 1)^3$ is computed by doing the following steps in succession:

- square,
- add 1,
- cube.

We say that $f(x) = (x^2 + 1)^3$ is the composite of the functions (from $\mathbb{R}$ to $\mathbb{R}$)

- square(x)=x^2,
- successor(x)=$x + 1$,
- cube(x)=x^3.

15.1 Notation for composite functions

In the present example we write

$$f(x) = \text{cube}(\text{successor}(\text{square}(x))),$$

or

$$f = \text{cube} \circ \text{successor} \circ \text{square}.$$

In general, if $f(x) = g(h(x))$ we write $f = g \circ h$ and say f is the *composite* of g and h.

Warning: Remember that $g \circ h$ means "do h first, then g." $g \circ h$ is usually different from $h \circ g$.

Example.
$\text{square}(\text{successor}(x)) = (x+1)^2 = x^2 + 2x + 1$
$\text{successor}(\text{square}(x)) = x^2 + 1$

Question 15.1 Let f, m and s be functions on the set of people defined by
$m(x) =$ mother of x
$f(x) =$ father of x
$s(x) =$ spouse of x.
What are the following?

(Note s is not actually a valid function on the set of people.)

$m \circ s(x)$	mother in law of x
$f \circ s(x)$	father in law of x
$m \circ m(x)$	grandmother (maternal) of x
$f \circ m(x)$	grandfather (maternal) of x
$s \circ s(x)$	x

Question 15.2 Write the following as composites of square(x), sqrt(x), successor(x) and cube(x).

(Assume that all of these have domain and codomain $\{x : x \in \mathbb{R} \text{ and } x \geq 0\}$.)

$$\begin{aligned}
\sqrt{1+x^3} &= \text{sqrt}(\text{successor}(\text{cube}(x))) = \text{sqrt} \circ \text{successor} \circ \text{cube}(x) \\
x^{\frac{3}{2}} &= \text{sqrt}((\text{cube}(x)) = \text{sqrt} \circ \text{cube}(x) \\
(1+x)^3 &= \text{cube}(\text{successor}(x)) = \text{cube} \circ \text{successor}(x) \\
(1+x^3)^2 &= \text{square}(\text{successor}(\text{cube}(x))) = \text{square} \circ \text{successor} \circ \text{cube}(x)
\end{aligned}$$

15.2 Conditions for composition

Composite functions do not always exist.

Example. If reciprocal : $\mathbb{R} - \{0\} \to \mathbb{R}$ is defined by $\text{reciprocal}(x) = \frac{1}{x}$ and predecessor : $\mathbb{R} \to \mathbb{R}$ is defined by $\text{predecessor}(x) = x - 1$, then reciprocal $\circ$ predecessor does not exist, because $\text{predecessor}(1) = 0$ is not a legal input for reciprocal.

To avoid this problem, we demand that the codomain of h be equal to the domain of g for $g \circ h$ to exist. This ensures that each output of h will be a legal input for g.

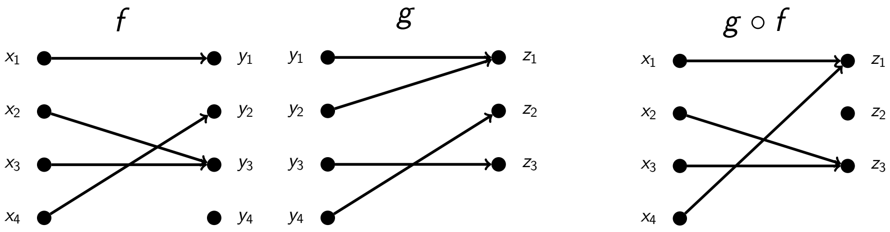
Let $h : A \to B$ and $g : C \to D$ be functions. Then $g \circ h : A \to D$ exists if and only if $B = C$.

Let $g : C \to D$ and $h : A \to B$ be functions.

The function $g \circ h$ exists if and only if $C = B$.

If it exists, $g \circ h : A \to D$ and is defined by $g \circ h(x) = g(h(x))$.

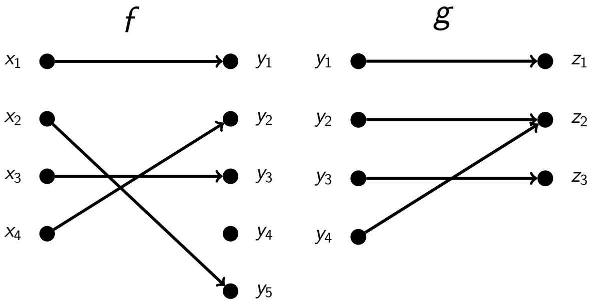
Question Let $f : A \to B$ and $g : C \to D$ be the functions pictured below.



Does $g \circ f$ exist?

codomain$(f) = \{y_1, y_2, y_3, y_4\}$ and domain$(g) = \{y_1, y_2, y_3, y_4\}$.
So $g \circ f$ does exist because codomain$(f) =$ domain(g).
$g \circ f : A \to D$

Question Let $f : A \to B$ and $g : C \to D$ be the functions pictured below.



Does $g \circ f$ exist?

codomain$(f) = \{y_1, y_2, y_3, y_4, y_5\}$ and domain$(g) = \{y_1, y_2, y_3, y_4\}$.
So $g \circ f$ does not exist because codomain$(f) \neq$ domain(g).

Example Let f, g and h be the functions

$f : \mathbb{R} \to \mathbb{Z}$ defined by $f(x) = \lfloor x \rfloor$ ("x rounded down")

$g : \mathbb{Z} \to \mathbb{R}$ defined by $g(x) = \sqrt{x'}$ where x' is the remainder when x is divided by 5.

$h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x^2 + 7$.

Does $g \circ f$ exist? Yes. $\text{codomain}(f) = \text{domain}(g)$ $g \circ f : \mathbb{R} \to \mathbb{R}$

Does $f \circ g$ exist? Yes. $\text{codomain}(g) = \text{domain}(f)$ $f \circ g : \mathbb{Z} \to \mathbb{Z}$

Does $g \circ h$ exist? No. $\text{codomain}(h) \neq \text{domain}(g)$

Does $g \circ f \circ g$ exist? Yes. $\text{codomain}(f \circ g) = \text{domain}(g)$

$g \circ f \circ g : \mathbb{Z} \to \mathbb{R}$.

15.3 The identity function

On each set A the function $i_A : A \to A$ defined by

$$i_A(x) = x,$$

is called the *identity function* (on A).

15.4 Inverse functions

Functions $f : A \to A$ and $g : A \to A$ are said to be inverses (of each other) if

$$f \circ g = g \circ f = i_A.$$

Example. square and sqrt are inverses of each other on the set $\mathbb{R}^{\geqslant 0}$ of reals $\geqslant 0$.

$$\text{sqrt}(\text{square}(x)) = x \text{ and } \text{square}(\text{sqrt}(x)) = x.$$

In fact, this is exactly what sqrt is supposed to do – reverse the process of squaring. However, this works only if we restrict the domain to $\mathbb{R}^{\geqslant 0}$. On $\mathbb{R}$ we do not have $\text{sqrt}(\text{square}(x)) = x$ because, for example,

$$\text{sqrt}(\text{square}(-1)) = \text{sqrt}(1) = 1.$$

This problem arises whenever we seek an inverse for a function which is not one-to-one. The squaring function on $\mathbb{R}$ sends both 1 and -1 to 1, but we want a single value 1 for sqrt(1). Thus we have to restrict the squaring function to $\mathbb{R}^{\geqslant 0}$.

15.5 Conditions for inversion

A function f can have an inverse without its domain and codomain being equal.

The inverse of a function $f : A \to B$ is a function $f^{-1} : B \to A$ such that

$$f^{-1} \circ f = i_A \quad \text{and} \quad f \circ f^{-1} = i_B.$$

Note that $f^{-1} \circ f$ and $f \circ f^{-1}$ are both identity functions but they have different domains.

Not every function has an inverse, but we can neatly classify the ones that do.

Let $f : A \to B$ be a function. Then $f^{-1} : B \to A$ exists if and only if f is one-to-one and onto.

Let $f : A \to B$.

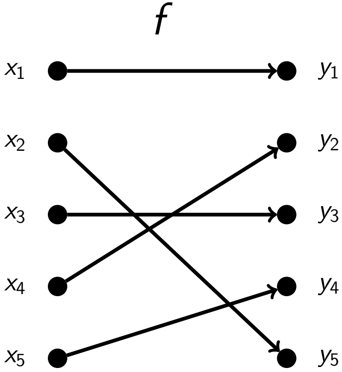
The function $f^{-1} : B \to A$ exists if and only if f is one-to-one and onto.

(Remember onto means range$(f) = B$.)

If it exists, $f^{-1} : B \to A$ is defined by $f^{-1}(y)$ equals the unique $x \in A$ such that $f(x) = y$.

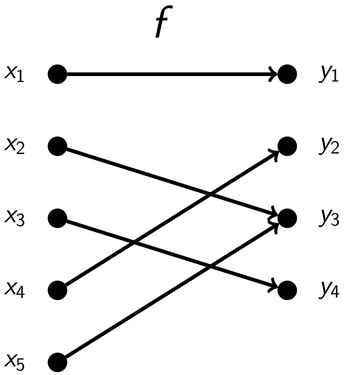
We have $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

Question Let $f : A \to B$ be the function pictured below.



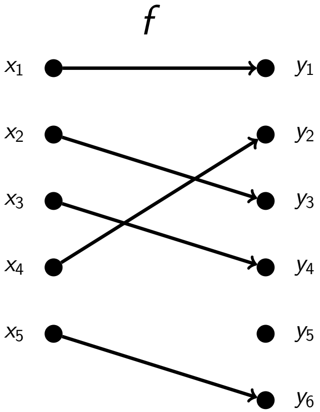
Does f^{-1} exist? Yes.

Question Let $f : A \to B$ be the function pictured below.



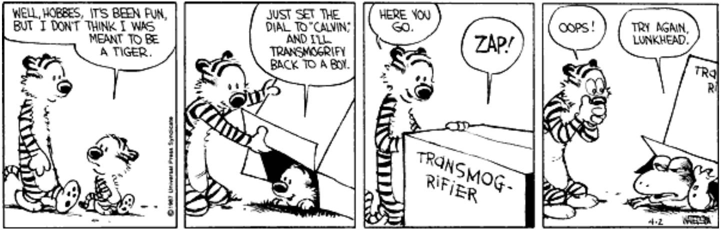
Does f^{-1} exist? No.
f is not one-to-one.

Question Let $f : A \to B$ be the function pictured below.



Does f^{-1} exist? No.

f is not onto.


WELL, HOBBES, IT'S BEEN FUN, BUT I DON'T THINK I WAS MEANT TO BE A TIGER.
JUST SET THE DIAL TO "CALVIN," AND I'LL TRANSMOGRIFY BACK TO A BOY.
HERE YOU GO.
ZAP!
TRANSMOG-RIFIER
OOPS!
TRY AGAIN, LUNKHEAD.

Question 15.4 What feature do

$\neg : \mathbb{B} \to \mathbb{B}$ defined by $\neg(x) = \neg x$;

$f(x) : \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$ defined by $f(x) = \frac{1}{x}$; and

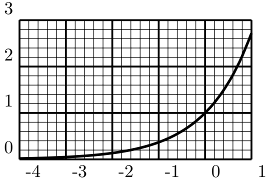
$g(x) : \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ defined by $g(x) = \frac{x}{x-1}$;

have in common?

They are their own inverses.

Example: e^x **and** $\log$

Consider $f : \mathbb{R} \to \mathbb{R}^{\geqslant 0} - \{0\}$ defined by $f(x) = e^x$. We know that e^x is one-to-one (e.g. because it is strictly increasing), and onto. So it has an inverse f^{-1} on $\mathbb{R}^{\geqslant 0} - \{0\}$.



Plot of $y = e^x$.

In fact, $f^{-1} = \log(y)$ where

$$\log : \mathbb{R}^{\geqslant 0} - \{0\} \to \mathbb{R}.$$

Now

$$e^{\log x} = x \quad \text{and} \quad \log(e^x) = x,$$

so $e^{\log x}$ and $\log(e^x)$ are both identity functions, but they have different domains.

The domain of $e^{\log x}$ is $\mathbb{R}^{\geqslant 0} - \{0\}$ (note log is defined only for reals > 0). The domain of $\log(e^x)$ is $\mathbb{R}$.

Question Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the function defined by $f((a, b)) = ab$. Does f^{-1} exist?

No. f is not one-to-one. e.g. $f((2, 3)) = f((1, 6))$

Question Let $g : \{x : x \text{ is a Monash student}\} \to \mathbb{N}$ be the function defined by $g(x)$ equals the ID number of x. Does g^{-1} exist?

No. g is not onto.

Question Let
$h : \{C : C \text{ is a circle in the plane with centre } (0, 0)\} \to \mathbb{R}^+$ be the function defined by $h(C)$ equals the area of C. Does h^{-1} exist?

Yes. h is one-to-one and onto.

15.6 Operations

An *operation* is a particular type of function, with domain $A \times A \times A \times \ldots \times A$ and codomain A, for some set A.

For example, the addition function $f(a, b) = a + b$ is called an *operation on* $\mathbb{R}$, because $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. (That is, addition is a function of two real variables, which takes real values.)

An operation with one variable is called *unary*, an operation with two variables is called *binary*, an operation with three variables is called *ternary*, and so on.

Examples

1. Addition is a binary operation on $\mathbb{R}$.
2. Successor is a unary operation on $\mathbb{N}$.
3. Intersection is a binary operation on $\mathcal{P}(A)$ for any set A.
4. Complementation is a unary operation on $\mathcal{P}(A)$ for any set A.