MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #4 and Additional Practice Questions

Tutorial Questions

1. (a) Define a sequence of integers $a_1, a_2, a_3, \ldots$ by $a_1 = 1$, $a_2 = 3$, and $a_t = a_{t-1} + a_{t-2}$ for each integer $t \geq 3$. (So the sequence goes $1, 3, 4, 7, 11, 18, \ldots$) How would you (informally) convince your friend Dwayne that every term in this sequence will be positive? How would you phrase this argument more formally.

   (b) Prove by induction that 3 divides $n^3 - 7n + 6$ for all integers $n \geq 0$.

2. (a) Are the following true or false?

   i. $\mathbb{N} \subseteq \mathbb{Q}$

   ii. $2 \subseteq \mathbb{N}$

   iii. $\{6, 7\} \subseteq \mathbb{N}$

   iv. $3 \in \mathbb{Q}$

   v. $\{3\} \in \mathbb{Q}$

   vi. $\mathbb{N} \subseteq \{x : x \in \mathbb{R}, x \geq 0\}$

   vii. $\{\} \subseteq \mathbb{N}$

   (b) Let $S = \{-1, 0, 1\}$. What is $\mathcal{P}(S)$?

   (c) Let $T = \{1, 2, \ldots, 10\}$. How many elements would $\mathcal{P}(T)$ have? Which of the following would be elements of $\mathcal{P}(T)$?

   i. $\emptyset$

   ii. $\{1, 4, 6\}$

   iii. 3

   iv. $\{2, 4, 12\}$

   v. $\{1, 2, \ldots, 10\}$

3. Let $A = \{1, 2\}$ and $B = \{-1, 0, 1\}$.

   (a) What is $A \cup B$?

   (b) What is $A \cap B$?

   (c) What is $A \times B$?

   (d) Is it true that, for any sets X, Y and Z, $(X \cup Y) \cap Z = X \cup (Y \cap Z)$? Why or why not?

   (e) Is it true that, for any sets Y and Z, $\mathcal{P}(Y) \cap \mathcal{P}(Z) = \mathcal{P}(Y \cap Z)$? Why or why not?

4. (a) Scrooge McDuck has created his own currency, duckbucks, which has only \$4 and \$7 notes. Using induction prove that, for any $n \geq 18$, n duckbucks can be made from \$4 and \$7 notes.

   (b) Dwayne doesn't understand induction but wants to be able to make change in duckbucks. Given your answer to (b), what (informal) instructions would you give him?

(See over for practice questions.)

Practice Questions

1. Let U be some universal set and let P, Q and R be subsets of U. Let

$$S_1 = P \cup (\overline{Q \cap R})$$
$$S_2 = (P \cup (P \cap Q)) \cap \overline{R}.$$

   (a) Draw a Venn diagram to determine whether it is always the case that $S_1 = S_2$.

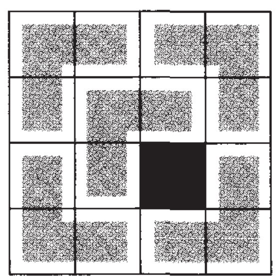
   (b) If this is not always the case, then does it ever happen? If so, when?

   (c) Let p, q and r be the propositions "$x \in P$", "$x \in Q$" and "$x \in R$". Find formulas in logic which mean "$x \in S_1$" and "$x \in S_2$". Find truth tables for these.

   (d) Do the truth tables contain the same information as you Venn diagrams? More? Less?

2. An L-tromino is like a domino, but made of three squares in the shape of an "L". Prove by induction that, for any n, if any one square is removed from a $2^n \times 2^n$ chess board, then the remaining squares can be completely covered by L-trominos.

   (The following picture shows an example on a 4×4 board.)



   Hint: A $2^{k+1} \times 2^{k+1}$ board can be split into four $2^k \times 2^k$ boards.

3. (a) How would you formally define intersections and unions of infinitely many sets?

   (b) Can you find a collection of infinitely many sets of integers $A_1, A_2, A_3, \ldots$ such that each set contains infinitely many integers, $A_{i+1} \subseteq A_i$ for $i = 1, 2, 3, \ldots$, and $A_1 \cap A_2 \cap A_3 \cap \cdots = \emptyset$?

4. As in 1(a) on the previous page, define a sequence of integers $a_1, a_2, a_3, \ldots$ by $a_1 = 1$, $a_2 = 3$, and $a_{t+1} = a_t + a_{t-1}$ for each integer $t \geq 3$. Prove by strong induction that $a_n \leq (\frac{7}{4})^n$ for each integer $n \geq 1$.