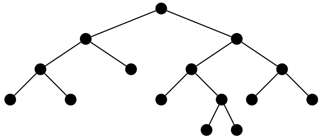
Lecture 35: Revision

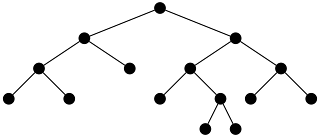


Prove this using the Handshaking lemma: If a binary tree has q question vertices then it has $q+1$ decision vertices.

ANS: The root has degree 2. The other $q-1$ question vertices have degree 3. All decision vertices have degree 1. Suppose there are x decision vertices. The sum of the degrees is $2+3(q-1)+x$. However, by the Handshaking Lemma this is twice the number of edges. And the number of edges is one less than the number of vertices since we have a tree. So

$$2+3(q-1)+x = 2(q+x-1).$$

Rearranging, we find that $x = q+1$, as required.



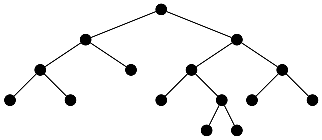
For any vertex v define $d(v)$ to be the distance (number of steps in the shortest walk) from the root to v.

> Define a relation R by uRv if $d(u) \leqslant d(v)$.
> Is R a partial order?

No. It is not antisymmetric since there are different vertices that have the same value of d.

> Define a relation R by uRv if $d(u) < d(v)$.
> Is R a partial order?

No. It is not reflexive since vertices aren't related to themselves.



Define a relation R by uRv if $d(u) < d(v)$ or $u = v$.
Is R a partial order?

Yes. [I'll leave the proof as an exercise].

Are any of the above relations total orders?

No. The first two weren't even partial orders, so they can't be total orders. This last one is not a total order since if I consider distinct u, v for which $d(u) = d(v)$, these vertices aren't related either way.

Is the pictured relation transitive?

$$\bullet \longrightarrow \bullet \longleftarrow \bullet$$

ANS: Yes. There is no occurrence of

$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$

without an arrow from the leftmost vertex to the rightmost vertex.

Any claim that you make about all the elements of an empty set is true!

Is the relation pRq defined by "p is older than q" antisymmetric?

ANS: Yes. It is impossible to have pRq and qRp since this would mean that p was older than q and q was older than p.

Is the relation pRq defined by "p is at least as old as q" antisymmetric?

ANS: No. Different people can be the same age.

Are either of the above relations partial orders?

ANS: No. The first one is not reflexive and the second is not antisymmetric.

Selections and arrangements

Selections of r elements from a set of n elements:

- Ordered selections without repetition $\dfrac{n!}{(n-r)!}$.
- Unordered selections without repetition $\dfrac{n!}{r!(n-r)!} = \dbinom{n}{r}$.
- Ordered selections with repetition n^r.
- Unordered selections with repetition $\dbinom{n+r-1}{n-1} = \dbinom{n+r-1}{r}$.

Selections and arrangements in graph theory

In each of the following sequence of questions the graph should have vertices $V_1, V_2, \ldots, V_n$.

> How many edges are possible in a simple graph on n vertices? How might I frame this as selection problem?

Each edge consists of an **unordered** selection of two vertices from the n possible (without repetition). Hence there are $\binom{n}{2}$ possible edges.

> Same question but now allowing loops as well.

This is now unordered selection **with** repetition allowed. So there are $\binom{n+2-1}{2} = \binom{n+1}{2}$ possible edges.

How many simple graphs on n vertices have e edges?

ANS: There are $\binom{n}{2}$ possible edges and we have to choose e of them (unordered selection without repetition). So $\binom{\binom{n}{2}}{e}$.

Same question, but allowing loops (but not multiple edges).

ANS: $\binom{\binom{n+1}{2}}{e}$ since there are $\binom{n+1}{2}$ possible edges and we have to choose e of them (unordered selection without repetition).

Same question, but allowing loops and multiple edges.

ANS: This is unordered selection **with** repetition allowed.
Again there are $\binom{n+1}{2}$ possible edges. So to choose e of them with repetition allowed there are $\binom{\binom{n+1}{2}+e-1}{e}$ possibilities.

Selections and arrangements for directed graphs

In each of the following questions the digraph should have vertices $V_1, V_2, \ldots, V_n$

How many possible edges are there in a digraph on n vertices
(a) if we don't allow loops?
(b) if we do allow loops?

ANS: We are now making ordered selections
(a) without repetition or (b) with repetition.

Hence in (a) the answer is $\frac{n!}{(n-2)!} = n(n-1)$.

In (b) it is n^2.

A recursive algorithm

Merge sort:

To sort a list we can

- Sort the first half.
- Sort the second half.
- Merge the two sorted halves.

Note that each of the first two steps is an instance of the same problem. So this algorithm uses itself recursively.

We need a stopping condition, which is that if a list has length 1 we don't need to sort it.

What next?

If you are interested in doing more mathematics (which you can do as an elective), then I recommend considering:

MTH2121 or MTH3121 Algebra and Number Theory I.
Includes: modular arithmetic, prime numbers, applications in cryptography including algorithms for secret communication, factoring numbers, testing whether a number is prime.

MTH3170 Network Mathematics
A whole unit on graph theory, include algorithms.