Faculty of Information Technology
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# FIT2014
# Solutions for Tutorial 1
# Logic and Proofs

**1.** We must show that, if $w \in$ PALINDROMES then $w \in \overline{\text{ODD-ODD}}$.

Suppose $w \in$ PALINDROMES. Then there is a string $x$ such that *either* $w = x\overleftarrow{x}$ *or* $w = xy\overleftarrow{x}$, where $\overleftarrow{x}$ denotes the reverse of $x$ and $y \in \{\text{a}, \text{b}\}$ is a single letter.

Suppose $w = x\overleftarrow{x}$. The numbers of a's and b's in $x\overleftarrow{x}$ are both even, since each is twice the number in $x$. So $w \in \overline{\text{ODD-ODD}}$.

Now suppose $w = xy\overleftarrow{x}$. Whichever letter in $\{\text{a}, \text{b}\}$ is *not* $y$ must appear an even number of times in $w$, by the same argument we have used previously. So that letter does not appear an odd number of times in $w$. So $w \in \overline{\text{ODD-ODD}}$.

*Alternative argument for the second case*, pointed out by an FIT2014 student in 2013:

Now suppose $w = xy\overleftarrow{x}$. Since the length of $w$ is odd, then it cannot have an both an odd number of a's and an odd number of b's (else its length would be even). So $w \in \overline{\text{ODD-ODD}}$.

**2.**

(a) $k$-th odd number $= 2k - 1$

(b) Inductive basis: when $k = 1$, the sum of the first $k$ odd numbers is just the first odd number, 1, which equals $1^2$, so it equals $k^2$.

(c)

$$\begin{aligned}
&\text{Sum of the first } k+1 \text{ odd numbers} \\
&\quad= 1 + 3 + \cdots + ((k+1)\text{-th odd number}) \\
&\quad= (1 + 3 + \cdots + (k\text{-th odd number})) + ((k+1)\text{-th odd number}) \\
&\quad= (\text{sum of the first } k \text{ odd numbers}) + ((k+1)\text{-th odd number}) \\
&\quad= (\text{sum of the first } k \text{ odd numbers}) + 2(k+1) - 1 \\
&\qquad\qquad\qquad (\text{using our formula from part (a)})
\end{aligned}$$

(d) Continuing from above,

$$\begin{aligned}
&\ldots \\
&\quad= k^2 + 2(k+1) - 1 \qquad\qquad (\text{by the Inductive Hypothesis})
\end{aligned}$$

(e) Continuing from above,

$$\begin{aligned}
&\ldots \\
&\quad= k^2 + 2k + 1 \\
&\quad= (k+1)^2.
\end{aligned}$$

This completes the Inductive Step (i.e., going from $k$ to $k+1$).

(f) So, by the Principle of Mathematical Induction, it is true <u>for all $k$</u> that the sum of the first $k$ odd numbers is $k^2$.

**3.** Base case: $n=1$:

The tree with one vertex has zero edges, which is $1-1$, so the claim is true for $n=1$.

Inductive step:

Suppose $k>1$, and that any tree with $k-1$ vertices has $(k-1)-1=k-2$ edges.

Let $T$ be any tree with $k$ vertices.

Now, every tree with $\geq 2$ vertices has a leaf, and removing any leaf from a tree gives another tree with one fewer vertex.

So, remove a leaf from $T$. Let $T^-$ be the smaller tree so obtained. It has $k-1$ vertices, so we can apply the Inductive Hypothesis to it. This tells us that $T^-$ has $k-2$ edges. Since we only deleted one edge when we deleted the leaf, this implies that $T$ has $(k-2)+1$ edges, i.e., $k-1$ edges. This completes the inductive step.

Therefore, by Mathematical Induction, it is true that, for all $n$, every tree on $n$ vertices has $n-1$ edges.

**4.**

Base case: $n=3$:

$3! = 6$, while $(3-1)^3 = 2^3 = 8$, so the inequality is true for $n=3$.

Inductive Step:

Suppose that $n! \leq (n-1)^n$ is true for a particular number $n$, where $n \geq 3$.

Let's look at what happens at $n+1$.

$$\begin{aligned}
(n+1)! &= (n+1)\cdot n! \qquad \text{(to express it in terms of a smaller case)} \\
&\leq (n+1)\cdot(n-1)^n \qquad \text{(by the Inductive Hypothesis, i.e., } n! \leq (n-1)^n\text{)} \\
&= (n+1)(n-1)(n-1)^{n-1} \qquad \text{(a slight rearrangement \ldots} \\
&\qquad\qquad \text{\ldots we are hoping to get } n+1 \text{ factors, all } \leq n \text{ \ldots)} \\
&= (n^2-1)(n-1)^{n-1} \qquad \text{(a little high-school algebra \ldots)} \\
&< n^2(n-1)^{n-1} \qquad \text{(and now we \textit{do} have } n+1 \text{ factors, all } \leq n\text{)} \\
&< n^{n+1} \\
&= ((n+1)-1)^{n+1}.
\end{aligned}$$

This last line is just to make it clear that the expression is of the required form.

So, by Mathematical Induction, it is true <u>for all $n$</u> that $n! \leq (n-1)^n$.

**5.**

1. $\mathbf{S_A} = \mathbf{B} \wedge \neg \mathbf{C}$
2. $\mathbf{S_B} = \mathbf{A} \to \mathbf{C}$
3. $\mathbf{S_C} = \neg \mathbf{C} \wedge (\mathbf{A} \vee \mathbf{B})$
4. Yes, since
$$(\mathbf{S_A} \wedge \mathbf{S_B} \wedge \mathbf{S_C}) \to (\mathbf{B} \wedge \neg \mathbf{A} \wedge \neg \mathbf{C})$$
is a tautology.

**6.**

(a) We prove it by constructing the truth table of each. It can be convenient to do this in stages.

| $P$ | $Q$ | $R$ | $Q \wedge R$ | $P \vee (Q \wedge R)$ |
|---|---|---|---|---|
| F | F | F | F | F |
| F | F | T | F | F |
| F | T | F | F | F |
| F | T | T | T | T |
| T | F | F | F | T |
| T | F | T | F | T |
| T | T | F | F | T |
| T | T | T | T | T |

| $P$ | $Q$ | $R$ | $P \vee Q$ | $P \vee R$ | $(P \vee Q) \wedge (P \vee R)$ |
|---|---|---|---|---|---|
| F | F | F | F | F | F |
| F | F | T | F | T | F |
| F | T | F | T | F | F |
| F | T | T | T | T | T |
| T | F | F | T | T | T |
| T | F | T | T | T | T |
| T | T | F | T | T | T |
| T | T | T | T | T | T |

The right-hand columns of each table are identical, so the two expressions (at the tops of those columns) are logically equivalent.

(b)

$$\begin{aligned}
P \wedge (Q \vee R) &= \neg\neg P \wedge (\neg\neg Q \vee \neg\neg R) \\
&= \neg\neg P \wedge \neg(\neg Q \wedge \neg R) \qquad \text{(by one of De Morgan's Laws)} \\
&= \neg(\neg P \vee (\neg Q \wedge \neg R)) \qquad \text{(by the other of De Morgan's Laws)} \\
&= \neg((\neg P \vee \neg Q) \wedge (\neg P \vee \neg R)) \qquad \text{(by part (a) of this question)} \\
&= \neg(\neg(P \wedge Q) \wedge \neg(P \wedge R)) \qquad \text{(by De Morgan, twice)} \\
&= \neg\neg(P \wedge Q) \vee \neg\neg(P \wedge R) \qquad \text{(by De Morgan, one last time)} \\
&= (P \wedge Q) \vee (P \wedge R)
\end{aligned}$$

Here we have used equality to stand for logical equivalence, which is normal.

**7.**

$$\begin{aligned}
(P_1 \wedge \cdots \wedge P_n) \Rightarrow C &= \neg(P_1 \wedge \cdots \wedge P_n) \vee C \\
&= (\neg P_1 \vee \cdots \vee \neg P_n) \vee C,
\end{aligned}$$

using De Morgan's Law.

Remark:

A disjunction of the form $\neg P_1 \vee \cdots \vee \neg P_n \vee C$, where $P_1, \ldots, P_n, C$ are each variables that can be True or False, is called a *Horn clause*. These play a big role in the theory of logic programming.

**8.**
(a) $L_{B,n} \vee L_{W,n} \vee L_{U,n}$

(b) If $L_{B,n}$ is true, then it's ok for vertex $n+1$ to be Black (since it then joins the black chain that includes vertex $n$, which must be ok as the position up to vertex $n$ is legal). It could also be Uncoloured, since adding a new uncoloured vertex next to an existing vertex can never make a legal position illegal. But vertex $n+1$ cannot be White, as it is then in a chain of its own which has no Uncoloured neighbour.
Similarly, if $L_{W,n}$ is true, then vertex $n+1$ can be White or Uncoloured, but it cannot be Black.
Lastly, if $L_{U,n}$ is true, then vertex $n+1$ can be in any of the three states, since if it is coloured then it forms a chain of one vertex that already has an uncoloured neighbour, namely vertex $n$.

(c) If $A_{B,n}$ is true, then vertex $n+1$ can be Uncoloured, since that never hurts legality. But it cannot be Black or White. If it were Black, then it would join the Black chain that contains vertex $n$ but does not yet have an uncoloured neighbour, so the position would remain almost legal but it wouldn't be legal. If vertex $n$ were White, it would become a single-vertex chain with no uncoloured neighbour, so the position would be illegal. (Furthermore, the Black chain containing vertex $n$ would not have an uncoloured neighbour, giving another reason for illegality, so the position is now not even *almost* legal.)
The same holds true for $A_{W,n}$: vertex $n+1$ can be Uncoloured, but not Black or White.
$A_{U,n}$ is impossible, since an almost legal position must have its final vertex coloured.

(d)
$L_{B,n+1}$ can be expressed as

$$(L_{B,n} \vee L_{U,n}) \wedge V_{B,n+1}.$$

$L_{W,n+1}$ can be expressed as

$$(L_{W,n} \vee L_{U,n}) \wedge V_{W,n+1}.$$

$L_{U,n+1}$ can be expressed as

$$(L_{B,n} \vee L_{W,n} \vee L_{U,n} \vee A_{B,n} \vee A_{W,n}) \wedge V_{U,n+1}.$$

$A_{B,n+1}$ can be expressed as

$$(L_{W,n} \vee A_{B,n}) \wedge V_{B,n+1}.$$

$A_{W,n+1}$ can be expressed as

$$(L_{B,n} \vee A_{W,n}) \wedge V_{W,n+1}.$$

**9.**

i. **taller(father(max), max)** $\wedge\neg$ **taller(father(max), father(claire))**

ii. $\exists X$ **taller(X, father(claire))**

iii. $\forall X \exists Y$ **taller(X, Y)**

iv. $\forall X$ **(taller(X, claire)** $\rightarrow$ **taller(X, max))**

# Supplementary exercises

**10.** (a)

(b)

Inductive basis:

Suppose $n \leq 100$. Applying `wc` to standard output of this length gives a one-line standard output stating the numbers of lines, words and characters, with the number of characters being $n$. This output has 1 line, 3 (or 4?) words and some small number of characters consisting of the digit 1, the digit 3, a couple of digits (at most) for $n$, and some number of spaces (say, 21 altogether, but the analysis is much the same if this number is different). Applying `wc` again gives one line with these numbers in it: 1, 3, 25, again alongside 21 spaces. Another application of `wc` gives the same result. So the claim is true for $n \leq 99$.

Inductive step:

Now suppose the claim is true when the file/string has $\leq k$ characters, where $k \geq 100$. Suppose we are given a file/string of $k+1$ characters. Applying `wc` gives a one-line standard output, giving numbers of lines, words and characters as $l$, $w$ and $k+1$, respectively, set out something like

$$l \qquad w \qquad k+1$$

For any number $x$, write digits($x$) for the number of digits in $x$. Our one-line output has some number $s$ of spaces, say $s = 21$; the number of non-space characters is

$$\text{digits}(l) + \text{digits}(w) + \text{digits}(k+1).$$

Now, $l \leq k+1$ and $w \leq k+1$.[1] Therefore the number of characters in the one-line output above is $\leq s + 3 \cdot \text{digits}(k+1)$. We claim this is $\leq k$ if $k$ is large enough.

> To see this, first try $k = 100$. Then
>
> $$\begin{aligned} s + 3 \cdot \text{digits}(k+1) &= 21 + 3 \cdot \text{digits}(100+1) \\ &= 21 + 3 \cdot \text{digits}(101) \\ &= 21 + 3 \cdot 3 \\ &= 30, \end{aligned}$$
>
> which is indeed $\leq k$. (Only minor changes are needed here if the actual value of $s$ is not 21; it will not be *much* different from 21.) Now, whenever $k$ increases by 1, digits($k+1$) either stays the same or increases by 1. But it only increases rarely, when $k+1$ becomes 100, then when it becomes 1000, and so on. So, given that $s + 3 \cdot \text{digits}(k+1) \leq k$ for $k = 100$ (and by a good margin), this inequality continues to hold for all higher $k$.[2]

Since this is now $\leq k$, the Inductive Hypothesis tells us that some further applications of `wc` will eventually give constant output.

Therefore the result follows for all $n$, by the Principle of Mathematical Induction.

**11.**

1. $\mathbf{S_I} = (\mathbf{K} \wedge \neg\mathbf{A}) \rightarrow \mathbf{I}$
2. $\mathbf{S_A} = (\mathbf{I} \wedge \mathbf{K}) \rightarrow \mathbf{A}$
3. $\mathbf{S_K} = \mathbf{K} \leftrightarrow ((\neg\mathbf{I} \wedge \neg\mathbf{A} \wedge \neg\mathbf{K}) \vee (\neg\mathbf{I} \wedge \mathbf{A} \wedge \mathbf{K}) \vee (\mathbf{I} \wedge \neg\mathbf{A} \wedge \mathbf{K}) \vee (\mathbf{I} \wedge \mathbf{A} \wedge \neg\mathbf{K}))$
4. No, since
$$(\mathbf{S_I} \wedge \mathbf{S_A} \wedge \mathbf{S_K}) \rightarrow (\mathbf{A} \wedge \mathbf{K} \wedge \neg\mathbf{I})$$
is not a tautology.
5. Yes, since
$$(\mathbf{S_I} \wedge \neg\mathbf{S_A} \wedge \mathbf{S_K}) \rightarrow (\mathbf{I} \wedge \mathbf{K} \wedge \neg\mathbf{A})$$
is a tautology.

[1] In fact, $w \leq (k+1)/2$, since every consecutive pair of words must have at least one space between them. But we don't need this better upper bound on $w$.

[2] Thanks to FIT2014 tutor Nathan Companez for part of this argument.