MAT1830

Lecture 13: Functions

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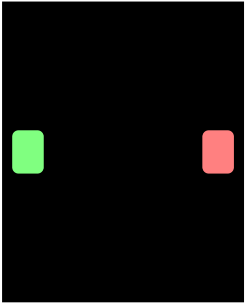
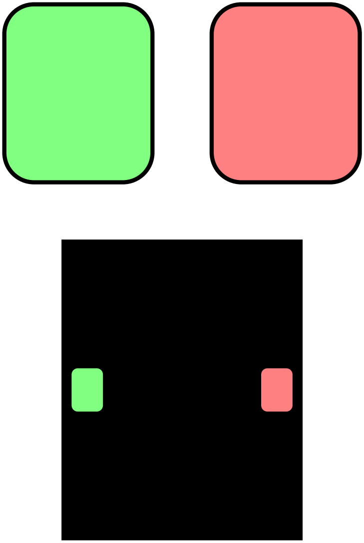
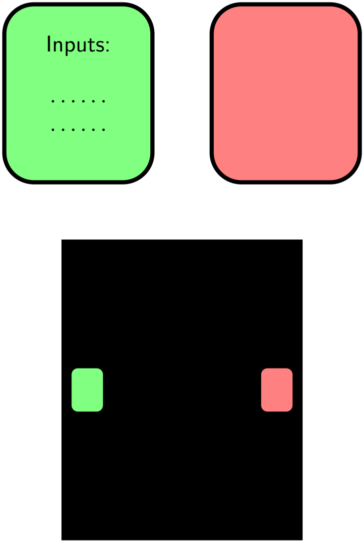
- Functions (subroutines) in programming are closely related to functions in the mathematical sense.
- In the case of functional programming languages (eg. Lisp, Haskell, Rust) they are exactly functions in the mathematical sense.
- Functions are used to define a lot of important concepts in maths and theoretical computer science.

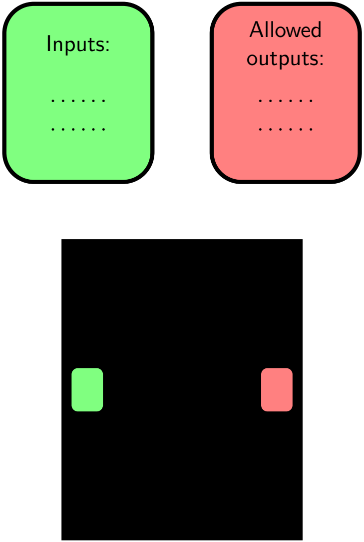
A function can be thought of as a “black box” which accepts inputs and, for each input, produces a single output.

[redacted]






Inputs:
. . . . . .
. . . . . .


Inputs:
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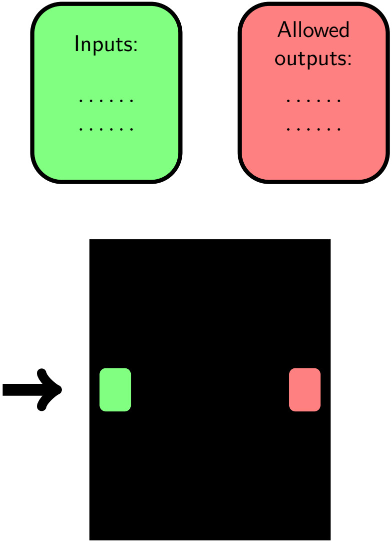
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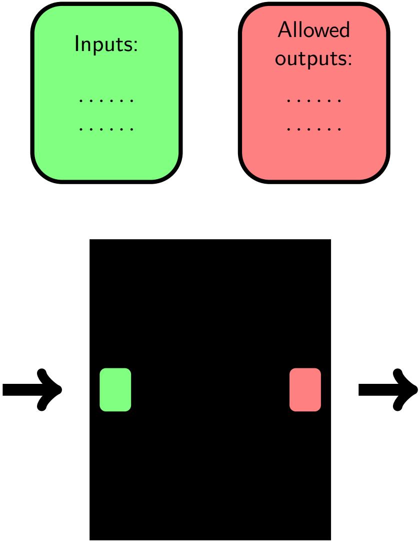
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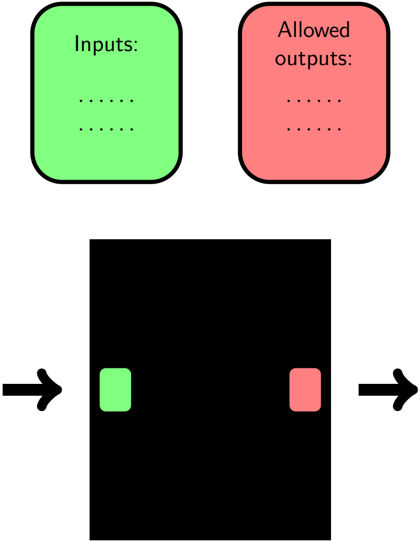
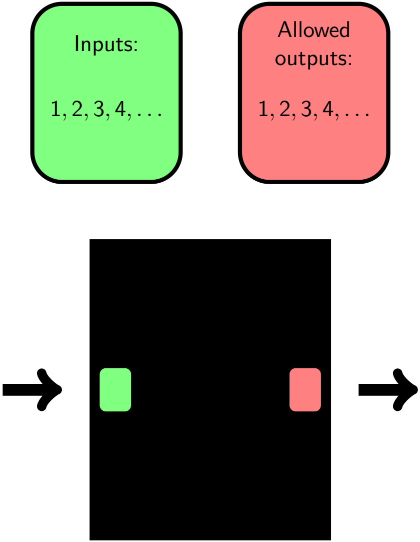


- Each input produces exactly one output.
  (Always the same output for a given input.)



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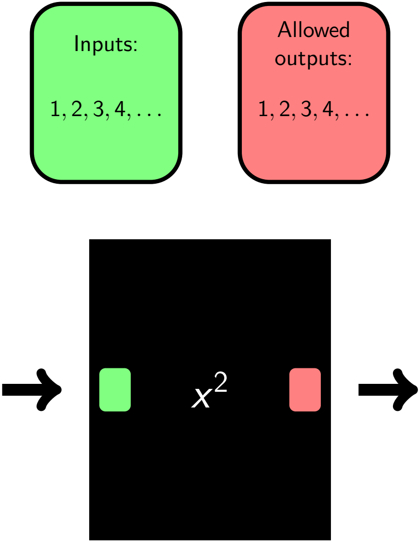

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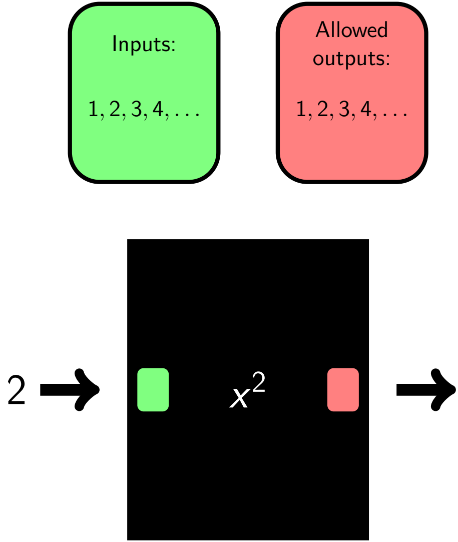

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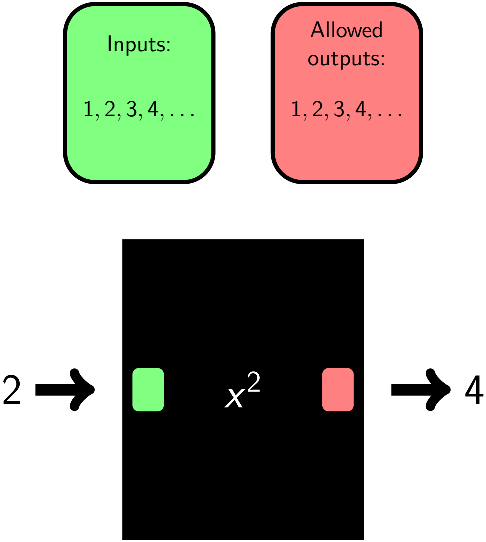

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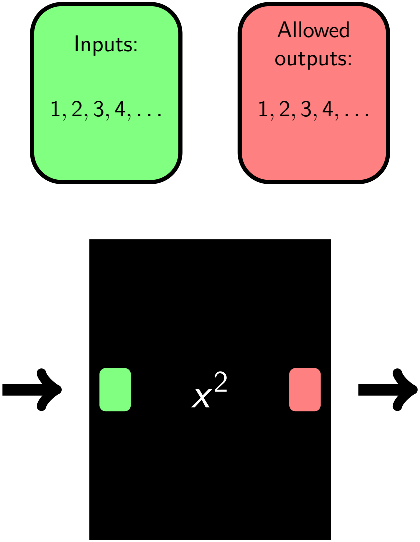

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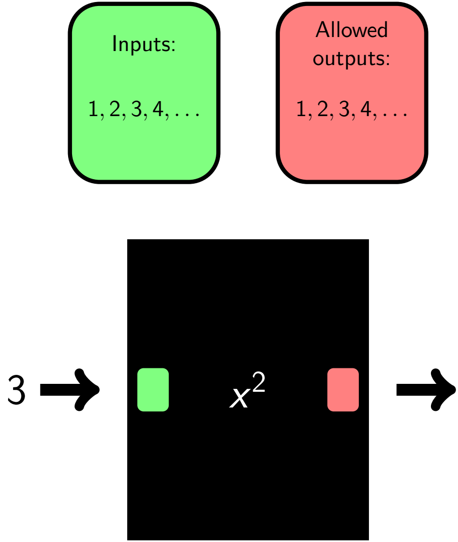

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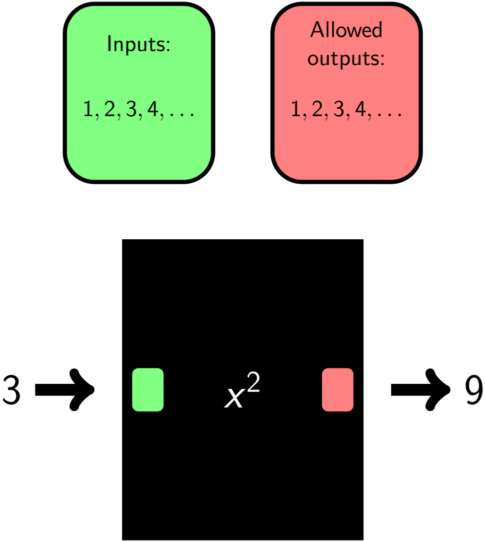

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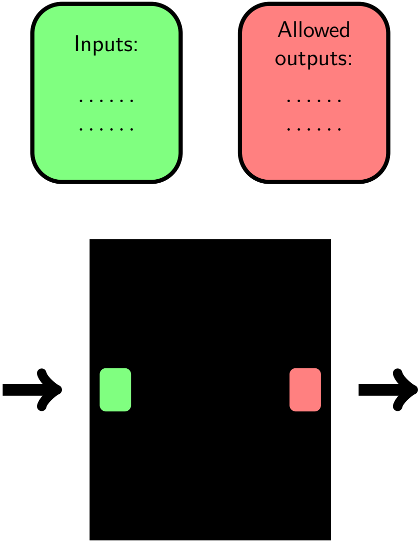

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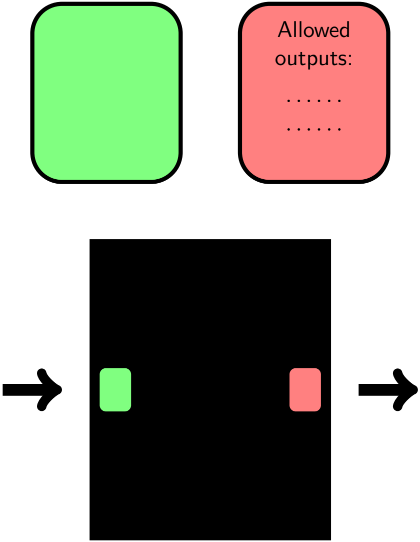

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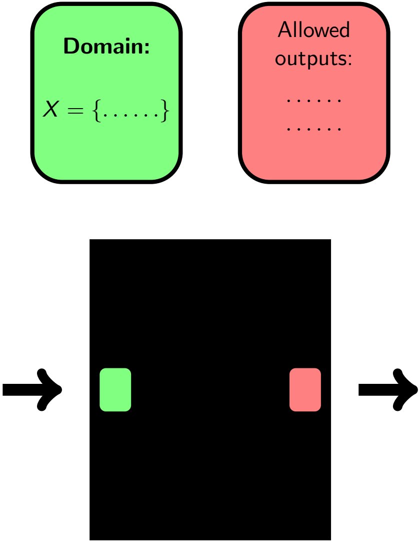

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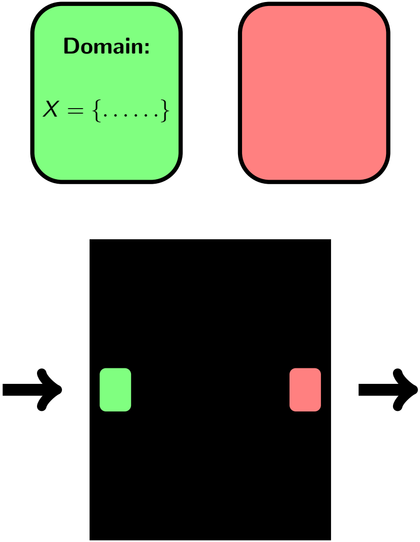

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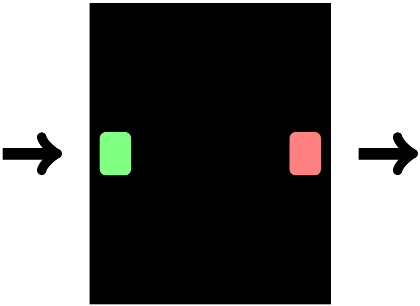
Domain:

$X = \{\ldots\ldots\}$

Codomain:

$Y = \{\ldots\ldots\}$



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x_3	y_3
$\vdots$	$\vdots$

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$$\{(x_1, y_1), (x_2, y_2), \\ (x_3, y_3), \ldots\ldots\ldots\}$$

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A set of ordered pairs from $X \times Y$ that contains exactly one ordered pair (x, y) for each $x \in X$.

Remember: The domain and codomain are part of the function and must always be defined.


THIS TRANSMOGRIFIER WILL TURN YOU INTO ANYTHING AT ALL.
TRANSMOG-RIFIER
ALL YOU DO IS SET THIS INDICATOR, AND THE MACHINE AUTOMATICALLY RESTRUCTURES YOUR CHEMICAL CONFIGURATION. YOU CAN BE AN EEL, A BABOON, A GIANT BUG, OR A DINOSAUR.
Eel
BaBOON
BUG
DINO-SAUR
WHAT IF YOU WANT TO BE SOMETHING ELSE?
I LEFT SOME ROOM. JUST WRITE IT ON THE SIDE.
WATTERSON

13.1 Defining functions via sets

Formally we represent a function f as a set X of possible inputs, a set Y so that every output of f is guaranteed to be in Y, and a set of (input,output) pairs from $X \times Y$. The vital property of a function is that each input gives exactly one output.

A function f consists of a *domain* X, a *codomain* Y, and a set of ordered pairs from $X \times Y$ which has exactly one ordered pair (x, y) for each $x \in X$.

When (a, b) is in this set we write $f(a) = b$.

The set of y values occurring in these pairs is the *range* of f.

Note that the range of a function is always a subset of its codomain but they may or may not be equal.

If the range of a function is equal to its codomain, we say the function is *onto*.

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"f is a function with domain X and codomain Y" is shortened to

$$f : X \to Y.$$

Example Let $f : \{0, 1, 2, 3\} \to \{0, 1, 2, 3, 4, 5, 6\}$ be defined by $f(x) = 2x$.

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x	$f(x)$
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1	2
2	4
3	6

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$\{(x, 2x) : x \in \mathbb{R}\}$

Arrow diagrams

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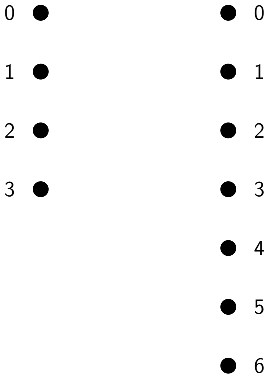

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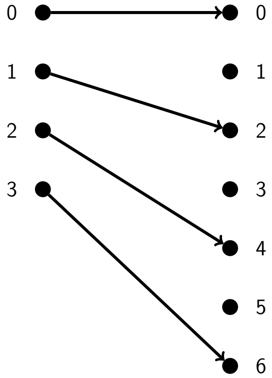

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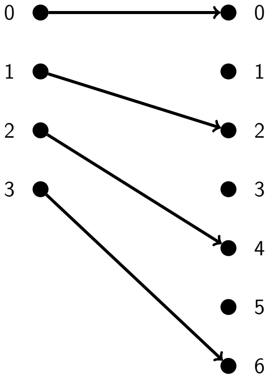

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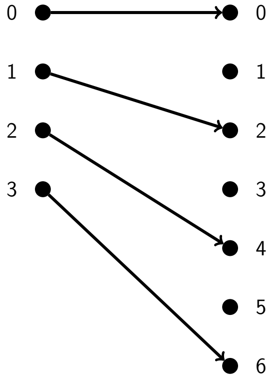

The range of f is $\{0, 2, 4, 6\}$. (So f is not onto.)

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Another reason is that "$\mathbb{R} \to \mathbb{R}$ functions", for example, make a nice class to consider.

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Question Does $\{(0, 7), (1, 4), (2, \pi), (2, 3)\}$ correspond to a function $f : \{0, 1, 2\} \to \mathbb{R}$?

No - it has two ordered pairs with first coordinate 2.

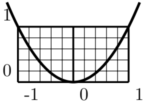
square : $\mathbb{R} \to \mathbb{R}$

Examples.

1. The squaring function square(x)= x^2 with domain $\mathbb{R}$, codomain $\mathbb{R}$, and pairs

$$\{(x, x^2) \ : \ x \in \mathbb{R}\},$$

which form what we usually call the *plot* of the squaring function.

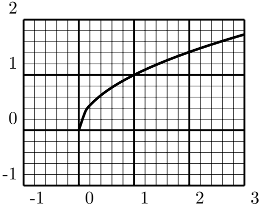


The range of this function (the set of y values) is the set $\mathbb{R}^{\geqslant 0}$ of real numbers $\geqslant 0$.

sqrt : $\mathbb{R}^{\geq 0} \to \mathbb{R}$

2. The square root function $\text{sqrt}(x) = \sqrt{x}$ with domain $\mathbb{R}^{\geqslant 0}$, codomain $\mathbb{R}$, and pairs

$$\{(x, \sqrt{x}) \ : \ x \in \mathbb{R} \text{ and } x \geqslant 0\}.$$



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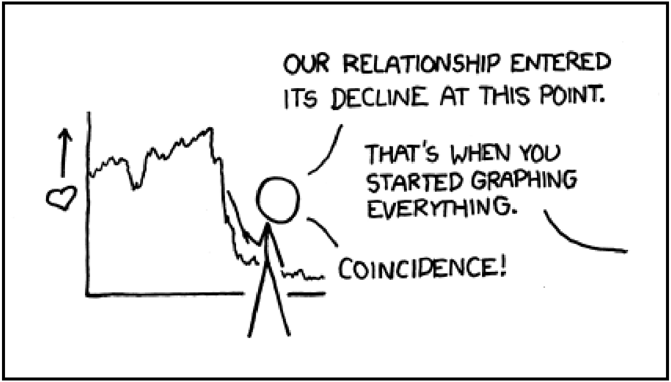
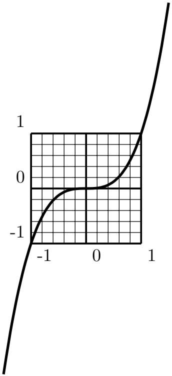

OUR RELATIONSHIP ENTERED ITS DECLINE AT THIS POINT.
THAT'S WHEN YOU STARTED GRAPHING EVERYTHING.
COINCIDENCE!

3. The cubing function $\text{cube}(x) = x^3$ with domain $\mathbb{R}$, codomain $\mathbb{R}$, and pairs

$$\{(x, x^3)\ :\ x \in \mathbb{R}\},$$



The range of this function is the whole of the codomain $\mathbb{R}$, so it is onto.

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13.2 Arrow notation

If f is a function with domain A and codomain B we write

$$f : A \to B,$$

and we say that f is from A to B.

For example, we could define

$$\text{square} : \mathbb{R} \to \mathbb{R}.$$

We could also define

$$\text{square} : \mathbb{R} \to \mathbb{R}^{\geqslant 0}.$$

Likewise, we could define

$$\text{cube} : \mathbb{R} \to \mathbb{R}.$$

However we could not define

$$\text{cube} : \mathbb{R} \to \mathbb{R}^{\geqslant 0},$$

because for some $x \in \mathbb{R}$, $\text{cube}(x)$ is negative. For example, $\text{cube}(-1) = -1$.

Question 13.2 Which of the following functions can be defined on the whole of $\mathbb{R}$, so that the function values also lie in $\mathbb{R}$?

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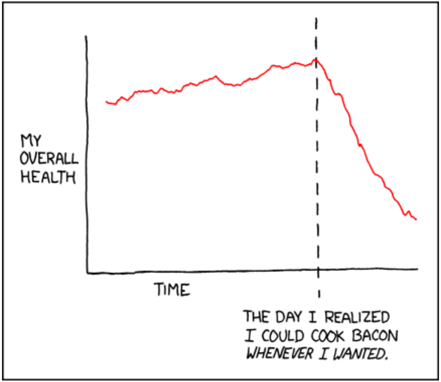
$\sqrt{x}$ No - undefined for $x < 0$.

$\sqrt[3]{x}$ Yes.


MY OVERALL HEALTH
TIME
THE DAY I REALIZED I COULD COOK BACON WHENEVER I WANTED.

13.3 One-to-one functions

A function $f : X \to Y$ is *one-to-one* if for each y in the range of f there is only one $x \in X$ such that $f(x) = y$.

For example, the function cube(x) is one-to-one because each real number y is the cube of exactly one real number x.

The function square: $\mathbb{R} \to \mathbb{R}$ is *not* one-to-one because the real number 1 is the square of two different real numbers, 1 and -1. (In fact each real $y > 0$ is the square of two different real numbers, $\sqrt{y}$ and $-\sqrt{y}$)

On the other hand, square : $\mathbb{R}^{\geqslant 0} \to \mathbb{R}$ *is* one-to-one because each real number y in $\mathbb{R}^{\geqslant 0}$ is the square of only one real number in $\mathbb{R}^{\geqslant 0}$, namely $\sqrt{y}$.

The last example shows that the domain of a function is an important part of its description, because changing the domain can change the properties of the function.

Question Is the function pictured below one-to-one?

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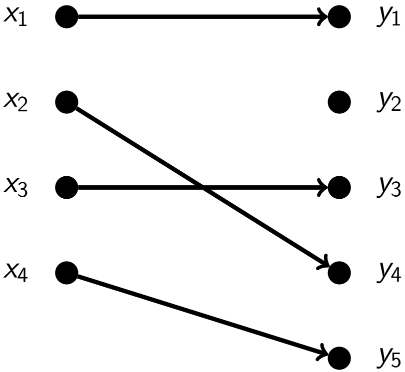

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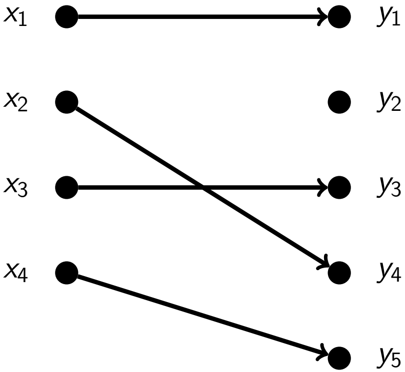

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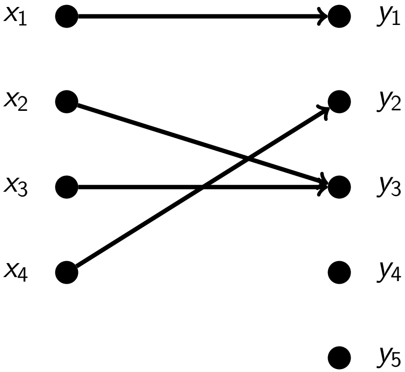

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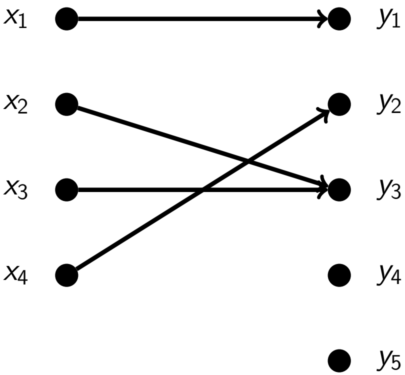

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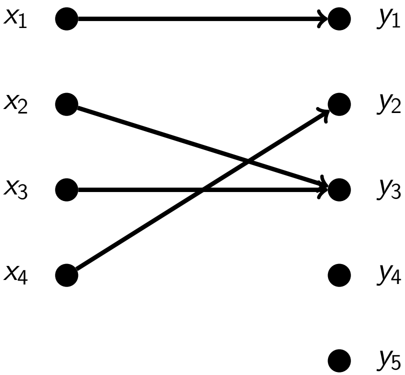

No. $f(x_2) = f(x_3)$.

13.4 Proving a function is one-to-one

There is an equivalent way of phrasing the definition of one-to-one: a function $f : X \to Y$ is one-to-one when, for all $x_1, x_2 \in X$,

$$f(x_1) = f(x_2) \;\Rightarrow\; x_1 = x_2.$$

This can be useful for proving that some functions are or are not one-to-one.

Example. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 6x + 2$ is one-to-one because

$$\begin{aligned} & & f(x_1) &= f(x_2) \\ \Rightarrow & & 6x_1 + 2 &= 6x_2 + 2 \\ \Rightarrow & & 6x_1 &= 6x_2 \\ \Rightarrow & & x_1 &= x_2. \end{aligned}$$

Example. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 1$ is not one-to-one because $f(-1) = 2$ and $f(1) = 2$ and so

$$f(-1) = f(1).$$

To show that a function $f : X \to Y$ is one-to-one we must show that, for all $x_1, x_2 \in X$,

$$\text{if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

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Question Is $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 3x - 2$ one-to-one?

Suppose that

$$f(x_1) = f(x_2) \qquad \text{for some } x_1, x_2 \in \mathbb{Z}.$$

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This shows that f is one-to-one.

To show that a function $f : X \to Y$ is *not* one-to-one we must show that there exist $x_1, x_2 \in X$ such that

$$f(x_1) = f(x_2) \text{ and } x_1 \neq x_2.$$

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No. $f(1) = 2$ and $f(-1) = 2$ (and obviously $1 \neq -1$)

Question 13.3 Find the range of the following functions, given that the domain of each is $\mathbb{R}$.

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e^x

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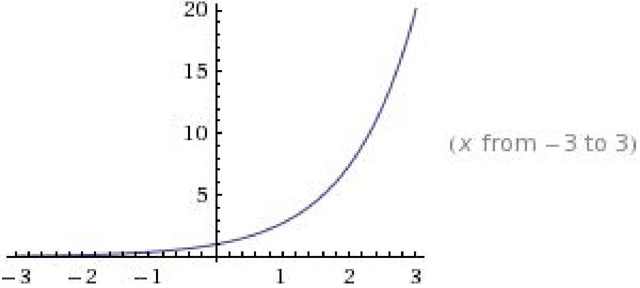

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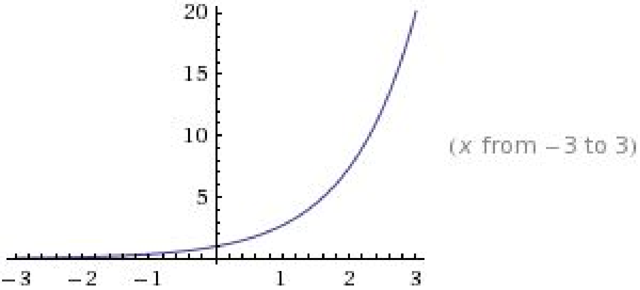

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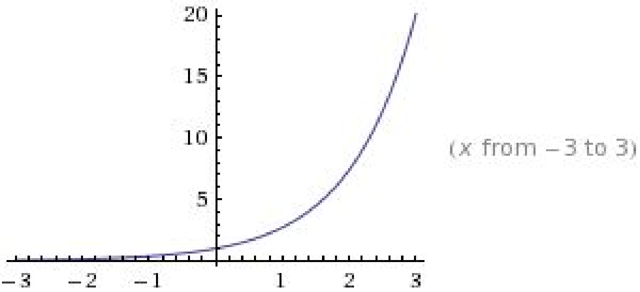

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The rest of these are left as an exercise.