Faculty of Information Technology
Semester 2, 2018

FIT2014
Tutorial 1
Logic and Proofs

ASSESSED PREPARATION: Question 3.
You must present a serious attempt at this entire question to your tutor at the start of your tutorial.

1. Let ODD-ODD be the language of strings, over the alphabet {a,b}, that contain an odd number of a's and an odd number of b's. Let $\overline{\text{ODD-ODD}}$ be its complement.

Prove that PALINDROMES $\subseteq \overline{\text{ODD-ODD}}$.

2. Prove the following statement, by mathematical induction:

(*) The sum of the first k odd numbers equals k^2.

(a) First, give a simple expression for the k-th odd number.
(b) Inductive basis: now prove the statement $(*)$ for $k = 1$.

Assume the statement $(*)$ true for a specific value k. This is our **Inductive Hypothesis**.

(c) Express the sum of the first $k + 1$ odd numbers ...

$$1 + 3 + \cdots + ((k + 1)\text{-th odd number})$$

... in terms of the sum of the first k odd numbers, plus something else.
(d) Use the inductive hypothesis to replace the sum of the first k odd numbers by something else.

(e) Now simplify your expression. What do you notice?

(f) When drawing your final conclusion, don't forget to briefly state that you are using the Principle of Mathematical Induction!

3. Prove, by induction on n, that for all $n \geq 1$, every tree on n vertices has $n - 1$ edges.

Use the fact that every tree has a leaf.

4. (a) Prove, by induction on n, that for all $n \geq 3$,

$$n! \leq (n-1)^n.$$

(b) [Challenge]
Can you use a similar proof to show that $n! \leq (n-2)^n$? What assumptions do you need to make about n? How far can you push this: what if 2 is replaced by a larger number? What is the best upper bound of the form $f(n)^n$ that you can find, where $f(n)$ is some function of n?

5. *Three boys, Adam, Brian and Claude, are caught, suspected of breaking a glass window. When the boys were questioned by police:*

Adam said: 'Brian did it; Claude is innocent'.

Brian said: 'If Adam is guilty then so is Claude'.

Claude said: 'I didn't do it; one of the others did'.

The police believed that all the boys were telling the truth, and therefore concluded that Brian broke the window and the others didn't.

Using the following propositions express the statements of the boys and the police conclusion in propositional logic.

A: **Adam broke the window.**

B: **Brian broke the window.**

C: **Claude broke the window.**

Assuming that all the boys were telling the truth, was the police conclusion logically valid?

6. Distributive Law for propositional logic:
(a) Prove that

$$P \vee (Q \wedge R) \quad \text{is logically equivalent to} \quad (P \vee Q) \wedge (P \vee R).$$

(b) Prove that

$$P \wedge (Q \vee R) \quad \text{is logically equivalent to} \quad (P \wedge Q) \vee (P \wedge R),$$

using part (a) and a rule named after a friend of Charles Babbage.

7. Prove that

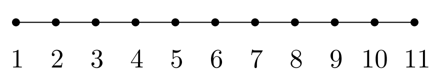
$$(P_1 \wedge \cdots \wedge P_n) \Rightarrow C \quad \text{is logically equivalent to} \quad \neg P_1 \vee \cdots \vee \neg P_n \vee C$$

8. One-dimensional Go.

This question uses some concepts from the ancient east Asian board game known as *Go* in the West, *Wéiqí* in China, *Go* or *Igo* in Japan, and *Baduk* in Korea. This game is over 2,000 years old, and is generally regarded as harder than Chess. Indeed, until very recently, computer programs for Go could not defeat human professionals (in contrast to Chess, where the best computer players have been stronger than human world champions since the late 1990s).

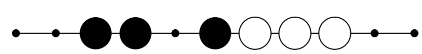
That changed dramatically early in 2016, with stunning performances by the program AlphaGo, created by Google DeepMind. (This company began as a start-up in London in 2010 and was acquired by Google in 2014.) In January 2016, it defeated the European champion Fan Hui in every game of a five-game match: `http://www.nature.com/news/google-ai-algorithm-masters-ancient-game-of-go-1.19234`. Then in March 2016 it defeated Lee Sedol of Korea, generally regarded as the best player in the world in recent times: `https://gogameguru.com/tag/deepmind-alphago-lee-sedol/`. The final score in that five-game match was 4-1 in favour of AlphaGo. In May 2017 it defeated the top-ranked player in the world, Ke Jie of China, 3-0. See `https://deepmind.com/research/alphago/` or `http://361points.com/articles/thoughtsonalphago/`.

Go is played on a graph, usually a square lattice (grid) of 19×19 vertices. But we will use much simpler graphs in this question, namely path graphs with n vertices and $n-1$ edges, where $n \geq 1$. For example, with $n = 11$ we get the following path graph with 11 vertices and 10 edges.

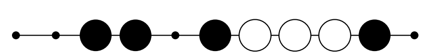


A *position* consists of a placement of black and white stones on some of the vertices of the graph. Each vertex may have a black stone, or a white stone (but not both), or be uncoloured (i.e., vacant). A position is *legal* if every vertex with a stone can be linked to an uncoloured vertex by a path consisting entirely of vertices with stones of that same colour (except for the uncoloured vertex at the end).

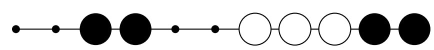
For example, the following position is legal, since each of its three "chains" of consecutive vertices of the same colour either starts or ends with an uncoloured vertex.



But the following position is illegal, since it has a chain of white vertices with black vertices at each end. (The position has four chains altogether, and three are ok. But it only takes one without an uncoloured neighbour to make the position illegal.)



We number the vertices on the path graph from 1 to n, from left to right. We say that a position on this path graph is *almost legal* if vertex n is coloured (i.e., has a stone) and its chain is not next to an uncoloured vertex, but every other chain is next to an uncoloured vertex. In other words, it is illegal, but the only chain making it illegal is the chain containing vertex n; all other chains are ok. The two positions given above are *not* almost legal: the first is legal (so it isn't *almost* legal), while the second is illegal but the illegality is not due to the last vertex (which in this case is uncoloured). The following position is almost legal. All its chains are ok except the last one on the right.



Let $V_{B,n}$, $V_{W,n}$, $V_{U,n}$, $L_{B,n}$, $L_{W,n}$, $L_{U,n}$, $A_{B,n}$, $A_{W,n}$ be the following propositions about a position on the n-vertex path graph.

$V_{B,n}$	Vertex n is Black.
$V_{W,n}$	Vertex n is White.
$V_{U,n}$	Vertex n is Uncoloured.
$L_{B,n}$	The position is legal, and vertex n is Black.
$L_{W,n}$	The position is legal, and vertex n is White.
$L_{U,n}$	The position is legal, and vertex n is Uncoloured.
$A_{B,n}$	The position is almost legal, and vertex n is Black.
$A_{W,n}$	The position is almost legal, and vertex n is White.

(a) Use the propositions $L_{B,n}$, $L_{W,n}$, $L_{U,n}$ (together with appropriate connectives) to write a logical expression for the proposition that the position is legal.

Now consider how legality and almost-legality on the n-vertex path graph are affected by extending the path to vertex $n+1$.

(b) If $L_{B,n}$ is true, what possible states (Black/White/Uncoloured) can vertex $n+1$ be in, if we want the position to be legal on the $n+1$-vertex path as well? Do the same for $L_{W,n}$ and $L_{U,n}$.

(c) If $A_{B,n}$ is true, what possible states can vertex $n+1$ be in, if we want the position to be legal on the $n+1$-vertex path?
Do the same for $A_{W,n}$.
Why is there no line for $A_{U,n}$ in the table?

(d) Construct a logical expression for $L_{B,n+1}$ using some of the propositions $V_{\_,n+1}, L_{\_,n}, A_{\_,n}$ in the above table. (In other words, you can only use the L-propositions and A-propositions for the n-vertex path graph, and the V-propositions for vertex $n+1$.)
Do the same for $L_{W,n+1}$, $L_{U,n+1}$, $A_{B,n+1}$, $A_{W,n+1}$.

9. Using the function symbol **father**, the predicate **taller**, and the constant symbols **claire** and **max**, convert the following sentences to First Order Logic. Assume that **taller(X,Y)** means **X** is taller than **Y** and the universe of discourse is "all people".

ii. Max's father is taller than Max but not taller than Claire's father.

ii. Someone is taller than Claire's father.

iii. Everyone is taller than someone else.

iv. Everyone who is taller than Claire is taller than Max.

Supplementary exercises

10. This question is based on Lab 0, Section 7, Exercise 3.

Let *program* be any program that can be run in Linux and produces standard output. Suppose we do *program* | `wc` as above, followed by a sequence of further applications of | `wc`.

$ *program* | `wc`

. . .

$ *program* | `wc` | `wc`

. . .

$ *program* | `wc` | `wc` | `wc`

. . .

⋮

(a) Determine how many pipes are required before the output ceases to change, and what that output will be.

(b) Prove by induction that, whatever *program* is (as long as it produces some standard output), continued application of | `wc` eventually produces this same fixed output.

This is probably best done by proving, by induction on n, that if | `wc` is applied repeatedly to a file of $\leq n$ characters, it will eventually produce the fixed output you found in part (a).

11. *Ingrid, Ann and Karen are considering going to a party. Ingrid says that she will go if Karen goes but Ann does not. Ann says she will go if both Ingrid and Karen go. Karen says she will go if and only if an even number of them go.*

Using the following propositions express the statements made by Ingrid, Ann and Karen.

I: **Ingrid will go to the party.**

A: **Ann will go to the party.**

K: **Karen will go to the party.**

Assuming everyone is telling the truth, is it logically valid to conclude that Ann and Karen will go to the party but Ingrid will not? If Ann was lying, then is it logically valid to conclude that Karen and Ingrid will go to the party but Ann will not?

12. A *vertex cover* in a graph G is a set VC of vertices such that every edge of G is incident with some vertex in VC.

A *clique* in a graph is a set of vertices that are pairwise adjacent. (I.e., every pair of vertices in the clique is linked by an edge.)

The *complement* of a graph G, written $\overline{G}$, is defined as follows. It has the same vertex set of G, and its edge set consists of every pair of vertices that are *not* adjacent in G.

Suppose we have a graph, and that **chosen** is a unary predicate that takes a vertex of our graph as its argument. This predicate therefore defines a subset of the vertex set of the graph (the "chosen vertices"). Suppose also that we have the following predicates, with the indicated meanings.

vertex(X)	**X** is a vertex in our graph
edge(X,Y)	there is an edge between vertices **X** and **Y**

(a) Write a statement in predicate logic, using the predicates **vertex**, **edge** and **chosen**, to say that the chosen vertices form a vertex cover.

(b) Write a statement in predicate logic, using the same predicates, to say that the chosen vertices form a clique.

(c) Prove that, for any k, the number of vertex covers of size k in G equals the number of cliques of size $n - k$ in $\overline{G}$.

(d) Give the relationship between the size of the smallest vertex cover in G and the size of the largest clique in $\overline{G}$.