Monash University
Faculty of Information Technology

Lecture 13
Chomsky Normal Form

Slides by David Albrecht (2011), some additions and modifications by Graham Farr (2013).

FIT2014 Theory of Computation

Overview

- Chomsky Normal Form
- CYK Parsing algorithm
- Pumping Lemma

Chomsky Normal Form

A **CFG** is said to be in **Chomsky Normal Form** if all the productions are in the form

Nonterminal → Nonterminal Nonterminal

(called a *live production*)

or

Nonterminal → terminal

(called a *dead production*)

For **any** context-free language **L**, the non-empty words in **L** can be generated by a grammar in **Chomsky Normal Form**.

Proof

- First eliminate all **ε productions**.
  - may need some new productions, to keep the effect of an empty production
- For each terminal, **a**, ensure that there is a production of the form **A → a,** and replace **a** in all other productions by **A**.
  - (Note **A** may need to be a new non-terminal symbol.)
- Eliminate all **unit productions**.
  - Given a rule **A → B**:
    - For each rule **B →** ***string*** create a new rule **A →** ***string***,
    - then delete **A → B**
- For any production that has more than 2 non-terminals on the right-hand side, split them into a sequence of productions that have only 2 non-terminals on the right-hand side.

- Consider the CFG

$$S \rightarrow bA \mid aB$$
$$A \rightarrow a \mid aS \mid bAA$$
$$B \rightarrow b \mid bS \mid aBB$$

- Then:

$$S \rightarrow B_1A \mid A_1B$$
$$A \rightarrow a \mid A_1S \mid B_1AA \qquad A_1 \rightarrow a$$
$$B \rightarrow b \mid B_1S \mid A_1BB \qquad B_1 \rightarrow b$$

- Then:

$$S \rightarrow B_1A \mid A_1B$$
$$A \rightarrow a \mid A_1S \mid B_1R_1 \qquad A_1 \rightarrow a$$
$$B \rightarrow b \mid B_1S \mid A_1R_2 \qquad B_1 \rightarrow b$$
$$R_1 \rightarrow AA$$
$$R_2 \rightarrow BB$$

Consequences

- **Cocke-Younger-Kasami (CYK) algorithm**
  - For each CFG and string **s**, we can decide whether or not **s** is generated by the CFG.
  - bottom-up parsing
- **Pumping Lemma for CFG**
  - is used to show there exists non-context-free languages.

CYK Algorithm

For each CFG and string **s**, we can decide whether or not **s** is generated by the CFG.

Proof (CYK Algorithm)

Idea of Proof:

- Find the Chomsky Normal Form for the non-empty words generated by the grammar.
- Add $\mathbf{S} \rightarrow \boldsymbol{\varepsilon}$ if $\boldsymbol{\varepsilon}$ can be generated by the CFG.
- If $\mathbf{s} = \boldsymbol{\varepsilon}$ then state that **s** can be generated if and only if $\mathbf{S} \rightarrow \boldsymbol{\varepsilon}$ is a production.
- Assume $\mathbf{s} = \mathbf{t_1 t_2 .... t_n}$ is non-empty.
- For each letter $\mathbf{t_k}$ find the non-terminals which can produce $\mathbf{t_k}$
- For each of the following pairs:

  $$\mathbf{t_1 t_2}\ ,\ \mathbf{t_2 t_3}\ ,\ \dots,\ \mathbf{t_{n-1} t_n}$$

  find the non-terminals that can produce the pair.
- For each of the following triples:

  $$\mathbf{t_1 t_2 t_3},\ \mathbf{t_2 t_3 t_4}\ ,\ \dots,\ \mathbf{t_{n-2} t_{n-1} t_n}$$

  find the non-terminals that can produce the triple.
- Continue, in this way to find the list of non-terminal that can produce $\mathbf{s} = \mathbf{t_1 t_2 .... t_n}$. If **S** is one of the non-terminals then **s** can be generated, otherwise **s** cannot be generated.

CYK Algorithm

Exercises:

Turn this sketch proof into a correct proof by induction.

Determine the complexity of the algorithm, in big-O notation.

Pumping Lemma

- Let **G** be any CFG in CNF with **k** non-terminal symbols and **w** is any word generated by **G** with length greater than 2^{k-1}.
- Then there exist strings **u**, **v**, **x**, **y**, and **z** such that
  - **w = uvxyz**
  - **v** and **y** are not both **ε**,
  - $|vxy| \leq 2^k$, and
  - all of $uv^2xy^2z, \ldots, uv^nxy^nz, \ldots$ are generated by **G**.

Proof

Chomsky Normal Form: derivation tree is *binary*.

If max path length of binary tree = ℓ,

then # leaves $\leq 2^{\ell}$.

In derivation tree for Chomsky Normal Form, leaves correspond to letters (terminal symbols), and each leaf is an only child (i.e., its parent has degree 2 in the graph, and has no other children).

- Terminals come only from productions of the form

$$\text{Non-terminal} \rightarrow \text{terminal}$$

So actually # leaves $\leq 2^{\ell-1}$.

Proof

Put k = # non-terminal symbols.

Let w be any string of length $> 2^{k-1}$.

Longest path in derivation tree for w has length $\geq k+1$.

If longest path had length $\leq k$, then # leaves $\leq 2^{k-1}$ (see prev. slide),
so $|w| \leq 2^{k-1}$, since leaves correspond to terminals.

Now, recall: for a path, # nodes = length + 1.

So # nodes in longest path $\geq k+2$.

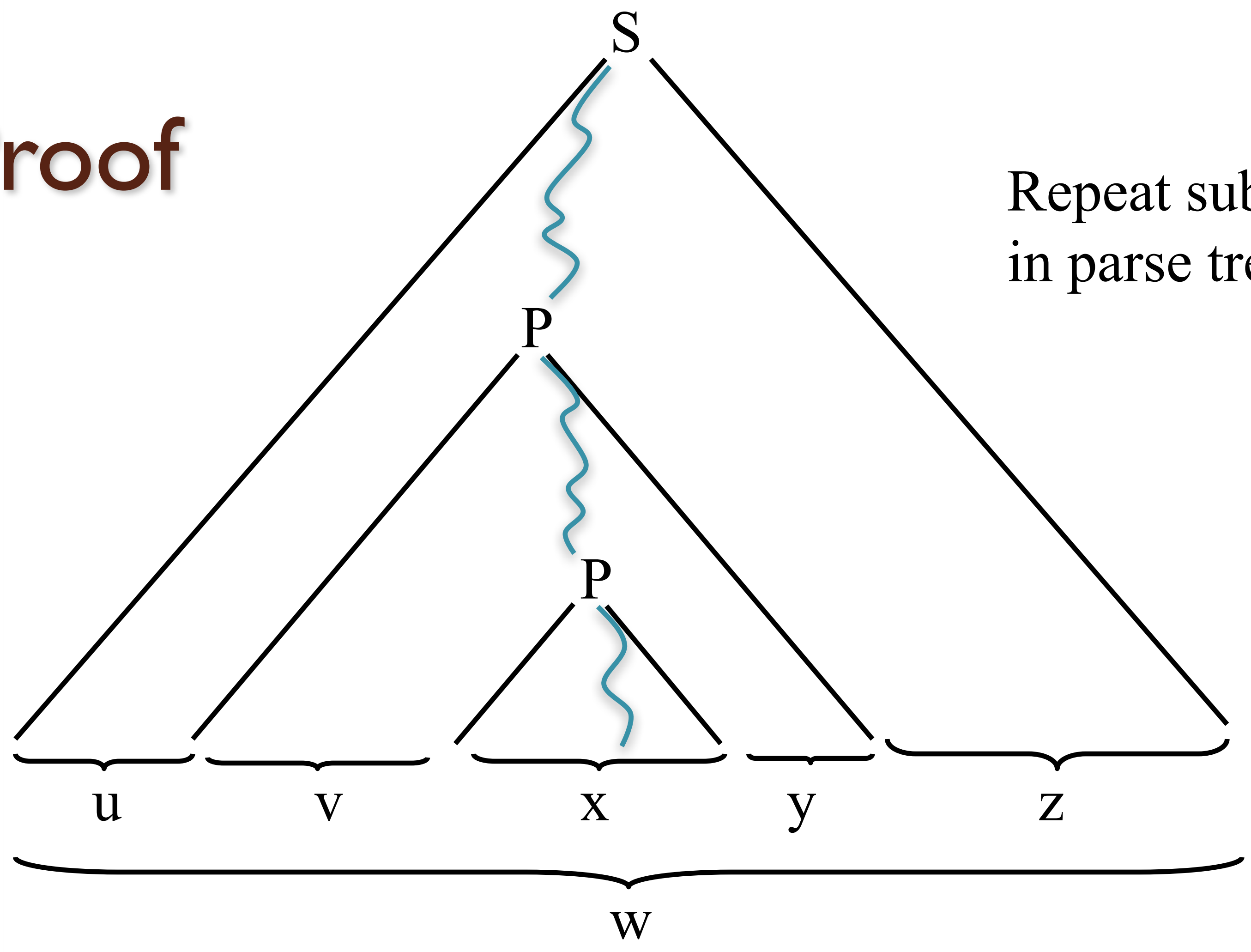
The last of these is a leaf (terminal node). All the others are non-terminals. So the path has $\geq k+1$ non-terminals.

But the number of different non-terminals is $\leq k$.

So the path contains two identical non-terminals.

Call them P.

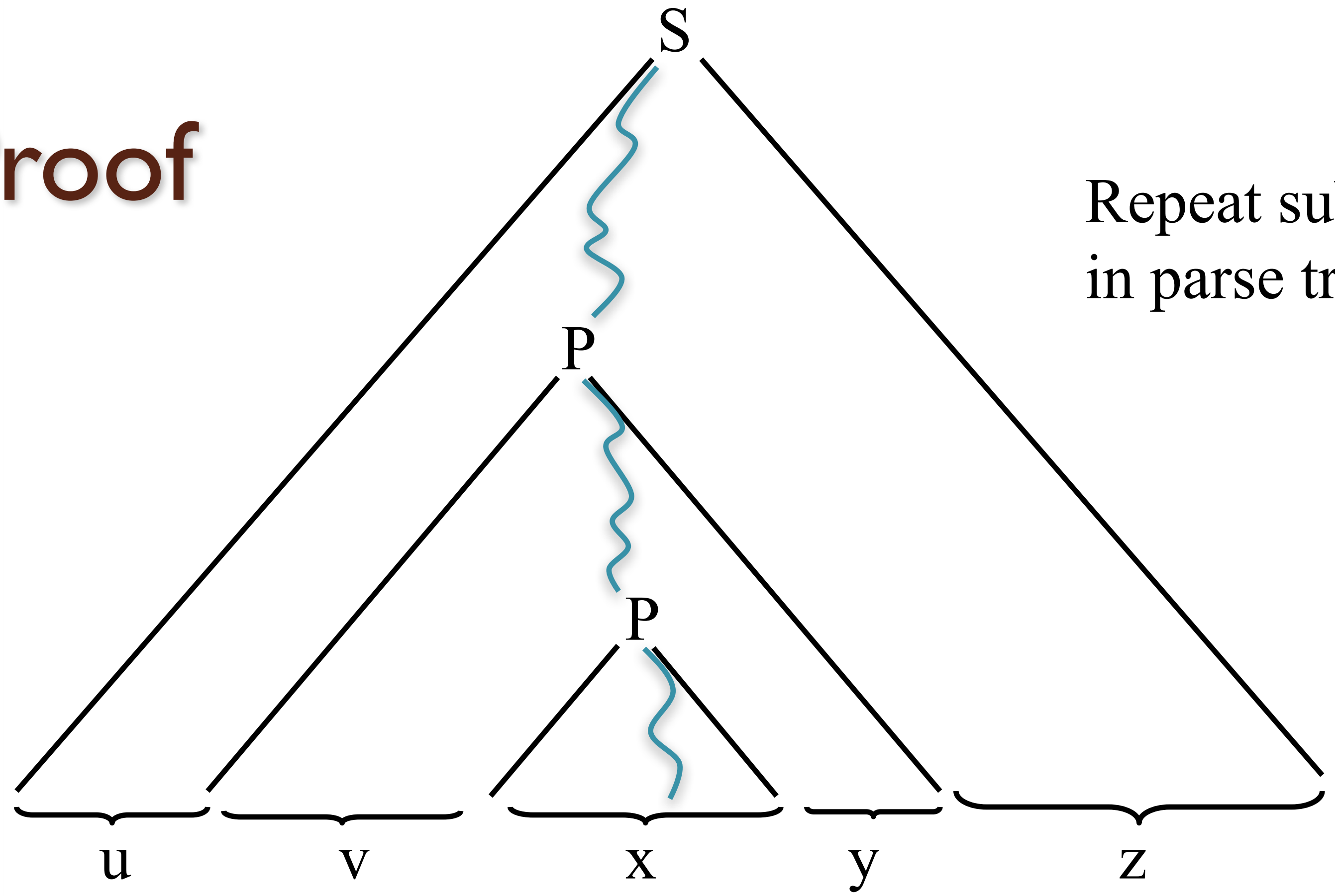
Proof



Repeat subtree in parse tree.

uvxyz

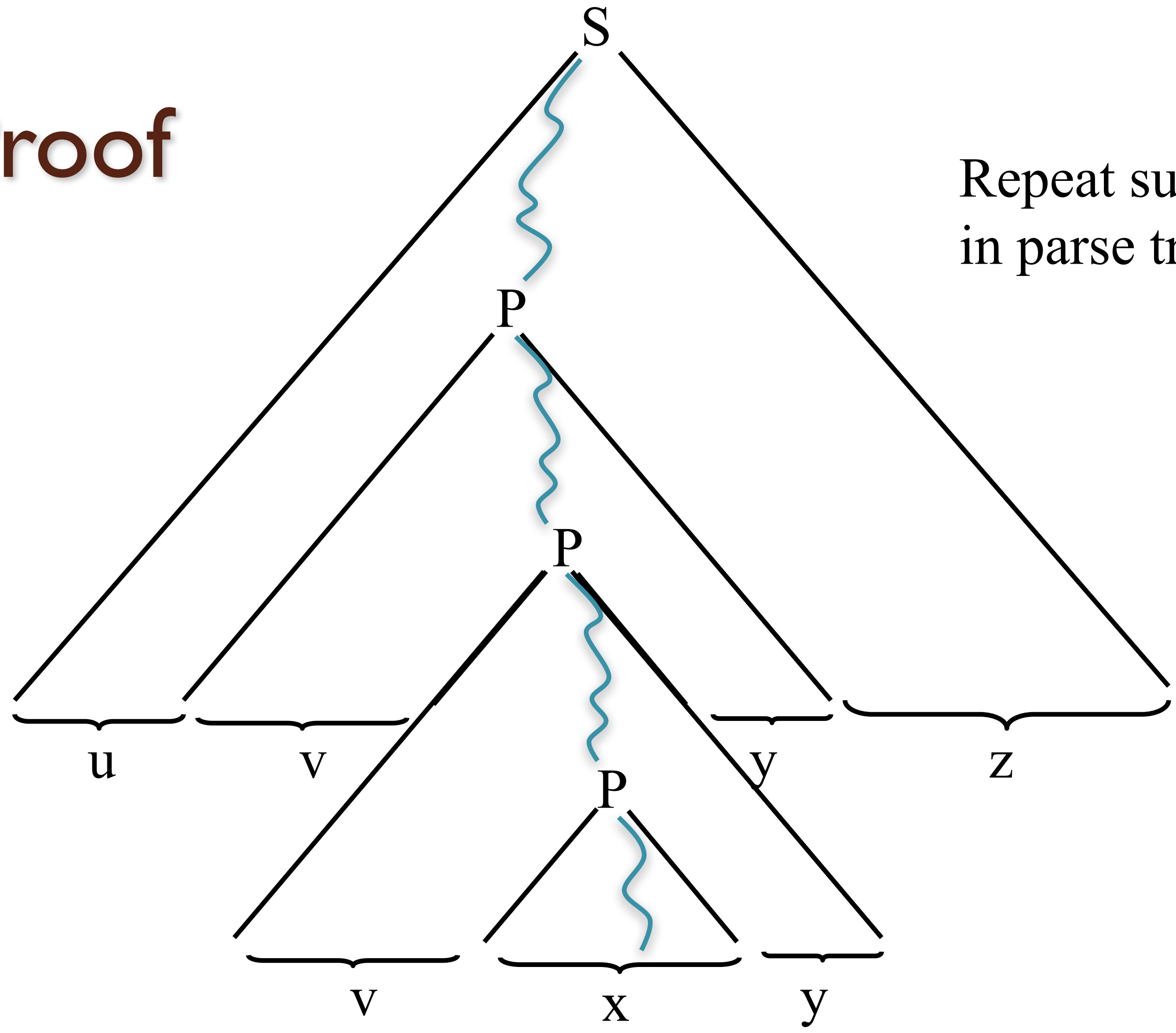
Proof



Repeat subtree in parse tree.

uvxyz

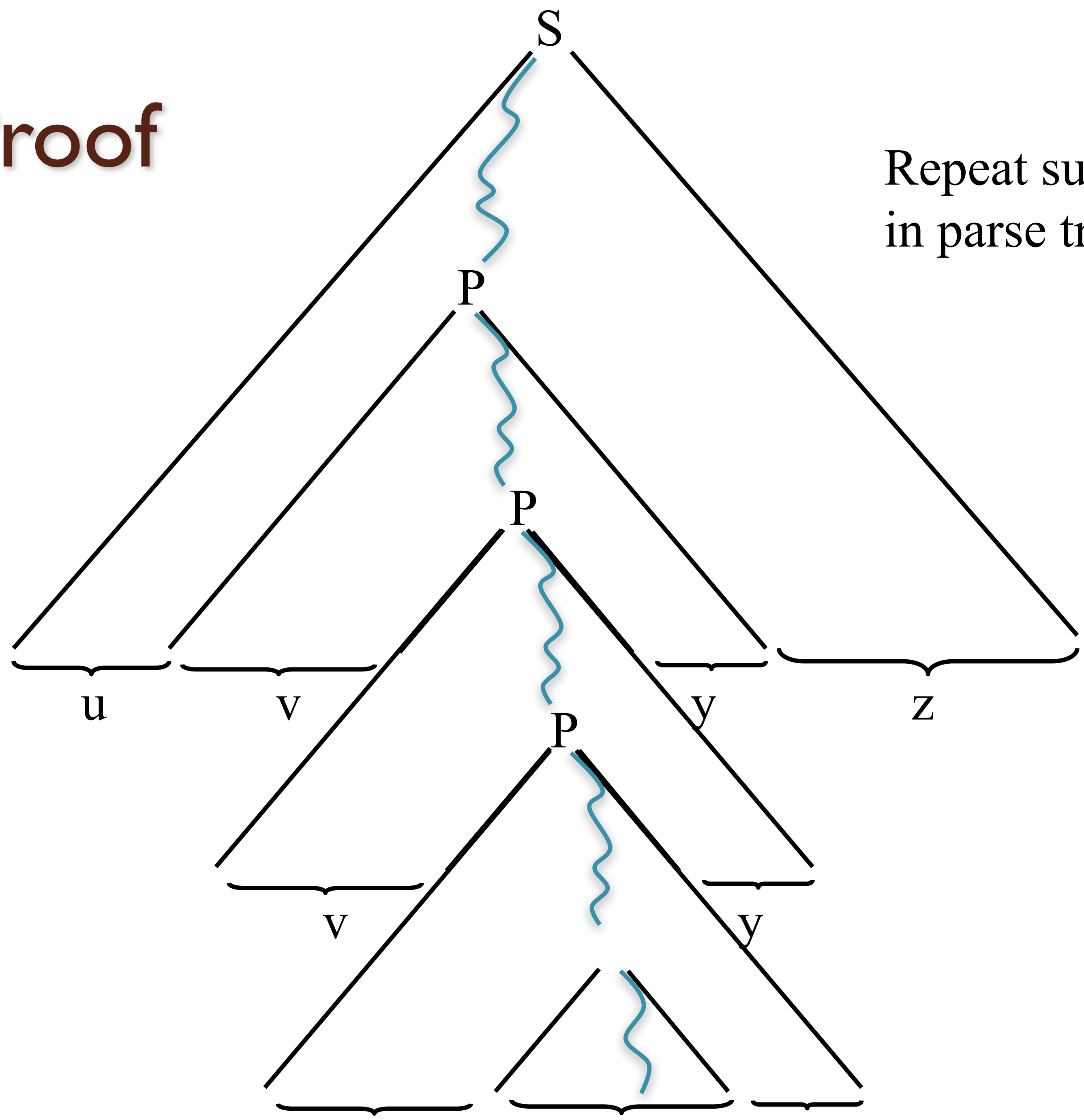
Proof



Repeat subtree in parse tree.

$uv^{2}xy^{2}z$

Proof



Repeat subtree in parse tree.

$uv^{\mathbf{3}}xy^{\mathbf{3}}z$

Proof

So we can generate

$uvxyz$, uv^2xy^2z, uv^3xy^3z, ……, uv^nxy^nz, …

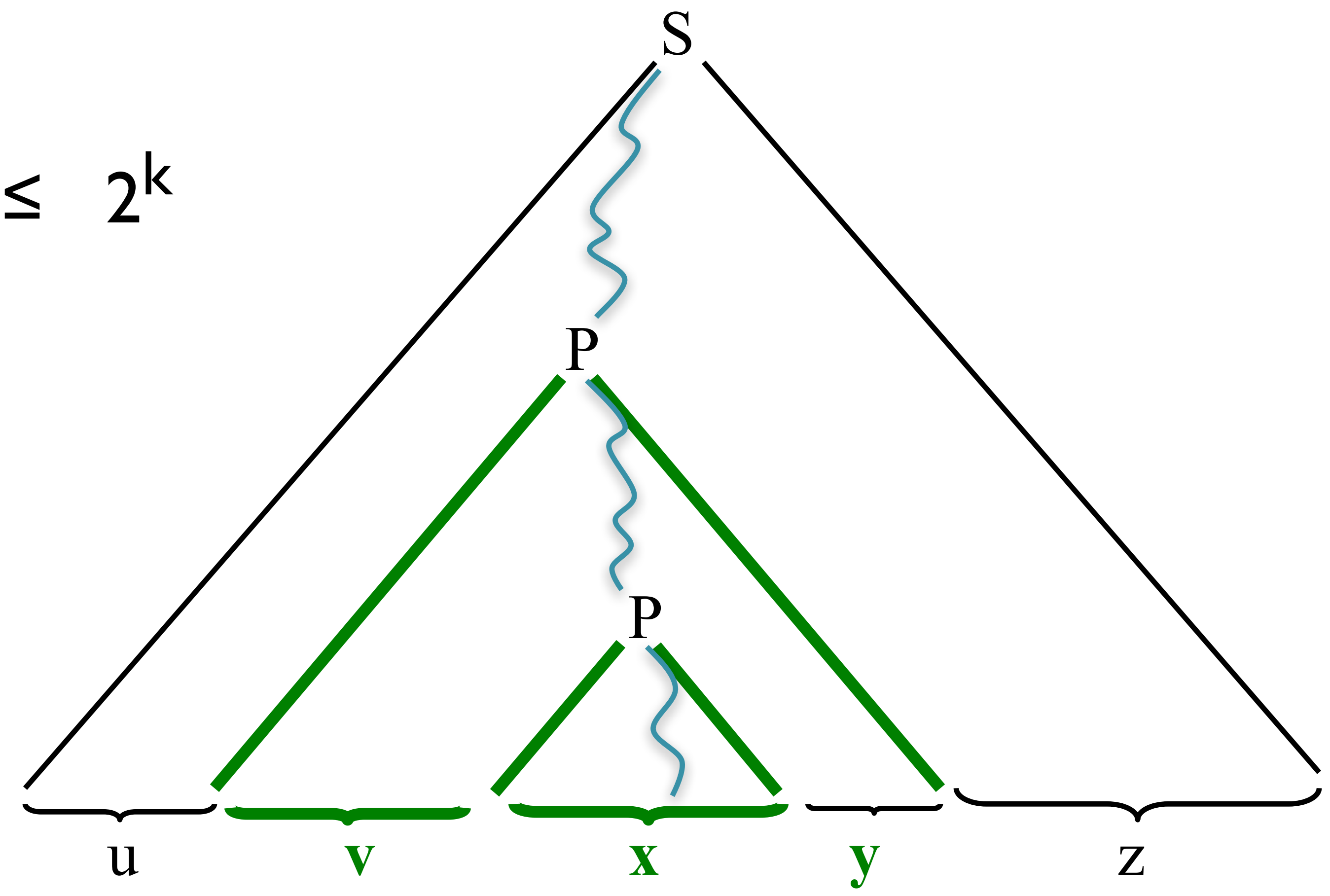
Note also:

We can choose the two repeated non-terminals P to be as far down the path as possible, so there are no other repeated non-terminals in the path from the higher P down to the leaf.

So the path from the upper P downwards has at most k non-terminal nodes apart from the upper P. So the word derived from P by this subtree has at most 2^k letters.

Proof

$|vxy| \leq 2^k$



$L = \{a^n b^n a^n : n \geq 0\}$

Assume **L** is a context free language.

Then there exists a CFG which generates **L**.

Convert this CFG to CNF.

Suppose it has **k non-terminal** symbols.

Take $\mathbf{N > 2^{k-1}/3}$

Let $\mathbf{w = a^N b^N a^N}$.

Then $\mathbf{length(w) > 2^{k-1}}$

- So, by the Pumping Lemma, there exist strings **u**, **v**, **x**, **y**, and **z**
- such that
  - **w = uvxyz**
  - **v** and **y** are not both ε, and
  - $\mathbf{uv^2xy^2z}$, …, $\mathbf{uv^{n}xy^{n}z}$,… are all in L.
- Case 1: **ab** is in **v** or **y**.
- Case 2: **ba** is in **v** or **y**.
- Case 3: **v** and **y** are all **a**'s or all **b**'s or one of them is ε.
- Consider: $\mathbf{uv^2xy^2z}$. Contradiction
- Therefore **L** is a non-context-free language.

Revision

- Know the uses of Chomsky Normal Form.
- Know and use the CYK algorithm.
- Know that there exist non-Context Free languages.
- Know an example of a language which is not a Context Free Language.
- Use Pumping Lemma to prove that certain languages are not context-free.
- Reading: Sipser, pp 108-111, 125-129.

Preparation

Read

•M.Sipser, "Introduction to the Theory of Computation", Chapter 3.