Monash University
Faculty of Information Technology

Lecture 17
Undecidable Problems

Slides by Graham Farr (2013-2017).

FIT2014 Theory of Computation

Overview

- Halting Problem (or Entscheidungsproblem)
- Proof of its undecidability
- Mapping reductions
- Other undecidable problems

Halting Problem

Input: Turing machine ***P***, input ***x***

Question: If ***P*** is run with input ***x***, does it eventually **halt**?

Theorem

The Halting Problem is undecidable.

Proved by:

Alonzo Church (1936): lambda calculus

Alan Turing (1936-37): Turing machines

Halting Problem

Theorem

The Halting Problem is undecidable.

Proof is by contradiction, and includes a more elaborate version of the Liar Paradox:

"This sentence is false."

Halting Problem is Undecidable

Proof: (by contradiction)

Assume there is a Decider, ***D***, for the Halting Problem.

So it can tell, for any ***P*** and ***x***, whether or not ***P*** eventually halts after being given input ***x***.

So it can tell, for any ***P***, whether or not ***P*** eventually halts after being given input ***P*** !

Construct another program (Turing machine) ***E*** as follows …

Halting Problem is Undecidable (cont'd)

E

> Input: ***P***
> Use ***D*** to determine what happens if ***P*** runs on itself.
> If ***D*** ***says, "P*** halts, with input ***P"***: loop forever.
> If ***D*** ***says, "P*** loops forever, with input ***P"***: Stop.

What happens when ***E*** is given itself as input?

If ***E*** halts, for input ***E***: then ***E*** loops forever, for input ***E***.

If ***E*** loops forever, for input ***E***: then ***E*** halts, for input ***E***.

Contradiction! Q.E.D.

YouTube film of proof: https://www.youtube.com/watch?v=92WHN-pAFCs

Other Undecidable Problems

DIAGONAL HALTING PROBLEM
Input: Turing machine P
Question: Does P eventually halt, for input P?

Above proof already shows this.

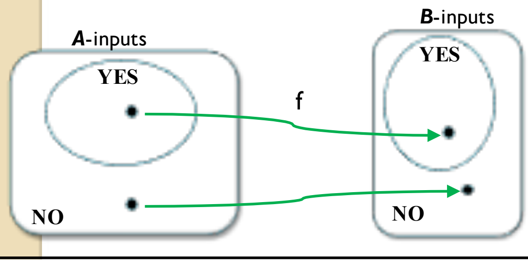
HALT FOR INPUT ZERO
Input: Turing machine P
Question: Does P eventually halt, for input 0?

HALT FOR INPUT ZERO is undecidable.

We'll prove this by **reduction** from the Diagonal Halting Problem.

Reductions

A **mapping reduction** from problem A to problem B is a computable function f : {A-inputs} ➔ {B-inputs} such that:
for every A-input x,
x is a YES-input for A if and only if $f(x)$ is a YES-input for B



Reductions

If there is a mapping reduction f from A to B, then:

- If B is decidable, then A is decidable.
  Decider for A:
  1. Input: x.
  2. Compute f(x).
  3. Call the Decider for B to determine if whether f(x) is a YES-input or a NO-input for B.
  4. *The same answer works for* A.
- If A is undecidable, then B is undecidable.
  - contrapositive of previous statement.

Back to showing that HALT FOR INPUT ZERO is undecidable …

Let M be any program, which we regard as an input to the Diagonal Halting Problem.
Define M' as follows:

M'

Input: x
Run M on input M

Observe:
- The construction of M' from M is computable.
- M halts on input M if and only if M' halts on input 0.

So, the function that sends M *to* M' is a **mapping reduction** from DIAGONAL HALTING PROBLEM to HALT FOR INPUT ZERO.

Therefore HALT FOR INPUT ZERO is undecidable.

Other Undecidable Problems

There's nothing special about zero, here.
So we get a whole lot of undecidability results.
For example:

HALT FOR INPUT 42
Input: Turing machine P
Question: Does P eventually halt, for input 42?

Proof of undecidability is virtually identical to the previous one …
Use a mapping reduction.

HALT FOR INPUT 42 is undecidabe.
Proof:
Let M be any program, which we regard as an input to the Diagonal Halting Problem.
Define M' as follows:

M'

Input: x
Run M on input M

Observe:
- The construction of M' from M is computable.
- M halts on input M if and only if M' halts on input 42.

So, the function that sends M *to* M' is a **mapping reduction** from DIAGONAL HALTING PROBLEM to HALT FOR INPUT 42.

Therefore HALT FOR INPUT 42 is undecidable. Q.E.D.

Other Undecidable Problems

ALWAYS HALTS
Input: Turing machine ***P***
Question: Does ***P*** always halt eventually, for any input?

Proof of undecidability is virtually identical to the previous one …

ALWAYS HALTS is undecidabe.

Proof:
Let ***M*** be any program, which we regard as an input to the Diagonal Halting Problem.
Define ***M'*** as follows:

M'

Input: ***x***   Run ***M*** on input ***M***

Observe:

- The construction of ***M'*** from ***M*** is computable.
- ***M*** halts on input ***M*** if and only if ***M'*** halts **for all x**.

So, the function that sends ***M*** to ***M'*** is a **mapping reduction** from DIAGONAL HALTING PROBLEM to **ALWAYS HALTS**.

Therefore **ALWAYS HALTS** is undecidable. Q.E.D.

Other Undecidable Problems

SOMETIMES HALTS
Input: Turing machine ***P***
Question: Is there some input for which ***P*** eventually halts?

SOMETIMES HALTS is undecidable.

Proof: by reduction from the Diagonal Halting Problem.

SOMETIMES HALTS is undecidabe.

Proof:
Let ***M*** be any program, which we regard as an input to the Diagonal Halting Problem.
Define ***M'*** as follows:

M'

Input: ***x***   Run ***M*** on input ***M***

Observe:

- The construction of ***M'*** from ***M*** is computable.
- ***M*** halts on input ***M*** if and only if ***M'*** halts **for some x**.

So, the function that sends ***M*** *to* ***M'*** is a **mapping reduction** from DIAGONAL HALTING PROBLEM to **SOMETIMES HALTS**.

Therefore **SOMETIMES HALTS** is undecidable. Q.E.D.

Other Undecidable Problems

NEVER HALTS
Input: Turing machine ***P***
Question: Does ***P*** always loop forever, for any input?

NEVER HALTS is undecidable.

Proof: by a more general type of reduction, from SOMETIMES HALTS.

If D is a decider for NEVER HALTS, then switching the outputs YES and NO gives a decider for SOMETIMES HALTS. But we now know that SOMETIMES HALTS is undecidable. Contradiction. So NEVER HALTS is undecidable too. Q.E.D.

Other Undecidable Problems

Input: Turing machine ***P*** and ***Q***
Question: Do ***P*** and ***Q*** always both halt, or both loop?
I.e., for all ***x***, ***P*** halts on input ***x*** iff ***Q*** halts on input ***x***.

Input: Turing machine ***P***
Question: If ***P*** is run on the input **"What's the answer?"**, does it output **"42"**?

Decidable or Undecidable?

Input: Turing machine ***P***, input ***x***.
Question: Does ***P*** accept ***x***?

Input: Turing machine ***P***, input ***x***, positive integer ***t***
Question: When ***P*** is run on ***x***, does it halt in $\leq t$ steps?

Input: Turing machine ***P***, positive integer ***s***.
Question: Does P have $\leq s$ states?

Input: Turing machine ***P***, positive integer ***k***
Question: Does ***P*** halt for some input of length $\leq k$.

Other Undecidable Problems

Input: a Turing machine P
Question: Is Accept(P) regular?
I.e., is P equivalent to a Finite Automaton

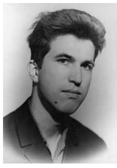
Input: a CFG
Question: is the language it generates regular?

Input: a CFG
Question: is there any string that it doesn't generate?
(over same alphabet)

Input: two CFGs.
Question: Do they define the same language?

Other Undecidable Problems

Input: a polynomial (in several variables)
Question: Does it have an integer root?
(Y. Matiyasevich, 1970)



Yuri Matiyasevich (b.1947
http://www-history.mcs.st-and.ac.uk/Biographies/Matiyasevich.htm

Post Correspondence Problem
(a problem about string matching; see Sipser, Section 5.2)

Emil Post (1897-1954
http://www-history.mcs.st-andrews.ac.uk/Biographies/Post.htm

Revision

- Know and understand the Halting Problem.
- Be able to use mapping reductions.
- Be able to show that some problems are undecidable
- Know examples of undecidable problems.

Reading

- Sipser, pp. 207-209, 216-220, 234-236.