Monash University
Faculty of Information Technology

Lecture 10
Context Free Grammars

Slides by David Albrecht (2011), modified by Graham Farr (2013).

FIT2014 Theory of Computation

Overview

- Inductive Definitions
- Context Free Grammars
- Parse Trees
- Derivations

Arithmetic Expressions

1. All integers are Arithmetic Expressions
2. If A and B are Arithmetic Expressions, so are:
   (i) A + B
   (ii) A - B
   (iii) A * B
   (iv) A / B
   (v) (A)

Production Rules

$S \rightarrow A$

$AE \rightarrow$ integer

$AE \rightarrow AE + AE$

$AE \rightarrow AE - AE$

$AE \rightarrow AE * AE$

$AE \rightarrow AE / AE$

$AE \rightarrow (AE)$

$A \rightarrow$ integer

$A \rightarrow A + A$

$A \rightarrow A - A$

$A \rightarrow A * A$

$A \rightarrow A / A$

$A \rightarrow (A)$

Backus-Naur Form

(a.k.a. Backus Normal Form)

John Backus (1924-2007)
http://www-history.mcs.st-and.ac.uk/Biographies/Backus.html

$S \rightarrow A$

$A \rightarrow \text{integer} \mid A + A \mid A - A \mid A * A \mid A / A \mid (A)$

Peter Naur (b. 1928)
http://datamuseum.dk/

Historical example: fragment of the BNF of ALGOL 60

4.1. Compound Statements and Blocks

4.1.1. Syntax

⟨unlabelled basic statement⟩ ::= ⟨assignment statement⟩|
    ⟨go to statement⟩|⟨dummy statement⟩|⟨procedure statement⟩
⟨basic statement⟩ ::= ⟨unlabelled basic statement⟩|⟨label⟩:
    ⟨basic statement⟩
⟨unconditional statement⟩ ::= ⟨basic statement⟩|⟨for statement⟩|
    ⟨compound statement⟩|⟨block⟩
⟨statement⟩ ::= ⟨unconditional statement⟩|
    ⟨conditional statement⟩
⟨compound tail⟩ ::= ⟨statement⟩ **end** |⟨statement⟩ ;
    ⟨compound tail⟩
⟨block head⟩ ::= **begin**⟨declaration⟩|⟨block head⟩ ;
    ⟨declaration⟩
⟨unlabelled compound⟩ ::= **begin** ⟨compound tail⟩
⟨unlabelled block⟩ ::= ⟨block head⟩ ; ⟨compound tail⟩
⟨compound statement⟩ ::= ⟨unlabelled compound⟩|
    ⟨label⟩:⟨compound statement⟩
⟨block⟩::=⟨unlabelled block⟩|⟨label⟩:⟨block⟩

from: J. W. Backus *et al.*, *Comm. ACM* **3** (5) (May 1960) 299-314.

Definitions

- A string is in **EQUAL** if it has an equal number of **a**' s and **b**' s.
- E.g.

  {ε, ab, ba, aabb, abab, abba, baba, …}

- An **a-type** string has one more **a** than **b**.
- A **b-type** string has one more **b** than **a**.

Definition of EQUAL

- A string is in **EQUAL** if it is
  - **ε**, or
  - an, **a** followed by a string of **b-type**, or
  - it is a **b** followed by a string of **a-type**.
- A string is of **a-type** if it is
  - an a, or
  - an **a** followed by a string in **EQUAL**, or
  - a **b** followed by two strings of **a-type**.
- A string is of **b-type** if it is
  - an **b**, or
  - a **b** followed by a string in **EQUAL**, or
  - an **a** followed by two strings of **b-type**.

$S \rightarrow \varepsilon$

$S \rightarrow aB$

$S \rightarrow bA$

$A \rightarrow a$

$A \rightarrow aS$

$A \rightarrow bAA$

$B \rightarrow b$

$B \rightarrow bS$

$B \rightarrow aBB$

Production Rules

$S \rightarrow \varepsilon$

$S \rightarrow bA$

$S \rightarrow aB$

$A \rightarrow a$

$A \rightarrow aS$

$A \rightarrow bAA$

$B \rightarrow b$

$B \rightarrow bS$

$B \rightarrow aBB$

Non-terminals

Terminals

A Context Free Grammar consists of:

1. An alphabet
   - The letters are called **terminals**
2. A set of symbols
   - We call these symbols **nonterminals**
   - One of these symbols is the **Start symbol**
   - **S** is often used as the start symbol.
3. A finite set of production rules of the form:

One nonterminal → finite string of terminals
and/or nonterminals

Definition

- The ***language generated*** by a Context Free Grammar (CFG)
  - consists of those strings which can be produced from the start symbol using the production rules.
- A language generated by a CFG is called a **Context Free Language** (CFL).

History

- Pāṇini (c520BC-c460BC)
  - studied *Sanskrit*

http://www-groups.dcs.st-and.ac.uk/history/Biographies/Panini.html

- Noam Chomsky (b. 1928)
  - studied *natural languages*

- John Backus
  - studied *programming languages*

Noam Chomsky,
during visit to Australia in 2011 to accept Sydney Peace Prize.

http://www.abc.net.au/news/2011-06-02/noam-chomsky/2741826

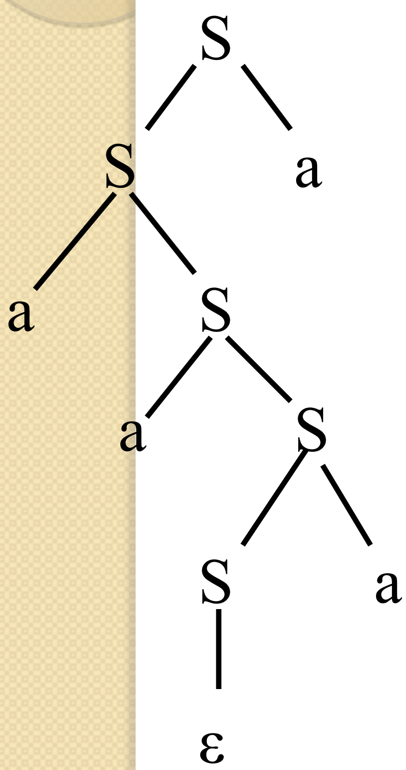
$S \rightarrow aS \mid Sa \mid \varepsilon$

1. $S \rightarrow Sa$
2. $S \rightarrow aS$
3. $S \rightarrow \varepsilon$

Derivation of aaaa

$$
\begin{aligned}
S &\Rightarrow Sa && \text{(Rule 1)} \\
&\Rightarrow aSa && \text{(Rule 2)} \\
&\Rightarrow aaSa && \text{(Rule 2)} \\
&\Rightarrow aaSaa && \text{(Rule 1)} \\
&\Rightarrow aa\varepsilon aa && \text{(Rule 3)} \\
&= aaaa
\end{aligned}
$$

Parse Tree



Derivation of aaaa

$S \Rightarrow Sa$ (Rule 1)

$\Rightarrow aSa$ (Rule 2)

$\Rightarrow aaSa$ (Rule 2)

$\Rightarrow aaSaa$ (Rule 1)

$\Rightarrow aa\varepsilon aa$ (Rule 3)

$= aaaa$

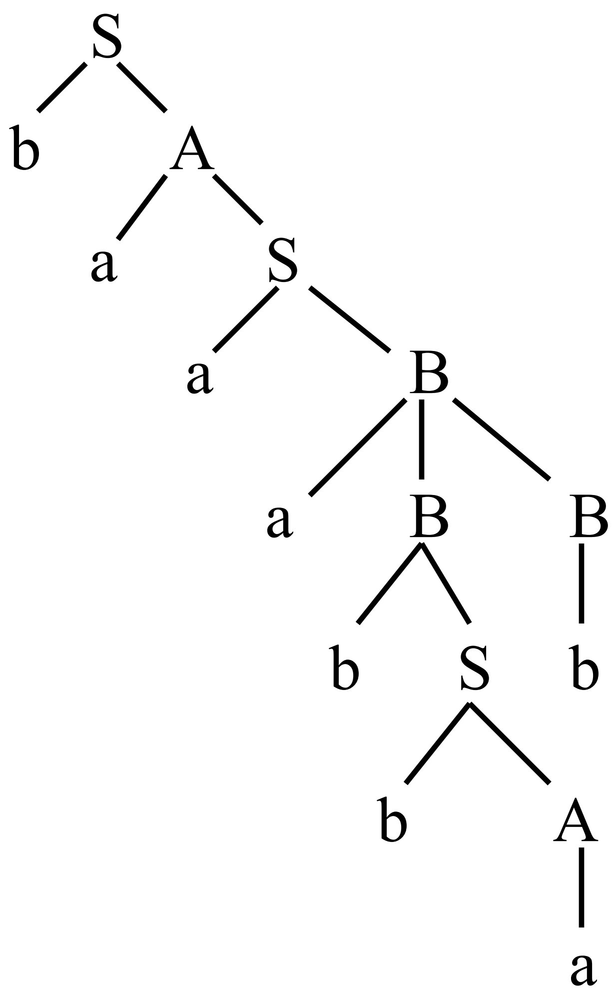
EQUAL

1. $S \rightarrow \varepsilon$
2. $S \rightarrow bA$
3. $S \rightarrow aB$
4. $A \rightarrow a$
5. $A \rightarrow aS$
6. $A \rightarrow bAA$
7. $B \rightarrow b$
8. $B \rightarrow bS$
9. $B \rightarrow aBB$

Derivation of baaabbab

$S \Rightarrow bA$	(Rule 2)
$\Rightarrow baS$	(Rule 5)
$\Rightarrow baaB$	(Rule 3)
$\Rightarrow baaaBB$	(Rule 9)
$\Rightarrow baaaBb$	(Rule 7)
$\Rightarrow baaabSb$	(Rule 8)
$\Rightarrow baaabbAb$	(Rule 2)
$\Rightarrow baaabbab$	(Rule 4)

Parse Tree



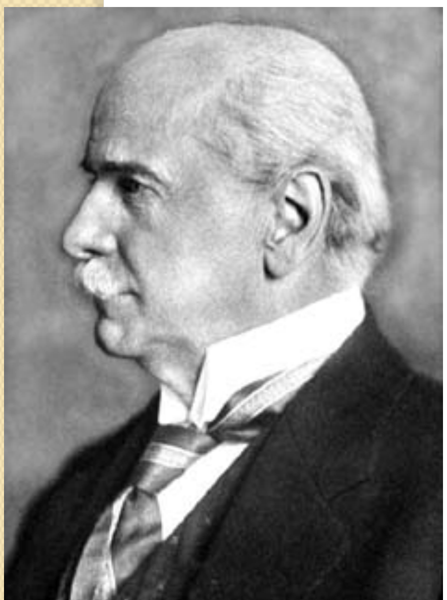
$$
\begin{aligned}
S &\Rightarrow bA \\
&\Rightarrow baS \\
&\Rightarrow baaB \\
&\Rightarrow baaaBB \\
&\Rightarrow baaaBb \\
&\Rightarrow baaabSb \\
&\Rightarrow baaabbAb \\
&\Rightarrow baaabbab
\end{aligned}
$$

PARENTHESES (a.k.a. the Dyck Language)

1. $S \rightarrow \varepsilon$
2. $S \rightarrow (S)$
3. $S \rightarrow SS$

Derivation of ()(())

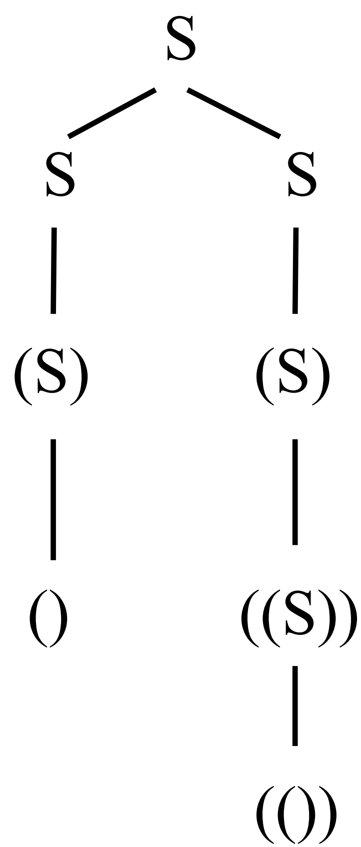
$$\begin{aligned} S &\Rightarrow SS && \text{(Rule 3)} \\ &\Rightarrow (S)S && \text{(Rule 2)} \\ &\Rightarrow (S)(S) && \text{(Rule 2)} \\ &\Rightarrow ()(S) && \text{(Rule 1)} \\ &\Rightarrow ()((S)) && \text{(Rule 2)} \\ &\Rightarrow ()(()) && \text{(Rule 1)} \end{aligned}$$



Walther von Dyck (1856-1934)

http://www-history.mcs.st-and.ac.uk/Biographies/Von_Dyck.html

Parse Tree



$$
\begin{aligned}
S &\Rightarrow SS \\
&\Rightarrow (S)S \\
&\Rightarrow (S)(S) \\
&\Rightarrow ()(S) \\
&\Rightarrow ()((S)) \\
&\Rightarrow ()(())
\end{aligned}
$$

Exercises

- Suppose we have two types of brackets, such as round and square: ( ) and [ ] . Find a context-free language for the set of all valid strings of such brackets.
- Find a context-free grammar for PALINDROMES
- For other languages we have met:
  - find context-free grammars for them, OR
  - if you think they don't have one, think about *why*.

A simple property of derivations

At any stage, the string to the left of the first nonterminal must be a prefix of the final (derived) string.

$\ldots\ldots.$

$$x_1 \ldots x_k \mathbf{A} B \ldots$$

$$\Rightarrow\ x_1 \ldots x_k \mathbf{aXY} B \ldots \qquad \text{(using } \mathbf{A} \rightarrow \mathbf{aXY}\text{)}$$

$\ldots\ldots$

$$\Rightarrow\ x_1 \ldots x_k a \ldots\ldots \qquad \text{(derived string)}$$

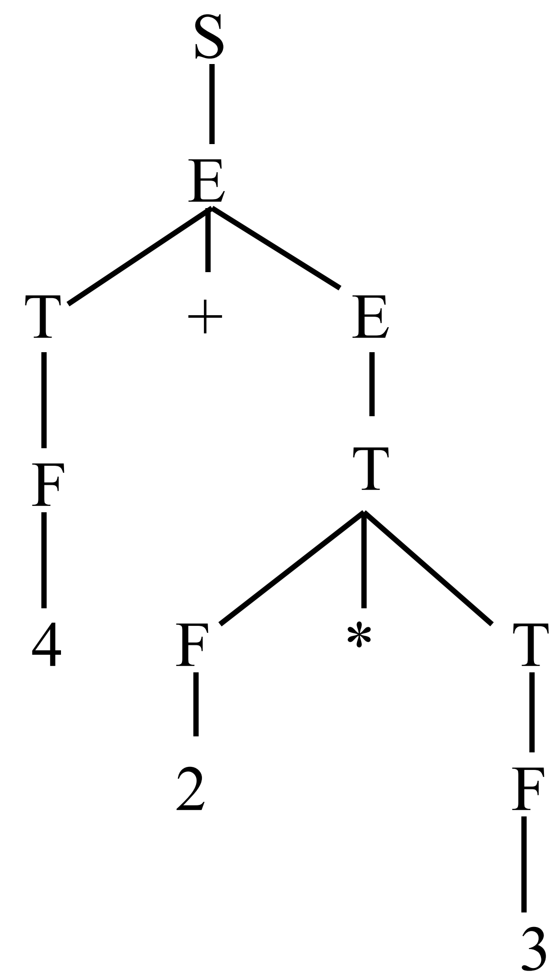
4 + 2*3

S → E

E → T + E | T - E | T

T → F * T | F / T | F

F → integer | (E)



$4 + 2*3$

$$S \rightarrow E$$
$$E \rightarrow T + E \mid T - E \mid T$$
$$T \rightarrow F * T \mid F / T \mid F$$
$$F \rightarrow \text{integer} \mid (E)$$

Leftmost Derivation

$$
\left\{
\begin{aligned}
\mathbf{S} &\Rightarrow \mathbf{E} \\
&\Rightarrow \mathbf{T} + E \\
&\Rightarrow \mathbf{F} + E \\
&\Rightarrow 4 + \mathbf{E} \\
&\Rightarrow 4 + \mathbf{T} \\
&\Rightarrow 4 + \mathbf{F}*T \\
&\Rightarrow 4 + 2*\mathbf{T} \\
&\Rightarrow 4 + 2*\mathbf{F} \\
&\Rightarrow 4 + 2*3
\end{aligned}
\right.
$$

4 + 2*3

$S \rightarrow E$

$E \rightarrow T + E \mid T - E \mid T$

$T \rightarrow F * T \mid F / T \mid F$

$F \rightarrow \text{integer} \mid (E)$

Rightmost Derivation

$$
\begin{aligned}
\mathbf{S} &\Rightarrow \mathbf{E} \\
&\Rightarrow T + \mathbf{E} \\
&\Rightarrow T + \mathbf{T} \\
&\Rightarrow T + F*\mathbf{T} \\
&\Rightarrow T + F*\mathbf{F} \\
&\Rightarrow T + \mathbf{F}*3 \\
&\Rightarrow \mathbf{T} + 2*3 \\
&\Rightarrow \mathbf{F} + 2*3 \\
&\Rightarrow 4 + 2*3
\end{aligned}
$$

Leftmost and rightmost derivations

- In a **Leftmost derivation**, the leftmost non-terminal is always replaced first.
- In a **Rightmost derivation**, the rightmost non-terminal is always replaced first.

Theorem

Whenever a string has a derivation, it also has a leftmost derivation of the same length.

Proof: see Tute 3.

Does the same hold for rightmost derivations?

A simple property of leftmost derivations

Whenever a production

$\mathbf{X} \rightarrow$ **terminals Non-terminal theRest**

is applied, the terminal letters on the left are appended to the current prefix to give a larger prefix of the derived string

…….

$x_1 \ldots x_k \mathbf{A} B \ldots$

$\Rightarrow x_1 \ldots x_k \mathbf{aXY} B \ldots$ (using $\mathbf{A} \rightarrow \mathbf{aXY}$)

……

$\Rightarrow x_1 \ldots x_k \mathbf{a} \ldots\ldots$ (derived string)

Revision

- Context Free Grammars
  - Definition. How to use them.
- Parse Trees
  - Definition. How to make them.
- Be able to construct leftmost and rightmost derivations.
- Read
  - M. Sipser, , "***Introduction to the Theory of Computation***", Chapter 2.