Monash University
Faculty of Information Technology

# Lecture 20
# Nondeterministic Polynomial time, and the class NP

Slides by Graham Farr (2012).

FIT2014 Theory of Computation

## Overview

- Deciding and Verifying
- Certificate
- The class NP
- Proving membership of NP
- Examples of languages in NP
- P $\subseteq$ NP
- The P-versus-NP problem
- Deciders for languages in NP
- Nondeterministic Polynomial-time Turing machines

## Deciding and Verifying

- *Deciding* if a string belongs to a language or not

versus

- *Verifying* that a string belongs to a language (if it does)

## Deciding and Verifying

- P is intended to contain languages which are *efficiently decidable*
  - i.e., you can efficiently solve the problem of deciding whether something is in the language or not
- Recall (Lecture 16, on Decidability):
  A ***decider*** for a language L is a TM that
  - halts for any input
  - accepts every x in L
  - rejects every x not in L
- P is the set of languages for which there is a polynomial-time decider.

## Deciding and Verifying

- Consider:
  { people who can kick a football }
  - How do you verify that a person can kick a football?
  - Give them a ball and get them to try to kick it.
  - This procedure is a decider.
    It enables you to *decide whether or not* they can kick a football.

## Deciding and Verifying

- Now consider:
  { university graduates }
  - How do you verify that a person is a graduate?
  - Can't do it just by meeting them, testing abilities etc. There is no efficient decider for this set.
  - But you can *verify* it if you have their degree certificate.
  - Hard to verify that someone is *not* a graduate.

## Deciding and Verifying

- A ***verifier*** for a language L is a TM that takes, as input, two strings x and y,
  - halts for any x, y
  - if x is in L, there exists y such that the TM accepts
  - if x is not in L, every y makes the machine reject.
- y is called a *certificate*
- x is accepted if and only if it has a certificate which can be verified.
- *Polynomial-time verifier*: a verifier with time complexity polynomial in length of x.

(i.e., $O(|x|^k)$, for some fixed k, where $|x|$ denotes length of string x)

## NP

- NP is the set of languages for which there is a polynomial-time *verifier*.
- **NP** stands for

  **N**on-deterministic **P**olynomial time

  (for reasons to be given later)
- NP is intended to contain languages for which membership can be efficiently verified, with the aid of an appropriate certificate.

## Proving membership of NP

To show a language is in NP, you need to:

- specify the *certificate*
- give a polynomial-time verifier (as an *algorithm*)
- *prove that it is a verifier* for the language
- prove that it is *polynomial time*.

## Proving membership of NP

Proof that { 3-colourable graphs } is in NP.

**Given:** graph G

**Certificate:** a function $f : V(G) \rightarrow$ { Red, White, Black}

**Verification:**

```
For each edge uv of G
{
    Look up f(u) and f(v).
    // … these are the colours given to the endpoints u, v of this edge
    Check that f(u) ≠ f(v).
    If so, continue. If not, Reject and halt.
    // … endpoints must get different colours
}
If loop completes with no edge rejected, then Accept and halt.
```

## Proving membership of NP

*Claim 1:*

This is a verifier for { 3-colourable graphs}.

*Proof:*

G is in { 3-colourable graphs}

if and only if

there exists a function $f : V(G) \rightarrow$ { Red, White, Black } such that, for each edge uv, we have $f(u) \neq f(v)$

if and only if

there exists a certificate such that our verifier accepts G.

*End of proof of Claim 1*

## Proving membership of NP

*Claim 2:*

Verifier takes polynomial time, in size of input.

*Proof:*

Main loop: # iterations = # edges = m, say.

For each edge: look up each endpoint in the certificate.

Suppose certificate is given as a list of colours, one for each vertex. The vertex gives the position in the list.

Looking up the colour of each endpoint takes $O(n)$ time, where $n :=$ # vertices.

Checking whether $f(u) \neq f(v)$ takes constant time.

So total time $\leq m \cdot n \cdot \text{constant} = O(mn)$.

So it takes polynomial time, in size of G.

*End of proof of Claim 2*

## Proving membership of NP

So we have proved that { 3-colourable graphs } is in NP.

Remarks:

Some of these time estimates are loose upper bounds.

Better estimates are often possible. (E.g., how long does it take to look something up in an array of size n ?)

But if our objective is to show that something is in NP, then all we need to show is that the time complexity of verification is bounded above by a polynomial (i.e., $O(n^k)$, for some fixed k ).

## Some languages in NP

For each of the examples we give, ask:

- What is the certificate?
- How do you verify it?

Examples:

- the set of 2-colourable graphs
- the set of 3-colourable graphs
- { (G, k) : G is a k-colourable graph }

## Some languages in NP

- the set of composite numbers
  - { x in N : there exist integers y, z such that $1 < y < x$, $1 < z < x$, and $x = y . z$ }
- SATISFIABILITY:

  the set of satisfiable Boolean expressions
- 2-SAT
  - exactly two literals in each clause
  - see the end of the previous lecture
- 3-SAT
  - exactly three literals in each clause

## Some languages in NP

- the set of Eulerian graphs
- the set of Hamiltonian graphs
  - A *Hamiltonian circuit* in a graph G is a circuit which includes each vertex exactly once. (note: a circuit doesn't repeat any vertex or edge)
  - A graph is *Hamiltonian* if it contains a Hamiltonian circuit.

## Some languages in NP

GRAPH ISOMORPHISM:

{ (G, H) : G is isomorphic to H }

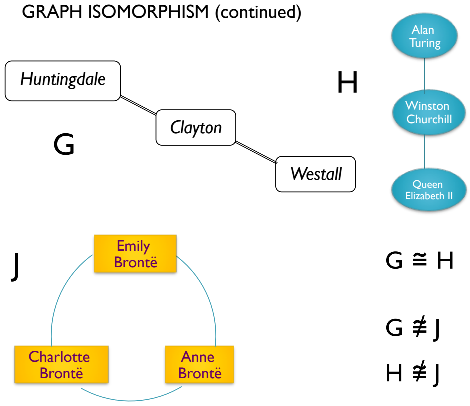
G is *isomorphic* to H if there is a bijection $f : V(G) \rightarrow V(H)$ such that, for all u, v in V(G), u is adjacent to v in G if and only if f(u) is adjacent to f(v) in H.

We write: $G \cong H$

Such a bijection is an *isomorphism*.

Informally: G and H are the same, apart from renaming vertices.

GRAPH ISOMORPHISM (continued)



## Some languages in NP

VERTEX COVER:

{ (G, k) : G has a vertex cover of size ≤ k }

A *vertex cover* in a graph G = (V, E) is a set X of vertices such that every edge has at least one endpoint in X.

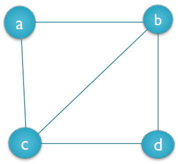
In this graph:

{a, b, c} is a vertex cover

{b, c} is a vertex cover

{a, b, c, d} is a vertex cover

{a, b} is NOT a vertex cover




## Some languages in NP

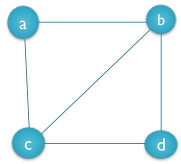
INDEPENDENT SET:

{ (G, k) : G has an independent set of size ≥ k }

An *independent set* in a graph G = (V, E) is a set X of vertices such that no edge has both endpoints in X.

In this graph:
{b} is an independent set
{a, d} is an independent set
{c, d} is NOT an independent set

- What is the relationship between vertex covers and independent sets?




## Some languages in NP

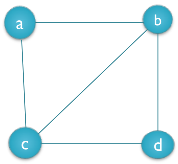
- CLIQUE:

{ (G, k) : G has a clique of size ≥ k }

A *clique* in a graph G = (V, E) is a set X of vertices such that every pair of vertices in X are adjacent.

In this graph:
{a} is a clique
{a, b} is a clique
{a, b, c} is a clique
{a, b, d} is NOT a clique

- What is the relationship between independent sets and cliques?




## P and NP

**Theorem**

P ⊆ NP

***Proof*** *(outline).*

For any L in P, there is a polynomial-time decider for L.

Turn this into a verifier:

Given string x, and any other string y as a certificate, we just run the decider on x.

Accept if the decider accepts. Reject if the decider rejects.

So we ignore the certificate.

Then prove that this is a polynomial-time verifier for L.

*End of proof*

## P and NP

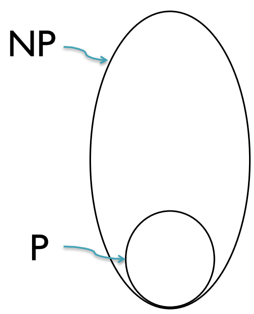
*Conjecture*

P ≠ NP

- The biggest open problem in Computer Science.
- One of the biggest open problems in Mathematics.
- Fame and glory await the solver …
- … and a Millennium Prize ($US 1 million) from the Clay Mathematics Institute

  http://www.claymath.org/millennium/
- But be careful: many false solutions have appeared, and continue to appear all the time.

## P and NP




In NP, not known to be in P:

SATISFIABILITY, 3-SAT,
HAMILTONIAN CIRCUIT,
3-COLOURABILITY,
VERTEX COVER,
INDEPENDENT SET, …

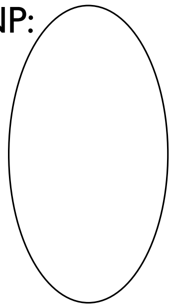
GRAPH ISOMORPHISM,
INTEGER FACTORISATION, …

In P:

2-SAT,
EULERIAN CIRCUIT,
2-COLOURABILITY,
PRIMES,
CONNECTED GRAPHS,
SHORTEST PATH,
Invertible matrices,
Context-free languages,
Regular languages, …

## P and NP

If P=NP:



SATISFIABILITY, 3-SAT,
HAMILTONIAN CIRCUIT,
3-COLOURABILITY,
VERTEX COVER,
INDEPENDENT SET, …

GRAPH ISOMORPHISM,
INTEGER FACTORISATION, …

2-SAT,
EULERIAN CIRCUIT,
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PRIMES,
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SHORTEST PATH,
Invertible matrices,
Context-free languages,
Regular languages, …

## P and NP

**Theorem**

Any language in NP can be decided in time $O(2^{n^K})$ for some K.

***Idea of Proof.***

Let L be any language in NP.

It has a polynomial-time verifier.

Construct from this a decider for L. How?

Decider does an exhaustive search of all possible certificates, to see if one of them gets the input accepted.

Prove it's a decider for L, and has the claimed time complexity.

## P and NP

The decider for L in detail:

Input: x

For each certificate y:

Call verifier on input x, certificate y.

If it accepts, then accept, else continue.

Accept x if the verifier accepts for some y;

Reject x if the verifier rejects for every y.

Decider for L? Clear from definition of a verifier.

Time complexity? If verifier has time complexity $O(n^k)$, then decider's time complexity is O( (# certificates) . $n^k$).

## P and NP

So: how many certificates?

At first sight: looks like infinitely many!

BUT in t steps, a Turing machine can examine at most t symbols in the certificate.

Our verifier has time complexity $O(n^k)$, which is $\leq c \cdot n^k$ (for sufficiently large n). So this verifier sees $\leq c \cdot n^k$ symbols in the certificate. Any symbols beyond that are ignored.

Assuming our usual alphabet {a, b}, the number of certificates that need to be checked is $\leq 2^{cn^k}$.

So, decider's total time complexity is $O(2^{cn^k} n^k)$.

## P and NP

So, decider's total time complexity is $O(2^{cn^k} n^k)$.

This is dominated by the exponential part, and in fact you can find a constant K a bit larger than k such that the time complexity is $O(2^{n^K})$.

So any language in NP can be decided in *exponential time*.

## Nondeterministic Turing machines

- All Turing machines so far have been *deterministic*
  - i.e., for each state and symbol, there is just one transition.
- So, for each state and for each symbol, the next action is completely *determined*: there is a specific next state, new symbol and direction. In fact, the entire computation is completely determined by the input.
- In a *nondeterministic Turing machine (NDTM)*: for a given state and symbol, there may be more than one possible transition. (Briefly mentioned in Lecture 14)
- *One* input may lead to *many* possible computations.
- Deterministic TMs are also NDTMs!

## Nondeterministic Turing machines

- The language accepted by a NDTM M is the set of input strings for which some computation leads to an Accept state.
- A NDTM M is a *decider* for a language L if
  - M halts on all inputs, and
  - the language accepted by M is L.
- A polynomial-time NDTM is a NDTM which, for any input x and any computation, halts in time $O(|x|^k)$, for some fixed k.

## Nondeterministic Turing machines

*Theorem*
L is in NP if and only if
some polynomial-time NDTM is a decider for L.

*Proof (outline):*

( ⇒ )

Suppose L has a verifier with time complexity $\leq c\,n^k$.

Construct a NDTM M as follows. On input x, M generates a string y of length $c\,n^k$, nondeterministically, and then just executes the verifier on x, y.

## Nondeterministic Turing machines

( ⇐ )

Let M be a polynomial-time NDTM that decides L.
Set up a way of encoding, as a string, the sequence of choices made at the nondeterministic steps of a computation. Use this string as a certificate …

*End of proof*

**NP** stands for **N**ondeterministic **P**olynomial time

Contrast with finite automata, where DFAs and NFAs define the same class of languages.

## Revision

- Sipser, sections 7.2-7.3.