

## Housekeeping

Assignment 5 is now available. It is due at the beginning of your support class in week 7 (10–13 April). Submission options for people missing their Fri 14 April tutorial are included on the assignment sheet.

People missing their Fri 14 April tutorial should feel free to attend any other tutorial in week 7 (a list will be posted on moodle).

Tutorial solutions 4 are also now available.

There's a link to the MLC video library under “additional resources” on the moodle page.

# MAT1830

## Lecture 16: Relations

## Relations - why should you care?

- ▶ Relations are used heavily in database theory in computer science.
- ▶ They are also used in theories of object orientation in programming.
- ▶ Relations can be thought of as a generalisation of functions.
- ▶ Like there is a functional programming paradigm there's a relational programming paradigm.

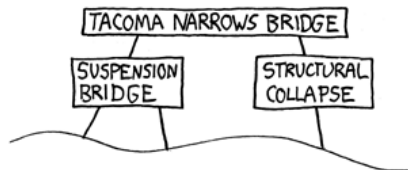
Roughly speaking, a binary relation on a set is something that tells us, for any two things in the set, that they are related or they are not related. (Order *is* important.)

You already know lots of examples of binary relations.

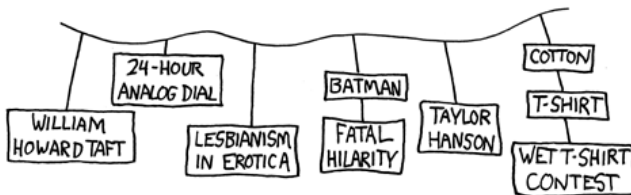
For example:

- ▶  $\leq$  on real numbers
- ▶ “divides” on natural numbers
- ▶ “is consecutive with” on integers

# THE PROBLEM WITH WIKIPEDIA:



[THREE HOURS OF  
FASCINATED CLICKING]



We could make a set  $\{(x, y) : x \text{ links to } y\}$ .

So, for example, (Tacoma Narrows Bridge, Suspension bridge) would be in the set and so would (Tacoma Narrows Bridge, Structural collapse), etc.

Mathematical objects can be related in various ways, and any particular way of relating objects is called a *relation* on the set of objects in question.

(This also applies to relations in the everyday sense. For example, “parent of” is a relation on the set of people.)

A *binary relation*  $R$  on a set  $A$  consists of  $A$  and a set of ordered pairs from  $A \times A$ .

When  $(a, b)$  is in this set we write  $aRb$ .

Similarly, a *ternary* relation on  $A$  would be defined by a set of ordered triples from  $A \times A \times A$ , and so on. (A *unary* relation on  $A$  is just a subset of  $A$ .)

A binary relation  $R$  on a set  $A$  consists of the set  $A$  together with a set of ordered pairs from  $A \times A$ .

If  $(x, y)$  is in the set then we write  $xRy$  and say “ $x$  is  $R$ -related to  $y$ ”.

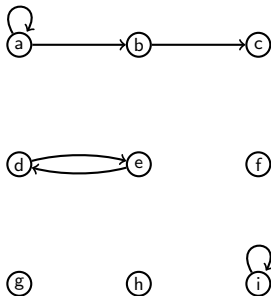
If  $(x, y)$  is not in the set then we write  $x \not R y$ .

Order matters. It might be that  $xRy$  but  $y \not R x$ .



## Arrow diagrams

**Example** Let  $R$  be the relation on  $\{a, b, c, d, e, f, g, h, i\}$  given by the set  $\{(a, a), (a, b), (b, c), (d, e), (e, d), (i, i)\}$ .



Is  $bRc$ ?    Yes.

Is  $eRf$ ?    No.

Is  $aRc$ ?    No.

Is  $dRe$ ?    Yes.

Is  $iRi$ ?    Yes.

## 16.1 Relations and functions

Any function  $f : X \rightarrow Y$  can be viewed as a relation  $R$  on  $X \cup Y$ . The relation is defined by

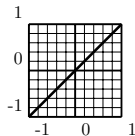
$$xRy \text{ if and only if } y = f(x).$$

However, not every relation is a function. Remember that a function must have exactly one output  $y$  for each input  $x$  in its domain. In a relation, on the other hand, an element  $x$  may be related to many elements  $y$ , or to none at all.

## 16.2 Examples

### 1. Equality on $\mathbb{R}$ .

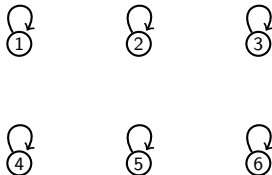
This is the relation consisting of the pairs  $(x, x)$  for  $x \in \mathbb{R}$ . Thus it is the following subset of the plane.



This relation is also a function (the identity function on  $\mathbb{R}$ ), since there is exactly one pair for each  $x \in \mathbb{R}$ .

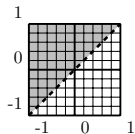
**Question** Give the set of ordered pairs for the relation “=” on  $\{1, 2, 3, 4, 5, 6\}$  and draw an arrow diagram for it.

$\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$



2. The  $<$  relation on  $\mathbb{R}$ .

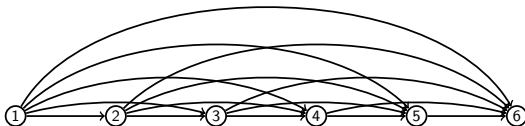
This relation consists of all the pairs  $(x, y)$  with  $x < y$ . It is the following shaded subset of the plane.



(The dashed line indicates that the points where  $x = y$  are omitted.)

**Question** Give the set of ordered pairs for the relation “ $<$ ” on  $\{1, 2, 3, 4, 5, 6\}$  and draw an arrow diagram for it.

$\{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6),$   
 $(3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6)\}$



TERMS I HAVE USED OR HEARD USED  
TO MAKE FUN OF THE INTERNET:

	NET	WEB	SPHERE	TUBES	BLAG
WORLD WIDE					
INTER-					
BLOGO-					
BLAGO-					
WEB-					

I HEARD ABOUT IT  
ON THE INTERBLAG!



**Question 16.1** Which of the following binary relations  $R$  satisfy  $\forall x \exists y (xRy)$ ?

$x \wedge y = \text{T}$  (for propositions  $x$  and  $y$ )

No - if  $x \equiv \text{F}$  there is no such  $y$ .

$x \subseteq y$  (for sets  $x$  and  $y$  of natural numbers)

Yes - take  $y = x$ , for example.

$x > y$  (for real numbers  $x$  and  $y$ )

Yes - take  $y = x - 1$ , for example.

$x$  divides  $y$  (for natural numbers  $x$  and  $y$ )

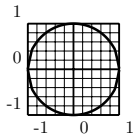
Yes -  $y = 2x$ , for example.



3. Algebraic curves.

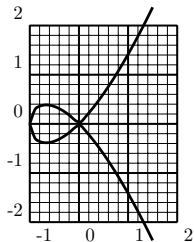
An algebraic curve consists of the points  $(x, y)$  satisfying an equation  $p(x, y) = 0$ , where  $p$  is a polynomial.

E.g. unit circle  $x^2 + y^2 - 1 = 0$ .



Notice that this relation is not a function, because there are two pairs with the same  $x$ , e.g.  $(0, 1)$  and  $(0, -1)$ .

Likewise, the curve  $y^2 = x^2(x + 1)$  is not a function.



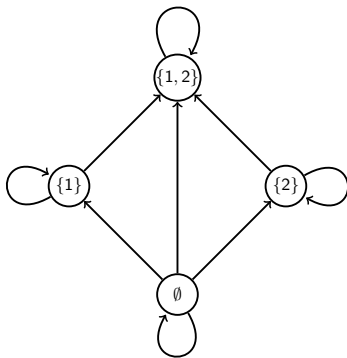
4. The subset relation  $\subseteq$ .

This consists of the ordered pairs of sets  $(A, B)$  such that  $A \subseteq B$ .  $A$  and  $B$  must both be subsets of some universal set  $U$ .

**Question** Give the set of ordered pairs for the relation " $\subseteq$ " on  $\mathcal{P}(\{1, 2\})$  and draw an arrow diagram for it.

Remember  $\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ .

$\{(\emptyset, \emptyset), (\emptyset, \{1\}), (\emptyset, \{2\}), (\emptyset, \{1, 2\}), (\{1\}, \{1\}), (\{1\}, \{1, 2\}),$   
 $(\{2\}, \{2\}), (\{2\}, \{1, 2\}), (\{1, 2\}, \{1, 2\})\}$



**Question 16.2** For each of the following binary relations on  $\mathbb{R}$ , find an equivalent relation.

$$(x = 0) \vee (y = 0)$$

$$xy = 0$$

$$(x = 0) \wedge (y = 0)$$

$$x^2 + y^2 = 0$$

HI, THIS IS  
YOUR SON'S SCHOOL.  
WE'RE HAVING SOME  
COMPUTER TROUBLE.



OH, DEAR - DID HE  
BREAK SOMETHING?

IN A WAY - )



DID YOU REALLY  
NAME YOUR SON  
Robert'); DROP  
TABLE Students;-- ?



OH, YES. LITTLE  
BOBBY TABLES,  
WE CALL HIM.

WELL, WE'VE LOST THIS  
YEAR'S STUDENT RECORDS.  
I HOPE YOU'RE HAPPY.



AND I HOPE  
YOU'VE LEARNED  
TO SANITIZE YOUR  
DATABASE INPUTS.

5. Congruence modulo  $n$ .

For a fixed  $n$ , congruence modulo  $n$  is a binary relation. It consists of all the ordered pairs of integers  $(a, b)$  such that  $n$  divides  $a - b$ .

## Congruence modulo $n$ (recap)

Remember  $a \equiv b \pmod{n}$  means that  $a$  and  $b$  have the same remainder when you divide them by  $n$ .

**Definition** We say  $a \equiv b \pmod{n}$  if  $n$  divides  $a - b$ .

For a fixed integer  $n \geq 2$ , congruence modulo  $n$  is a binary relation.

**Question** Which integers are congruent to 1 modulo 7?

Integers in the set  $\{\dots, -20, -13, -6, 1, 8, 15, 22, \dots\}$ .

This is the set  $\{7k + 1 : k \in \mathbb{Z}\}$ .

**Question** Which integers are congruent to 2 modulo 5?

Integers in the set  $\{\dots, -13, -8, -3, 2, 7, 12, 17, \dots\}$ .

This is the set  $\{5k + 2 : k \in \mathbb{Z}\}$ .

Numbers with the same parity (even or odd) are congruent modulo  $2$ .

Decimal numbers ending in the same digit are congruent modulo  $10$ .

The time (in hours) can be thought of as a number modulo  $24$ .

An angle (in degrees) can be thought of as a number modulo  $360$ .



### 16.3 Properties of congruence

As the symbol  $\equiv$  suggests, congruence mod  $n$  is a lot like equality. Numbers  $a$  and  $b$  which are congruent mod  $n$  are not necessarily equal, but they are “equal up to multiples of  $n$ ,” because they have equal remainders when divided by  $n$ .

Because congruence is like equality, congruence  $a \equiv b \pmod{n}$  behave a lot like equations. In particular, they have the following three properties.

1. Reflexive property.

$$a \equiv a \pmod{n}$$

for any number  $a$ .

2. Symmetric property.

$$a \equiv b \pmod{n} \Rightarrow b \equiv a \pmod{n}$$

for any numbers  $a$  and  $b$ .

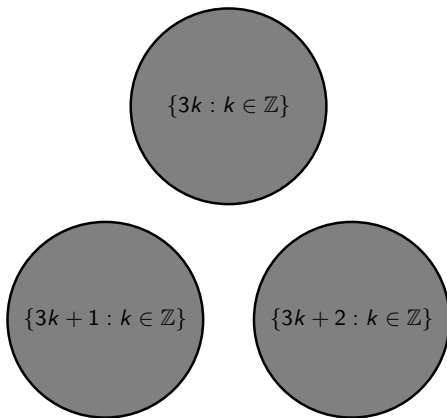
3. Transitive property.

$$a \equiv b \pmod{n} \text{ and } b \equiv c \pmod{n} \Rightarrow$$
$$a \equiv c \pmod{n}$$

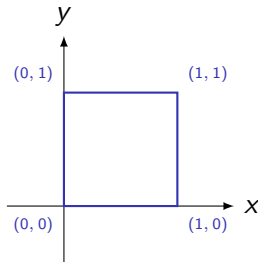
for any numbers  $a, b$  and  $c$ .

These properties are clear if one remembers that  $a \equiv b \pmod{n}$  means  $a$  and  $b$  have the same remainder on division by  $n$ .

**Question** Let  $R$  be the binary relation on  $\mathbb{Z}$  defined by  $xRy$  if and only if  $x \equiv y \pmod{3}$ . Roughly, what would an arrow diagram for  $R$  look like?



**Question 16.3** Use logic symbols and the  $\leq$  relation to write a relation between real numbers  $x$  and  $y$  which says that the point  $(x, y)$  lies in the square with corners  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$  and  $(1, 1)$ .



$$(0 \leq x) \wedge (x \leq 1) \wedge (0 \leq y) \wedge (y \leq 1)$$