Faculty of Information Technology, Monash University

FIT2004, S2/2016

Week 9: Bellman-Ford and Floyd-Warshall Algorithms

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ACKNOWLEDGMENTS

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Overview

Continuation of shortest-path algorithms:

- Bellman-Ford algorithm
  - Shortest paths in graphs with negative weights
  - Detecting reachable negative-weight cycles
- Floyd-Warshall Algorithm
  - All pairs shortest paths/distances problem
  - Transitive closure of a graph

Recommended reading

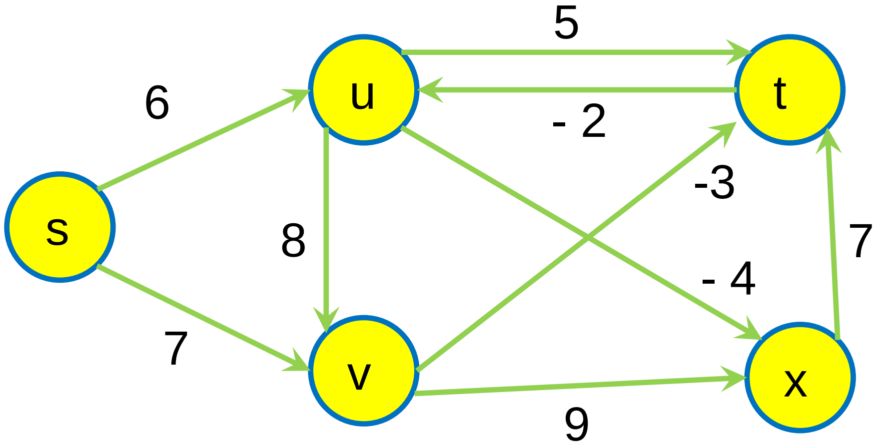
- Cormen et al. Introduction to Algorithms.
  - Section 24.1 Bellman-Ford Algorithm

- http://www.csse.monash.edu.au/~lloyd/tildeAlgDS/Graph/

- http://www.csse.monash.edu.au/~lloyd/tildeAlgDS/Graph/Directed/

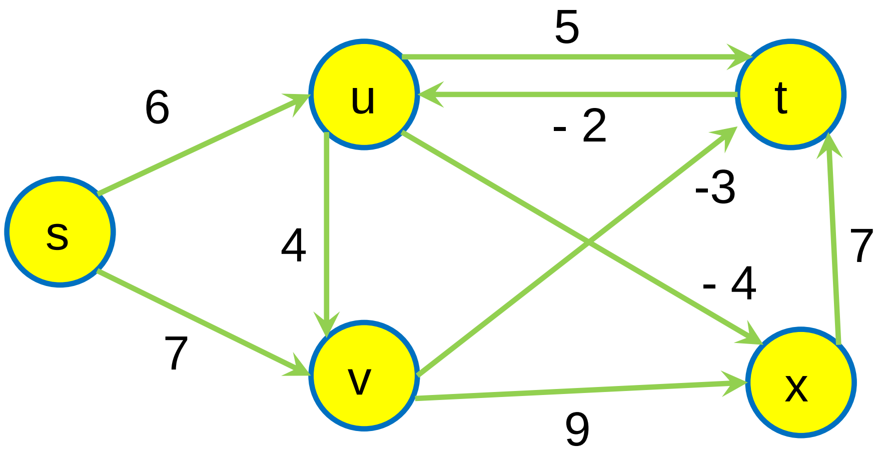
Shortest path (negative weights)

- What is the shortest distance from s to x in this graph?
- What will be the shortest distance from s to x if Dijkstra's algorithm is used on this graph?
- We saw that Dijkstra's algorithm cannot handle graph with negative weights. How to compute shortest paths on such graphs?
  - Bellman-Ford Algorithm



Shortest path (negative weights)

- What is the shortest distance from s to x on **this** graph?
- If there is a negative cycle in the graph, the notion of shortest path/distance does not make sense.
- Bellman-Ford algorithm returns
  - shortest distances from s to all vertices in the graph if there are no negative cycles
  - an error if there is a negative cycle reachable from s (i.e., can be used to detect negative cycles)



Bellman-Ford Algorithm

Initialize:

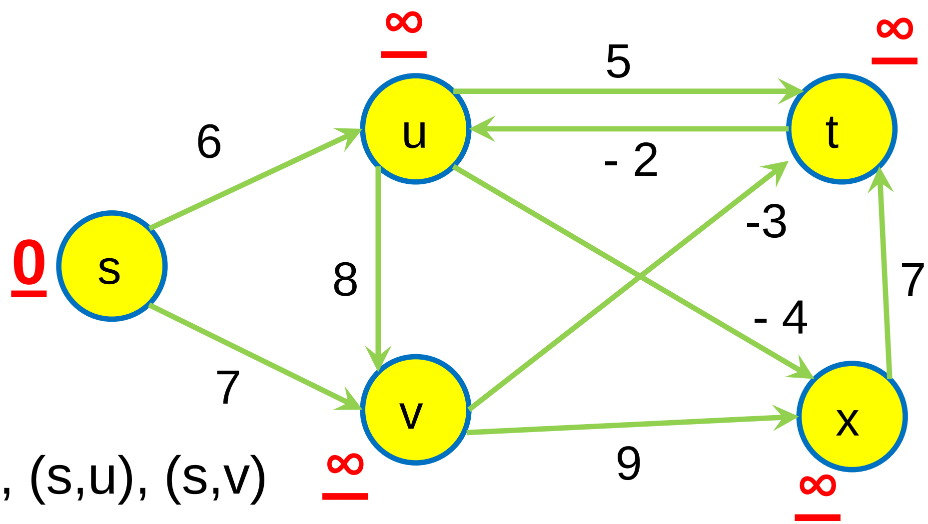
- For each vertex a in the graph
  - dist(s,a) = ∞
- dist(s,s) = 0

Consider the following operation:

- For each edge (a, b, w) in the graph // the order does not matter
  - dist(s, b) = min(dist(s,b) , dist(s,a) + w)

What is dist(s,u)?

What is dist(s,x)?



Assume the following order:

(u,t), (u,v), (u,x), (t,u), (v,t), (v,x), (x,t), (s,u), (s,v)

Bellman-Ford Algorithm

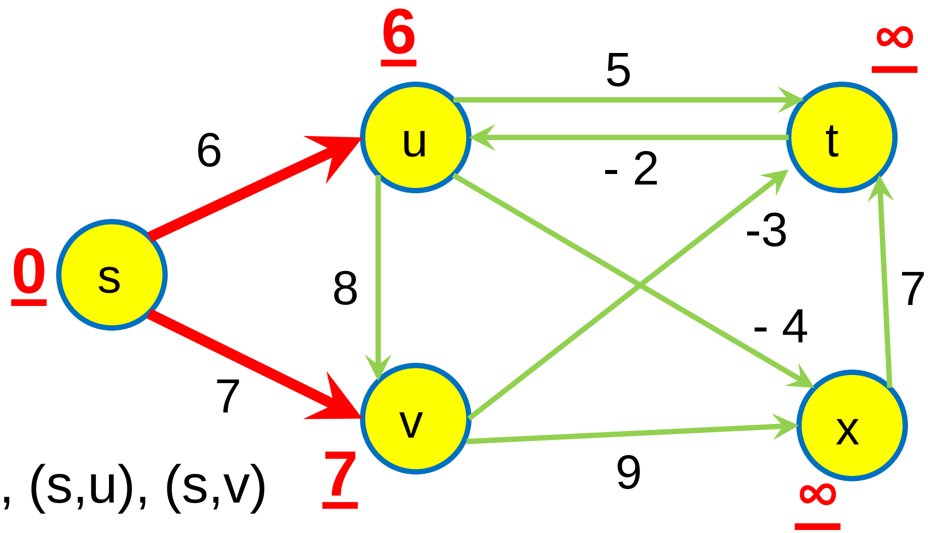
Initialize:

- For each vertex a in the graph
  - dist(s,a) = ∞
- dist(s,s) = 0

Consider the following operation:

- Repeat 2 times
  - For each edge (a, b, w) in the graph // the order does not matter
    - dist(s, b) = min(dist(s,b) , dist(s,a) + w)

What is dist(s,u)?
What is dist(s,x)?



Assume the following order:

(u,t), (u,v), (u,x), (t,u), (v,t), (v,x), (x,t), (s,u), (s,v)

Bellman-Ford Algorithm

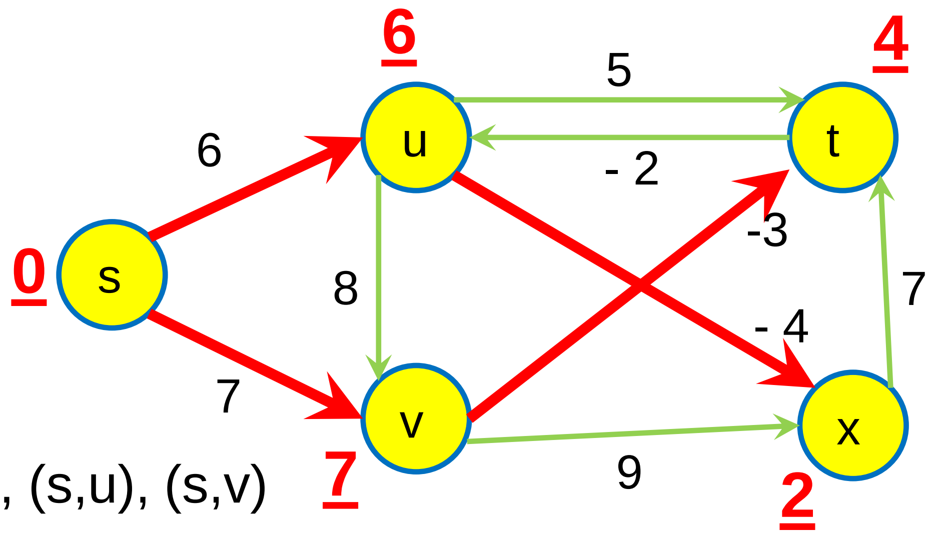
Initialize:

- For each vertex a in the graph
    - dist(s,a) = ∞
- dist(s,s) = 0

Consider the following operation:

- Repeat 3 times
    - For each edge (a, b, w) in the graph // the order does not matter
        - dist(s, b) = min(dist(s,b) , dist(s,a) + w)

What is dist(s,u)?
What is dist(s,x)?



Assume the following order:

(u,t), (u,v), (u,x), (t,u), (v,t), (v,x), (x,t), (s,u), (s,v)

Bellman-Ford Algorithm

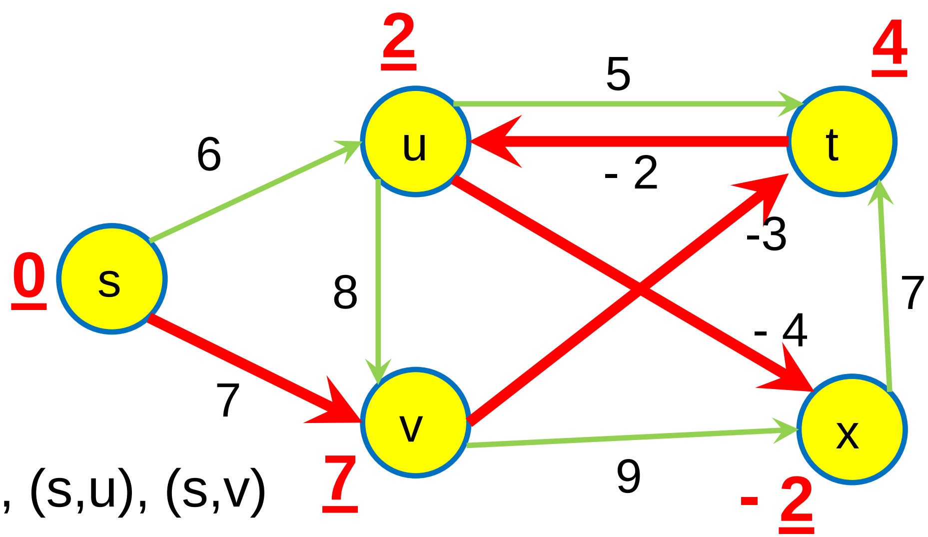
Initialize:

- For each vertex a in the graph
  - dist(s,a) = ∞
- dist(s,s) = 0

Consider the following operation:

- Repeat 4 times
  - For each edge (a, b, w) in the graph // the order does not matter
    - dist(s, b) = min(dist(s,b) , dist(s,a) + w)

What is dist(s,u)?
What is dist(s,x)?



Assume the following order:

(u,t), (u,v), (u,x), (t,u), (v,t), (v,x), (x,t), (s,u), (s,v)

Bellman-Ford Algorithm

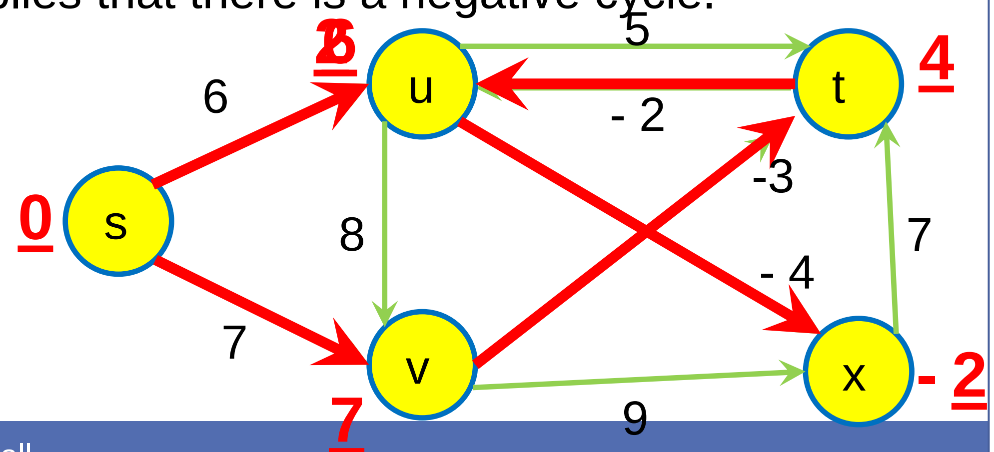
```
# STEP 1: Initializations
dist[1...V] = infinity
pred[1...V] = Null
dist[s] = 0
# STEP 2: Iteratively estimate dist[v] (from source s)
for i = 1 to V-1:
        for each edge <u,v,w> in the whole graph:
                est = dist[u] + w
                if est < dist[v]:
                        dist[v] = est
                        pred[v] = u

# STEP 3: Checks and returns false if a negative weight cycle
# is along the path from s to any other vertex
for each edge <u,v,w> in the whole graph:
        if dist[u]+w < dist[v] :
                return error; # negative edge cylce found in this graph

return dist[...], pred[...]
```

Time Complexity:

O(VE)

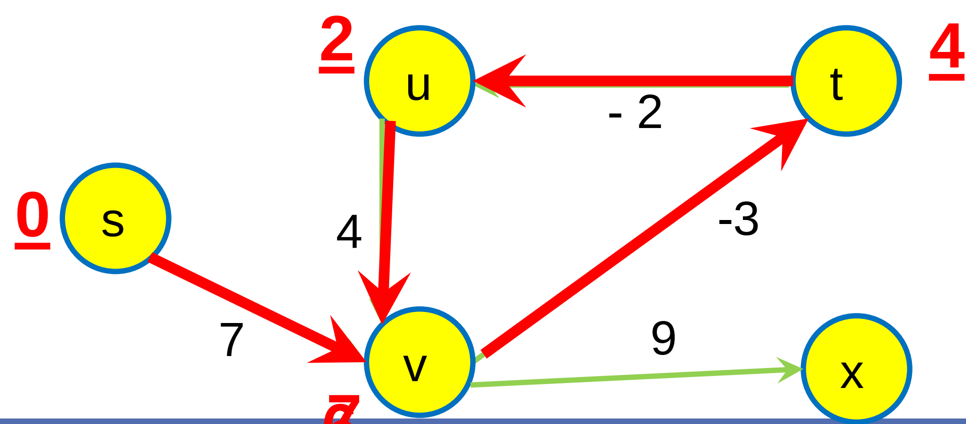
Bellman-Ford Algorithm: Correctness

- We established that the negative cycles do not make sense
- Can a shortest path from s to t have a non-negative cycle (i.e., with weight zero or more)?
  - No, because the path that avoids this cycle will have smaller or equal distance
- What is the maximum number of edges in a shortest path between two vertices?
  - V - 1
- Since the estimates are updated as many times as the maximum possible path length (V - 1), the shortest path must converge because:
  - The first iteration guarantees the shortest distances to all vertices considering paths of length at most 1 edge
  - The second iteration guarantees the shortest distances to all vertices considering paths of length at most 2 edges
  - …
  - The (V - 1)-th iteration guarantees the shortest distances to vertices considering paths of length at most (V-1) edges
- If V-th iteration reduces the distance of a vertex, this means that there is a shortest path with length V which implies that there is a negative cycle.



Bellman-Ford Algorithm: Negative Cycles

- If V-th iteration reduces the distance of a vertex, this means that there is a shortest path with length V which implies that there is a negative cycle.
- Consider the graph with vertices s, u, v, and t and assume we have run (V-1 = 3) iterations.
- In the 4^{th} iteration, the weight of at least one vertex will be reduced (due to the presence of a negative cycle).
- Important: Bellman-Ford Algorithm detects negative cycles only if such cycle is reachable from the source vertex
  - E.g., if x is the source vertex, the algorithm will not detect the negative cycle
- If the goal is to detect the existence of a negative cycle in the graph, one solution is to call Bellman-Ford algorithm for each vertex in the graph.



All-Pairs Shortest Distances

Problem

- Return shortest distances between **all** pairs of vertices in the graph.

For unweighted graphs:

- For each vertex v in the graph
  - Call Breadth-First Search for v

Time complexity:

$$O(V(V+E)) = O(V^2 + EV)$$

Since E can be as large as $O(V^2)$ → $O(V^3)$ for dense graphs

For weighted graphs (with non-negative weights):

- For each vertex v in the graph
  - Call Dijkstra's algorithm for v

Time complexity:

$$O(V(V \log V + E \log V)) = O(V^2 \log V + EV \log V)$$

For dense graphs → $O(V^3 \log V)$

For weighted graphs (allowing negative weights):

- For each vertex v in the graph
  - Call Bellman-Ford algorithm for v

Time complexity:

$$O(V(VE)) = O(V^2 E)$$

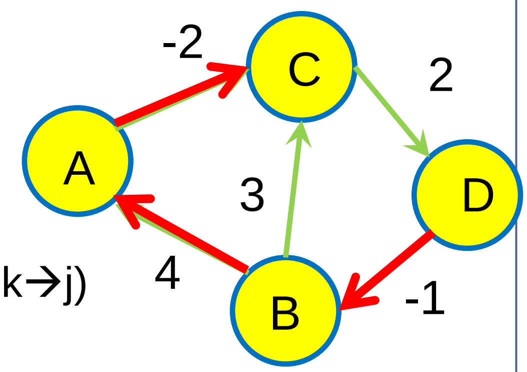
For dense graphs → $O(V^4)$

Can we do better?

- Yes, Floyd-Warshall Algorithm returns all-pairs shortest distances in $O(V^3)$ for graphs allowing negative weights.

Floyd-Warshall Algorithm

- Create the adjacency matrix called dist[][]
- For each vertex k in the graph
  - For each pair of vertices i and j in the graph
    - If dist(i → k → j) is smaller than the current dist(i→j)
      - Update/create shortcut i →j with weight equal to dist(i→k→j)
        i.e., update dist[i][j] = dist[i][k] + dist[k][j]



Assume that the outer for-loop will access vertices in the order A, B, C, D

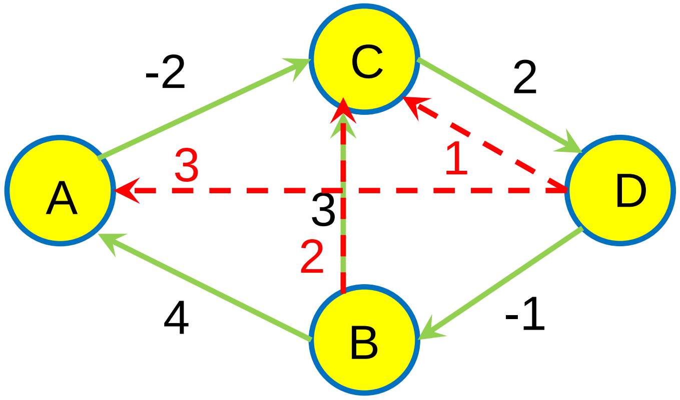
First iteration of outer loop (i.e., k is A):

Which shortcut(s) i→j is/are updated after the execution of the **inner** for-loop?

Second iteration of outer loop (i.e., k is B):

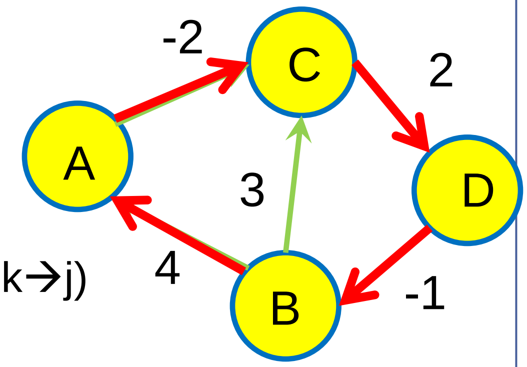
Which shortcut(s) i→j is/are updated after the execution of the **inner** for-loop?

	A	B	C	D
A	0	∞	-2	∞
B	4	0	~~3~~ 2	∞
C	∞	∞	0	2
D	~~∞~~ 3	-1	~~∞~~ 1	0



Floyd-Warshall Algorithm

- Create the adjacency matrix called dist[][]
- For each vertex k in the graph
  - For each pair of vertices i and j in the graph
    - If dist(i → k → j) is smaller than the current dist(i→j)
      - Update/create shortcut i →j with weight equal to dist(i→k→j)
      - i.e., update dist[i][j] = dist[i][k] + dist[k][j]



Assume that the outer for-loop will access vertices in the order A, B, C, D

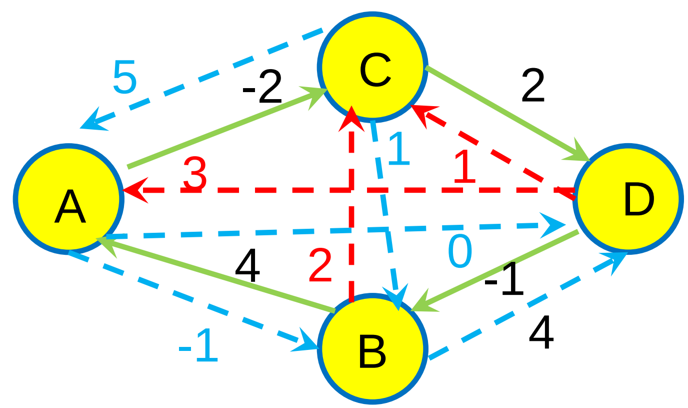
Third iteration of outer loop (i.e., k is C):

Which shortcut(s) i→j is/are updated after the execution of the **inner** for-loop?

Fourth iteration of outer loop (i.e., k is D):

Which shortcut(s) i→j is/are updated after the execution of the **inner** for-loop?

	A	B	C	D
A	0	~~∞~~ -1	-2	~~∞~~ 0
B	4	0	2	~~∞~~ 4
C	~~∞~~ 5	~~∞~~ 1	0	2
D	3	-1	1	0



Floyd-Warshall Algorithm

```
dist[][] = E #  Create adjacency matrix using E
for vertex k in 1..V:
# Invariant: dist[i][j] corresponds to the shortest path from i
to j considering the intermediate vertices 1 to k-1
   for vertex i in 1..V:
      for vertex j in 1..V:
         dist[i][j] = min(dist[i][j], dist[i][k] + dist[k][j])
```

Time Complexity:

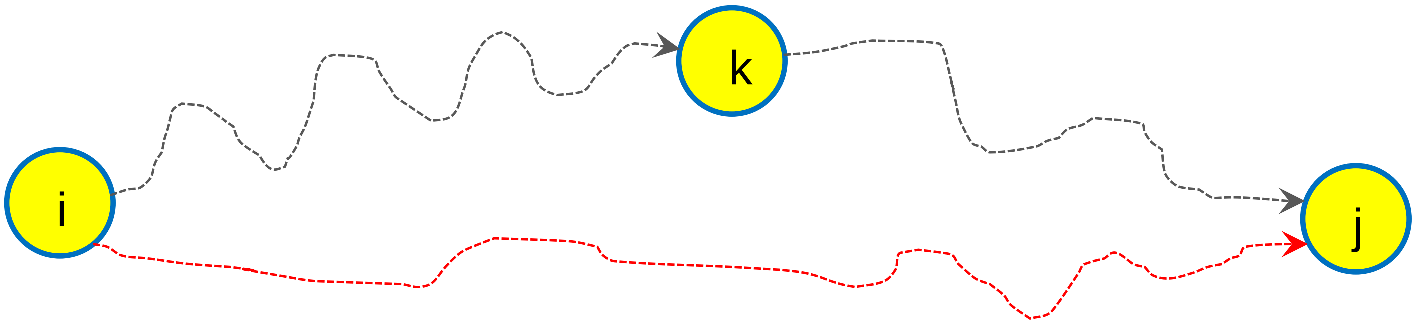
$O(V^3)$

Space Complexity:

$O(V^2)$

Floyd-Warshall Algorithm: Correctness

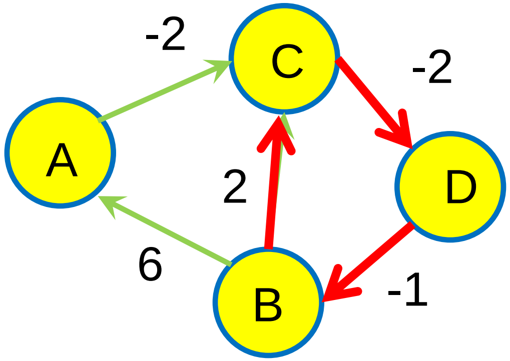
- The invariant in the code is central to the algorithm's correctness.
- At the start of the k-th iteration of the outer loop, dist[i][j] corresponds to the shortest path from i to j considering the intermediate vertices 1 to k-1.
  - This is certainly true initially (for k = 1) when no intermediate vertices are allowed.
  - Now there is no point in visiting any vertex more than once on any shortest path.
  - If the k-th vertex is to improve on the known path from vertex i to vertex j then it can only be by going from i to k and then k to j (possibly via vertices in 1 to k-1)
  - Thus, minimum of dist(i→k→j) and dist(i→j) gives the minimum distance from i to j considering the intermediate vertices 1 to k.



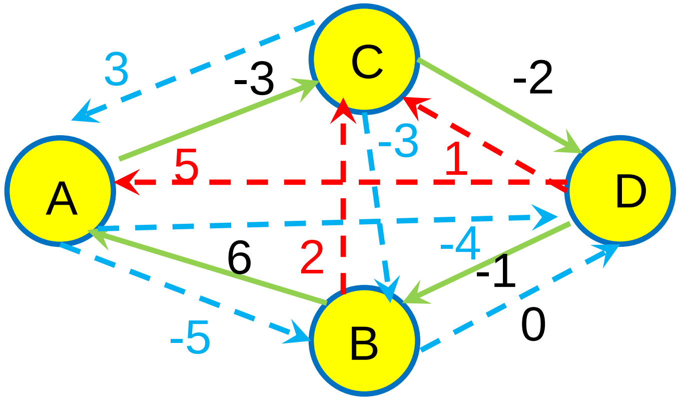
A wiggly arrow between two vertices x and y denote the shortest path between x and y considering the intermediate vertices 1 to k-1

Floyd-Warshall Algorithm: Negative Cycles

- If there is a negative cycle, there will be a vertex v such that dist[v][v] is negative.
- Look at the diagonal of the adjacency matrix and return error if a negative value is found

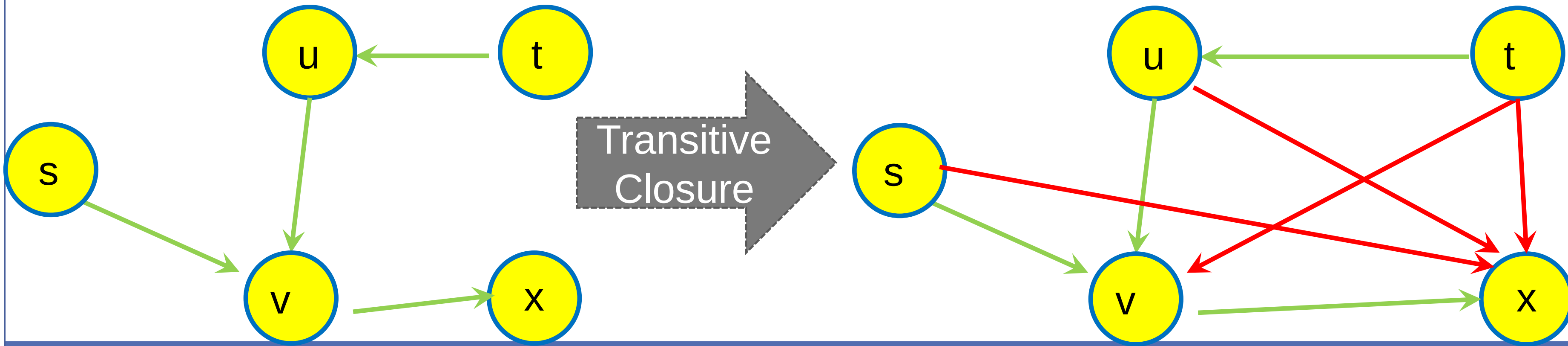


	A	B	C	D
A	0	2	-3	-4
B	6	-1	2	0
C	3	-3	-1	-2
D	5	-1	1	0



Transitive Closure of a Graph

- Given a graph G = (V,E), its transitive closure is another graph (V,E') that contains the same vertices V but contains an edge between any two vertices u and v such that there is a path between u and v in the original graph.
- **Applications:** What are the pairs of vertices (u,v) in the graph such that one can reach from u to v.
  - E.g., given flights between different cities, can I go from city A to city B (regardless of the number of flights I need to take), or where can/cannot I go from city A.
- **Solution:** Assign each edge a weight 1 and then apply Floyd-Warshall algorithm. If dist[i][j] is not infinity, this means there is a path from i to j in the original graph. (Or just maintain True and False as shown next)



Floyd-Warshall Algorithm for Transitive Closure

```
# Modify Floyd-Warshall Algorithm to compute Transitive Closure
# initialization
for vertex i in 1..V:
        for vertex j in 1..V:
            if there is an edge between i and j or i == j:
                TC[i][j] = True
            else:
                TC[i][j] = False
for vertex k in 1..V:
# Invariant: TC[i][j] corresponds to the existence of path from i to j considering the
intermediate vertices 1 to k-1
    for vertex i in 1..V:
        for vertex j in 1..V:
            TC[i][j] = TC[i][j] or (TC[i][k] and TC[k][j])
```

Time Complexity:

$O(V^3)$

Space Complexity:

$O(V^2)$

Summary

Take home message

- Dijkstra’s algorithm works only for graphs with non-negative weights
- Bellman-Ford computes shortest paths in graphs with negative weights in O(VE) and can also detect the negative cycles that are reachable
- Floyd-Warshall Algorithm computes all-pairs shortest paths and transitive closure in $O(V^3)$

Things to do (this list is not exhaustive)

- Go through recommended reading and make sure you understand why the algorithms are correct
- Implement Bellman-Ford and Floyd-Warshall Algorithms

Coming Up Next

- Minimum spanning trees and topological sort