Housekeeping

Assignment 6 and tutorial sheet 6 are now available.

Tutorial solutions 5 will be available tonight.

Assignment 5 is due at the beginning of your support class next week (10–13 April). Submission options for people missing their Fri 14 April tutorial are included on the assignment sheet.

People missing their Fri 14 April tutorial should feel free to attend any other tutorial next week (a list is now up on moodle).

MAT1830

Lecture 18: Order Relations

A *partial order relation* R on a set A is a binary relation with the following three properties.

1. Reflexivity.
$$aRa$$
$$\text{for all } a \in A.$$

2. Antisymmetry.
$$aRb \text{ and } bRa \Rightarrow a = b$$
$$\text{for all } a, b \in A.$$

3. Transitivity.
$$aRb \text{ and } bRc \Rightarrow aRc$$
$$\text{for all } a, b, c \in A.$$

The relation $\subseteq$ on sets is a typical example of a partial order relation.

A *total order relation* R on a set A is a partial order relation that also has the property aRb or bRa for all $a, b \in A$.

For a binary relation R on a set A.

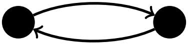
Antisymmetry: For all $x, y \in A$, if xRy and yRx then $x = y$.

This definition is useful for proofs but I think the contrapositive is more intuitive.

Antisymmetry (equivalent defn): For all $x, y \in A$, if $x \neq y$ then it is not the case that xRy and yRx.

Antisymmetry (For a binary relation R on a set A.)

I never see:



To prove R is antisymmetric, show that...

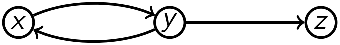
For all $x, y \in A$, if xRy and yRx then $x = y$.

To prove R is not antisymmetric, show that...

There are some $x, y \in A$ such that $x \neq y$, xRy and yRx.

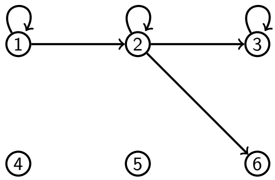
Warning Antisymmetric does not mean “not symmetric”!

An example which is neither symmetric nor antisymmetric:



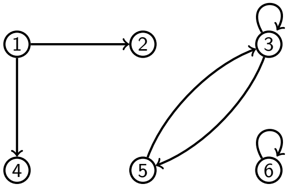
Technically, “$=$” is both symmetric and antisymmetric.

Question Let R be the relation on A pictured below. Is R antisymmetric?



Yes. For all $x, y \in A$, if xRy and yRx then $x = y$.

Question Let S be the relation on A pictured below. Is S antisymmetric?

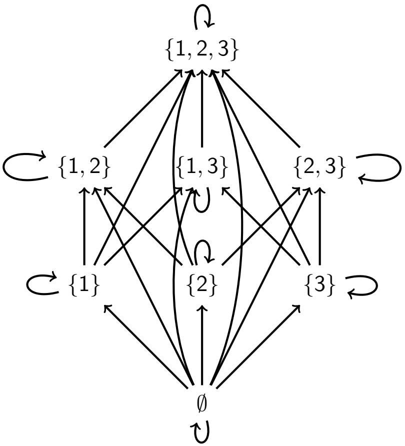


No. $3S5$ and $5S3$ (and $3 \neq 5$).

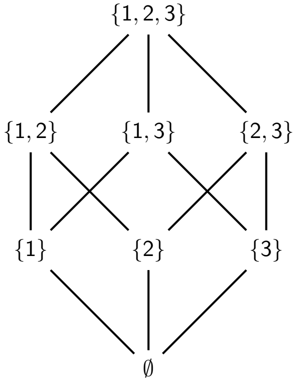
Hasse Diagrams

Example The relation $\subseteq$ on $\mathcal{P}(\{1,2,3\})$ is a partial order relation.

Arrow diagram



Hasse diagram

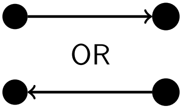


Definition A binary relation R on a set A is a *total order relation* if

- it is a partial order relation; and
- for any $x, y \in A$ we have xRy or yRx.

Everywhere I see:

I actually see:



Example $\leq$ on $\mathbb{R}$ is a total order relation (because for any $x, y \in \mathbb{R}$ we have that $x \leq y$ or $y \leq x$).

Example $\subseteq$ on $\mathcal{P}(\{1, 2, 3\})$ is not a total order relation (for example, $\{1\} \not\subseteq \{2, 3\}$ and $\{2, 3\} \not\subseteq \{1\}$).

18.1 Examples

1. The $\leqslant$ relation on $\mathbb{R}$.
   This is obviously reflexive ($a \leqslant a$), antisymmetric ($a \leqslant b$ and $b \leqslant a \Rightarrow a = b$) and transitive ($a \leqslant b$ and $b \leqslant c \Rightarrow a \leqslant c$).

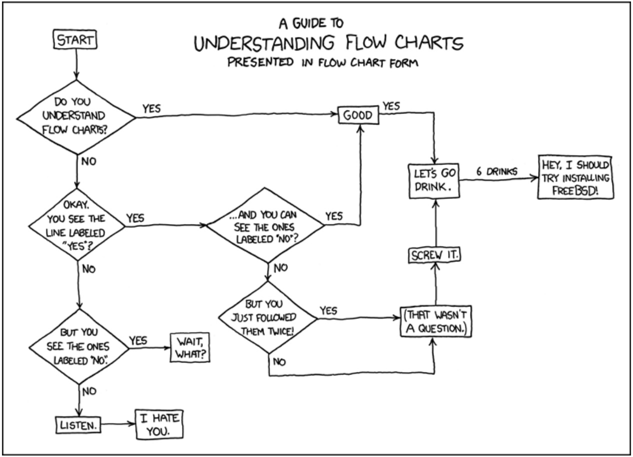
   It also has the property that for any a and b we have $a \leqslant b$ or $b \leqslant a$. This makes it a total order relation.

2. Alphabetical order of words.
   Words on the English alphabet are ordered by comparing the leftmost letter at which they differ. The word with the earlier letter at this position is earlier in alphabetical order. This is another total order.

3. Alphabetical order of functions on $\mathbb{N}$.
   Functions on $\mathbb{N}$ can also be totally ordered by comparing their values at the least natural number where they differ. The function with the lower value is put lower in the ordering.

4. Ordering functions on $\mathbb{R}$.
   There is no known total ordering of functions on $\mathbb{R}$. They can be partially ordered in various ways. For example, we can say that $f \leqslant g$ if $f(x) \leqslant g(x)$ for all x.
   With this ordering there are many pairs of functions, e.g. square and cube, neither of which is $\leqslant$ the other.

5. Divisibility on $\mathbb{N}$.
   The relation "a divides b" is reflexive, antisymmetric and transitive, hence a partial order. It is not a total order because, e.g., neither of the numbers 2,3 divides the other.

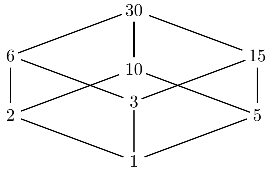

A GUIDE TO
UNDERSTANDING FLOW CHARTS
PRESENTED IN FLOW CHART FORM
START
DO YOU UNDERSTAND FLOW CHARTS?
YES
GOOD
YES
NO
OKAY. YOU SEE THE LINE LABELED "YES"?
YES
...AND YOU CAN SEE THE ONES LABELED "NO"?
YES
NO
NO
BUT YOU SEE THE ONES LABELED "NO".
YES
WAIT, WHAT?
NO
LISTEN.
I HATE YOU.
BUT YOU JUST FOLLOWED THEM TWICE!
YES
(THAT WASN'T A QUESTION.)
NO
SCREW IT.
LET'S GO DRINK.
6 DRINKS
HEY, I SHOULD TRY INSTALLING FREEBSD!

18.2 Remarks

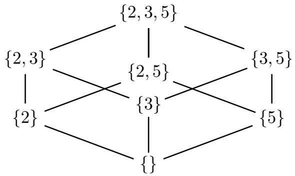
1. The set {divisors of 30}, ordered by the "divides" relation, has the same structure as the power set of $\{2, 3, 5\}$, ordered by $\subseteq$.

Here are the two sets, with lines connecting elements to those lower in the ordering.

Divisors of 30:



Subsets of $\{2, 3, 5\}$:



2. Such a correspondence between divisors and subsets of the prime factors occurs for any number with no repeated prime factor. It has a couple more interesting properties too:

- The gcd corresponds to $\cap$.
  E.g. $\gcd(6, 10) = 2$, $\{2, 3\} \cap \{2, 5\} = \{2\}$.

- The *least common multiple* (lcm) corresponds to $\cup$.
  E.g. $\text{lcm}(6, 10) = 30$, $\{2, 3\} \cup \{2, 5\} = \{2, 3, 5\}$.

3. In fact, every partial order relation has the structure of an ordering of sets by $\subseteq$. For each element a, consider the set $\{s : sRa\}$ of elements "less than or equal to a". Then

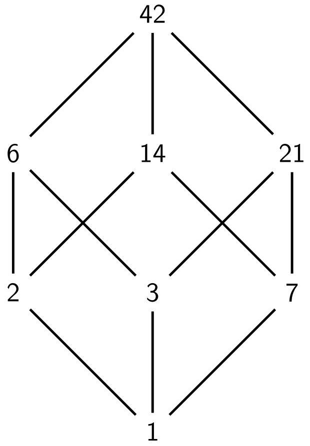
$$aRb \Leftrightarrow \{s : sRa\} \subseteq \{s : sRb\}.$$

4. The usual ordering of real numbers is almost an alphabetical ordering of their decimals. However, there are cases where two different decimals represent the same number, e.g.

$$0.4999\cdots = 0.50000\cdots$$

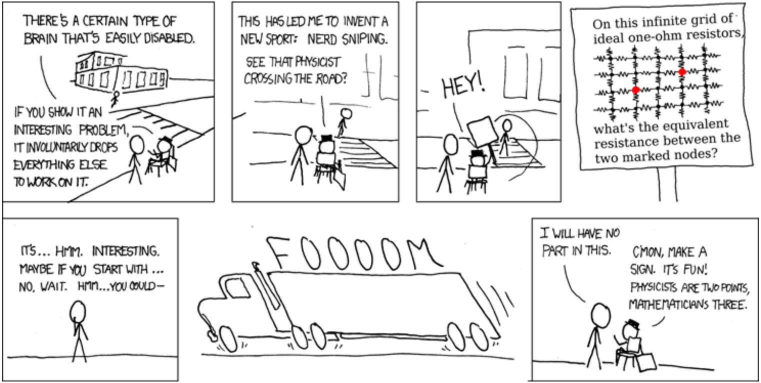
Question 18.2 Draw a Hasse diagram for the set of divisors of 42.

The set of divisors is $\{1, 2, 3, 6, 7, 14, 21, 42\}$.



Question 18.2 (cont) Why does the Hasse diagram for the set of divisors of 42 look like the diagram for the set of the divisors of 30?

Because $42 = 2 \times 3 \times 7$ and $30 = 2 \times 3 \times 5$. They both look like the Hasse diagram for the subsets of $\{a, b, c\}$ ($a = 2$, $b = 3$, $c = 7$ for 42, and $a = 2$, $b = 3$, $c = 5$ for 30).


THERE'S A CERTAIN TYPE OF BRAIN THAT'S EASILY DISABLED.
IF YOU SHOW IT AN INTERESTING PROBLEM, IT INVOLUNTARILY DROPS EVERYTHING ELSE TO WORK ON IT.
THIS HAS LED ME TO INVENT A NEW SPORT: NERD SNIPING.
SEE THAT PHYSICIST CROSSING THE ROAD?
HEY!
On this infinite grid of ideal one-ohm resistors, what's the equivalent resistance between the two marked nodes?
IT'S... HMM. INTERESTING. MAYBE IF YOU START WITH ... NO, WAIT. HMM...YOU COULD—
FOOOOM
I WILL HAVE NO PART IN THIS.
C'MON, MAKE A SIGN. IT'S FUN! PHYSICISTS ARE TWO POINTS, MATHEMATICIANS THREE.

18.3 Well-ordering

A *well-order relation* R on a set A is a total order relation that also has the property that each nonempty subset of A has a least element.

The set $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is totally ordered by $\leqslant$ and every nonempty subset of $\mathbb{N}$ has a least element. So $\leqslant$ on $\mathbb{N}$ is a well-order relation.

The well-ordering of $\mathbb{N}$ is the basis of proofs by induction. (We previously used well-ordering when we said "there is no infinite descending sequence of positive integers.")

It is also the basis of the "alphabetical" total ordering of functions on $\mathbb{N}$ – to order two functions f and g we need to look at the least element of $\{n : f(n) \neq g(n)\}$.

The set $\mathbb{R}$ of reals is also totally ordered by $\leqslant$, but not well-ordered. For example, the subset of reals > 0 has no least member.

Definition A binary relation R on a set A is a *well-order relation* if

- it is a total order relation; and
- every non-empty $B \subseteq A$ has a least element.

We could write this second condition formally as

- for every non-empty $B \subseteq A$ there is an $\ell \in B$ such that $\ell R y$ for all $y \in B$.

Example $\leq$ on $\mathbb{N}$ is a well-order relation (because every non-empty set of natural numbers has a least element).

Example $\leq$ on $\mathbb{R}$ is not a well-order relation (for example, the set $\{x \in \mathbb{R} : x > 2\}$ has no least element).

Question 18.3 Let R be a binary relation on $\mathbb{N} \times \mathbb{N}$ defined by $(m_1, n_1)R(m_2, n_2)$ if and only if either

- $m_1 < m_2$; or
- $m_1 = m_2$ and $n_1 \leq n_2$.

Question Is this a partial ordering? Yes.

Question Is this a total ordering? Yes.

Question Is this a well ordering?

Let S be a nonempty subset of $\mathbb{N} \times \mathbb{N}$.

Let $a = \min\{m : (m, n) \in S\}$.

Now let $b = \min\{n : (a, n) \in S\}$.

(a, b) is the least element in S.

So R is a well ordering, since every nonempty subset of $\mathbb{N} \times \mathbb{N}$ has a least element.