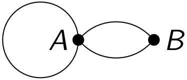
Lecture 31: Degree

The *degree* of a vertex A in a graph G is the number of times A occurs in the list of edges of G.

For example, if G is



then the list of edges is $AA,\ AB,\ AB$, and hence degree$(A) = 4$.

Intuitively speaking, degree(A) is the number of "ends" of edges occurring at A. In particular, a loop at A contributes 2 to the degree of A.

The handshaking lemma

> In any graph,
> sum of degrees $= 2 \times$ number of edges.

The reason for the name is that if each edge is viewed as a handshake,

then at each vertex V

$$\text{degree}(V) = \text{number of hands.}$$

Hence

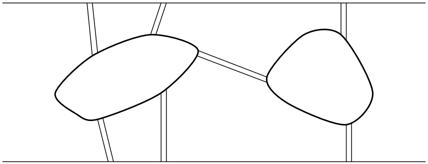
$$\begin{aligned} \text{sum of degrees} &= \text{total number of hands} \\ &= 2 \times \text{number of handshakes.} \end{aligned}$$

An important consequence

The handshaking lemma implies that *in any graph the sum of degrees is even* (being $2\times$something). Thus it is impossible, for example, for a graph to have degrees 1,2,3,4,5.

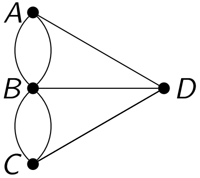
The seven bridges of Königsberg

In 18th century Königsberg there were seven bridges connecting islands in the river to the banks as follows.



The question came up: is it possible for a walk to cross all seven bridges without crossing the same bridge twice?

An equivalent question is whether there is a trail which includes all edges in the following graph.



Euler's solution

Euler (1737) observed that the answer is no, because

1. Each time a walk enters and leaves a vertex it "uses up" 2 from the degree.
2. Hence if all edges are used by the walk, all vertices except the first and last must have even degree.
3. The seven bridges graph in fact has four vertices of odd degree.

Euler's theorem

The same argument shows in general that

> A graph with > 2 odd degree vertices has no trail using all its edges.

And a similar argument shows

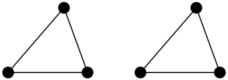
> A graph with odd degree vertices has no *closed* trail using all its edges.

(Because in this case the first and last vertex are the same, and its degree is "used up" by a closed trail as follows: 1 at the start, 2 each time through, 1 at the end.)

The converse theorem

If, conversely, we have a graph G whose vertices all have even degree, is there a closed trail using all the edges of G?

Not necessarily. For example, G might be *disconnected*:



We say a graph is *connected* if any two of its vertices are connected by a walk (or equivalently, by a trail or a path). We call a trail using all edges of a graph an *Euler trail*. Then we have

> A connected graph with no odd degree vertices has a closed Euler trail.

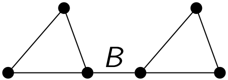
Such a closed trail can be constructed as follows:

1. Starting at any vertex V_1, follow a trail t_1 as long as possible.
2. The trail t_1 eventually returns to V_1, because it can leave any other vertex it enters. (Immediately after the start, V_1 has one "used" edge, and hence an odd number of "unused" edges. Any other vertex has an even number of "unused" edges.)
3. If t_1 does not use all edges, retrace it to the first vertex V_2 where t_1 meets an edge not in t_1.
4. At V_2 add a "detour" to t_1 by following a trail out of V_2 as long as possible, not using edges in t_1. As before, this trail eventually returns to its starting point V_2, where we resume the trail t_1. Let t_2 be the trail t_1 plus the detour from V_2.
5. If t_2 does not use all the edges, retrace t_2 to the first vertex V_3 where t_2 meets an edge not in t_2. Add a detour at V_3, and so on.

Since a graph has only a finite number of edges, this process eventually halts. The result will be a closed trail which uses all the edges (this requires the graph to be connected, since any unused edge would be connected to used ones, and thus would have eventually been used).

Bridges

A *bridge* in a connected graph G is an edge whose removal disconnects G. E.g. the edge B is a bridge in the following graph.



The construction of an Euler trail is improved by the doing the following (*Fleury's algorithm*).

- Erase each edge as soon as it is used.
- Use a bridge in the remaining graph only if there is no alternative.

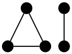
It turns out, when this algorithm is used, that it is not necessary to make any detours. The improvement, however, comes at the cost of needing an algorithm to recognise bridges.

Questions

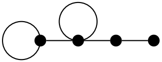
31.1 For each of the following sequences, construct a simple graph whose vertices have those degrees, or explain why no such graph exists.

- $1, 2, 3, 4$
- $1, 2, 1, 2, 1$
- $1, 2, 2, 2, 1$

Using the Handshaking Lemma, we can rule out the second option since the sum of the degrees in any graph must be even, and $1 + 2 + 1 + 2 + 1 = 7$, which is odd.

The third sequence is achieved by  or by 

The first sequence is impossible for a simple graph. Since each vertex can have at most one edge to each other vertex, the largest possible degree would be 3. However, it can be achieved by multigraphs such as

 or 

Questions

31.2 A graph G has adjacency matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Decide, without drawing the graph, whether G is connected or not.

ANS: The graph is not connected. We can tell this from the adjacency matrix because there is no edge between $\{V_1, V_2, V_3\}$ and $\{V_4, V_5\}$.

Questions

31.3 A graph H has adjacency matrix $\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$

What are the degrees of its vertices? Does H have a closed Euler trail?

ANS: The degree of a vertex is the number of edges that are connected to it (where we count loops as 2). Hence the degree of V_1 is 4, the degree of V_2 is 3, the degree of V_3 is 4, the degree of V_4 is 3, the degree of V_5 is 4.
Does H have a closed Euler trail? No! To have a closed Euler trail, every degree must be even. Here we have two vertices of odd degree, so there is no closed Euler trail.

Actually there is another reason why there is no Euler trail in H. H is disconnected! There is no edge between $\{V_1, V_3, V_5\}$ and $\{V_2, V_4\}$.