Specialist Mathematics 3/4 Bound Reference

Sam Murphy

2023

Theorem 1. $\{Sam\} \cap \{Bitches\} = \emptyset$

Proof. {Sam} $\in$ Samuel Murphy
{Bitches} $\in$ All Females

A person "Person" is denoted as being a certain sex by the notation: Person_M or Person_F

Samuel Murphy = $(\text{Samuel Murphy})_M$

A female takes the form of Person_r. $\therefore$ {Bitches} contains all Person_F, and exclusively all Person_F.

The set Sam contains one element $(\text{Samuel Murphy})_M$, which does not take the form Person_F $\therefore$ the sets Sam and Bitches contain no elements in common.
$\therefore$ {Sam} $\cap$ {Bitches} $= \emptyset$ $\square$

Contents

1 Unit 1/2 Assumed Knowledge

1.1 Circular Functions

1.1.1 Radians

1.1.2 Sine, Cosine Graphs

1.1.3 Tangent Graphs

1.1.4 Trigonometric Identities

1.2 Solving Triangles

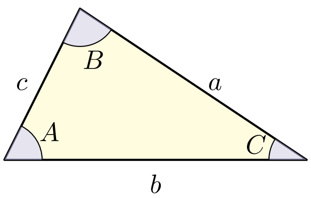
1.2.1 Pythagoras Theorem

1.2.2 Trigonometric Ratios

1.2.3 Sine Rule

The sine rule is used to find unknown sides or angles in a non-right triangle in 2 situations:

- One side and two angles are given
- Two sides and one **non-included** angle are given



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

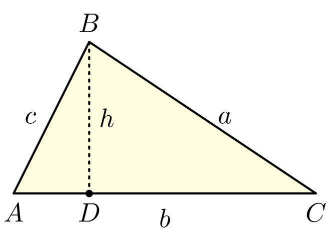
OR

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Note

Side a is always opposite angle A, side b is always opposite angle B and side c is always opposite angle C.

Proof of the Sine Rule



Consider $\triangle ABD$.

$$\sin A = \frac{h}{c}$$

$\therefore h = c \sin A$ **(1)**

Consider $\triangle BCD$.

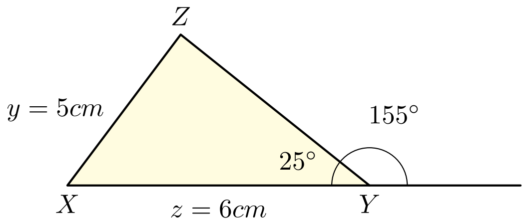
$$\sin C = \frac{h}{a}$$

$\therefore h = a \sin C$ **(2)**

(1) = **(2)** $\therefore c \sin A = a \sin C$

$$\implies \frac{c}{\sin C} = \frac{a}{\sin A} \quad \text{(Sine rule)} \quad \square$$

Two sides and Non-included Angle (Special Case)
Use the same rule to find the magnitude of $\angle ZXY$ given $\angle Y = 25°$, $y = 5cm$ and $z = 6cm$.



$$\angle X + \angle Z = 155°$$

$$\frac{\sin 25}{5} = \frac{\sin \angle Z}{6} \implies \angle Z = 30.47°$$
$$\therefore \angle ZXY = 155 - 30.47 = 124.53°$$

1.2.4 Cosine Rule

1.3 Non-Linear Relations

1.3.1 Parametric Curves

A parametric curve in the plane is given by a pair of equations:

$$x = f(t) \text{ and } y = g(t)$$

Where t is called the **parameter** of the curve.

Description When asked to describe any parametric curve, you need to consider the direction which a particle following that parametric curve would travel, as well as the point at which the motion starts/ends.

1.4 Circles

1.4.1 Equations

Circle Equations	
Cartesian	*Parametric*
$x^2 + y^2 = r^2$	$x = r\cos(t)$ $y = r\sin(t)$

1.4.2 Inequalities

1.5 Ellipses

1.5.1 Equations

1.5.2 Inequalities

1.6 Algorithms

1.6.1 Karatsuba's Multiplication Algorithm

To find the product of a pair of two-digit numbers $m = 10a + b$ and $n = 10c + d$.

Karatsuba's Multiplication Algorithm	
1	Calculate ac. Call the result F.
2	Calculate bd. Call the result G.
3	Calculate $(a + b)(c + d)$. Call the result H.
4	Calculate H-F-G. Call the result K.
5	Calculate 100F+10K+G. The result is mn.

Example
Use Karatsuba's multiplication algorithm to calculate 23×31.

$$\begin{gathered}
23 = 2 \times 10 + 3,\ 31 = 3 \times 10 + 1 \\
a = 2, b = 3, c = 3, d = 1 \\
ac = 2 \times 3 = 6 = F \\
bd = 3 \times 1 = 3 = G \\
(a + b)(c + d) = (2 + 3)(3 + 1) = 20 = H \\
H - F - G = 20 - 6 - 3 = 11 = K \\
mn = 100F + 10K + G = 100(6) + 10(11) + 3 = 713
\end{gathered}$$

1.6.2 Decimal → Binary

Binary uses only the digits 0 and 1 to represent numbers. The positions of digits correspond to different powers of 2.
Example

Algorithm for Decimal to Binary Conversion

1. Input n
2. Let q be the quotient when n is divided by 2.
3. Let r be the remainder when n is divided by 2.
4. Record r.
5. Let n have the value of q.
6. If $n > 0$, then repeat from Step 2.
7. Write the recorded values of r in reverse order.

Convert the decimal number 237 into binary form.

$$
\begin{gathered}
n = 237, q = 118, r = 1 \\
n = 118, q = 59, r = 0 \\
n = 59, q = 29, r = 1 \\
n = 29, q = 14, r = 1 \\
n = 14, q = 7, r = 0 \\
n = 7, q = 3, r = 1 \\
n = 3, q = 1, r = 1 \\
n = 1, q = 0, r = 1 \\
\therefore 237 \to 11101101
\end{gathered}
$$

1.6.3 The Euclidian Algorithm

Can be used to determine the Highest Common Factor (HCF) of a pair of integers.

Euclidean Division

If a and b are integers with $b > 0$, then there are unique integers q and r such that $a = qb + r$ where $0 \leq r < b$

Note

Here q is the *quotient* and r is the *remainder* when a is divided by b.

Theorem. *Let a and b be two integers with $b \neq 0$. If $a = qb + r$, where q and r are integers, then HCF(a, b)=HCF(b, r).*

Example

Find the highest common factor of 72 and 42.

$$\begin{gathered}
72 = 1 \times 42 + 30 \\
HCF(72, 42) = HCF(42, 30) \\
42 = 1 \times 30 + 12 \\
HCF(42, 30) = HCF(30, 12) \\
30 = 2 \times 12 + 6 \\
HCF(30, 12) = HCF(12, 6) = 6 \\
\therefore HCF(72, 42) = 6
\end{gathered}$$

1.6.4 Iteration and Selection

Assignment of Values to Variables
Variables are strings of one or more letters that act as placeholders for different values that can change. Assigning a value to a variable is denoted by a right-to-left arrow ($\leftarrow$). For example, $x \leftarrow 3$ means 'assign the value of 3 to variable x.'

Consider the following instructions:
Step 1: $x \leftarrow 3$
Step 2: $y \leftarrow 2x + 1$
Step 3: $x \leftarrow 4$
After following the above steps, we have $x = 4$ and $y = 7$.

Iteration
Iteration is the construction of a loop that allows the controlled repetition of algorithmic steps.

Example
An initial investment of $100 000 is invested at an interest rate of 5% p.a. compounded annually.
(a) Write an algorithm to find the value of the investment at the end of each year for the first five years.

1	$V \leftarrow 000$ and $n \leftarrow$.
2	Print n and Print V.
3	$V \leftarrow V \times 1.05$ and $n \leftarrow n + 1$.
4	Print n and Print V
5	Repeat from step 3 while $n < 5$.

(b) Demonstrate the algorithm with a table of values.

n	V
0	100 000
1	105 000
2	110 250
3	115 762.5
4	121 550.63
5	127 628.16

Selection
Decision-making constructs allow us to specify whether certain steps should be followed based on some condition. For example, we can use an instruction such as 'If... then...". This is called selection.

Example

$$\text{For } n \in \mathbb{N},\ \text{define } t_n = \begin{cases} 2n+4 & \text{if n is even} \\ n+3 & \text{if n is odd} \end{cases}$$

(a) Write an algorithm to generate the first N terms of this sequence.

1	$n \leftarrow$
2	If n is even, $t \leftarrow +4$. Otherwise, $t \leftarrow n+3$
3	Print n and Print t.
4	$n \leftarrow n+1$
5	Repeat steps 2 to 4 while $n < N$.

(b) Demonstrate the algorithm for N=6 with a table of values.

n	t
1	4
2	8
3	6
4	12
5	8
6	16

1.6.5 Pseudocode

If-Then Block

This construct provides a means of making decisions within an algorithm. Certain instructions are only followed if a given condition is satisfied.

The basic template for an If-Then block is as follows:

if *condition* **then**
  follow these instructions

Example

Using pseudocode, write an algorithm to find the maximum of two numbers a and b.

input a, b
if $a \geq b$ **then**
  print a
else
  print b

The For Loop

This construct provides a means of repeatedly executing the same set of instructions in a controlled way.

for condition **do**
  follow these instructions

<u>Example</u>

Consider the sequence: $1^2, 2^2, 3^2, 4^2, ..., n^2$.
Using pseudocode, write an algorithm to calculate:

(a) the sum of the terms in this sequence.

input n
sum$\leftarrow$ 0
for i from 1 to n **do**
  sum$\leftarrow$sum+i^2
  print sum

(b) the product of the terms in this sequence.

input n
product$\leftarrow$ 1
for i from 1 to n **do**
  product$\leftarrow$product$\times i^2$
  print product

While Loop

This construct provides another means of controlled, repeated execution of instructions. It does this by performing iterations indefinitely, as long as some condition remains true.

while condition **do**
  perform these instructions

<u>Example</u>

Consider the sequence defined by the rule:
$x_{n+1} = 5x_n + 4$, where $x_1 = 3$.
Write an algorithm that will determine the smallest value of n for which $x_n > 10000$.
Show a desk check to test the operation of the algorithm.

A *desk check* is a manual verification of the algorithm's function with a table of values.

$n \leftarrow 1$
$x \leftarrow 3$
while $x \leq 100$ **do**
  $n \leftarrow n + 1$
  $x \leftarrow 5x + 4$
print n and print x

n	x_n
1	3
2	19
3	99
4	499
5	2499
6	12499

Functions
A function takes one or more input values and returns an output value. Functions can be defined and then used in other algorithms.

Example
Define the function $f(x) = 3x + 2$.
define f(x):
  $y \leftarrow 3x + 2$
  return y

The Quotient and Remainder Functions

1.7 Complex Numbers

1.7.1 Definition

Imaginary Numbers
The imaginary number i has the property that $i^2 = -1$. By declaring $i = \sqrt{-1}$ we can find square roots of all negative numbers.

$$\begin{aligned}\sqrt{-4} &= \sqrt{4 \times (-1)} \\ &= \sqrt{4} \times \sqrt{-1} \\ &= 2i\end{aligned}$$

Complex Numbers
A complex number is an expression in the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

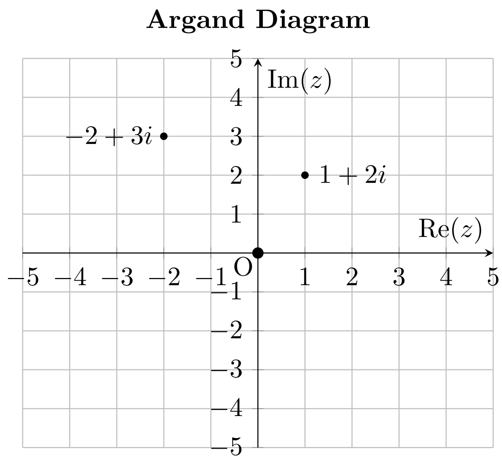
$$\begin{aligned}\text{Let } z = a + bi,\ a, b \in \mathbb{R} \\ \text{Real Part: } Re(z) = a \\ \text{Imaginary Part: } Im(z) = b\end{aligned}$$

The Imaginary Part of a complex number is the *coefficient* of i **only**. Do not include i when stating the imaginary part.

1.7.2 Equality of Complex Numbers

$$\begin{aligned}\text{Let } z_1 = a + bi \text{ and } z_2 = c + di,\ a, b, c, d \in \mathbb{R}. \\ \text{If } z_1 = z_2,\ \text{then } a = c, b = d\end{aligned}$$

1.7.3 Argand Diagram



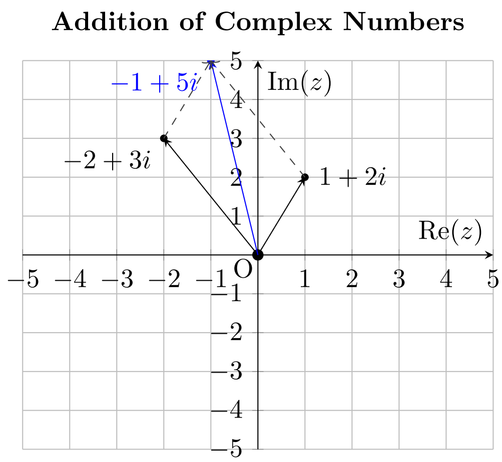
An Argand Diagram is a geometric representation of the set of complex numbers. In a vector sense, a complex number has two dimensions: the real and the imaginary parts. Therefore a plane is required to represent $\mathbb{C}$.

The horizontal axis represents $\mathrm{Re}(z)$ for $z \in \mathbb{C}$, and the vertical axis represents $\mathrm{Im}(z)$, for $z \in \mathbb{C}$.

A complex number $a + bi$ is at the point (a, b) on the Argand Diagram. A complex number in the form $a + bi$ is said to be in **cartesian form**.

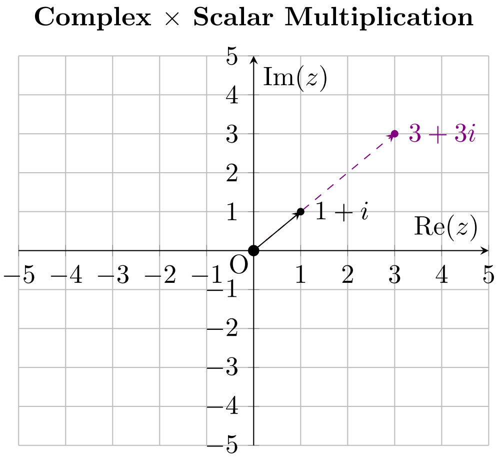
Geometric Representations of Basic Operations

Addition



Addition of complex numbers on an Argand Diagram acts like vector addition. To do this, you form a parallelogram with the two complex numbers and the origin. The fourth point is the sum of the two complex numbers. This is illustrated to the left.

Scalar Multiplication

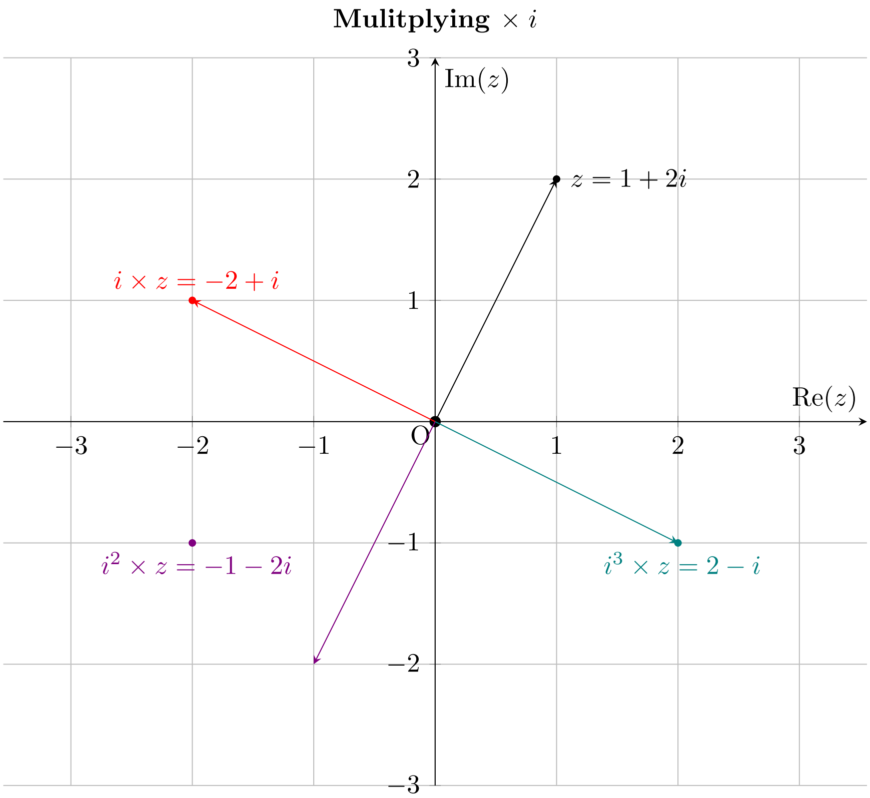


Multiplying a complex number by a scalar k extends the distance of the complex number from the origin on the Argand Diagram by a factor of k.

You can draw a straight line through z, kz, and the origin.

The Argand Diagram on the left shows the multiplication of $z = 1 + i$ by a factor of $k = 3$.

Multiplication by i



By multiplying by i, a complex number is rotated 90° counterclockwise through the Argand Diagram about the origin. **Distance from the origin is conserved.**

1.7.4 Addition, Multiplication of Complex Numbers

$$\text{Let } z_1 = a + bi,\ z_2 = c + di$$

Addition

$$z_1 + z_2 = (a + c) + (b + d)i$$

Subtraction

$$z_1 - z_2 = (a - c) + (b - d)i$$

Multiplication by a real constant

$$kz_1 = k(a + bi) = ka + kbi$$

Example
Evaluate:

(a) $(2 - 5i) + (-4 + 3i)$

$$\begin{aligned} &= (2 - 4) + (-5 + 3)i \\ &= -2 - 2i \\ &= -2(1 - i) \end{aligned}$$

(b) $5(5 - 3i) - (2 - i)$

$$\begin{aligned} &= (25 - 15i) - (2 - i) \\ &= (25 - 2) + (-15 + 1)i \\ &= 23 - 14i \end{aligned}$$

Multiplication of Complex Numbers

$$\begin{aligned} (a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (bc + ad)i \end{aligned}$$

Example

$$\begin{aligned} (5 + 2i)(3 - 4i) &= (5 \times 3 - 2 \times (-4)) + (2 \times 3 + 5 \times (-4))i \\ &= (15 + 8) + (6 - 20)i = 23 - 14i \end{aligned}$$

Powers of i

$i^{4n} = 1,\ i^{4n+1} = i,\ i^{4n+2} = -1,\ i^{4n+3} = -i$

The Sum of Two Squares

$$\begin{aligned} i^2 &= -1 \\ \therefore a^2 + b^2 &= a^2 - i^2b^2 \\ &= a^2 - (bi)^2 \\ &= (a + bi)(a - bi) \end{aligned}$$

1.7.5 Conjugates

If $z = a + bi$ then it's conjugate is $\overline{z} = a - bi$.

Properties of Conjugates
$z\overline{z} = \|z\|^2$
$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
$\overline{z_1 z_2} = \overline{z_1} \times \overline{z_2}$
$\overline{kz} = k\overline{z}$
$z + \overline{z} = 2Re(z)$

<u>Example</u>
Let $z_1 = 3 + 2i$ and $z_2 = 5 - 3i$

(a) Show that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

$$
\begin{aligned}
z_1 + z_2 &= (3 + 2i) + (5 - 3i) \\
&= 8 - i \\
&\underline{\text{LHS}} \\
\overline{z_1 + z_2} &= \overline{8 - i} \\
&= 8 + i \\
&\underline{\text{RHS}} \\
\overline{z_1} &= 3 - 2i \\
\overline{z_2} &= 5 + 3i \\
\overline{z_1} + \overline{z_2} &= 8 + i = \text{LHS} \quad \square
\end{aligned}
$$

(b) Show that $\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2}$

$$
\begin{aligned}
z_1 \times z_2 &= (3 + 2i) \times (5 - 3i) \\
&= (15 + 6) + (10 - 9)i \\
&= 21 + i \\
\therefore \underline{\text{LHS}} &= \overline{21 + i} = 21 - i \\
&\underline{\text{RHS}} \\
\overline{z_1} &= 3 - 2i \\
\overline{z_2} &= 5 + 3i \\
\therefore \underline{\text{RHS}} &= (15 + 6) + (-10 + 9)i \\
&= 21 - i = \text{RHS} \quad \square
\end{aligned}
$$

1.7.6 Division of Complex Numbers

$$\frac{3 + 2i}{i} \times \frac{i}{i} = \frac{(3 + 2i)i}{-1} = \frac{3i - 2}{-1} = 2 - 3i$$

In the above example, by multiplying the division of the two complex numbers by $\frac{i}{i}$, you make the denominator real, hence allowing us to find the quotient.

$$\frac{4 + 2i}{3 - 2i} \times \frac{3 + 2i}{3 + 2i} = \frac{12 + 8i + 6i + 4i^2}{9 + 4} = \frac{8 + 14i}{13} = \frac{8}{13} + \frac{14}{13}i$$

In the above example, we multiply both the numerator and denominator by the conjugate of the denominator. This makes the denominator a real number, allowing us to simplify the complex division.

<u>Example</u>

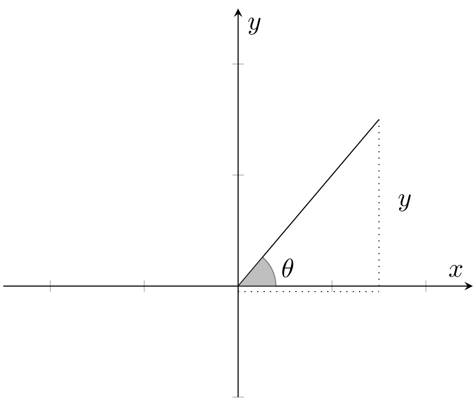
Solve for z: $(2-i)z = 42i$

$$\begin{aligned} z &= \frac{42i}{2-i} \\ &= \frac{42i}{2-i} \times \frac{2+i}{2+i} \\ &= \frac{84i + 42i^2}{4+1} = \frac{84i - 42}{5} \\ z &= -\frac{42}{5} + \frac{84}{5}i \quad \checkmark \end{aligned}$$

1.7.7 Polar Form

Polar Coordinates
Consider the point $P(x, y)$ on the cartesian plane. The point P always makes a distance r from the origin and an angle θ from the positive x-axis.

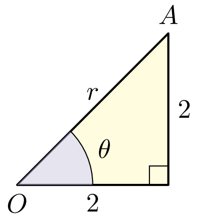
Cartesian Plane



$$\begin{aligned} \cos(\theta) &= \frac{x}{r} \implies x = r\cos(\theta) \\ \sin(\theta) &= \frac{y}{r} \implies y = r\sin(\theta) \\ \tan(\theta) &= \frac{y}{x} \\ r &= \sqrt{x^2 + y^2} \end{aligned}$$

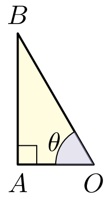
Example
Convert the following cartesian coordinates into polar form.

(a) A(2,2)



$$r = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$
$$\tan(\theta) = \frac{2}{2} = 1 \therefore \theta = \frac{\pi}{4}^c$$
$$\therefore A(2\sqrt{2}, \frac{\pi}{4}^c)$$

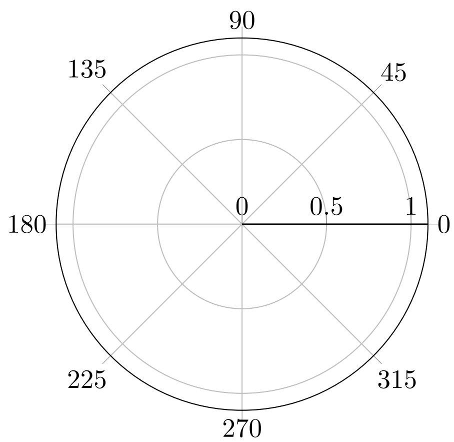
(b) B(-1, $\sqrt{3}$)



$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$
$$\tan(\theta) = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$
$$\therefore \mathit{Base\ Angle} = \frac{\pi}{3} \therefore \theta = \frac{2\pi}{3}$$
$$\therefore B(2, \frac{2\pi}{3}^c)$$

The Polar Plane

content



Polar Form of Complex Numbers

A complex number of form $z = a + bi$ is said to be in **cartesian form**. This can be converted to polar form as follows:

$$z = a + bi$$
On an Argand Diagram, $a = r\cos\theta$ and $b = r\sin\theta$.
$$\therefore z = r\cos\theta + r\sin\theta i$$
$$= r(\cos\theta + i\sin\theta)$$
$$= r \operatorname{cis}\theta \text{ (Polar Form)}$$

$r = |z| = \sqrt{a^2 + b^2}$

$\theta = \operatorname{Arg}(z) = \arctan\left(\frac{b}{a}\right)$

Principle Value of Argument
$\text{Arg}(z) \in (-\pi, \pi]$

Example
Express the following complex numbers in polar form.

(a) $z = 1 + \sqrt{3}i$

$$r = \sqrt{(1)^2 + (\sqrt{3})^2}$$
$$= 2 = |z|$$

$$\tan\theta = \frac{\sqrt{3}}{1}$$
$$\theta = \frac{\pi}{3}$$
$$\therefore z = 2\,\text{cis}\left(\frac{\pi}{3}\right)$$

(b) $z = 2 - 2i$

$$r = \sqrt{2^2 + (-2)^2}$$
$$= \sqrt{8} = 2\sqrt{2} = |z|$$

$$\tan\theta = \frac{-2}{2} = -1$$
$$\theta = \frac{-\pi}{4}$$
$$\therefore z = 2\sqrt{2}\,\text{cis}\left(\frac{-\pi}{4}\right)$$

Example
Express $z = 2cis\left(\frac{-2\pi}{3}\right)$ in cartesian form.

$$|z| = 2 = \sqrt{a^2 + b^2} \qquad z = a + bi$$
$$z = 2\left(\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right)$$
$$z = 2(-\frac{1}{2} + i\frac{-\sqrt{3}}{2})$$
$$z = -1 - \sqrt{3}i$$

1.7.8 Multiplication and Division in Polar Form

If $z_1 = r_1\,\text{cis}\,\theta_1$ and $z_2 = r_2\,\text{cis}\,\theta_2$

Then $z_1 \times z_2 = r_1 \times r_2 \times \text{cis}(\theta_1 + \theta_2)$

and $\frac{z_1}{z_2} = \frac{r_1}{r_2} \times \text{cis}(\theta_1 - \theta_2)$.

Example
If $z_1 = 3\,\text{cis}\left(\frac{\pi}{2}\right)$, $z_2 = 2\,\text{cis}\left(\frac{5\pi}{6}\right)$, find z_1z_2 in polar form.

$$z_1z_2 = (3 \times 2) \times \text{cis}\left(\frac{\pi}{2} + \frac{5\pi}{6}\right) = 6\,\text{cis}\left(\frac{4\pi}{3}\right) = 6\,\text{cis}\left(\frac{-2\pi}{3}\right)$$

Example

If $z_1 = -\sqrt{3} + i$ and $z_2 = 2\sqrt{3} + 2i$, find $\frac{z_1}{z_2}$ in polar form.

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{-\sqrt{3}+i}{2\sqrt{3}+2i} \times \frac{2\sqrt{3}-2i}{2\sqrt{3}-2i} \\
&= \frac{-2\times 3 + 2\sqrt{3}i + 2\sqrt{3}i + 2}{(2\sqrt{3})^2 - (2i)^2} \\
&= \frac{-4+4\sqrt{3}i}{12+4} = \frac{-1+\sqrt{3}i}{4}
\end{aligned}$$

1.7.9 de Moivre's (de Movies) Theorem

$$\begin{aligned}
\text{Let } z &= r\operatorname{cis}\theta \\
\therefore z^2 &= r\operatorname{cis}\theta \times r\operatorname{cis}\theta \\
&= r^2\operatorname{cis}(2\theta) \\
\therefore z^3 &= r^3\operatorname{cis}(3\theta) \\
\therefore z^5 &= r^5\operatorname{cis}(5\theta) \\
\dots \\
z^n &= r^n\operatorname{cis}(n\theta) \text{ (de Moivre's Theorem)}
\end{aligned}$$

Example

Evaluate $(2\sqrt{3} - 2i)^7$. Give your answer in **cartesian form**.

$$\begin{aligned}
r &= \sqrt{(2\sqrt{3})^2 + (2)^2} = \sqrt{4} \\
\cos\theta &= \frac{2\sqrt{3}}{4} \;\therefore \theta = \frac{\pi}{6} \\
\sin\theta &= \frac{-1}{2} \;\therefore \theta = -\frac{\pi}{6} \\
\therefore \theta &= -\frac{\pi}{6} \\
z &= 4\operatorname{cis}\left(-\frac{\pi}{6}\right) \\
z^7 &= 4^7\operatorname{cis}\left(-\frac{7\pi}{6}\right) \\
&= 16384\operatorname{cis}\left(\frac{5\pi}{6}\right) \\
&= 16384\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right] = 16384\left[-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right] \\
&= -8192\sqrt{3} + 8192i
\end{aligned}$$

1.7.10 Factorisation of Polynomials

Complex Quadratics

Example

Solve for z.

(a)$2z^2 + 5z + 4 = 0$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{-7}}{4}$$
$$= \frac{-5 \pm \sqrt{7}i}{4}$$

Alternatively...

$2z^2 + 5x + 4 = 0$

$$2\left[z^2 + \frac{5}{2}z + 2\right] = 0$$
$$2\left[\left(z + \frac{5}{4}\right)^2 - \frac{25}{16} + 2\right] = 0$$
$$2\left[\left(z + \frac{5}{4}\right)^2 + \frac{7}{16}\right] = 0$$
$$2\left(z + \frac{5}{4}\right)^2 + \frac{14}{16} = 0$$
$$\left(z + \frac{5}{4}\right)^2 + \frac{7}{16} = 0$$
$$\left(z + \frac{5}{4}\right)^2 = -\frac{7}{16}$$
$$z + \frac{5}{4} = \pm\frac{i\sqrt{7}}{4}$$
$$z = -\frac{5}{4} \pm \frac{i\sqrt{7}}{4}$$
$$z = \frac{-5 \pm \sqrt{7}i}{4}$$

(b)$z^2 - 6z + 14 = 0$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{-20}}{2}$$
$$= \frac{6 \pm i\sqrt{20}}{2}$$
$$= \frac{6 \pm 2i\sqrt{5}}{2}$$
$$= 3 \pm i\sqrt{5}$$

Complex Polynomials

Factor Theorem. *Let $a \in \mathbb{C}$, then $z-\alpha$ is a factor of polynomial $P(z)$* ***if and only if*** *$P(z)=0$.*

Factor Theorem Example
$P(z) = z^2 + 4 = (z+2i)(z-2i)$
So $z-2i$ is a factor $\therefore P(2i) = (2i)^2 + 4 = -4+4 = 0$.
$z+2i$ is a factor $\therefore P(-2i) = (-2i)^2 + 4 = -4+4 = 0$.

Fundamental theorem. *For $n \geq 1$, every polynomial of degree n can be expressed as a product of n linear factors over $\mathbb{C}$. Therefore, every polynomial of degree n has n solutions.*

Example
Show that $z=1$ is a solution of $z^3+z^2+3z-5=0$ and then find the other solutions.

$$\begin{array}{r|l} & z^2 + 2z + 5 \\ \hline z-1 & z^3 + z^2 + 3z - 5 \\ & -z^3 + z^2 \\ \hline & 2z^2 + 3z \\ & -2z^2 + 2z \\ \hline & 5z - 5 \\ & -5z + 5 \\ \hline & 0 \end{array}$$

$$z^3 + z^2 + 3z - 5 = (z-1)(z^2+2z+5)$$

Consider z^2+2z+5

$$z = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$\therefore (z-1)(z+1+2i)(z+1-2i)$$

The other two solutions are:

$z = -1+2i$ and
$z = -1-2i$

1.7.11 Conjugate Root Theorem

Conjugate Root Theoreom. *Let $P(z)$ be a polynomial with* ***real*** *coefficients. If $a+bi$ is a solution then the conjugate $a-bi$ is also a solution.*

Example
Do the following polynomials obey the conjugate root theorem?
(a) $P(z) = z^2 + z + 1$
Yes, as all coefficients of z are real numbers.

(b) $P(z) = z^3 - i(z+5i)$
No, as the coefficient of z is non-real, and thus it does not follow the conjugate root theorem.

Example

If $2 - i$ is a root of $z^2 + az + b$, find a & b where $a, b \in \mathbb{R}$.

$$\begin{aligned} &P(z) \text{ has real coefficients} \\ z = 2 - i \text{ is a soln.} &\therefore z - 2 + i \text{ is a factor of } P(z) \\ z = 1 + i \text{ is a soln.} &\therefore z - 2 - i \text{ is a factor of} P(z) \end{aligned}$$

$$\begin{aligned} P(z) &= (z - 2 + i)(z - 2 - i) \\ &= z^2 - 2z - iz - 2z + 4 + 4i + iz - 2i - i^2 \\ &= z^2 - 4z + 4 + 1 \\ &= z^2 - 4z + 5 \end{aligned}$$

$$\begin{aligned} &z^2 + az + b = z^2 - 4z + 5 \\ &\text{Equating coefficients of z:} \\ &a = -4 \text{ and } b = 5 \end{aligned}$$

1.7.12 Modulus of Complex Numbers

Definition of Modulus

For $z = a + bi$, the modulus of z is denoted by $|z|$ and is defined by:

$$|z| = \sqrt{a^2 + b^2}$$

This can be thought of as the distance of the complex number z from the origin of the Argand Diagram.

Properties of Modulus
$\lvert z_1 \times z_2 \rvert = \lvert z_1 \rvert \times \lvert z_2 \rvert$
$\lvert \frac{z_1}{z_2} \rvert = \frac{\lvert z_1 \rvert}{\lvert z_2}$
$\lvert z_1 + z_2 \rvert \leq \lvert z_1 \rvert + \lvert z_2 \rvert$

1.7.13 Subsets of the Complex Plane

Distance between Complex Numbers

Let $z_1 = a_1 + b_1 i$, and $z_2 = a_2 + b_2 i$.
The distance between z_1 and z_2 is given by:

$$|z_2 - z_1| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$$

Circles in the Complex Plane

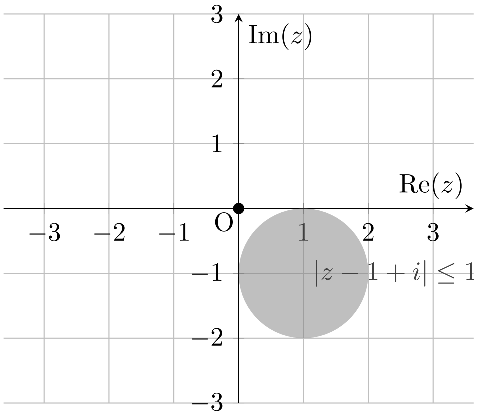
Let $z = x + yi$ then $|z| = r$.

$$\begin{aligned} |x+yi| &= r \\ \sqrt{x^2+y^2} &= r \\ x^2+y^2 &= r^2 \end{aligned}$$

So, $|z| = r$ represents the sets of complex numbers that lie on the circle with centre $(0,0)$ and radius r.

The general rule for circles in the complex plane:
Let w be a fixed complex number and let $r > 0$. The equation $|z - w| = r$ defines the circle of radius r around centre w.

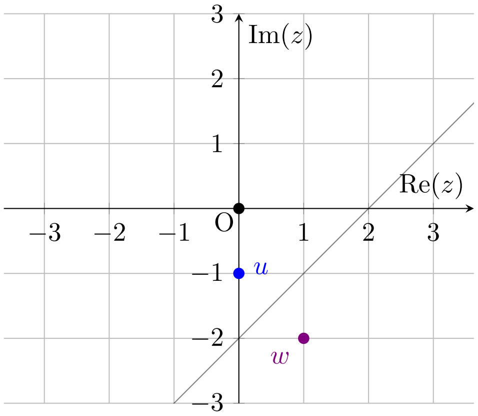
Circle in Complex Plane



Lines in the Complex Plane

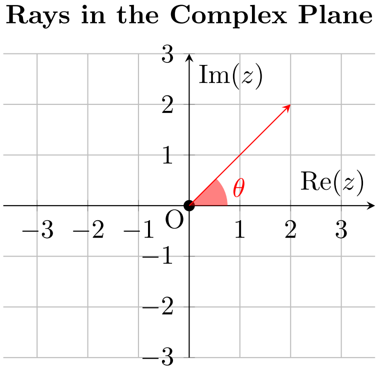
Let $u, w \in \mathbb{C}$.
Therefore, $|z-u| = |z-w|$ defines the subset of $\mathbb{C}$ that are equidistant from u and w. This is a straight line.

Line in the complex plane



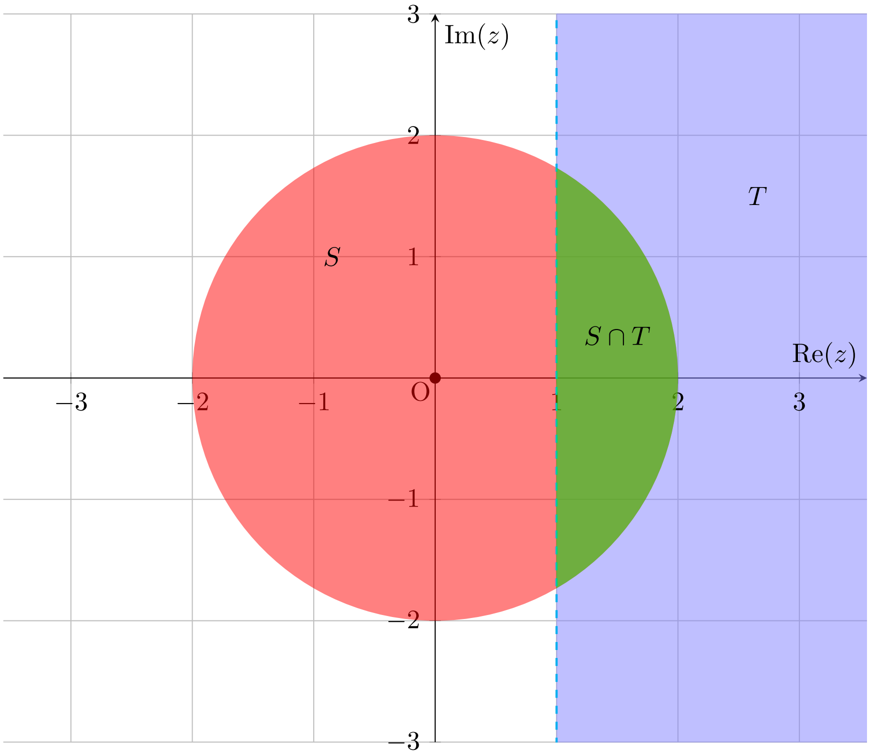
Rays in the Complex Plane

Let θ be a fixed angle. The equation $\text{Arg}(z) = \theta$ defines a ray extending from the origin at an angle θ measured counterclockwise from the horizontal axis. $-\pi < \theta \leq \pi$.



Example

Define sets S and T of complex numbers by:
$S = \{z : |z| \leq 2\}$ and $T = z : Re(z) > 1$
Sketch $S \cap T$



1.8 Vectors

1.8.1 Introduction to Vectors

Directed Line Segments

A directed line segment from point A to point B is denoted by $\overrightarrow{AB}$ or $\underset{\sim}{a}$.

Vectors in 2D

Let $\underset{\sim}{v} = x\underset{\sim}{i} + y\underset{\sim}{j}$ where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are unit vectors in the x and y directions respectively.

Magnitude or length $= |\underset{\sim}{v}| = \sqrt{x^2 + y^2}$

Unit vector $= \hat{\underset{\sim}{v}} = \frac{1}{|\underset{\sim}{v}|} \cdot \underset{\sim}{v}$

Position Vectors

Point $A(2, 1)$ has the position vector $\overrightarrow{OA} = 2\underset{\sim}{i} + \underset{\sim}{j}$

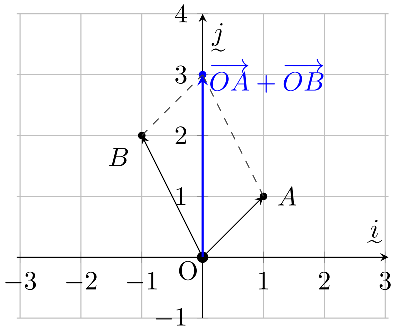
Point $B(3, 3)$ has the position vector $\overrightarrow{OB} = 3\underset{\sim}{i} + 3\underset{\sim}{j}$

1.8.2 Vector Operations

Vector Addition

$$\begin{aligned} \overrightarrow{OA} &= \underset{\sim}{i} + \underset{\sim}{j} \text{ and } \overrightarrow{OB} = -\underset{\sim}{i} + 2\underset{\sim}{j} \\ \therefore \overrightarrow{OA} + \overrightarrow{OB} &= (\underset{\sim}{i} + \underset{\sim}{j}) + (-\underset{\sim}{i} + 2\underset{\sim}{j}) \\ &= 3\underset{\sim}{j} \end{aligned}$$

Vector Addition

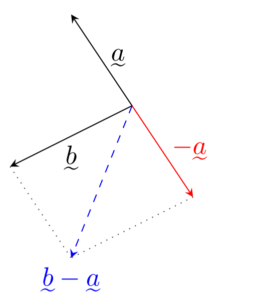


Vector Subtraction

$\overleftarrow{OB} - \overleftarrow{OA} = \overleftarrow{OB} + (-\overleftarrow{OA})$

To perform vector subtraction, we simply flip the vector that we are subtracting 180° and perform vector addition on these two new vectors.

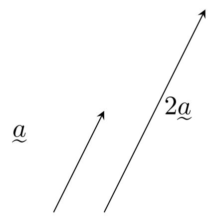
Vector Subtraction



Scalar Multiplication

Multiplying a vector by a real number changes the length of the vector (magnitude).

Scalar multiplication of Vectors



1.8.3 Parallel, Position, and Zero Vectors

Zero Vector
The zero vector ($\underset{\sim}{0}$) has no magnitude and no direction.

Parallel Vectors
Parallel vectors have the same or *exact* opposite direction.

> **Note**
>
> Two non-zero vectors are parallel if $\underset{\sim}{u} = k\underset{\sim}{v}$ where $k \in \mathbb{R} \setminus \{0\}$.

Position Vectors
For any point $P(x, y)$ there is a position vector from the origin $\overrightarrow{OP} = \begin{bmatrix} x \\ y \end{bmatrix}$

1.8.4 Rectangular Components

Standard Unit Vectors

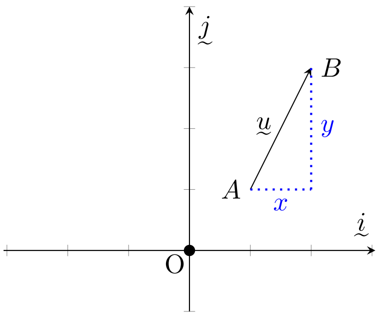
> **Note**
>
> A unit vector is a vector of length 1 unit in the same direction as a given vector.
>
> The unit vector of $\underset{\sim}{a}$ is denoted by $\hat{\underset{\sim}{a}}$.
>
> $|\underset{\sim}{a}| \times \hat{\underset{\sim}{a}} = \underset{\sim}{a} \quad \therefore \hat{\underset{\sim}{a}} = \frac{1}{|\underset{\sim}{a}|} \times \underset{\sim}{a}$

$\underset{\sim}{i}$ is the unit vector in the positive x direction and $\underset{\sim}{j}$ is the unit vector in the positive y direction.

Vector Components

$$\underset{\sim}{u} = \overrightarrow{AB} = \begin{bmatrix} x \\ y \end{bmatrix} = x\underset{\sim}{i} + y\underset{\sim}{j}$$

$$|\underset{\sim}{u}| = \sqrt{x^2 + y^2}$$



1.8.5 Linear Combination of Non-Parallel Vectors

If two non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are not parallel and $m\underset{\sim}{a} + n\underset{\sim}{b} = p\underset{\sim}{a} + q\underset{\sim}{b}$ then $m = p$ and $n = q$.

1.8.6 Linear Dependence

A set of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are linearly dependent **if and only if** there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\underset{\sim}{a} = \lambda\underset{\sim}{b}$.

$$\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} \qquad \underset{\sim}{b} = 4\underset{\sim}{i} - 6\underset{\sim}{j}$$

$$\underset{\sim}{b} = 2\underset{\sim}{a} \qquad \therefore \text{ linear dependence}$$

If no $\lambda \in \mathbb{R} \setminus \{0\}$ exists such that $\underset{\sim}{a} = \lambda\underset{\sim}{b}$, the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are said to be linearly independent.

Linear Dependence of 3 Vectors

Let $\underset{\sim}{a}, \underset{\sim}{b}$ be non-zero vectors that are not parallel. $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ are linearly dependent **if and only if** there exists $m, n \in \mathbb{R}$ such that $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$

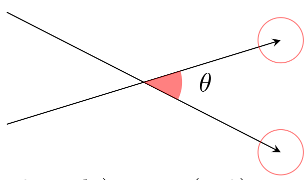
1.8.7 Scalar Product

If $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j}$, then $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2$

Geometric Description of Dot Product

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}| \times |\underset{\sim}{b}| \times \cos\theta$$

θ is defined as the angle where both 'hats' of the vectors are.



When $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, the vectors are perpendicular (incident at $90°$ angle) as $cos(90°) = 0$

Simplifying Expressions of Dot Products

Example

Expand and simplify:

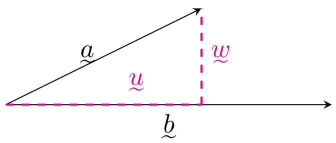
(a) $\underset{\sim}{a} \cdot (\underset{\sim}{a} + \underset{\sim}{b}) - \underset{\sim}{a} \cdot \underset{\sim}{b}$

$$= \underset{\sim}{a}^2 + \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{a}^2 = |\underset{\sim}{a}|^2$$

(b) $(\underset{\sim}{a} \cdot \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b})$

$$= \underset{\sim}{a}^2 - \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{a} - \underset{\sim}{b}^2 = \underset{\sim}{a}^2 - \underset{\sim}{b}^2 = |\underset{\sim}{a}|^2 - |\underset{\sim}{b}|^2$$

1.8.8 Scalar and Vector Projections/Resolute



It is often useful to decompose vector $\underset{\sim}{a}$ into the sum of two vectors, where one is perpendicular to $\underset{\sim}{b}$ and the other parallel to $\underset{\sim}{b}$.

In the example left, $\underset{\sim}{u}$ is parallel to $\underset{\sim}{b}$ and $\underset{\sim}{w}$ is perpendicular to $\underset{\sim}{b}$.

$$\underset{\sim}{a} = \underset{\sim}{u} + \underset{\sim}{w} \text{ where } \underset{\sim}{w} \perp \underset{\sim}{b}, \quad \underset{\sim}{u} \parallel \underset{\sim}{b}$$

$$\begin{aligned}
\underset{\sim}{u} &= k\underset{\sim}{b} \text{ as } \underset{\sim}{u} \parallel \underset{\sim}{b} \therefore k \in \mathbb{R} \\
\underset{\sim}{w} &= \underset{\sim}{a} - \underset{\sim}{u} \\
&= \underset{\sim}{a} - k\underset{\sim}{b}
\end{aligned}$$

$$\begin{aligned}
(\underset{\sim}{a} - k\underset{\sim}{b}) \cdot \underset{\sim}{b} &= 0 \text{ as } \underset{\sim}{w} \perp \underset{\sim}{b} \\
\underset{\sim}{a}\underset{\sim}{b} - k\underset{\sim}{b}^2 &= 0 \\
\underset{\sim}{a} \cdot \underset{\sim}{b} &= k\underset{\sim}{b}^2 \\
k &= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2}
\end{aligned}$$

$$\begin{aligned}
\therefore \underset{\sim}{u} &= k\underset{\sim}{b} \\
&= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2} \cdot \underset{\sim}{b}
\end{aligned}$$

Vector Resolute (Vector Projection)

The vector resolute of $\underset{\sim}{a}$ in direction $\underset{\sim}{b}$ is given by:

$$\underset{\sim}{u} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2} \cdot \underset{\sim}{b} \quad \text{OR} \quad \underset{\sim}{u} = \underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} \cdot \underset{\sim}{b}$$

Scalar Resolute
The scalar resolute does not include the direction of the resolution. It is given by:

$$\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}$$

<u>Example</u>
Let $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j}$.
(a) Find the scalar resolute of $\underset{\sim}{a}$ in direction $\underset{\sim}{b}$

$$\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|} = \frac{1(1) + 3(-1)}{\sqrt{1^2 + (-1)^2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$$

(b) Find the vector resolute of $\underset{\sim}{a}$ in direction $\underset{\sim}{b}$

$$\begin{aligned}
\underset{\sim}{u} &= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2} \cdot \underset{\sim}{b} \\
&= \frac{1(1) + 3(-1)}{(\sqrt{1^2 + (-1)^2})^2} \cdot \underset{\sim}{b} \\
&= \frac{-2}{2} \cdot \underset{\sim}{b} = -1(\underset{\sim}{i} - \underset{\sim}{j}) \\
\underset{\sim}{u} &= -\underset{\sim}{i} + \underset{\sim}{j}
\end{aligned}$$

$$\begin{aligned}
\underset{\sim}{a} &= \underset{\sim}{u} + \underset{\sim}{w} \\
\therefore \underset{\sim}{w} &= \underset{\sim}{a} - \underset{\sim}{u} \\
&= \underset{\sim}{i} + 3\underset{\sim}{j} - (-\underset{\sim}{i} + \underset{\sim}{j}) \\
&= 2\underset{\sim}{i} + 2\underset{\sim}{j}
\end{aligned}$$

(c) Find the vector resolute of $\underset{\sim}{b}$ in direction $\underset{\sim}{a}$

$$\begin{aligned}
\underset{\sim}{u} &= \frac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{|\underset{\sim}{a}|^2} \cdot \underset{\sim}{a} \\
&= \frac{3(-1) + 1(1)}{(\sqrt{10})^2} \cdot \underset{\sim}{a} \\
&= -\frac{1}{5}\underset{\sim}{a} \\
&= -\frac{1}{5}\underset{\sim}{i} - \frac{3}{5}\underset{\sim}{j}
\end{aligned}$$

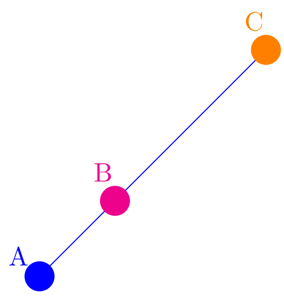
$$\begin{aligned}
\underset{\sim}{b} &= \underset{\sim}{u} + \underset{\sim}{w} \\
\therefore \underset{\sim}{w} &= \underset{\sim}{b} - \underset{\sim}{u} \\
&= \underset{\sim}{i} - \underset{\sim}{j} - (-\frac{1}{5}\underset{\sim}{i} - \frac{3}{5}\underset{\sim}{j}) \\
\underset{\sim}{w} &= \frac{6}{5}\underset{\sim}{i} - \frac{2}{5}\underset{\sim}{j}
\end{aligned}$$

1.8.9 Co-linearity

3 points are co-linear if $\overrightarrow{AB} = k\overrightarrow{AC}$ or $\overleftarrow{AB} = k\overleftarrow{BC}$.

Note

In order to be co-linear, the two vectors being compared must have a common point.

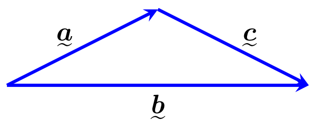


1.8.10 Vector Proofs

1. Draw a diagram
2. Write down what you know
3. Write down what to show
4. LHS and RHS of show and make them equal

Example

Using your knowledge of vectors, prove the cosine rule.



Know: $\underset{\sim}{c} = \underset{\sim}{b} - \underset{\sim}{a}$
Show: $|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$$\begin{aligned}
\text{LHS} &= |\underset{\sim}{c}|^2 \\
&= \underset{\sim}{c} \cdot \underset{\sim}{c} \\
&= (\underset{\sim}{b} - \underset{\sim}{a}) \cdot (\underset{\sim}{b} - \underset{\sim}{a}) \quad \text{as } \underset{\sim}{c} = \underset{\sim}{b} - \underset{\sim}{a} \\
&= \underset{\sim}{b} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{a} \\
&= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2\underset{\sim}{a} \cdot \underset{\sim}{b} \\
&= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta \\
\text{LHS} &= \text{RHS} \quad \square
\end{aligned}$$

1.8.11 Vector Functions and Vector Kinematics

Vector Function

Consider the vector function $\underset{\sim}{r}(t) = t\underset{\sim}{i} + (t+2)\underset{\sim}{j}$

$\underset{\sim}{r}(t)$ represents a family of vectors defined by different values of t.

Finding Cartesian Equation

$$\begin{aligned}
\underset{\sim}{r}(t) &= x\underset{\sim}{i} + y\underset{\sim}{j} \\
\underset{\sim}{r}(t) &= t\underset{\sim}{i} + (t+2)\underset{\sim}{j} \\
x = t \text{ and } y &= t + 2 \ \therefore\ y = x + 2
\end{aligned}$$

Example

The position vector $\underset{\sim}{r}(t)$ of a particle moving relative to the origin at time t seconds is given by:

$$\underset{\sim}{r}(t) = 2\cos(t)\underset{\sim}{i} - 3\sin(t)\underset{\sim}{j}$$

Find the cartesian equation and state the domain and range.

$$x = 2\cos(t) \implies \cos(t) = \frac{x}{2}$$
$$y = -3\sin(t) \implies \sin(t) = -\frac{y}{3}$$

$$\sin^2(t) + \cos^2(t) = 1$$
$$\left(\frac{x}{2}\right)^2 + \left(-\frac{y}{3}\right)^2 = 1$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \text{ (Ellipse)}$$

Domain

$x = 2\cos(t) \to 2\cos(t) \in [-2, 2]$

$\therefore x_{cartesian} \in [-2, 2]$

Range

$y = -3\sin(t) \to -3\sin(t) \in [-3, 3]$

$\therefore y_{cartesian} \in [-3, 3]$

Position, Velocity, Acceleration as Vector Functions

We can describe position, velocity, and acceleration as vector functions. The general forms of these functions are written below.

Position Vector	$\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$
Velocity Vector	$\underset{\sim}{v}(t) = \dot{x}(t)\underset{\sim}{i} + \dot{y}(t)\underset{\sim}{j}$
Acceleration Vector	$a(t) = \dot{\underset{\sim}{v}}(t) = \ddot{\underset{\sim}{r}} = \ddot{x}(t)\underset{\sim}{i} + \ddot{y}(t)\underset{\sim}{j}$
Distance from O	$\lvert\underset{\sim}{r}(t)\rvert = \sqrt{(x(t))^2 + (y(t))^2}$
Speed	$\lvert\underset{\sim}{v}(t)\rvert = \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2}$
Magnitude of Acceleration	$\lvert\underset{\sim}{a}(t)\rvert = \sqrt{(\ddot{x}(t))^2 + (\ddot{y}(t))^2}$

Note

$\dot{f}(t)$ denotes the first derivative of function f.

$\ddot{f}(t)$ denotes the second derivative of function f.