Mathematical Methods 3/4 Bound Reference

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Contents

1 AOS1 - Functions, Relations, and Graphs

1.1 Distinguishing between a Function and Relation

1.1.1 Relations

A relation is a set of ordered pairs $(x_1, y_1), (x_2, y_2)$ etc.

1.2 Key Features and Properties of a Function or Relation

1.2.1 Set Notation

Set notation is used to state/specify the domain and range of functions and relations.

> $A \subseteq B$; A is a subset of B, i.e. all values in A are contained in B
>
> $B \setminus A$; all values that are in set B but not set A
>
> $A \cup B$; all the values in A **or** B
>
> $A \cap B$; al the values in A **and** B

Common Sets
$\mathbb{R}$ = real numbers = all numbers over the interval $(-\infty, \infty)$ both rational and irrational.
$\mathbb{Q}$ = rational numbers = all real numbers that can be represented as a fraction.
$\mathbb{Z}$ = integers = all whole numbers
$\mathbb{N}$ = natural numbers = positive integers
$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

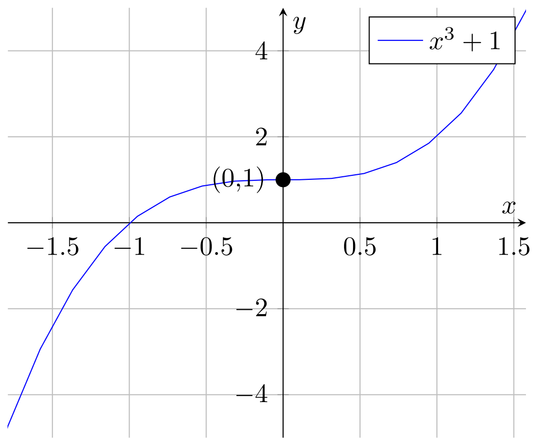
1.2.2 Interval Notation

$$x \in \mathbb{R}^+ = [0, \infty) = \{x : 0 \leq x < \infty\}$$

[means an endpoint is included in the interval.
(means an endpoint is not included in the inteval.

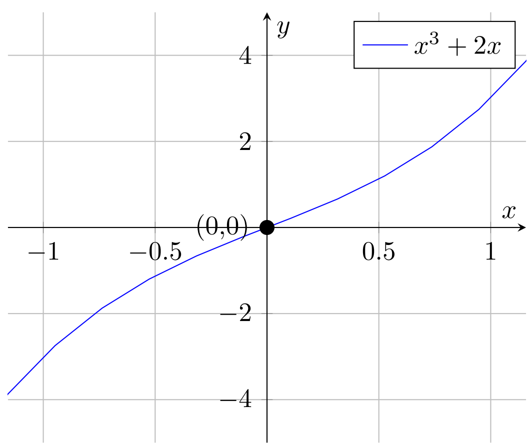
1.2.3 Stationary Points of Inflection

Stationary points of inflection are points of inflection at which the tangent line at that point has a gradient of 0.



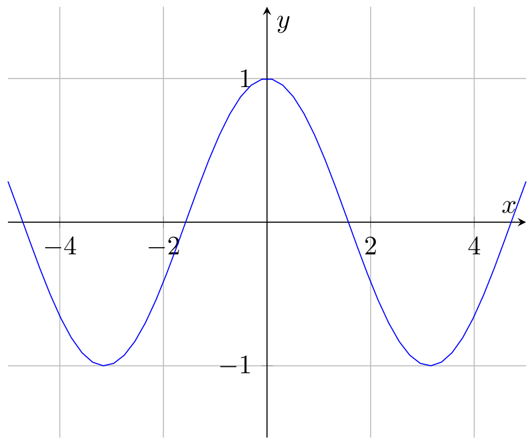
1.2.4 Non-stationary Points of Inflection

Non-stationary points of inflection are points of inflection at which the tangent line at that point has a non-zero gradient.



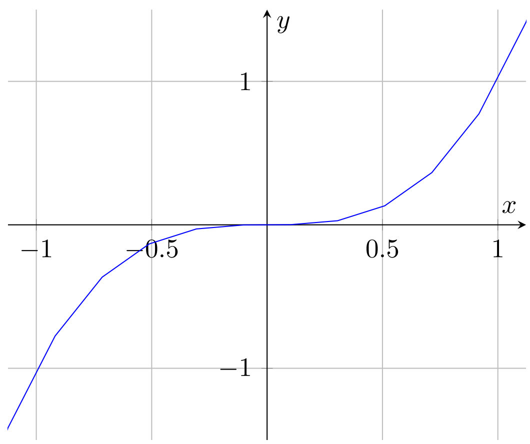
1.2.5 Symmetry of Functions - Even Functions

A function is called 'even' if it is symmetrical around the y-axis. $f(x) = f(-x)$ for all values of x.



1.2.6 Symmetry of Functions - Odd Functions

A function is called 'odd' if it is symmetrical about the origin. $f(x) = -f(-x)$ for all values of x.



1.2.7 Symmetry of Polynomials

Even function	$f(x) = ax^4 + cx^2 + e$ about $x = 0$. $(0, e)$ will be a turning point of the quartic.
Odd function	$f(x) = bx^3 + dx$ about $(0, 0)$. $(0, 0)$ is the point of inflection of the cubic.
Reflective symmetry	$f(x) = a(x-h)^4 + c(x-h)^2 + e$ about $x = h$. (h, e) will be a turning point of the quartic.
Rotational symmetry	$f(x) = b(x-h)^3 + d(x-h) + e$ about (h, e). (h, e) is the point of inflection of the cubic.

1.2.8 Features of Sine and Cosine Functions

$$y = a\sin(nx) + c \text{ and } y = a\cos(nx) + c$$

Amplitude
The amplitude is the maximum displacement of a sine or cosine function from the mean value. It is the value of a, regardless of its sign. That is, $y = 2\sin(x)$ and $y = -2\sin(x)$ both have an amplitude of 2.

Period The domain of a single repetition of the function. For sine and cosine, this is given by $\frac{2\pi}{n}$. The value of n also represents the number of cycles that can be fit into the original period of the function.

Mean Value
The centre of a circular function. A sine or cosine function will oscillate about this value, and it is given by the value of c.

Maximums and Minimums
Sine and cosine have maximum values of +1 and minimum values of -1. These will be changed under transformations. Maximum = mean position + amplitude and Minimum = mean position - amplitude.

1.3 Types of Function

1.3.1 Natural Power Functions

$$f(x) = a[b(x-h)]^n - k$$

Positive Even n; Turning-point Form
For even powers of x, such as $y = x^2$ and $y = x^4$, the values of y are the same for positive and negative values of x. That is $x^2 = (-x)^2$. Therefore, reflectional symmetry occurs about $x = 0$.
Quadratic

$$f(x) = x^2 \text{ or } f(x) = a(x-h)^2 + k \quad \textbf{Turning Point at } (h, k)$$

Quartic

$$f(x) = x^4 \text{ or } f(x) = a(x-h)^4 + k \quad \textbf{Turning Point at } (h, k)$$

Positive Odd n; Point of Inflection Form
For odd powers of x, such as $y = x$ and $y = x^3$, the values of y are opposite for positive and negative values of x. That is, $-x^3 = (-x)^3$. Therefore, there is rotational (180deg turn) symmetry about $(0, 0)$.
Linear

$$f(x) = x \ \textbf{ or } \ f(x) = a(x-h) + k \quad \textbf{Point on line at } (h, k)$$

Cubic

$$f(x) = x^3 \ \textbf{ or } \ f(x) = a(x-h)^3 + k \quad \textbf{Point of Inflection at } (h, k)$$

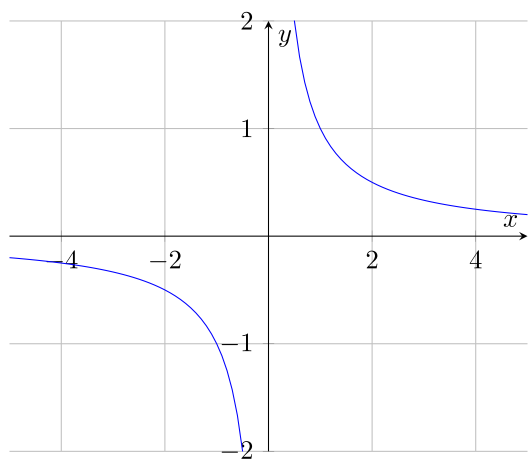
1.3.2 Negative Power Functions

$$f(x) = a[b(x-h)]^{-n} + k = \frac{a}{[b(x-h)]^n} + k$$

Negative power functions have **asymptotes**. These are lines which the graph will approach but never reach.
The vertical asymptote of a negative power function is at $x = h$. Therefore, the **maximal domain** of a negative power function is $\mathbb{R} \setminus \{h\}$

The **range** depends on if its an **even** or **odd** negative power function. Negative odd power functions have a range of $\mathbb{R} \setminus \{k\}$, whereas negative even power functions have a range of (k, ∞) for $a > 0$ or $(-\infty, k)$ for $a < 0$.

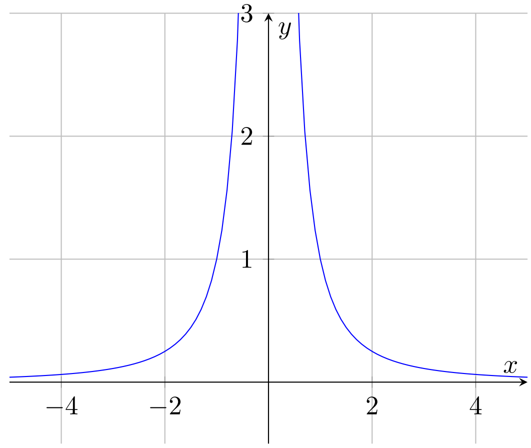
Hyperbola (Odd negative power functions)
$f(x) = x^{-1} = \frac{1}{x}$ forms the shape of a hyperbola.



$$f(x) = \frac{a}{x-h} + k \text{ shows the transformation of a hyperbola.}$$

The equation of a hyperbola can also be written as $y = \frac{ax+b}{cx+d}$. In these cases, the numerator should be divided by the denominator to be written as $y = \frac{A}{x+B} + C$ to allow easier sketching.

Truncus (Even negative power functions)
$f(x) = x^{-2} = \frac{1}{x^2}$ forms the shape of a truncus.



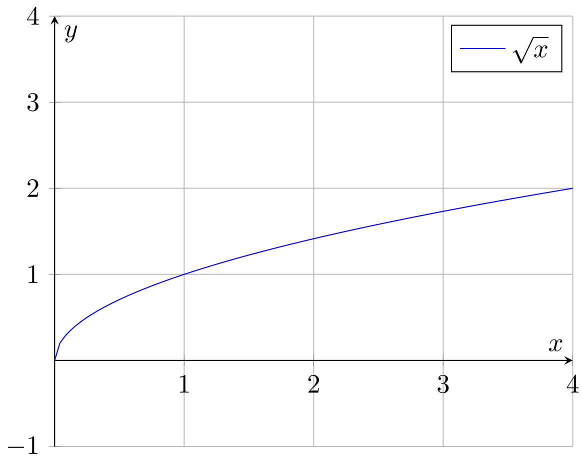
$$f(x) = \frac{a}{(x-h)^2} + k \text{ shows the transformation of a truncus.}$$

1.3.3 Root Funtions

$$f(x) = a[b(x-h)]^{\frac{1}{n}} + k = a\sqrt[n]{b(x-h)} + k$$

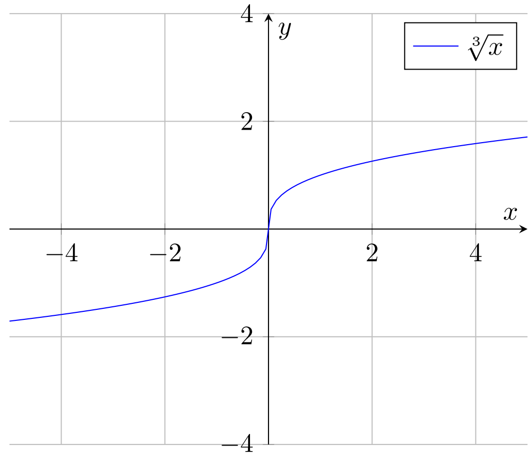
Square Root Function

$$f(x) = \sqrt{x} \ \textbf{ or } \ f(x) = a\sqrt{x-h} + k \quad \textbf{Endpoint at } (h, k)$$



Cube Root Function

$$f(x) = \sqrt[3]{x} \ \ \textbf{or} \ \ f(x) = a\sqrt[3]{x-h}+k \quad \textbf{Point of inflection at } (h,k)$$

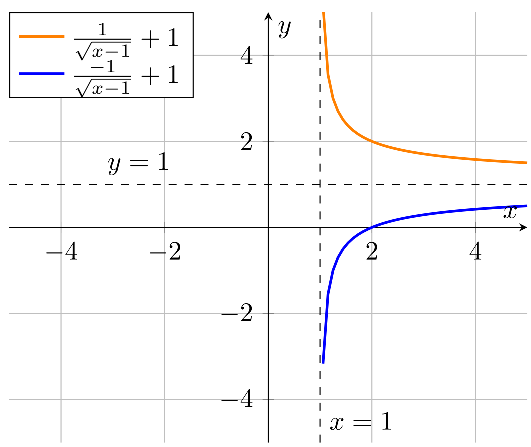


1.3.4 Reciprocal Root Functions

$$f(x) = \frac{a}{\sqrt[n]{b(x-h)}} + k$$

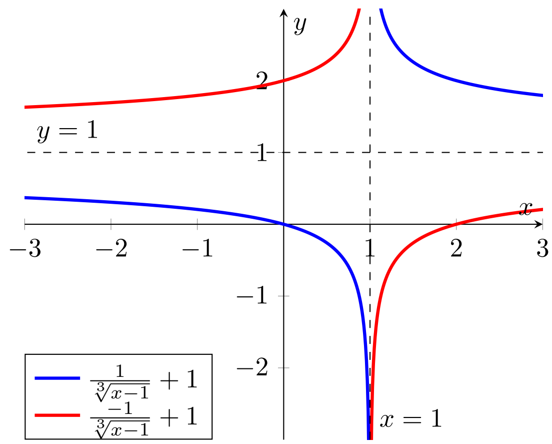
Reciprocal Square Root

$$f(x) = \frac{1}{\sqrt{x}} \ \ \textbf{or} \ \ f(x) = \frac{a}{\sqrt{x-h}} + k \quad \textbf{Asymptotes at } x = h,\ y = k$$



Reciprocal Cube Root

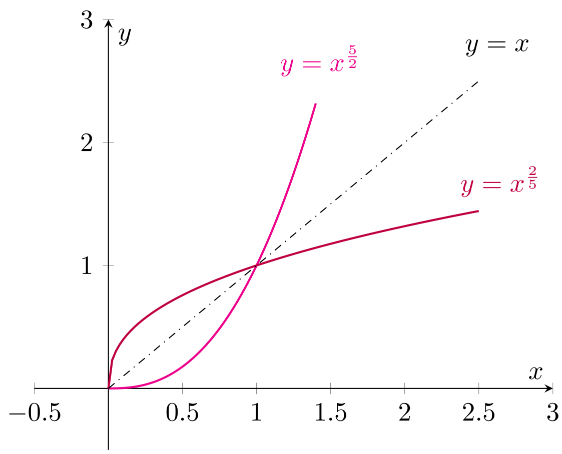
$$f(x) = \frac{1}{\sqrt[3]{x}} \textbf{ or } f(x) = \frac{a}{\sqrt[3]{x-h}} + k \quad \textbf{Asymptotes at } x = h,\ y = k$$



1.3.5 Rational Power Functions

$$f(x) = a[b(x-h)]^{\frac{p}{q}}$$

If $p > q$ then the polynomial shape will dominate as $\frac{p}{q} > 1$.
If $p < q$ then the root shape will dominate as $0 < \frac{p}{q} < 1$.



Polynomial Domination ($p > q$)
For even values of p, the function will be in turning-point form (act like a quadratic or quartic). For odd values of p, the function will be in point of inflection form (act like a cubic).

Root Domination ($p < q$)
For even values of p, the function with exhibit square-root-like behaviour. For odd values of p, the function will exhibit cube-root-like behaviour.

Negative Rational Power Functions

$$f(x) = a[b(x-h)]^{-\frac{p}{q}}$$

If $p > q$ the negative power shape dominates as $-\frac{p}{q} < -1$.
If $p < q$ the reciprocal root shape dominates as $-1 < -\frac{p}{q} < 0$.
If $p = q$ then $y = \frac{1}{x}$.
As $\frac{p}{q} \to 1$ the graph of $x^{\frac{p}{q}} \to \frac{1}{x}$ and approaches a hyperbola.

Domain
For even values of q the domain is (h, ∞) for $b > 0$ and $(-\infty, h)$ for $b < 0$.
For odd values of q the domain is $\mathbb{R} \setminus \{h\}$.

For even values of p, the function will exhibit truncus-like behvaiour. For odd values of p, the function will exhibit hyperbola-like behaviour.

1.3.6 Exponential and Natural Log Functions (Euler's Number)

Euler's Number
Compound growth is a repeated percentage increase. That is, multiplying a number by $(1 + r\%)$ over and over. If we assume 100% growth we get $(1 + \frac{1}{n})^n$. If we increase the number of successive compoundings to infinity we get the following limit:

$$\lim_{x \to \infty} \left(1 + \frac{1}{n}\right)^n \approx 2.71828$$

This is what we call e or Euler's number. e is the maximum proportion of growth when we compound at 100% as much as possible.

General Exponential Function

$$f(x) = a^x, \quad a \in \mathbb{R}^+$$

$$f(x) = Aa^{n(x-b)} + c, \quad a \in \mathbb{R}^+$$

Range and Domain
$x \in \mathbb{R}$
$y \in (c, \infty), \quad A > 0$
$y \in (-\infty, c), \quad A < 0$

Natural Exponential Function

$$f(x) = e^x, \quad e \approx 2.71828$$

$$\int e^x dx = e^x + c \quad \frac{d}{dx}(e^x) = e^x$$

Since we do not always compound at 100% we need to modify e accordingly. By taking a power of e we can use it to modify the rate or the time or both of compounding. e^x can be interpreted as:

- the proportion of growth after x time periods with 100% continuous growth, or
- the proportion of growth with a continuous rate of x in one time period, or
- the proportion of growth where $x = rate \times time$.

General Logarithm Functions

$$f(x) = \log_a(x), \;\; a > 1$$

$$f(x) = A\log_a(n(x-b)) + c, \;\; a > 1$$

Range and Domain
$x \in (0, \infty)$ for $n > 0$
$x \in (-\infty, 0)$ for $n < 0$
$y \in \mathbb{R}$

Natural Logarithm Function, $\log_e(x)$
Logarithms are the index of a power. Therefore, the natural logarithm is the index of the natural exponential. The natural logarithm can thus be determined as:

- the time taken to reach a certain proportion of growth with a 100% continuous growth rate, or
- the continuous rate needed to reach a certain proportion of growth in one unit of time, or
- the product of the time and the continuous rate.

More specifically, we can use the natural logarithm to convert to continuous growth rates:

$$growth = a^{time} = \left(e^{\log_e(a)}\right)^{time} = \left((e^{rate})^{time}\right) = e^x, \quad \log_e(e^x) = x$$

$$growth = a^{rate} = \left(e^{\log_e(a)}\right)^{rate} = \left((e^{time})^{rate}\right) = e^x, \quad \log_e(e^x) = x$$

1.3.7 Circular Functions

Sine and Cosine Functions

$$f(x) = \sin(x) \ \ \textbf{or} \ \ f(x) = a\sin(n(x-b)) + c$$

$$f(x) = \cos(x) \ \ \textbf{or} \ \ f(x) = a\cos(n(x-b)) + c$$

Period
The period of $\sin(nx)$ and $\cos(nx)$ is given by $\frac{2\pi}{n}$.

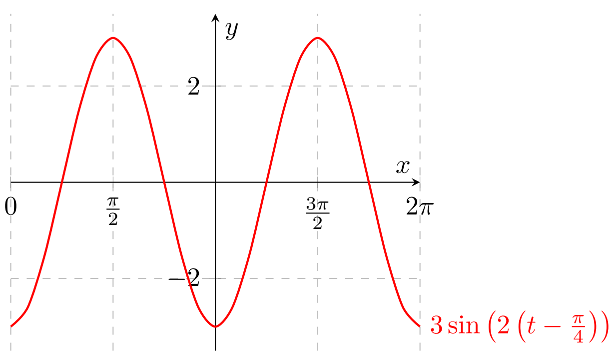
Stationary Points
Sine and cosine have a maximum of +1 and minimum values of -1. These values will change under transformations as outlined in the general forms above. The maximums and minimums can be determined as follows:

Maximum	Minimum
$c + a$ for $a > 0$	$c - a$ for $a > 0$
$c - a$ for $a < 0$	$c + a$ for $a < 0$

Sketching Sinusoidal Curves
To sketch a sine or cosine function, sketch the minimum, mean and maximum values at 4 points. For $\sin(x)$ or $\cos(x)$ we use $(0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi)$.

<u>VCAA 2004 Exam 1 Question 4a</u>
On the set of axes, sketch the graph of the function with rule $y = 3\sin\left(2\left(t - \frac{\pi}{4}\right)\right), -\pi \leq t \leq \pi$.

$\theta = 2\left(t - \frac{\pi}{4}\right)$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$t - \frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
t	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$
$\sin(\theta)$	0	1	0	-1	0
$3\sin(\theta)$	0	3	0	-3	0



Graph Equivalence
The graph of cosine is:

- reflection in the y-axis and translation of $\frac{\pi}{2}$ units right of the graph of the sine.
- reflection in the x-axis and translation of $\frac{\pi}{2}$ units right of the graph of the sine.
- translation of $\frac{\pi}{2}$ units left of the graph of the sine.

The Tangent Function

$$f(x) = \tan(x) \ \textbf{ or } \ f(x) = a\tan(n(x-b)) + c$$

Period
The period of $\tan(x)$ is π. For $\tan(nx)$, the period is given by $\frac{\pi}{2}$.

Domain and Asymptotes
Since $\tan(x) = \frac{\sin(x)}{\cos(x)}$, $\tan(x)$ is undefined whenever $\cos(x)$ is also undefined. $\tan(nx)$ is undefined at $x = \frac{\pi}{2n}$. These are the asymptotes of the function and their x-coordinates can be found by adding and subtracting $\frac{\pi}{n}$ to/from $\frac{\pi}{2n}$.

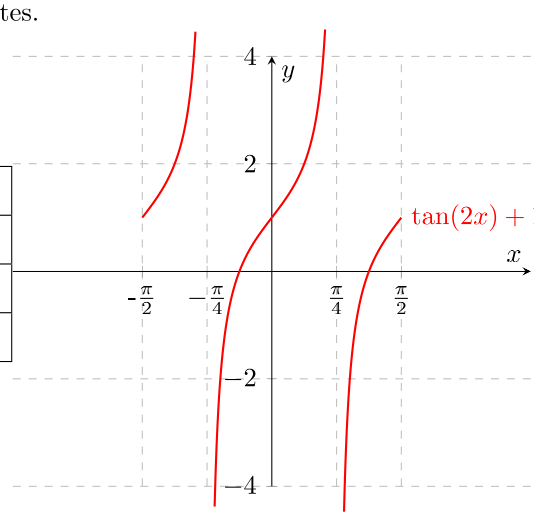
The domain can be written using unions of each period of the tangent, or using set rejection. For $0 \leq x \leq 2\pi$, the domain of $\tan(x)$ is $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right]$ or $[0, 2\pi] \setminus \{\frac{\pi}{2}, \frac{3\pi}{2}\}$. A similar process can be followed for the domain of $\tan(nx)$.

Stationary Points
There are no stationary points for tangent functions. There is a point of inflection but it is not stationary (not horizontal, i.e. gradient at point of inflection is $\neq 0$).

Sketching Tangent Graphs
To sketch a tangent function, we use the angles where the tangent is horizontal or vertical on the unit circle $(-\frac{\pi}{2}, 0, \frac{\pi}{2})$ and their related tangent values (0 and undefined) as well as $\pm\frac{\pi}{4}$ for another point to use as a scale between the zeroes and asymptotes whose tangents are ± 1.

<u>VCAA 2017 NHT Exam 1 Question 4a</u>
Let $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$, where $f(x) = \tan(2x) + 1$. Sketch the graph of f on the axes below. Label any asymptotes with the appropriate equation, and label the end points and the axis intercepts with their coordinates.

Angle: $2x$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
x	$-\frac{\pi}{4}$	$-\frac{\pi}{8}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$
$\tan(2x)$	Undefined	-1	0	1	Undefined
$\tan(2x)+1$	Undefined	0	1	2	Undefined



1.3.8 Sum and Difference Functions

Sum and difference functions are functions that are comprised of basic functions added or subtracted from each other. Polynomials are the sum and difference of power functions with positive integer powers.

Sum Functions
The ordinates, or y-values, of a sum function are defined as the sum of the ordinates of the constituent functions. The domain of a sum function is defined as the set of x-values over which both functions are defined.

$$(f+g)(x) = f(x) + g(x), \;\; d_{f+g} = d_f \cap d_g$$

Difference Functions
Difference functions can be written as a sum function with the reflection of the function in the x-axis.

$$(f-g)(x) = f(x) - g(x) = f(x) + (-g(x)), \;\; d_{f-g} = d_f \cap d_g$$

VCAA 2013 Exam 2 Question 5
If $f : (-\infty, 1) \to \mathbb{R}, f(x) = 2\log_e(1-x)$ and $g : [-1, \infty) \to \mathbb{R}, g(x) = 3\sqrt{x+1}$, find the maximal domain of (f+g)(x).

The maximal domain of the function $(f+g)(x)$ is $d_f \cap d_g = (-\infty, 1) \cap [-1, \infty) = [-1, 1)$

Sketching Sum and Difference Functions
To sketch the graph of a sum or difference function, we add the ordinates (y-values) of each coordinate from both graphs together. We can speed up this process by applying some generalisations:

- The sum function will have an x-intercept when the two functions have the same value but with opposite signs.
- If one graph has an x-intercept, then the sum function's ordinate will equal the other function's at that point.
- If either graph has a vertical asymptote, the sum function will also have that asymptote.

1.3.9 Product Functions

Product functions consist of basic functions multiplied together. Polynomials are the product of linear and power functions with positive integers.

The ordinates of a product function are defined as the product of the ordinates of the constituent functions. The domain of the product function is the set of x-values over which both functions are defined.

$$(fg)(x) = f(x) \times (x), \quad d_{fg} = d_f \cap d_g$$

Modified VCAA 2013 Exam 2 Question 5

If $f : (-\infty, 1) \to \mathbb{R}, f(x) = 2\log_e(1-x)$ and $g : [-1, \infty) \to \mathbb{R}, g(x) = 3\sqrt{x+1}$, find the maximal domain of the product function $(fg)(x)$.

The maximal domain of the function $(fg)(x)$ is $d_f \cap d_g = (-\infty, 1) \cap [-1, \infty) = [-1, 1)$.

Sketching Product Functions

The ordinates of a product function can be found by taking the product of the ordinates of the two constituent functions. This can be expedited via some generalisations like the sum and difference functions:

- The product function has an x-intercept when either function has an x-intercept.
- If one function's ordinate is 1, the product function's ordinate equals the other function's at that point.
- If one function's ordinate is -1, the product function's ordinate equals the opposite of the other function's ordinate at that point.
- If the two graphs are both positive or both negative, the product function will be positive.
- If the two graphs have different signs, the product function will be negative.

1.3.10 Composite Functions

Composite functions are the result of nesting a function inside another. The notation for this is below.

$$f \circ g(x) = f(g(x)), \quad d_{f \circ g} = d_g \setminus x : g(x) \notin d_f, \quad r_g \subset d_f$$

Effect of the Outside Function $f(x)$

The outside function gives the general shape of the graph of the composite function. That is, $f(x)$ determines the general shape of the graph $y = f(g(x))$.

Effect of the Inside Function $g(x)$

The inside function gives the sections of the outside function to graph.
For $g(x) > 0$, the graph of $y = f(g(x))$ is similar to the graph of $y = f(x)$ on the right of the y-axis.
For $g(x) < 0$, the graph of $y = f(g(x))$ is similar to the graph of $y = f(x)$ on the left of the y-axis.

The smaller the value of $g(x)$, the graph of $y = f(g(x))$ is similar to $y = f(x)$ closer to the y-axis and vice-versa.

If $g(x)$ is sine, cosine, or tangent, the graph of $y = f(g(x))$ will loop horizontally.
If $g(x)$ is an even function ($g(x) = g(-x)$), then $f(g(x))$ will also be an even function.
If $g(x)$ has an asymptote at $x = a$, then the graph of $y = f(g(x))$ will also have an asymptote at $x = a$.

1.3.11 Piecewise Functions

A piecewise function is a function that takes the shape of different functions in different sections of its domain.

$$f(x) = \begin{cases} x^2 & x \leq 0 \\ \frac{100-x}{100} & 0 < x \leq 100 \\ 2x + 1 & 100 < x \end{cases}$$

1.4 Transformations of the Plane and Functions

1.4.1 Transformations of the Plane

In geometry, we can transform shapes (and the plane) by using transformations. Three such transformations are dilations (enlargement or shrinking), reflections (mirror across a line), and translations (sliding). In Coordinate Geometry, the same transformations can be applied to points and graphs.

Dilations
Dilations involve multiplying the x or y values by a scaling factor. That is, stretch it in a direction by a factor (e.g. dilated in the y-direction by a factor of 2).

Dilations can be either phrased as:

- dilation by a factor of a in the x/y direction, or
- dilation by a factor of a away from the y/x axis.

Reflections
Reflections involve multiplying the x or y values by a negative. By multiplying the x-value by a negative, the number reflects in the y-axis. By multiplying the y-value by a negative, reflection in the x-axis occurs.

Translations Translations involve adding a number to the x or y values. That is, moving every point uniformally by the same amount in the same direction.

1.4.2 Constructing a Function from a List of Transformations

To transform a function from a given list of transformations, we can apply all the the transformations to each point on the graph by applying them generally to x and y then modify the function accordingly.

Since the transformations are based on multiplication and addition, the order of operations needs to be considered.
For the function $y = f(x)$ we can see there is a $y =$ term, so vertical transformations can be applied directly to the function. That is, the new y, $y' = ay + d = af(x) + d$.
However, there is no $x =$ term, so the horizontal transformations need to be rearranged to substitute into the original x. That is, for the new x, $x' = bx + c$, we need to rearrange to $x = \frac{1}{b}(x' - c)$.

VCAA 2002 Exam 1 Question 5a

The graph of the function with rule $y = \frac{1}{x}$ is transformed as follows:

- a dilation by a factor of $\frac{1}{2}$ from the y-axis
- a reflection in the y-axis
- a translation of +3 units parallel to the x-axis
- a translation of +1 unit parallel to the y-axis

Find the equation of the rule of the transformed function.

$$(x,y) \to \left(\frac{1}{2}x, y\right) \to \left(-\frac{1}{2}x, y\right) \to \left(-\frac{1}{2}x+3, y\right) \to \left(-\frac{1}{2}x+3, y+1\right) = (x', y')$$

$$x' = -\frac{1}{2}x + 3 \implies x = -2(x'-3)$$

$$y' = y+1 = \frac{1}{x}+1 = \frac{1}{-2(x'-3)}+1 \implies y = \frac{1}{-2(x-3)}+1 \text{ is the transformed function.}$$

1.4.3 Describing the Effect of Transformations

When a function has been transformed, the x and y values have been manipulated in some manner. It is useful to describe these transformations specifically as it makes sketching graphs of transformed functions much easier.

VCAA 2009 Exam 2 Question 12

A sequence of transformations that maps the curve $y = \sqrt{x}$ to the curve $y = 1 - 3\sqrt{2x+\pi}$ is...

let $y' = 1 - 3\sqrt{2x'+\pi} = 1 - 3\sqrt{x} = -3 \times y + 1$ $x = 2x' + \pi \implies x' = \frac{1}{2}x - \frac{\pi}{2}$

Hence it is evident that the order of transformations is:

- Dilation by a factor of 3 from the x-axis
- Dilation by a factor of $\frac{1}{2}$ from the y-axis
- Reflection in the x-axis
- Translation of 1 unit up and $\frac{\pi}{2}$ units left

1.4.4 Inverse Transformations

If you are able to transform $y = f(x) \rightarrow y = Af(n(x-h)) + k$ then inverse transformations reverse the process. Inverse transformations must be applied in the reverse order to that in which they were originally applied.

Transformation	Inverse Transformation
Dilation by a factor of A parallel to y-axis	Dilation by a factor of $\frac{1}{A}$ parallel to the y-axis
Dilation by a factor of A from the x-axis	Dilation by a factor of $\frac{1}{A}$ from the x-axis
Dilation by a factor n parallel to the x-axis	Dilation by a factor $\frac{1}{n}$ parallel to the x-axis.
Dilation by a factor n from the y-axis	Dilation by a factor $\frac{1}{n}$ from the y-axis.
Reflection in an axis	Reflection in the same axis
Translation of h units in positive x-direction	Translation of h units in negative x-direction.
Translation of k units in positive y-direction	Translation of k units in negative y-direction.

1.4.5 Equivalent Transformations

Since functions can be algebraically manipulated, it is possible to show that different transformations will have the same result for particular graphs. These do not necessarily apply to all functions and different transformations are equivalent for different functions.

Examples

$y = (3x)^2 = 9x^2$

Dilation by a factor of $\frac{1}{3}$ from the y-axis is equivalent to a dilation by a factor of 9 from the x-axis.

$y = (3x)^3 = 27x^3$

Dilation by a factor of $\frac{1}{3}$ from the y-axis is equivalent to a dilation by a factor of 27 from the x-axis.

$y = 3x$

Dilation by a factor of $\frac{1}{3}$ from y-axis is equivalent to a dilation by a factor of 3 from the x-axis.

$y = \log_3(x) - 2 = \log_3(\frac{x}{9})$

Translation 2 units down is equivalent to a dilation by a factor of 9 from the y-axis.

$y = \log_2(x) + 3 = \log_2(8x)$

Translation of 3 units up is equivalent to a dilation by a factor of $\frac{1}{8}$ from the y-axis.

1.4.6 Monoparametric Transformations

Dilations

$$y = Af(x)$$

"A dilation by a factor of A from the x-axis" or "a dilation by a factor of A parallel to the y-axis". The x-intercepts do not change.

$$y = f(nx)$$

"A dilation by a factor of $\frac{1}{n}$ from the y-axis" or "a dilation by a factor of $\frac{1}{n}$ parallel to the x-axis". The y-intercepts don't change.

Reflections

$$y = -f(x)$$

"A reflection in the x-axis."
Intersections between the original function and its reflection occur on the x-axis.

$$y = f(-x)$$

"A reflection in the y-axis."
Intersections between the original function and its reflection occur on the y-axis.

Translations

$$y = f(x) + c$$

"A translation of c units up (for $c > 0$)" or "a translation of c units down (for $c < 0$)".
Shape of the curve does not change.

$$y = f(x + b)$$

"A translation of b units left (for $b > 0$)" or "a translation of b units right (for $b < 0$)".
Shape of the curve does not change.

1.5 Applications of Functions

1.5.1 Modelling Data and Practical Situations

When we collect a set of data, we often want to describe as accurately as possible a function that could generate the same points or at least a good approximation of them. To do this, we need to know what shape the points take, which is best achieved by plotting the data. If the shape is recognisable, we can fit a model to the data easily but some datasets require a combination (sum, difference, or product) of functions to more accurately describe the data.

Features that Enable the Recognition of Possible Models

Positive Power Model $y = ax^n$	Proportional to the n-th power.
Negative Power Model $y = \frac{a}{x^n}$	Inversely proportional to the n-th power (exhibits asymptotic behaviour).
Linear Model $y = mx + b$	Constantly increasing or decreasing
Polynomial Model	Takes the shape of a polynomial
Sinusoidal Model $y = a\sin(bx)$ or $y = a\cos(bx)$	Oscillation between two values
Exponential Model $y = ae^{bx}$	Increasing or decreasing exponentially (decreasing has asymptotic behaviour)
Logarithmic Model $y = a\log_e(bx)$	Slow growth or decay, or data ranges over orders of magnitude.

1.5.2 Applications of Exponential Functions

$$A(t) = A_0 \times a^{kt} + C$$

Modelling Growth if $a > 1$ and $k > 1$	Modelling Decay if $0 < a < 1$ or $k < 0$
Cell Growth	Radioactive Decay
Population Growth	Population Decay
Compound interest	Cooling temperatures

Initial Value
The initial value occurs when $t = 0$. That is, $A(0) = A_0 + C$.

Example
A population, P, after t years is modelled by $P(t) = 100 \times 1.5^t + 20$. The initial population was $P(0) = 100 \times 1.5^0 + 20 = 100 + 20 = 120$.

Rate of Growth/Decay
The rate of growth or decay is the base of the exponential. A multiplier in the index changes the frequency with which the rate is applied. For e^{bx}, the continuous rate is b.

Example
An investment is compounding annually such that the amount of the investment

$\$A$, after t years is given by $A(t) = 5000 \times 1.067^t$. $A(0) = 5000$. The interest rate of the investment is $1 + r\% = 1.067 \implies r\% = 0.067 = 6.7\%$.

Half-Life
The time it takes for there to be half of the initial value due to a decaying model. That means solving the following:

$$\frac{A_0 + C}{2} = A_0 \times a^{kt} + C \implies a^{kt} = \frac{1}{2} \text{ for } t$$

Example
A population of sheep is decaying exponentially. At a particular time, there were 1000 sheep observed. Six months later, 800 sheep remained. Determine a function, $A(t)$, that models the population of the sheep t months after the first observation.

$$A_0 = 1000, \quad A = 1000 \times e^{bt}$$
$$800 = 1000 \times e^{6b} \implies e^{6b} = \frac{4}{5} \implies b = \frac{1}{6}\log_e\left(\frac{4}{5}\right) \approx -0.0372$$
$$\therefore A = 1000 \times e^{-0.0372t}$$

The continuous rate at which the population of sheep is decaying is 3.72%. Determine the half-life of the population.

$$e^{-0.0372t} = 0.5 \implies -0.0372t = \log_e(0.5) \implies t = 18.63 \approx 19 months$$

Long-Run Value
The Long-run value is the value that a decaying model will approach in the long run. As $t \to \infty$, $A \to C$, the horizontal asymptote.

Example
A population of wild rabbits is in decline. There is an estimate of 760 rabbits when the decline is first notices, 752 after one day, and 716 after two days. An equation to model the rabbit population is in the form $R(t) = a \times b^t + c$ where R is the number of rabbits after t days. Find the long-run value.

$$790 = a \times b^0 + c \implies c = 790 - a$$
$$752 = a \times b^1 + c \implies 752 = ab + 790 - a \implies a(1-b) = 38$$
$$716 = a \times b^2 + c \implies 716 = ab^2 + 790 - a \implies a(1-b^2) = 74$$
$$\implies a(1-b)(1+b) = 74 \implies 38(1+b) = 74 \implies 1 + b = \frac{37}{19} \implies b = \frac{18}{19} \approx 0.95$$
$$a(1-0.95) = 38 \implies a = 760, \quad c = 790 - 760 = 30$$
$$\therefore R(t) = 760 \times 0.95^t + 30$$

In the long run, there is expected to be 30 rabbits left.

1.5.3 Applications of Sine and Cosine Functions

VCAA 2001 Exam 2 Question 1 (modified)
The temperature T degrees Celsius in a greenhouse at t hours after midnight for a typical November day is modelled by the formula $T = 25 - 4\cos(\frac{\pi(t-3)}{12}),\ \text{for } 0 \leq t \leq 24$. Find the period, amplitude and maximum and minimum temperatures of the greenhouse.

NOTCOMPLETE

1.5.4 Determining the Equation of a Function

There are multiple methods to determine a function's equation.

- Directly substitute any known coordinates in to the function with variable parameters to create simultaneous equations to solve.
- Directly substitute into a particular form of the function.
- Use transformations from the graph of the untransformed function to the given graph.
- Use knowledge of the function to determine the parameters.

Example
Find the values of a and b such that the graph of $f(x) = ae^{bx}$ passes through the points $(\log_e(32), 32)$ and $(\log_e(5), 500)$. Then express $f(x)$ in the form e^{mx+n}.

AOS2 - Algebra, Number, and Structure

1.6 Polynomial Equations

1.6.1 Real Solutions

A polynomial with real coefficients of up to degree n will have up to n real solutions.

1.6.2 Null Factor Law

If $a \times b = 0$ then $a = 0 \text{and/or} b = 0$.
For the polynomial $P(x) = 0$, if you can factorise $P(x)$ you can use the null factor law to solve each factor equal to 0.
Rational root theorem and factor theorem can be used to factorise polynomials.

1.6.3 Quadratic Formula

Quadratic equations can be solved in three ways: by factorising using null factor law, by completing the square and solving a power equation, or by using the quadratic formula.

For a quadratic equation in the form $ax^2 + bx + c = 0$, the solutions are found by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic Equations using Substitutions
Like factoring, some equations can be rewritten as a quadratic equation and thus solved via quadratic methods.

Example

$$\begin{aligned}
(2x+1)^2 + 5(2x+1) + 6 = 0 \\
\text{Let } y = 2x+1 \\
y^2 + 5y + 6 = 0 \\
y = \frac{-5 \pm \sqrt{5^2 - 4(1)(6)}}{2} = \frac{-5 \pm 1}{2} \\
y = 2x+1 = -\frac{4}{2} = -2 \implies 2x = -3 \implies x = -\frac{3}{2} \\
y = 2x+1 = -\frac{6}{2} = -3 \implies 2x = -4 \implies x = -2
\end{aligned}$$

1.6.4 The Discriminant and Number of Solutions

The discriminant describes the number and types of solutions a quadratic equation has. $\Delta = b^2 - 4ac$.

Number of Real Solutions	Condition
2	$\Delta > 0$
1	$\Delta = 0$
0	$\Delta < 0$

If the discriminant is a perfect square (or fraction), then the solutions to the quadratic are **rational**.

1.7 Inequalities

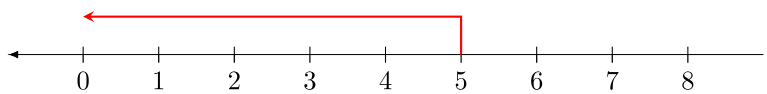
1.7.1 Single-Variable Linear Inequalities

Solving a Single-Variable Linear Inequality
To solve a single-variable inequality, we solve for the pronumeral as we would a linear equation. Similar to linear equations, whatever you do to one side of the inequality must be done to the other side.

Example

$$\begin{aligned} x + 5 &\leq 2 \\ x &\leq -3 \end{aligned}$$

Sketching a Single-Variable Linear Inequality
Once we have solved for the pronumeral we can present it graphically. Single-variable linear inequalities are sketched on a number line.



1.8 Inverse Relations and Functions

1.9 Exponential Equations

1.10 Trigonometric Equations

1.11 Power Equations

1.12 Composite Function Equations

1.13 Solutions to Equations

1.14 Newton's Method

1.15 Equations of the Form $f(x) = g(x)$

1.16 Literal Equations

1.17 Systems of Simultaneous Linear Equations

AOS3 - Calculus

AOS4 - Data analysis, probability, and statistics