VCE Mathematics 3/4 Bound Reference

SAM MURPHY

2023

Theorem 1. $\{Sam\} \cap \{Bitches\} = \emptyset$

Proof. {Sam} $\in$ Samuel Murphy
{Bitches} $\in$ All Females

A person "Person" is denoted as being a certain sex by the notation: Person_M or Person_F

Samuel Murphy = $(\text{Samuel Murphy})_M$

A female takes the form of Person_r. $\therefore$ {Bitches} contains all Person_F, and exclusively all Person_F.

The set Sam contains one element $(\text{Samuel Murphy})_M$, which does not take the form Person_F $\therefore$ the sets Sam and Bitches contain no elements in common.
$\therefore \{\text{Sam}\} \cap \{\text{Bitches}\} = \emptyset$ ■

Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$		$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$	
$\frac{d}{dx}\left((ax+b)^n\right) = an(ax+b)^{n-1}$		$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$	
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$		$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$	
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$		$\int \frac{1}{x} dx = \log_e(x) + c,\ x > 0$	
$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$		$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$	
$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$		$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$	
$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$			
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$	Newton's method	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
trapezium rule approximation	$Area \approx \frac{x_n - x_0}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right]$		

Probability

$\Pr(A) = 1 - \Pr(A')$		$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	
$\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$			
mean	$\mu = \mathrm{E}(X)$	variance	$\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$
binomial coefficient	$\binom{n}{x} = \dfrac{n!}{x!(n-x)!}$		

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
binomial	$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$	$\mu = np$	$\sigma^2 = np(1-p)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)\,dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)\,dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}(\hat{P}) = p$
standard deviation	$\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

END OF FORMULA SHEET

Mensuration

area of a circle segment	$\frac{r^2}{2}(\theta - \sin(\theta))$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$	sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
volume of a pyramid	$\frac{1}{3}Ah$	cosine rule	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Algebra, number and structure (complex numbers)

$z = x + iy = r(\cos(\theta) + i\sin(\theta)) = r\,\mathrm{cis}(\theta)$	$\lvert z\rvert = \sqrt{x^2 + y^2} = r$	
$-\pi < \mathrm{Arg}(z) \le \pi$	$z_1 z_2 = r_1 r_2 \,\mathrm{cis}(\theta_1 + \theta_2)$	
$\frac{z_1}{z_2} = \frac{r_1}{r_2}\mathrm{cis}(\theta_1 - \theta_2)$	de Moivre's theorem	$z^n = r^n \mathrm{cis}(n\theta)$

Data analysis, probability and statistics

for independent random variables $X_1, X_2 \ldots X_n$	$\mathrm{E}(aX_1 + b) = a\,\mathrm{E}(X_1) + b$ $\mathrm{E}(a_1X_1 + a_2X_2 + \ldots + a_nX_n)$ $= a_1\mathrm{E}(X_1) + a_2\mathrm{E}(X_2) + \ldots + a_n\mathrm{E}(X_n)$	
	$\mathrm{Var}(aX_1 + b) = a^2\mathrm{Var}(X_1)$ $\mathrm{Var}(a_1X_1 + a_2X_2 + \ldots + a_nX_n)$ $= a_1^2\mathrm{Var}(X_1) + a_2^2\mathrm{Var}(X_2) + \ldots + a_n^2\mathrm{Var}(X_n)$	
for independent identically distributed variables $X_1, X_2 \ldots X_n$	$\mathrm{E}(X_1 + X_2 + \ldots + X_n) = n\mu$	
	$\mathrm{Var}(X_1 + X_2 + \ldots X_n) = n\sigma^2$	
approximate confidence interval for μ	$\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$	
distribution of sample mean $\bar{X}$	mean	$\mathrm{E}(\bar{X}) = \mu$
	variance	$\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}$

Calculus

$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \ne -1$
$\frac{d}{dx}(e^{ax}) = ae^{ax}$	$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$
$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$	$\int \frac{1}{x} dx = \log_e\lvert x\rvert + c$
$\frac{d}{dx}(\sin(ax)) = a\cos(ax)$	$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$
$\frac{d}{dx}(\cos(ax)) = -a\sin(ax)$	$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$
$\frac{d}{dx}(\tan(ax)) = a\sec^2(ax)$	$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$
$\frac{d}{dx}(\cot(ax)) = -a\,\mathrm{cosec}^2(ax)$	$\int \mathrm{cosec}^2(ax)\,dx = -\frac{1}{a}\cot(ax) + c$
$\frac{d}{dx}(\sec(ax)) = a\sec(ax)\tan(ax)$	$\int \sec(ax)\tan(ax)\,dx = \frac{1}{a}\sec(ax) + c$
$\frac{d}{dx}(\mathrm{cosec}(ax)) = -a\,\mathrm{cosec}(ax)\cot(ax)$	$\int \mathrm{cosec}(ax)\cot(ax)\,dx = -\frac{1}{a}\mathrm{cosec}(ax) + c$
$\frac{d}{dx}(\sin^{-1}(ax)) = \frac{a}{\sqrt{1-(ax)^2}}$	$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$
$\frac{d}{dx}(\cos^{-1}(ax)) = \frac{-a}{\sqrt{1-(ax)^2}}$	$\int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$
$\frac{d}{dx}(\tan^{-1}(ax)) = \frac{a}{1+(ax)^2}$	$\int \frac{a}{a^2 + x^2} dx = \tan^{-1}\left(\frac{x}{a}\right) + c$
	$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c,\ n \ne -1$
	$\int \frac{1}{ax+b} dx = \frac{1}{a}\log_e\lvert ax+b\rvert + c$

Calculus – continued

product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
integration by parts	$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$
Euler's method	If $\frac{dy}{dx} = f(x, y)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h \times f(x_n, y_n)$.
arc length parametric	$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
surface area Cartesian about x-axis	$\int_{x_1}^{x_2} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$
surface area Cartesian about y-axis	$\int_{y_1}^{y_2} 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$
surface area parametric about x-axis	$\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
surface area parametric about y-axis	$\int_{t_1}^{t_2} 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Kinematics

acceleration	$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$	
constant acceleration formulas	$v = u + at$	$s = ut + \frac{1}{2}at^2$
	$v^2 = u^2 + 2as$	$s = \frac{1}{2}(u+v)t$

Vectors in two and three dimensions

$\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$	$\lvert\underset{\sim}{r}(t)\rvert = \sqrt{x(t)^2 + y(t)^2 + z(t)^2}$
	$\dot{\underset{\sim}{r}}(t) = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$
for $\underset{\sim}{r}_1 = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{r}_2 = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$	vector scalar product $\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = \lvert\underset{\sim}{r}_1\rvert\lvert\underset{\sim}{r}_2\rvert\cos(\theta) = x_1x_2 + y_1y_2 + z_1z_2$
	vector cross product $\underset{\sim}{r}_1 \times \underset{\sim}{r}_2 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1z_2 - y_2z_1)\underset{\sim}{i} + (x_2z_1 - x_1z_2)\underset{\sim}{j} + (x_1y_2 - x_2y_1)\underset{\sim}{k}$
vector equation of a line	$\underset{\sim}{r}(t) = \underset{\sim}{r}_1 + t\underset{\sim}{r}_2 = (x_1 + x_2t)\underset{\sim}{i} + (y_1 + y_2t)\underset{\sim}{j} + (z_1 + z_2t)\underset{\sim}{k}$
parametric equation of a line	$x(t) = x_1 + x_2t \quad y(t) = y_1 + y_2t \quad z(t) = z_1 + z_2t$
vector equation of a plane	$\underset{\sim}{r}(s,t) = \underset{\sim}{r}_0 + s\underset{\sim}{r}_1 + t\underset{\sim}{r}_2 = (x_0 + x_1s + x_2t)\underset{\sim}{i} + (y_0 + y_1s + y_2t)\underset{\sim}{j} + (z_0 + z_1s + z_2t)\underset{\sim}{k}$
parametric equation of a plane	$x(s,t) = x_0 + x_1s + x_2t,\ y(s,t) = y_0 + y_1s + y_2t,\ z(s,t) = z_0 + z_1s + z_2t$
Cartesian equation of a plane	$ax + by + cz = d$

Circular functions

$\cos^2(x) + \sin^2(x) = 1$	
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \mathrm{cosec}^2(x)$
$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$	$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$	$\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$
$\sin(2x) = 2\sin(x)\cos(x)$	
$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$	$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$
$\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$	$\cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$

END OF FORMULA SHEET

Contents

1 Useful Rules, Values and Identities

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \tan\frac{\pi}{4} = 1$$
$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$
$$\sin\frac{\pi}{6} = \frac{1}{2} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

1.1 TI Nspire Shortcut Keys

Shortcut	Outcome
ctrl+A	Selects all
ctrl+C	Copy
ctrl+I	Insert page
ctrl+N	New doc
ctrl+O	Open
ctrl+1	Page down (END)
ctrl+space	Underscore
Pi	π
Theta	θ
Infinity	∞

Add a Problem If you want to add a new problem to the document, go doc+4+1

1.2 Functions

Odd Function $f(-x) = -f(x)$. Odd functions have rotational symmetry about the origin.

Even Function $f(-x) = f(x)$. Even functions are symmetrical about the y-axis.

Sum Functions $(f+g)(x) = f(x) + g(x)$. dom $(f+g) =$ dom $f \cap$ dom g

Product Functions $(fg)(x) = f(x)g(x)$. dom $(fg) =$ dom $f \cap$ dom g

Composite Functions $g \circ f(x) = g(f(x))$. dom $(g \circ f) =$ dom$f$

Inverse Functions $f(f^{-1}(x)) = x$

2 Silly Blunders

u-sub For definite integrals, do not forget to change the limits with the rule of u.

Linear combinations of random variables - standard deviation For calculating $\text{sd}(aX \pm bY)$ **always** add the variances first through $a^2\text{Var}(X) + b^2\text{Var}(Y)$. Only ever find standard deviation through the square root of the variance.

Kinematics - acceleration Do not forget to multiply $\frac{dv}{dx}$ by v again.

Integration notation Don't be an idiot. Remember the dx on an integral. (CornChip Blunder)

Complex polynomials When it asks for a solution for z, don't write $z - 2$, write $z = 2$. Solutions are equations for z that lead to $P(z) = 0$. Do not list factors when it asks for solutions. (CornChip Blunder)

Cuboid A cuboid is not a cube. You're sped.

Decimals Questions ask for decimal answers. Check this.

Percentages Probability or sample statistic questions may ask for a percentage proportion. DO NOT GIVE THE PROBABILITY (0.69), GIVE THEM THE PERCENTAGE (69%).

MCQs about Stats Some MCQs may ask about the variance of a variable or the standard deviation. Give them which one they ask for!!

Domains of Inverse Usually when they ask you to find an inverse function, they want you to find its domain as well. Watch out for this!

Magnitude of Scalar Resolute The expression $\frac{1}{|\underset{\sim}{b}|}|\underset{\sim}{b} \cdot \underset{\sim}{a}|$ is the **magnitude** of the scalar resolute, not just the scalar resolute.

3 Unit 1/2 Assumed Knowledge

3.1 Sequences and Series

Arithmetic Sequences Sequences formed by adding a fixed amount to successive terms.

$$t_n = a + (n-1)d$$

a is the first term in the sequence, d is the common difference.

Arithmetic Series The sum of an arithmetic sequence.

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

Geometric Sequences Sequences formed by multiplying successive terms by a fixed ratio.

$$t_n = ar^{n-1}$$

a is the first term, r is the common ratio $r = \frac{t_k}{t_{k-1}}$ for $k > 1$

Geometric Series The sum of terms in a geometric sequence.

$$S_n = \frac{a(1-r^n)}{1-r}$$

Infinite Geometric Series

$$S_\infty = \frac{a}{1-r} \qquad -1 < r < 1$$

3.2 Non-Linear Relations

3.2.1 Circles

$$(x-h)^2 + (y-k)^2 = r^2$$

Represents a circle of radius r centred on (h, k).

3.2.2 Ellipses

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Ellipse with centre (h,k) and $2a$ is the horizontal axis length, b is the vertical axis length.

3.2.3 Hyperbolas

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \qquad \text{Asymptotes: } y-k=\pm\frac{b}{a}(x-h)$$

Hyperbola with centre (h,k) and $2a$ is the horizontal distance between foci, b is a vertical dilation. The tangents

Exam-style Question: VCAA 2007 Spesh E2 A1

> A hyperbola has equation $\dfrac{(x-2)^2}{a^2}-\dfrac{4(y+3)^2}{a^2}=1$, where $a\in\mathbb{R}\backslash\{0\}$. The product of the gradients of the asymptotes is:

$$\frac{(x-2)^2}{a^2}-\frac{4(y+3)^2}{a^2}=\frac{(x-2)^2}{a^2}-\frac{(y+3)^2}{\left(\frac{a}{2}\right)^2}=1$$

Asymptotes are $y+3=\pm\frac{\frac{a}{2}}{a}(x-2)=\pm\frac{1}{2}(x-2)$ and the product of the gradients is $\frac{1}{2}\times\left(-\frac{1}{2}\right)=\frac{-1}{4}$.

4 Functions

4.1 Tech

Defining a function Make sure you use := to define functions! Alternatively, menu+1+1

4.1.1 fMax(rule,iv,lower,upper) and fMin(rule,iv,lower,upper)

This finds the maximum and minimum values of a function over an interval. iv denotes the independent variable of the rule.

4.1.2 tangentLine(rule,iv,x)

This finds the equation of the line tangent to the graph of the rule at the point with x-coordinate x. iv denotes the independent variable of the rule.

4.1.3 normalLine(rule,iv,x)

This finds the equation of the normal line tangent to the graph of the rule at the point with x-coordinate x. iv denotes the independent variable of the rule.

4.1.4 scriptedmath\poi(f(x),x)

Gives stationary points, axes intercepts, straight-line asymptotes, and endpoints.

4.1.5 scriptedmath\inv(f(x),pointX)

Finds $f^{-1}(x)$. If the function is not one-to-one it will restrict domain to contain the point where x=pointX.

4.2 Average Value and Average Rate

Average value The average value of a continuous function f over $[a,b]$ is given by:

$$f_avg=\frac{1}{b-a}\int_a^b f(x)dx$$

Average rate of change The average rate of change of a function f over $[a,b]$ is given by:

$$A(x)=\frac{f(b)-f(a)}{b-a}$$

4.3 Transformations

$(x, y) \to (x + a, y)$	Translation a units right
$(x, y) \to (x - a, y)$	Translation a units left
$(x, y) \to (x, y + a)$	Translation a units up
$(x, y) \to (x, y - a)$	Translation a units down
$(x, y) \to (x, ay)$	Dilation from x-axis by factor a
$(x, y) \to (ax, y)$	Dilation from y-axis by factor a
$(x, y) \to (-x, y)$	Reflection in y-axis
$(x, y) \to (x, -y)$	Reflection in x-axis

Exam-style Question: 2010 NEAP Methods E2 A12

The graph of f where $f(x) = x^{-\frac{3}{2}}$ undergoes a dilation of factor 4 parallel to x-axis. The same result could be achieved had f undergone a dilation of factor:
A) $\frac{1}{8}$ from y-axis B) 8 from y-axis C) $\frac{1}{8}$ from x-axis D) 8 from the x-axis E) $\frac{1}{16}$ from the x-axis.

$$\begin{aligned} f\left(\frac{x}{4}\right) = \left(\frac{x}{4}\right)^{-\frac{3}{2}} &= 8x^{-\frac{3}{2}} \quad \text{As } \left(\frac{1}{4}\right)^{-\frac{3}{2}} = 8 \\ &= 8f(x) \implies \text{a factor of 8 from x-axis} \end{aligned}$$

4.4 Composite Functions

Domain The domain of a composite function $f(g(x))$ is all the inputs x which are in the domain of g such that $g(x) \in \text{dom} f$

4.5 Exponential Functions

Exam-style Question: 2006 VCAA Methods E2 B3e

Find the values of k, where $k \in \mathbb{R}^+$, for which the equation $3 - ke^x - e^{-x} = 0$ has one or more solutions for x.

Solving $3 - ke^x - e^{-x} = 0$ for x

Let $u = e^x$

$3 - ku - (u)^{-1} = 0 \equiv 3u - ku^2 - 1 = 0$

$u = \dfrac{-3 \pm \sqrt{9 - 4k}}{2k} = e^x \implies x = \ln\left(\dfrac{-3 \pm \sqrt{9 - 4k}}{2k}\right)$

$\star$ Argument of a logarithm must be real and positive

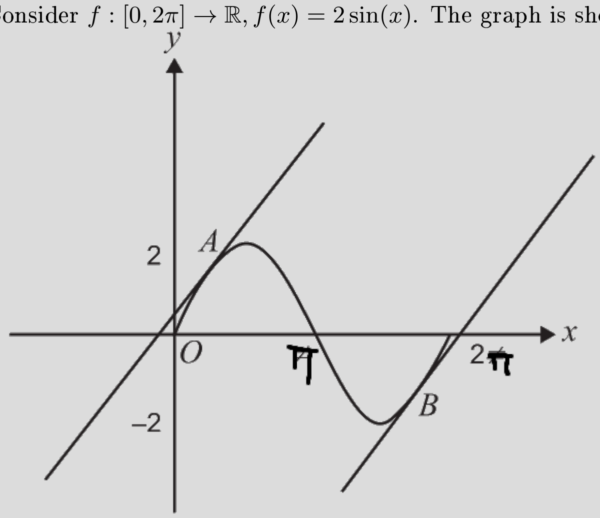
$9 - 4k \geq 0 \implies k \leq \dfrac{9}{4}$

$\therefore 0 < k \leq \dfrac{9}{4} \because k \in \mathbb{R}^+$

4.6 Tangents

Exam-style Question: 2006 VCAA Methods E2 B1

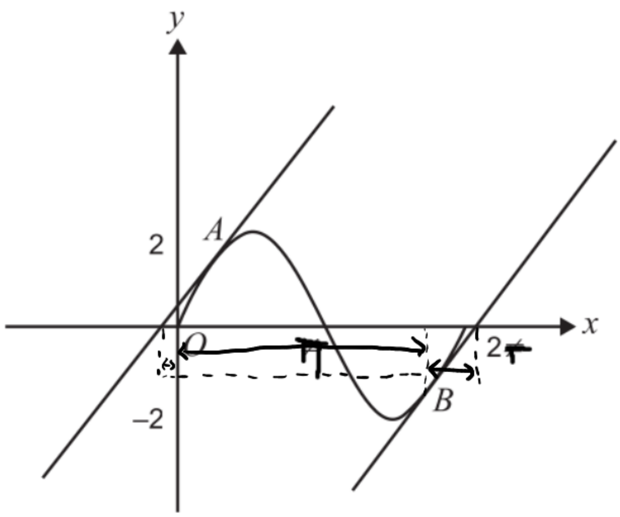
Consider $f : [0, 2\pi] \to \mathbb{R}, f(x) = 2\sin(x)$. The graph is shown below with tangents drawn at A and B.



The tangent to the curve at $x = \frac{\pi}{3}$ is given by $y = x + \frac{3\sqrt{3}-\pi}{3}$.

The x-intercept of the tangent at A is $\left(\frac{\pi-3\sqrt{3}}{3}, 0\right)$

The two tangents to the curve at points A and B have gradient 1. A translation of m units right in the positive x direction takes the tangent at A to the tangent at B. Find m.



Tangent at A cuts x-axis at $x = \frac{\pi}{3} - \sqrt{3}$.

Period of function is 2π

Tangent at B will be another $\frac{\pi}{3} - \sqrt{3}$ units right after cycle

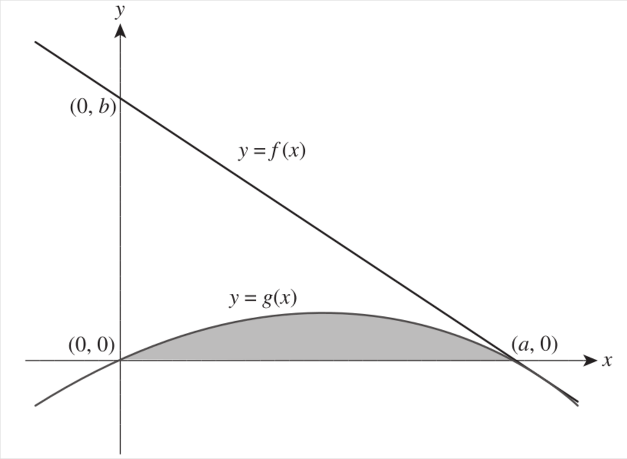
$\therefore m = 2\pi + 2\left(\frac{\pi - 3\sqrt{3}}{3}\right)$

$= \frac{4\pi}{3} + 2\sqrt{3}$

4.7 Areas

Exam-style Question: 2021 NEAP Methods MCQ20

The graph below shows a linear function f and a quadratic function g, where $f(x) \geq g(x)$ for $x \in \mathbb{R}$



Given that $f(a) = g(a)$ and $f(x) = g(x)$ has one solution, the area of the shaded region in terms of a and b can be expressed as:

Maximum area occurs when parabola is tangential to the line at $x = a$

$$g'(x) = k(2x - a) \rightarrow g'(a) = ak$$

$$\begin{aligned} m_{tangent} &= -\frac{b}{a} = g'(a) \\ ak &= \frac{-b}{a} \\ k &= -\frac{b}{a^2} \end{aligned}$$

$$\begin{aligned} \therefore g(x) &= -\frac{b}{a^2}x(x-a) \\ \text{max area } &= \int_0^a -\frac{b}{a^2}x(x-a)dx \\ &= \frac{ab}{6} \end{aligned}$$

5 Vectors and Kinematics

5.1 Vectors Tech

Create a vector ctrl+(creates a matrix which we can use for vectors.

5.1.1 norm(a)

Menu+7+7+1 give the norm (magnitude) of a vector.

5.1.2 unitV(a)

Menu+7+C+1 creates a unit vector in direction of a given vector.

5.1.3 dotP(a,b)

Menu+7+C+3 gives the dot product of two vectors.

5.1.4 crossP(a,b)

Menu+7+C+2 gives the cross product of two vectors.

5.1.5 sam_vectors\vang(a,b)

gives the angle between two vectors.

5.1.6 sam_vectors\vecres(a,b)

Put in vectors and it will find the vector resolute of $\underset{\sim}{a}$ in both the direction of and perpendicular to $\underset{\sim}{b}$.

5.1.7 sam_vectors\lindep(a,b,c,p)

Find a value for pronumeral p that makes the set of vectors a, b, c linearly dependent. In the first example below, we use it to find a value of p for which the vectors are linearly dependent. In the second example it is used to verify that a set of known vectors is linearly dependent. It returns a constant solution, so therefore they must be dependent.

$$lindep\left(\begin{bmatrix}-1 & 6 & -3\end{bmatrix},\begin{bmatrix}2 & -8 & 5\end{bmatrix},\begin{bmatrix}3 & 2 & \left|1-p^2\right|\end{bmatrix},p\right)$$

$$\{p, p=\sqrt{5}$$
$$\{p, p=-\sqrt{5}$$

Done

$$lindep\left(\begin{bmatrix}-1 & 6 & -3\end{bmatrix},\begin{bmatrix}2 & -8 & 5\end{bmatrix},\begin{bmatrix}3 & 2 & 4\end{bmatrix},x\right)$$

$$\{x, x=\mathbf{c1}$$

Done

5.2 Angles Between Vectors

Exam-style Question: 2006 VCAA Spesh E2 B2

> Point A has position vector $\underset{\sim}{a} = -\underset{\sim}{i} - 4\underset{\sim}{j}$, point B has position vector $\underset{\sim}{b} = 2\underset{\sim}{i} - 5\underset{\sim}{j}$, point C has position vector $\underset{\sim}{c} = 5\underset{\sim}{i} - 4\underset{\sim}{j}$ and point D has position vector $\underset{\sim}{d} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$. It is known that $\cos(\angle ADC) = \frac{4}{5}$. Find the cosine of $\angle ABC$ and **hence show** that $\angle ADC$ and $\angle ABC$ are supplementary (add to π).

$$\overrightarrow{BA} = -3\underset{\sim}{i} + \underset{\sim}{j} \qquad \overrightarrow{BC} = 3\underset{\sim}{i} + \underset{\sim}{j}$$

$$\begin{aligned}\text{Let}\theta &= \angle ABC \\ \cos\theta &= \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|} \\ &= -\frac{4}{5}\end{aligned}$$

$$\begin{aligned}\arccos\left(-\frac{4}{5}\right) &= \frac{\pi}{2} + \arcsin\left(\frac{4}{5}\right) \\ \therefore \arccos\left(-\frac{4}{5}\right) + \arccos\left(\frac{4}{5}\right) &= \frac{\pi}{2} + \arcsin\left(\frac{4}{5}\right) + \arccos\left(\frac{4}{5}\right) \\ &= \frac{\pi}{2} + \frac{\pi}{2} = \pi\end{aligned}$$

5.3 Linear Dependence

$$x_1\underset{\sim}{a_1} + x_2\underset{\sim}{a_2} + \ldots + x_n\underset{\sim}{a_n} = 0 \;\not\Rightarrow\; x_i = 0 \forall i \in [1, n]$$

A set of vectors is said to be linearly dependent if at least one of its members can be expressed as a linear combination of other vectors in the set. Alternatively, a set of vectors are linearly dependent if a vector could be removed from the set and not affect the set's span.

Linear combination A vector $\underset{\sim}{w}$ is a linear combination of $\underset{\sim}{u}$ and $\underset{\sim}{v}$ if it can be expressed in form $\underset{\sim}{w} = k_1\underset{\sim}{u} + k_2\underset{\sim}{v}$ for $k_1, k_2 \in \mathbb{R}$.

General Case A set $S \in \mathbb{R}^n$ of n vectors $\underset{\sim}{a}_1, \underset{\sim}{a}_2, ..., \underset{\sim}{a}_n$ is linearly dependent **if and only if** there exists $k_1, k_2, ..., k_n \in \mathbb{R}$ (not all equal 0) such that $k_1\underset{\sim}{a}_1 + k_2\underset{\sim}{a}_2 + ... + k_n\underset{\sim}{a}_n = 0$

5.4 Scalar Product

The scalar product of vectors $\underset{\sim}{a}, \underset{\sim}{b}$ is given by:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$$

where θ is the angle between the two vectors.

Properties of the Scalar Product

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{a} \qquad \underset{\sim}{a} \cdot \left(\underset{\sim}{b} + \underset{\sim}{c}\right) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$$

$$k\left(\underset{\sim}{a} \cdot \underset{\sim}{b}\right) = \left(k\underset{\sim}{a}\right) \cdot \underset{\sim}{b} = \underset{\sim}{a} \cdot \left(k\underset{\sim}{b}\right) \qquad \underset{\sim}{a} \cdot 0 = 0$$

Parallel Vectors For parallel vectors $\underset{\sim}{a}, \underset{\sim}{b}$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{cases} |\underset{\sim}{a}||\underset{\sim}{b}| & \text{if } \underset{\sim}{a}, \underset{\sim}{b} \text{ are in the same direction} \\ -|\underset{\sim}{a}||\underset{\sim}{b}| & \text{if } \underset{\sim}{a}, \underset{\sim}{b} \text{ are in opposite directions} \end{cases}$$

5.5 Vector and Scalar Resolutes

We can decompose a vector $\underset{\sim}{a}$ into the sum of two vectors in perpendicular directions.

Vector Resolute in General The Vector resolute of $\underset{\sim}{a}$ in direction $\underset{\sim}{b}$ is given by:

$$\underset{\sim}{u} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\underset{\sim}{b} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|^2}\underset{\sim}{b} = \left(\underset{\sim}{a} \cdot \frac{\underset{\sim}{b}}{|\underset{\sim}{b}|}\right)\left(\frac{\underset{\sim}{b}}{|\underset{\sim}{b}|}\right) = \left(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}\right)\hat{\underset{\sim}{b}}$$

Scalar Resolute in General The scalar resolute is the signed length of the vector resolute of $\underset{\sim}{a}$ in direction $\underset{\sim}{b}$

$$\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}$$

5.6 Vector Proofs

5.6.1 Collinearity

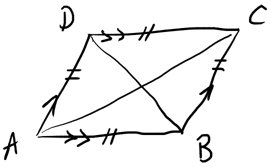
Three distinct points A, B, C are collinear if and only if $\overrightarrow{AC} = k\overrightarrow{AB}$ for some $k \in \mathbb{R}\setminus\{0\}$.

Miscellaneous Facts for Proof

Parallel Vectors For $k \in \mathbb{R}^+$, the vector $k\underset{\sim}{a}$ is in the same direction as $\underset{\sim}{a}$ and has magnitude $k|\underset{\sim}{a}|$ and the vector $-k\underset{\sim}{a}$ is in the opposite direction to $\underset{\sim}{a}$ and has magnitude $k|\underset{\sim}{a}|$.

Two non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel if and only if $\underset{\sim}{b} = k\underset{\sim}{a}$ for some $k \in \mathbb{R}\setminus\{0\}$.

Prove that the diagonals of a rhombus are perpendicular using a vector method.



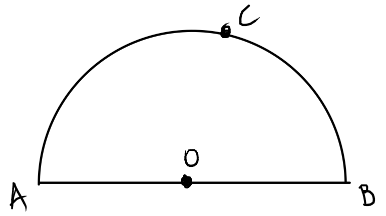
$|\vec{AB}| = |\vec{AD}| = |\vec{DC}| = |\vec{CB}|$

$\vec{AB} = \vec{DC}$ and $\vec{AD} = \vec{BC}$

$$\begin{aligned}\vec{AC}\cdot\vec{BD} &= (\vec{AB}+\vec{BC})\cdot(\vec{BC}+\vec{CD})\\ &= (\vec{AB}+\vec{BC})\cdot(\vec{BC}-\vec{AB})\\ &= |\vec{BC}|^2 - |\vec{AB}|^2\\ &= 0 \quad \text{by definition}\end{aligned}$$

$\vec{AC}\cdot\vec{BD} = 0 \iff \vec{AC} \perp \vec{BD}$ ∎

Prove that the angle subtended by a diameter at a point on a circle is a right angle.



Let O be the centre of the circle and let $\vec{AB}$ be the diameter. Let C be a point on the circle's circumference other than A or B.

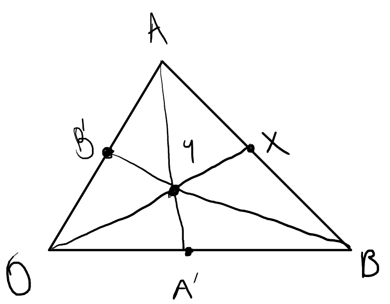
Let $\underset{\sim}{a} = \vec{OA}$, $\underset{\sim}{c} = \vec{OC}$, $\vec{OB} = -\underset{\sim}{a}$ $\Rightarrow |\vec{OA}| = |\vec{OB}| = |\vec{OC}| = r$ where r = radius of the circle.

$\vec{AC} = \vec{AO} + \vec{OC} = \underset{\sim}{c} - \underset{\sim}{a}$ and $\vec{BC} = \vec{BO} + \vec{OC} = \underset{\sim}{a} + \underset{\sim}{c}$

$$\begin{aligned}\vec{AC}\cdot\vec{BC} &= (\underset{\sim}{c}-\underset{\sim}{a})\cdot(\underset{\sim}{a}+\underset{\sim}{c})\\ &= |\underset{\sim}{c}|^2 - |\underset{\sim}{a}|^2 = r^2 - r^2 = 0 \quad \text{by definition}\end{aligned}$$

Hence $\vec{AC} \perp \vec{BC} \Rightarrow \angle ACB$ is a right angle ∎

Prove that the medians of a triangle are concurrent.



Let $\underset{\sim}{a} = \vec{OA}$, $\underset{\sim}{b} = \vec{OB}$

Show that $|AY| : |YA'| = |BY| : |YB'| = 2:1$

$\vec{AY} = \lambda \vec{AA'}$ $\quad \vec{BY} = \mu \vec{BB'}$ for $\lambda, \mu \in \mathbb{R}$

$\vec{AA'} = \frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a}$ $\qquad \vec{BB'} = -\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{a}$

$\Rightarrow \vec{AY} = \lambda(\frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a})$ $\qquad \vec{BY} = \mu(\frac{1}{2}\underset{\sim}{a} - \underset{\sim}{b})$

$\vec{BY}$ can also be defined as:

$\vec{BY} = \vec{BA} + \vec{AY} = -\underset{\sim}{b} + \underset{\sim}{a} + \lambda(\frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a})$ $\quad \therefore -\mu\underset{\sim}{b} + \frac{\mu}{2}\underset{\sim}{a} = (1-\lambda)\underset{\sim}{a} + (\frac{\lambda}{2} - 1)\underset{\sim}{b}$

Equating coefficients of independent vectors $\underset{\sim}{a}, \underset{\sim}{b}$:

$\frac{\mu}{2} = 1 - \lambda$ ① $\qquad -\mu = \frac{\lambda}{2} - 1$ ②

$$\begin{aligned}2\times① + ② : \quad 0 &= 2 - 2\lambda + \frac{\lambda}{2} - 1\\ 1 &= \frac{3}{2}\lambda\\ \lambda &= \frac{2}{3}\end{aligned}$$

$\lambda = \frac{2}{3} \rightarrow$ ① gives $\mu = \frac{2}{3}$

$\therefore |\vec{AY}| : |\vec{YA'}| = |\vec{BY}| : |\vec{YB'}| = 2:1$

By symmetry, the intersection of $\vec{AA'}$ and $\vec{OX}$ must also divide $\vec{AA'}$ into a ratio of $2:1$, and therefore this intersection is Y.

Hence the three medians are concurrent at centroid Y. ∎

5.7 Vector Equations

Vector Equation of Line given by Point and Direction $\underset{\sim}{r}(\lambda) = \underset{\sim}{a} + \lambda \underset{\sim}{d}, \lambda \in \mathbb{R}$. **Note** that there is no unique vector equation for a given line ℓ, as we can choose any point with position vector $\underset{\sim}{a}$ as our starting point and any vector $\underset{\sim}{d} \parallel \ell$.

Vector Equation of Line given by Two Points $\underset{\sim}{r}(\lambda) = \underset{\sim}{a} + \lambda \left(\underset{\sim}{b} - \underset{\sim}{a}\right), \lambda \in \mathbb{R}$ where $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are the position vectors of the two points.

Converting from Vector to Cartesian Equation Consider the vector equation as a set of parametrics, where the coefficient of $\underset{\sim}{i}$ is equal to x, coefficient of $\underset{\sim}{j}$ is equal to y, etc. and then solving to get the cartesian form.

Changing from Cartesian to Vector Equation The direction vector of the line $\underset{\sim}{d}$ is given by $\underset{\sim}{d} = (\text{run})\underset{\sim}{i} + (\text{rise})\underset{\sim}{j}$.

Lines in 3D Lines in three dimensions can be described in three ways, where $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ is the position vector of a point A on the line and $\underset{\sim}{d} = d_1\underset{\sim}{i} + d_2\underset{\sim}{j} + d_3\underset{\sim}{k}$ is a vector parallel to the line.

Vector	Parametric	Cartesian
$\underset{\sim}{r}(\lambda) = \underset{\sim}{a} + \lambda \underset{\sim}{d}$	$x = a_1 + d_2\lambda$ $y = a_2 + d_2\lambda$ $z = a_3 + d_3\lambda$	$\frac{x-a_1}{d_1} = \frac{y-a_2}{d_2} = \frac{z-a_3}{d_3}$

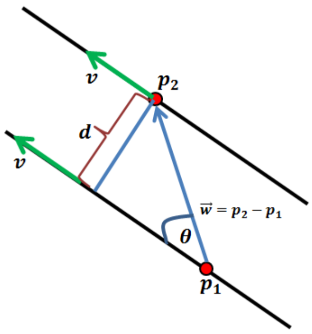
Parallel and Perpendicular Lines For two lines $\ell_1 : \underset{\sim}{r}_1(\lambda) = \underset{\sim}{a}_1 + \lambda \underset{\sim}{d}_1, \lambda \in \mathbb{R}$ and $\ell_2 : \underset{\sim}{r}_2(\gamma) = \underset{\sim}{a}_2 + \gamma \underset{\sim}{d}_2, \gamma \in \mathbb{R}$:

- $\ell_1 \parallel \ell_2 \iff \underset{\sim}{d}_1 \parallel \underset{\sim}{d}_2$
- $\ell_1 \perp \ell_2 \iff \underset{\sim}{d}_1 \perp \underset{\sim}{d}_2 \iff \underset{\sim}{d}_1 \cdot \underset{\sim}{d}_2 = 0$

Distance between two lines (non-parallel)

Distance between two parallel lines Let $\underset{\sim}{v}$ be the vector in the direction of the lines. Now let p_1 be a point on line 1 and p_2 be a point on line 2. The distance between the lines, as shown in the diagram below, can be calculated using the cross product.

$$\begin{aligned} d &= |\underset{\sim}{w}| \sin\theta \\ &= \left|\underset{\sim}{\hat{v}} \times \underset{\sim}{w}\right| \\ &= \left|\underset{\sim}{\hat{v}} \times \left(\underset{\sim}{p_2} - \underset{\sim}{p_1}\right)\right| \end{aligned}$$



5.8 Intersections and Skew Lines

5.8.1 Lines in 2D Space

A pair of lines in 2D space can

- intersect
- be parallel and distinct
- coincide

To check whether two parallel lines coincide choose a point on one line and see if it is on the other.

5.8.2 Lines in 3D Space

Skew Lines Two lines are skew lines if they are not parallel **and** do not intersect. Two lines are skew lines **if and only if** they do not lie in the same plane.

Coincident and Parallel Lines We can determine whether 3D lines are coincident or parallel and distinct by similar methods as in 2 dimensions. If parallel lines share a point they must coincide.

Intersecting Lines Two lines $\ell_1 : \underset{\sim}{r}_1(\lambda) = \underset{\sim}{a}_1 + \lambda \underset{\sim}{d}_1$ and $\ell_2 : \underset{\sim}{r}_2(\gamma) = \underset{\sim}{a}_2 + \gamma \underset{\sim}{d}_2$ have a common point if $\exists \lambda, \gamma \in \mathbb{R} \ni \underset{\sim}{r}_1(\lambda) = \underset{\sim}{r}_2(\gamma)$.

Concurrence of 3 lines A point of concurrence is where 3 or more lines meet.

Angle Between 2 Lines The angle between two lines can be found using the scalar product of their direction vectors. The angle is either θ or $\pi - \theta$, whichever lies in the interval $[0, \frac{\pi}{2}]$.

5.9 Vector Product (Cross Product)

Geometric Relationship The relationship between the two vectors in a cross multiplication and the result is called 'mutually perpendicular'.

Properties of the Vector Product

- Direction of $\underset{\sim}{a} \times \underset{\sim}{b}$ is perpendicular to the plane containing $\underset{\sim}{a}$ and $\underset{\sim}{b}$.
- $|\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta$
- $\underset{\sim}{a} \parallel \underset{\sim}{b} \implies |\underset{\sim}{a} \times \underset{\sim}{b}| = 0$
- $\underset{\sim}{a} \perp \underset{\sim}{b} \implies |\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}|$
- Vector product is not associative $\left(\underset{\sim}{a} \times \underset{\sim}{b}\right) \times \underset{\sim}{c} \neq \underset{\sim}{a} \times \left(\underset{\sim}{b} \times \underset{\sim}{c}\right)$
- Vector product is not commutative $\underset{\sim}{a} \times \underset{\sim}{b} \neq \underset{\sim}{b} \times \underset{\sim}{a}$
- Distributes over addition $\underset{\sim}{a} \times \left(\underset{\sim}{b} + \underset{\sim}{c}\right) = \underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{a} \times \underset{\sim}{c}$
- $\underset{\sim}{a} \times \underset{\sim}{b} = -\left(\underset{\sim}{b} \times \underset{\sim}{a}\right)$

Vector Product in Component Form Let $\underset{\sim}{a} = a_1\hat{\underset{\sim}{i}} + a_2\hat{\underset{\sim}{j}} + a_3\hat{\underset{\sim}{k}}$ and $\underset{\sim}{b} = b_1\hat{\underset{\sim}{i}} + b_2\hat{\underset{\sim}{j}} + b_3\hat{\underset{\sim}{k}}$, then:

$$\begin{aligned}
\underset{\sim}{a} \times \underset{\sim}{b} &= \left(a_1\hat{\underset{\sim}{i}} + a_2\hat{\underset{\sim}{j}} + a_3\hat{\underset{\sim}{k}}\right) \times \left(b_1\hat{\underset{\sim}{i}} + b_2\hat{\underset{\sim}{j}} + b_3\hat{\underset{\sim}{k}}\right) \\
&= (a_2b_3 - a_3b_2)\hat{\underset{\sim}{i}} - (a_1b_3 - a_3b_1)\hat{\underset{\sim}{j}} + (a_1b_2 - a_2b_1)\hat{\underset{\sim}{k}}
\end{aligned}$$

We can also use the determinant of a 3×3 matrix to find the components of a vector product.

$$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \hat{\underset{\sim}{i}} & \hat{\underset{\sim}{j}} & \hat{\underset{\sim}{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\underset{\sim}{i}} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\underset{\sim}{j}} + \begin{vmatrix} a_2 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{\underset{\sim}{k}}$$

Area from Vector Product $|\underset{\sim}{a} \times \underset{\sim}{b}|$ equals the area of the parallelogram spanned by $\underset{\sim}{a}$ and $\underset{\sim}{b}$

5.10 Vector Equations of Planes

Equations of Planes A plane Π can be described in 3-dimensional space with 2 vectors

- Position vector $\underset{\sim}{a}$ of a point on the plane
- A vector $\underset{\sim}{n}$ that is normal to the plane

Let $\underset{\sim}{r}$ be the position vector of a point P on the plane, then the vector $\overrightarrow{AP} = \underset{\sim}{r} - \underset{\sim}{a}$ lies on the plane and therefore $\overrightarrow{AP} \perp \underset{\sim}{n}$.

$$\begin{aligned}
(\underset{\sim}{r} - \underset{\sim}{a}) \cdot \underset{\sim}{n} &= 0 \\
\underset{\sim}{r} \cdot \underset{\sim}{n} &= \underset{\sim}{a} \cdot \underset{\sim}{n}
\end{aligned}$$

The above equation is known as the **vector equation** of the plane.

If we let $\underset{\sim}{r} = x\hat{\underset{\sim}{i}} + y\hat{\underset{\sim}{j}} + z\hat{\underset{\sim}{k}}$ and $\underset{\sim}{n} = n_1\hat{\underset{\sim}{i}} + n_2\hat{\underset{\sim}{j}} + n_3\hat{\underset{\sim}{k}}$ we can obtain the **cartesian equation** for the plane.

$$\begin{gathered}
\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n} \\
n_1x + n_2y + n_3z = k \text{ where } k = \underset{\sim}{a} \cdot \underset{\sim}{n}
\end{gathered}$$

We can also describe planes with the use of parameters. For example, if $n_1x + n_2y + n_3z = k$, with $a_3 \neq 0$ describes a plane, it can be described by the following parametrics:

$$x = \lambda \qquad y = \gamma \qquad z = \frac{k - n_1\lambda - n_2\gamma}{n_3} \qquad \lambda, \gamma \in \mathbb{R}$$

We can determine the equations of planes with:

- a point on the plane and a normal vector at that point
- three points on the plane which are not collinear, or
- two lines on the plane that intersect

Example: Equation of a Plane given Point and Normal Vector

A plane Π is such that the vector $-\underset{\sim}{\hat{i}}+5\underset{\sim}{\hat{j}}-3\underset{\sim}{\hat{k}}$ is normal to it at point A with the position vector $\underset{\sim}{a}=-3\underset{\sim}{\hat{i}}+4\underset{\sim}{\hat{j}}+6\underset{\sim}{\hat{k}}$ on the plane. Find the cartesian and vector equations that describe Π.

$$\begin{aligned}
\underset{\sim}{r}\cdot\underset{\sim}{n} &= \underset{\sim}{a}\cdot\underset{\sim}{n}\\
\underset{\sim}{r}\cdot\left(-\underset{\sim}{\hat{i}}+5\underset{\sim}{\hat{j}}-3\underset{\sim}{\hat{k}}\right) &= \left(-3\underset{\sim}{\hat{i}}+4\underset{\sim}{\hat{j}}+6\underset{\sim}{\hat{k}}\right)\cdot\left(-\underset{\sim}{\hat{i}}+5\underset{\sim}{\hat{j}}-3\underset{\sim}{\hat{k}}\right)\\
\underset{\sim}{r}\cdot\left(-\underset{\sim}{\hat{i}}+5\underset{\sim}{\hat{j}}-3\underset{\sim}{\hat{k}}\right) &= 3+20-18=5\\
\therefore \underset{\sim}{r}\cdot\left(-\underset{\sim}{\hat{i}}+5\underset{\sim}{\hat{j}}-3\underset{\sim}{\hat{k}}\right) &= 5
\end{aligned}$$

$$\begin{aligned}
\underset{\sim}{r} &= x\underset{\sim}{\hat{i}}+y\underset{\sim}{\hat{j}}+z\underset{\sim}{\hat{k}}\\
\left(x\underset{\sim}{\hat{i}}+y\underset{\sim}{\hat{j}}+z\underset{\sim}{\hat{k}}\right)\cdot\left(-\underset{\sim}{\hat{i}}+5\underset{\sim}{\hat{j}}-3\underset{\sim}{\hat{k}}\right) &= 5\\
-x+5y-3z &= 5
\end{aligned}$$

Example: Equation of Plane Given 3 Points

Consider the plane containing the points $A(0,1,1), B(2,1,0)$ and $C(-2,0,3)$. (a) Find cartesian equation of the plane.

$$\overrightarrow{AB}=2\underset{\sim}{\hat{i}}-\underset{\sim}{\hat{k}} \qquad \text{and} \qquad \overrightarrow{AC}=-2\underset{\sim}{\hat{i}}-\underset{\sim}{\hat{j}}+2\underset{\sim}{\hat{k}}$$

$$\begin{aligned}
\overrightarrow{AB}\times\overrightarrow{AC} &= -\underset{\sim}{\hat{i}}-2\underset{\sim}{\hat{j}}-2\underset{\sim}{\hat{k}}\\
\therefore \underset{\sim}{n} &= -\underset{\sim}{\hat{i}}-2\underset{\sim}{\hat{j}}-2\underset{\sim}{\hat{k}}
\end{aligned}$$

Use previous method to find plane equation to be:

$$-x-2y-2z=-4$$

(b) Find the axis intercepts of the plane

$$\begin{aligned}
&\text{x-intercept: Let } y=z=0 \implies x=4\\
&\text{y-intercept: Let } x=z=0 \implies 2y=4 \implies y=2\\
&\text{z-intercept: Let } x=y=0 \implies 2z=4 \implies z=2
\end{aligned}$$

Example: Equation of Plane Given 2 Intersecting Lines

Find a vector and cartesian equation of the plane containing the two lines:

$$\begin{aligned}
\ell_1 : \underset{\sim}{r}_1(\lambda) &= 5\underset{\sim}{\hat{i}}+2\underset{\sim}{\hat{j}}+\lambda\left(2\underset{\sim}{\hat{i}}+\underset{\sim}{\hat{j}}+\underset{\sim}{\hat{k}}\right) \qquad \text{and}\\
\ell_2 : \underset{\sim}{r}_2(\gamma) &= -3\underset{\sim}{\hat{i}}+4\underset{\sim}{\hat{j}}+6\underset{\sim}{\hat{k}}+\gamma\left(\underset{\sim}{\hat{i}}-\underset{\sim}{\hat{j}}-2\underset{\sim}{\hat{k}}\right)
\end{aligned}$$

$$\underset{\sim}{a} = 5\underset{\sim}{\hat{i}} + 2\underset{\sim}{\hat{j}} \text{ is on the plane}$$

Let $\underset{\sim}{n}$ be the a vector perpendicular to both $\underset{\sim}{d_1}$ and $\underset{\sim}{d_2}$

$$\begin{aligned}\underset{\sim}{n} &= \underset{\sim}{d_2} \times \underset{\sim}{d_2} \\ &= -\underset{\sim}{\hat{i}} + 5\underset{\sim}{\hat{j}} - 3\underset{\sim}{\hat{k}}\end{aligned}$$

$$\begin{aligned}\therefore \underset{\sim}{r} \cdot \left(-\underset{\sim}{\hat{i}} + 5\underset{\sim}{\hat{j}} - 3\underset{\sim}{\hat{k}}\right) &= \underset{\sim}{a} \cdot \underset{\sim}{n} \\ &= -1 \times 5 + 2 \times 5 + -3 \times 0 \\ \therefore \underset{\sim}{r} \cdot \left(-\underset{\sim}{\hat{i}} + 5\underset{\sim}{\hat{j}} - 3\underset{\sim}{\hat{k}}\right) &= 5 \\ \implies -x + 5y - 3z = 5\end{aligned}$$

5.11 Distances, Angles, Intersections of Planes

5.11.1 Distances

Point to Plane distance The distance from a point P to a plane Π is given by

$$d = \left|\overrightarrow{PQ} \cdot \underset{\sim}{\hat{n}}\right|$$

where Q is any point on the plane and $\hat{n}$ is a unit vector normal to the plane

Plane to Origin Distance A plane that does not pass through the origin has a vector equation $\underset{\sim}{r} \cdot \underset{\sim}{n} = k$ for $k \neq 0$. Take the point M as the closest point on the plane to the origin. $\overrightarrow{OM} = m\underset{\sim}{n}$ where $|m|$ is the distance of the plane to the origin.

$$\begin{aligned}\left(m\underset{\sim}{\hat{n}}\right) \cdot \underset{\sim}{n} &= k \\ m\left(\underset{\sim}{\hat{n}} \cdot \underset{\sim}{n}\right) &= m|\underset{\sim}{n}| = k \\ \therefore m &= \frac{k}{|\underset{\sim}{n}|}\end{aligned}$$

Distance Between Parallel Planes To find the distance between planes Π_1 and Π_2 we choose any point P on Π_1 and find its nearest distance to Π_2, we can also use each plane's distance from origin.

Example: Distance Between Parallel Planes

Consider the planes given by:
$$\Pi_1 : 2x - y + 2z = 5 \qquad \Pi_2 : 2x - y + 2z = -2$$
Find the distance between the two planes.

First, we find the distance of each plane from the origin.

$$\underset{\sim}{n} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k} \implies |\underset{\sim}{n}| = 3$$

$$\begin{aligned}
\text{Signed distance from origin of } \Pi_1 &= \frac{5}{|\underset{\sim}{n}|} = \frac{5}{3} \\
\text{Signed distance from origin of } \Pi_2 &= \frac{-2}{|\underset{\sim}{n}|} = \frac{-2}{3} \\
\therefore \text{Distance between planes} &= \frac{5}{3} - \left(\frac{-2}{3}\right) \\
&= \frac{7}{3}
\end{aligned}$$

Distance Between Skew Lines Given two skew lines it can be shown that there is a unique line segment PQ joining them that is perpendicular to both lines. Consider the lines defined by $\underset{\sim}{r}_1(\lambda) = \underset{\sim}{a}_1 + \lambda \underset{\sim}{d}_1, \lambda \in \mathbb{R}$ and $\underset{\sim}{r}_2(\gamma) = \underset{\sim}{a}_2 + \gamma \underset{\sim}{d}_2, \gamma \in \mathbb{R}$. The distance between these skew lines is given by:
$$d = \left|\left(\underset{\sim}{a}_2 - \underset{\sim}{a}_1\right) \cdot \underset{\sim}{\hat{n}}\right| \qquad \underset{\sim}{\hat{n}} = \frac{\underset{\sim}{d}_1 \times \underset{\sim}{d}_2}{|\underset{\sim}{d}_1 \times \underset{\sim}{d}_2|}$$

5.11.2 Intersections and Angles

Two planes that are not parallel will intersect along a line. To find the angle between the planes choose a point P common to both planes and find the angle θ between the normal vectors. The answer will either be θ or $\pi - \theta$, whichever is in the interval $[0, \frac{\pi}{2}]$.

Intersection of a Line and a Plane The angle between a line and a plane is 90°-θ where θ is the angle between the line and a normal to the plane. To find the point of intersection between a plane and line, we need to find a value for t for which $\underset{\sim}{r}(t)$ represents a point on the line. That is, $\underset{\sim}{r}(t) \cdot \underset{\sim}{n} = k$. This will help us find a t value and then by subbing back in we find the point of intersection.

Example: Intersection of Line and Plane

Consider the line represented by $\underset{\sim}{r}(\lambda) = 3\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k} + \lambda(\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k})$ and the plane represented by $\underset{\sim}{r} \cdot (\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}) = 2$. Find the point of intersection and angle of intersection between the line and plane.

<u>Point of Intersection</u>

$$\begin{aligned}
\underset{\sim}{r}(\lambda) \cdot (\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}) &= 2 \\
\left(3\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k} + \lambda\left(\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)\right) \cdot (\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}) &= 2 \\
(3 + \lambda) + (-1 + 2\lambda) + 2(-1 - \lambda) &= 2 \\
\therefore \lambda &= 2 \\
\underset{\sim}{r}(2) = 5\underset{\sim}{i} + 3\underset{\sim}{j} - 3\underset{\sim}{k}
\end{aligned}$$

Therefore the point of intersection is (5,3,-3).

<u>Angle of Intersection</u>
$\underset{\sim}{d} = \underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ is parallel to the line and $\underset{\sim}{n} = \underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$ is normal to the plane.

$$\begin{aligned}
\underset{\sim}{d} \cdot \underset{\sim}{n} &= |\underset{\sim}{d}||\underset{\sim}{n}| \cos\theta \\
1 &= 6 \cos\theta \\
\theta &= 80.4°
\end{aligned}$$

Therefore the angle of intersection is $90° - 80.4°$ as 80.4° is the angle with the normal. Hence the angle of intersection is 9.6°.

5.11.3 Intersection of Planes

Example: Equation of the Line of Intersection

Let Π_1 and Π_2 be planes represented by:

$$\Pi_1 : \underset{\sim}{r} \cdot (\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}) = 6 \qquad \Pi_2 : \underset{\sim}{r} \cdot (2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}) = 4$$

Find a vector equation to describe the line of intersection of these planes.

Consider the cartesian equations of two planes:

$$x + y - 3z = 6 \quad (1) \qquad 2x - y + z = 4 \quad (2)$$
$$(1) + (2) \to 3x - 2z = 10 \quad (3)$$
$$\text{Let } x = \lambda \implies z = \frac{3\lambda - 10}{2} \text{ (from (3))} , y = \frac{7\lambda - 18}{2} \text{ (from (2))}$$

This gives us the parametric equations for the line of intersection:

$$x = \lambda, y = \frac{7\lambda - 18}{2}, z = \frac{3\lambda - 10}{2}$$

Converting to vector form:

$$\underset{\sim}{r}(\lambda) = -9\underset{\sim}{j} - 5\underset{\sim}{k} + \lambda\left(\underset{\sim}{i} + \frac{7}{2}\underset{\sim}{j} + \frac{3}{2}\underset{\sim}{k}\right)$$

5.12 Parametric Vector Equations

5.13 SUVAT and Constant Acceleration

$$v = u + at \qquad s = ut + \frac{1}{2}at^2 \qquad v^2 = u^2 + 2as \qquad s = \frac{1}{2}(u + v)t$$

5.14 Terminal (Limiting) Velocity

Exam-style Question: VCAA 2017 Spesh E2 B2c

After two seconds of falling, a skydiver's acceleration is affected by air resistance such that their acceleration is given by:

$$a = g - 0.01v^2$$

Find the limiting (terminal) velocity in ms^{-1} that they would reach.

$$\text{Let } a = 0$$
$$g = 0.01v^2$$
$$v = 10\sqrt{g}$$

5.15 Velocity Functions of Time

Exam-style Question: VCAA 2007 Spesh E2 B5

A car travelling at $20ms^{-1}$ passes a stationary police car, and then decelerates so that its velocity vms^{-1}, at time t seconds after passing the police car, is given by $v = 20 - 2\arctan(t)$.
b) Explain why v will never equal 16.

$$\begin{aligned} &t \to \infty, \arctan(t) \to \frac{\pi}{2} \\ &v \to \left(20 - 2 \cdot \frac{\pi}{2}\right)^{+} = (20-\pi)^{+} \\ &v > 20 - \pi > 16 \end{aligned}$$

5.16 Velocity-Time Graphs

Displacement is given by the signed area bounded by the graph and the t-axis.

Acceleration is given by the gradient.

Distance is given by the total area bounded by the graph and the t-axis.

5.17 Vector Functions

5.17.1 Describing Particle's Path

Consider $\underset{\sim}{r}(\lambda) = x(\lambda)\underset{\sim}{i} + y(\lambda)\underset{\sim}{j}, \lambda \in \mathbb{R}$. $\underset{\sim}{r}(\lambda)$ represents a family of vectors defined by the parameter λ. Solve the components as normal parametrics to obtain the equation of the path.

5.17.2 Describing Curves with Vector Functions

Vector functions to describe curves are not unique.

5.18 Position Vectors as a Function of Time

Consider the vector function $\underset{\sim}{r}(t) = \cos(t)\underset{\sim}{i} + \sin(t)\underset{\sim}{j}, t \geq 0$.

- At $t = 0$, $\underset{\sim}{r}(t) = \underset{\sim}{i}$ so the particle starts at $(1, 0)$
- The particle moves at a constant speed along $x^2 + y^2 = 1$
- The particle moves counterclockwise (sub in successive t values)
- The period of the motion is 2π; it takes 2π units of time to complete one circle.

5.19 Vector Calculus

5.19.1 Properties of Vector Derivatives

$$\frac{d}{dt}\left(\underset{\sim}{c}\right) = 0 \qquad \text{For constant vector } \underset{\sim}{c}$$

$$\frac{d}{dt}\left(k\underset{\sim}{r}(t)\right) = k\frac{d}{dt}\left(\underset{\sim}{r}(t)\right), k \in \mathbb{R}$$

$$\frac{d}{dt}\left(\underset{\sim}{r_1}(t) + \underset{\sim}{r_2}(t)\right) = \frac{d}{dt}\left(\underset{\sim}{r_1}(t)\right) + \frac{d}{dt}\left(\underset{\sim}{r_2}(t)\right)$$

$$\frac{d}{dt}\left(f(t) \cdot \underset{\sim}{r}(t)\right) = f(t)\frac{d}{dt}\left(\underset{\sim}{r}(t)\right) + \underset{\sim}{r}(t)\frac{d}{dt}\left(f(t)\right), f : \mathbb{D} \to \mathbb{R}$$

5.19.2 Vector Antidifferentiation

Consider:

$$\int \underset{\sim}{r}(\lambda)d\lambda = \int x(\lambda)\underset{\sim}{i} + y(\lambda)\underset{\sim}{j} + z(\lambda)\underset{\sim}{k}d\lambda = \left(\int x(\lambda)d\lambda\right)\underset{\sim}{i} + \left(\int y(\lambda)d\lambda\right)\underset{\sim}{j} + \left(\int z(\lambda)d\lambda\right)\underset{\sim}{k} = X(\lambda)\underset{\sim}{i} + Y(\lambda)\underset{\sim}{j} + Z(\lambda)\underset{\sim}{k} + \underset{\sim}{c}, \underset{\sim}{c} \in \mathbb{R}^3$$

Where $\frac{dX}{d\lambda} = x(\lambda)$ and $\frac{dY}{d\lambda} = y(\lambda)$ and $\frac{dZ}{d\lambda} = z(\lambda)$ and $\underset{\sim}{c}$ is a constant vector.

5.19.3 Velocity and Acceleration Along a Curve

Consider a particle with position vector $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$.

Velocity $\underset{\sim}{v} = \dot{\underset{\sim}{r}}(t) = \dot{x}(t)\underset{\sim}{i} + \dot{y}(t)\underset{\sim}{j}$

Acceleration $\underset{\sim}{a}(t) = \ddot{\underset{\sim}{r}}(t) = \ddot{x}(t)\underset{\sim}{i} + \ddot{y}(t)\underset{\sim}{j}$

Distance Between Points on Vector Function Curve $|\underset{\sim}{r}(t_1) - \underset{\sim}{r}(t_0)|$

Distance traversed along the curve If $\underset{\sim}{r}(\lambda) = x(\lambda)\underset{\sim}{i} + y(\lambda)\underset{\sim}{j}$ describes the path of a particle, the distance traversed along the curve from $\lambda = a$ to $\lambda = b$ is given by:

$$\int_a^b \sqrt{\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2}\, d\lambda$$

6 Complex Numbers

6.1 Tech

6.1.1 Lines in the Complex Plane

Define z:=x+y·**i**, then Menu+3+1 to get the solve() function and then input your complex relation, and solve for y.

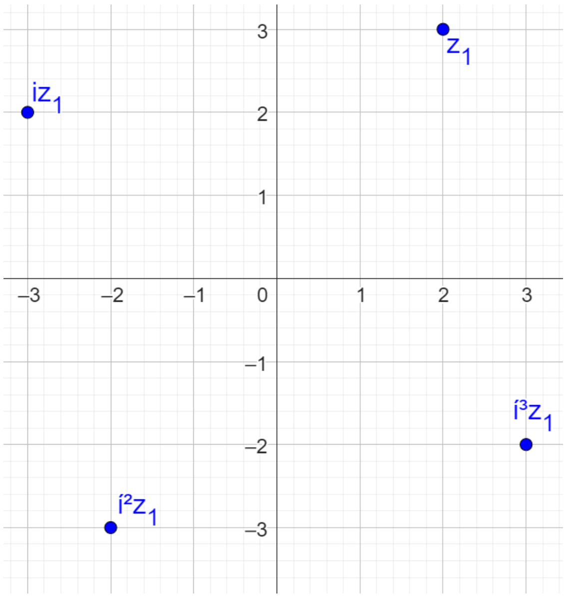
6.1.2 Relations with Arguments in the Complex Plane

For relations in form $\text{Arg}(z) = a$ we define z:=x+y·**i**, and then use Menu+2+9+4 to input the angle() function. We then menu+3+1 solve(angle(z)=a,y) to get the relation.

6.2 Operations of Complex Numbers in Cartesian Form

For $z_1 = a + bi$ and $z_2 = c + di$ for $a, b, c, d \in \mathbb{R}$

$$z_1 + z_2 = (a + c) + (b + d)i \qquad z_1 z_2 = (ac - bd) + (ad + bc)i$$
$$z_1 - z_2 = (a - c) + (b - d)i \qquad kz = ka + kbi$$



Multiplication by i $z \times i, z \in \mathbb{C}$ gives a $\frac{\pi}{2}$ rotation in the counterclockwise direction about the origin.

$$i^{4n} = 1 \qquad i^{4n+1} = i \qquad i^{4n+2} = -1 \qquad i^{4n+3} = -i$$

6.3 Properties of Conjugates

For $z = a + bi$, $\bar{z} = a - bi$

$$\overline{z_1 \mp z_2} = \bar{z}_1 + \bar{z}_2 \qquad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2 \qquad \overline{kz} = k\bar{z}, k \in \mathbb{R} \qquad z\bar{z} = |z|^2 \qquad z + \bar{z} = 2\text{Re}(z)$$

Multiplicative Inverse for Division If $z = a + bi \neq 0$ then

$$z^{-1} = \frac{a - bi}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$$

For division:

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = \frac{z_1 \bar{z}_2}{|z_2|^2}$$

6.4 Polar Form Operations

Conjugate $z = r\,\text{cis}\,\theta \implies \bar{z} = r\,\text{cis}\,(-\theta)$. On an Argand diagram this can be represented as a reflection of z in the Real axis.

Addition and Subtraction You need to convert to cartesian form to complete these operations using $\text{cis}\,\theta = (\cos\theta + i\sin\theta)$.

Scalar Multiplication If $k \in \mathbb{R}^+$ then $\text{Arg}(z) = Arg(kz)$. If $k \in \mathbb{R}^-$ then:

$$\text{Arg}(kz) = \begin{cases} \text{Arg}(z) - \pi & 0 < \text{Arg}(z) \leq \pi \\ \text{Arg}(z) + \pi & -\pi < \text{Arg}(z) \leq 0 \end{cases}$$

Complex Multiplication Let $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \qquad \text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) + 2k\pi, k \in \{-1, 0, 1\}$$

Complex Division Let $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, $r_2 \neq 0$.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2) \qquad \text{Arg}(z_1 z_2) = \text{Arg}(z_1) - \text{Arg}(z_2) + 2k\pi, k \in \{-1, 0, 1\}$$

$\text{Arg}(\frac{1}{z}) = -\text{Arg}(z)$ given $z \notin \mathbb{R}^-$

6.5 Quadratics over Complex Numbers

Sum of Squares Since $i^2 = -1$, we can rewrite a sum of squares as a difference of squares using i. $z^2 + a^2 = z^2 - (ai)^2 = (z + ai)(z - ai)$

6.6 Polynomials over Complex Numbers

Remainder Theorem Let $\alpha \in \mathbb{C}$. When a polynomial $P(z)$ is divided by $z - \alpha$, the remainder is $P(\alpha)$

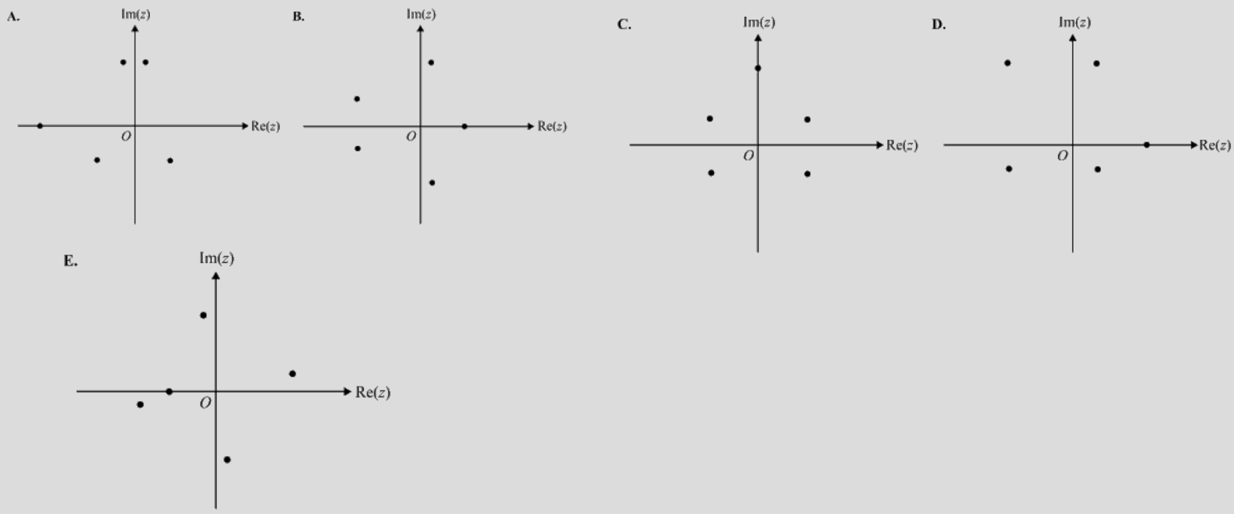
Factor Theorem Let $\alpha \in \mathbb{C}$. Then $z - \alpha$ is a factor of $P(z)$ **if and only if** $P(a) = 0$.

Conjugate Root Theorem Let $P(z)$ be a polynomial with real coefficients. If $a + bi$ is a solution to $P(z) = 0$, with $a, b \in \mathbb{R}$, then the complex conjugate $a - bi$ is also a solution.

The Fundamental Theorem of Algebra Every polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ of degree n, where $n \geq 1$ and the coefficients a_i are complex numbers, has at least one linear factor in the complex number system.

Exam-style Question: VCAA 2007 Spesh E2 A6

Which one of the following diagrams would represent the roots of $z^5, z^2 - z + c = 0$ in the complex plane for $z \in \mathbb{C}$ and $c \in$



$\mathbb{R}$?

As the coefficients of z are all real numbers, therefore the roots will exist in conjugate pairs. Thus it is **B**.

6.7 Solving with De Moivre

Equations of form $z^n = a, a \in \mathbb{C}$ can be solved with de Moivre's theorem.

Example: de Moivre's theorem to solve basic case

Let $z = r\operatorname{cis}\theta$ and $a = q\operatorname{cis}\phi$. Solve $z^n = a$.

$$\begin{aligned} z &= r\operatorname{cis}\theta, a = q\operatorname{cis}\phi & \operatorname{cis}(n\theta) &= \operatorname{cis}\phi \\ z^n &= a & n\theta &= \phi + 2k\pi, k \in \mathbb{Z} \\ r^n \operatorname{cis}(n\theta) &= q\operatorname{cis}\phi & \theta &= \frac{1}{n}(\phi + 2k\pi), k \in \mathbb{Z} \\ r &= \sqrt[n]{q} \end{aligned}$$

We then use the information to begin finding solutions.

Complex Roots

- For $n \in \mathbb{N}$ and $a \in \mathbb{C}$, the solutions $z^n = a$ are called the nth-roots of a
- Solutions lie evenly spaced on a circle about the origin of radius $|a|^{\frac{1}{n}}$.
- There are n solutions and they are spaced equally around the circle by intervals of $\frac{2\pi}{n}$

Roots of Unity For $n \in \mathbb{N}$ the solutions of $z^n = 1$ are called the nth-roots of unity.

- Solutions lie on the unit circle
- There are n solutions and they are spaced evenly by $\frac{2\pi}{n}$ around the circle
- $z = 1$ is always a solution

Sum of the nth-roots of Unity $z^n = 1$ has a set of solutions defined by the geometric sequence with ratio $w = \operatorname{cis}\left(\frac{2\pi}{n}\right)$. The sum of the nth-roots of unity is always 0.

$$1, w, w^2, w^3, w^4, ..., w^{n-1}, w = \operatorname{cis}\left(\frac{2\pi}{n}\right), n \in \mathbb{N}$$

$$\sum_{i=0}^{n-1} w^i = 0 \text{ For } w = \operatorname{cis}\left(\frac{2\pi}{n}\right), w^n = 1$$

6.8 Subsets of Complex Plane

6.8.1 Circles

A circle is defined as the set of points equidistant from a complex number.
For $z, z_1 \in \mathbb{C}$ we can form a circle of radius r about z_1 by:

$$(z - z_1)\overline{(z - z_1)} = r^2 \implies |z - z_1|^2 = r^2 \implies |z - z_1| = r$$

Combination of Moduli As $|z| = |\bar{z}|$, a linear combination of the moduli $a|z| + b|\bar{z}| = r(a + b)$ will be a circle of radius r.

Find Equation of Circle given Endpoints of Diameter If z_1 and z_2 are the endpoints of the diameter of a circle, then the equation of a circle can be determined in **diametric form**:

$$(z - z_1)\overline{(z - z_2)} + (z - z_2)\overline{(z - z_1)} = 0$$

Equation from Three Points Given three complex numbers that lie on a circle, the centre z_0 can be found by finding the intersection of the perpendicular bisectors of the points.

$$r = |z_0 - z_1| = |z_0 - z_2| = |z_0 - z_3| \qquad z, z_1, z_2, z_3 \in \mathbb{C}$$

6.8.2 Lines

Perpendicular Bisectors The set of points equidistant on the complex plane from two complex numbers form a line.

$$|z - z_1| = |z - z_2| \qquad z, z_1, z_2 \in \mathbb{C}$$

Converting Perpendicular Bisector to Cartesian Form First find midpoint of the line between z_1 and z_2. Determine the slope between the points and take its negative reciprocal. Substitute in to $y = m(x - x_1) + y_1$.

Exam-style Question: VCAA Spesh 2017 E2 B4f

The equation of the line passing through the two roots of $z^2+4z+16=0, z\in\mathbb{C}$ can be expressed as $|z-a|=|z-b|$. Find b in terms of a.

$$\text{Let } a=\text{Re}(a)+\text{Im}(a)i \qquad b=\text{Re}(b)+\text{Im}(b)i$$
$$\frac{\text{Re}(a)+\text{Re}(b)}{2}=-2 \qquad \text{Im}(b)=\text{Im}(a)$$
$$\implies \frac{a+b}{2}=-2+\text{Im}(a)i$$
$$\therefore b=-4-a+2\text{Im}(a)i=-4-(\text{Re}(a)+\text{Im}(a)i)+2\text{Im}(a)i=-4-\text{Re}(a)-\text{Im}(a)i+2\text{Im}(a)i$$
$$\therefore b=-4-(\text{Re}(a)-\text{Im}(a)i)=-4-\bar{a}$$

6.8.3 Rays

A ray extending at angle θ from positive direction of the $\text{Re}(z)$ axis originating from $z_1\in\mathbb{C}$ is defined by:

$$\text{Arg}(z-z_1)=\theta \qquad z,z_1\in\mathbb{C},\theta\in(-\pi,\pi)$$

A ray is not defined for $z=0+0i$ as there is no unique angle for $\text{Arg}(0+0i)$

Exam-style Question: VCAA 2019 Spesh A5 (modified) - Intersections of Rays

Let $z=x+yi$ for $x,y\in\mathbb{R}$. The rays $\text{Arg}(z-2)=\frac{\pi}{4}$ and $\text{Arg}\left(z-(5+i)\right)=\frac{5\pi}{6}$ for $z\in\mathbb{C}$, intersect at (a,b). Find the value of b.

$$\text{Arg}(z-2)=\frac{\pi}{4}\implies y=x-2, x>2 \qquad \text{Arg}\left(z-(5+i)\right)=\frac{5\pi}{6}\implies y=\frac{-1}{\sqrt{3}}(x-5)+1, x<5$$

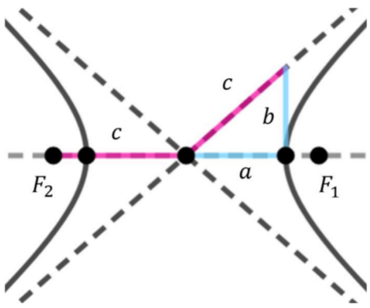
Consider (a,b)

$$\therefore b=a-2 \qquad b=\frac{-1}{\sqrt{3}}(a-5)+1$$
$$\therefore a=\sqrt{3}+2, b=\sqrt{3}$$

6.8.4 Hyperbolas

$$|z-z_1|-|z-z_2|=2a \qquad z,z_1,z_2\in\mathbb{C},a\in\mathbb{R}$$

z_1,z_2 are the foci of the ellipse, and a is the semi-major axis length.



Find Cartesian Form The centre of the hyperbola z_0 is the midpoint of the foci, that is $z_0=\frac{z_1+z_2}{2}$. The linear eccentricity c of the hyperbola is given $c=|z_0-z_1|=|z_0-z_2|$. The semi-major axis length a, the semi-minor axis length b and eccentricity c are related by $c^2=a^2+b^2 \implies b=\sqrt{c^2-a^2}$. We then use this to write a horizontal (left below) or vertical (right below) hyperbola.

$$\frac{(x-\text{Re}(z_0)^2)}{a^2}-\frac{(y-\text{Im}(z_0))^2}{b^2}=1 \qquad \frac{(y-\text{Im}(z_0))^2}{a^2}-\frac{(x-\text{Re}(z_0))^2}{b^2}=1$$

6.8.5 Intersections of Subsets

Exam-style Question: 2006 NEAP Spesh E2 A8

> The values of $\mathrm{Re}(z), z \in \mathbb{C}$ such that z belongs to $S \cap T$ where $S = \{z : |z-1| \leq 1\}$ and $T = \{z : \mathrm{Re}(z) + \mathrm{Im}(z) = 1\}$ are

Subset T represents a line with the equation $x+y=1$ or $y=1-x$. Subset S represents a circle with equation $(x-1)^2+y^2 \leq 1$. Substituting $1-x$ into the equation for S and solving in CAS gives $1-\frac{\sqrt{2}}{2} \leq x \leq \frac{\sqrt{2}}{2}+1$

7 Logic and Proof

7.1 Converses, Negations, Contrapositives

For a conditional statement $P \implies Q$:

Negation P and $\neg Q$

de Morgan's Law $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$
$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
$\neg(P \implies Q) \equiv P \wedge \neg Q$

Converse $Q \implies P$

Contrapositive $\neg Q \implies \neg P$

7.2 Proof of Equivalent Statements

To prove $P \iff Q$ both $P \implies Q$ and $Q \implies P$ have to be shown to be true.

7.3 Proof by Contradiction

First, assume the statement is false, then show that this assumption leads to mathematical nonsense. Therefore we conclude that the assumption was wrong, and therefore the statement must be true.

7.4 Quantifiers and Counterexamples

$$\forall: \text{ 'For all'} \qquad \exists: \text{ 'There exists'}$$

To prove universal statements ($\forall$) we need to provide a general argument that holds for every case in the set. To disprove universal statements, we need to provide a counterexample where it does not hold.
To prove an existence statement ($\exists$) we need to show that it holds for at least one member in the set. To disprove these statements, we prove that its negation is true.

7.4.1 Negations with Quantifiers

Recall de Morgan's laws, apply these as normal and then flip the quantifier symbol ($\forall$ or $\exists$).

7.5 Product and Sum Notation

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + ... + n^2 \qquad \prod_{i=1}^{n}(2i-1) = 1 \times 3 \times 5 \times ... \times (2n-1)$$

7.6 Direct Proof

Exam-style Question: 1000 Spesh Questions Doc Proof 3

Theorem. *There are no integers a and b such that $4a+8b=34$.*

Proof. $4a+8b=34 \implies 4(a+2b)=34$ and hence the statement is false as $\nexists k \in \mathbb{Z} \ni 34 = 4k$ ∎

7.7 Proof by Contradiction

To prove $P \implies Q$ by contradiction, we suppose the negation of the statement $\neg(P \implies Q)$ is true and show this results in nonsense.

Exam-style Question: 1000 Spesh Questions Doc Proof 2

Theorem. $\sqrt{3}+\sqrt{7}<2\sqrt{5}$

Proof. Suppose that $\sqrt{3}+\sqrt{7}\geq 2\sqrt{5}$. By squaring both sides we get $3+2\sqrt{21}+7\geq 20$.

$$\begin{aligned} 10+2\sqrt{21} &\geq 20 \\ \sqrt{21} &\geq 5 \\ 21 &\geq 25 \end{aligned}$$

$21\geq 25$ is a contradiction, and thus $\sqrt{3}+\sqrt{7}<2\sqrt{5}$ ■

Exam-style Question: 1000 Spesh Questions Doc Proof 43

Theorem. $\cos\theta+\sin\theta\leq\sqrt{2}$

Proof. Suppose that $\cos\theta+\sin\theta>\sqrt{2}$.

$$\begin{aligned} (\cos\theta+\sin\theta)^2 &> 2 \\ \cos^2\theta+2\cos\theta\sin\theta+\sin^2\theta &> 2 \\ 1+\sin 2\theta &> 2 \\ \sin 2\theta &> 1 \end{aligned}$$

As $\sin 2\theta\in[-1,1]$, a contradiction has been reached, thus $\cos\theta+\sin\theta\leq\sqrt{2}$. ■

Exam-style Question: 1000 Spesh Questions Doc Proof 51

Theorem. *The square root of an irrational number m is also irrational.*

Proof. Suppose that $\sqrt{m}=\frac{a}{b} \qquad a,b\in\mathbb{Z} \qquad m\notin\mathbb{Q}$.

$$b^2m=a^2$$

The RHS is an integer, but LHS is not since m is by definition irrational. Hence a contradiction has been reached. Hence the square root of the irrational number m is also irrational. ■

7.8 Arithmetic Mean-Geometric Mean Inequality

For $x,y\geq 0$, the arithmetic mean is greater than or equal to the geometric mean.

$$\frac{x+y}{2}\geq\sqrt{xy}$$

Example: AM-GM Inequality

Theorem. *For $a,b>0$ and $ab=24$, prove that $2a+3b\geq 24$.*

Proof. Assume $ab=24$, using the AM-GM inequality we find:

$$
\begin{aligned}
2a + 3b &= \frac{4a+6b}{2} \\
&\geq \sqrt{(4a)(6b)} = \sqrt{24ab} \\
&= \sqrt{24 \times 24} \\
&= 24
\end{aligned}
$$

$\therefore 2a + 3b \geq 24$ as required. ■

7.9 Proof by Induction

Exam-style Question: University of Waikato Induction Problems Set Proof 18

Theorem. *$5^{2n+1} + 2^{2n+1} = 7k, k \in \mathbb{Z}$ for all integers $n \geq 0$.*

Proof. ◊ *Hypothesis*:

$$P(n) : 5^{2n+1} + 2^{2n+1} = 7k, k \in \mathbb{Z}$$

◊ *Base case*:

$$P(0) : 5 + 2 = 7 = 7(1) \therefore \text{ holds}$$

◊ *Inductive hypothesis*:

$$\text{Assume } P(k) \text{ holds for all } k \in \mathbb{Z}^+ \cup \{0\} : 5^{2k+1} + 2^{2k+1} = 7m, m \in \mathbb{Z}$$

◊ *Inductive step*:

$$P(k+1) : 5^{2k+3} + 2^{2k+3} = 7x, x \in \mathbb{Z}$$

$$
\begin{aligned}
LHS = 5^2 \cdot 5^{2k+1} + 2^2 \cdot 2^{2k+1} = 25 \cdot 5^{2k+1} + 4 \cdot 2^{2k+1} &= 4\left(5^{2k+1} + 2^{2k+1}\right) + 21 \cdot 5^{2k+1} \\
&= 4(7m) + 7(3 \cdot 5^{2k+1}) \\
&= 7(4m + 3 \cdot 5^{2k+1}) = 7x
\end{aligned}
$$

Therefore $(\forall k \in \mathbb{Z}^+ \cup \{0\}) P(k) \implies P(k+1)$
By principle of mathematical induction, $P(n)$ holds for $n \geq 0$. ■

Exam-style Question: 1000 Spesh Questions Doc Induction Proof 11

Theorem. $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{2(n+2)}$ *for $n \geq 1$*

Proof. ◊ *Hypothesis*:

$$P(n) : \sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{2(n+2)}$$

◊ *Base case*:

$$
\begin{aligned}
P(1) : \text{LHS} = \quad & \sum_{r=1}^{1} \frac{1}{(r+1)(r+2)} = \frac{1}{6} \\
& \text{RHS} = \frac{1}{2(1+2)} = \frac{1}{6}
\end{aligned}
$$

LHS=RHS $\therefore P(1)$ holds

◊ *Inductive hypothesis*:

$$\text{Assume } P(k) \text{ holds for all } k \in \mathbb{Z} \cap [1, \infty) : \sum_{r=1}^{k} \frac{1}{(r+1)(r+2)} = \frac{k}{2(k+2)}$$

◊ *Inductive step*:

$$P(k+1) : \sum_{r=1}^{k+1} \frac{1}{(r+1)(r+2)} = \frac{k+1}{2(k+3)}$$

$$\begin{aligned} \text{LHS} = \sum_{r=1}^{k+1} \frac{1}{(r+1)(r+2)} &= \sum_{r=1}^{k} \frac{1}{(r+1)(r+2)} + \frac{1}{(k+2)(k+3)} = \frac{k+1}{2(k+2)} + \frac{1}{(k+2)(k+3)} \\ &= \frac{1}{k+2}\left(\frac{k}{2} + \frac{1}{k+3}\right) \\ &= \frac{1}{2(k+2)}\left(\frac{k^2+3k+1}{2(k+3)}\right) = \frac{1}{2(k+2)}\left(\frac{(k+1)(k+2)}{k+3}\right) \\ &= \frac{k+1}{2(k+3)} = \text{RHS} \end{aligned}$$

Therefore $(\forall k \in \mathbb{Z} \cap [1, \infty)) P(k) \implies P(k+1)$ and therefore by the principle of mathematical induction, $P(n)$ holds for all integers $n \geq 1$.

■

8 Circular Functions

8.1 Tech

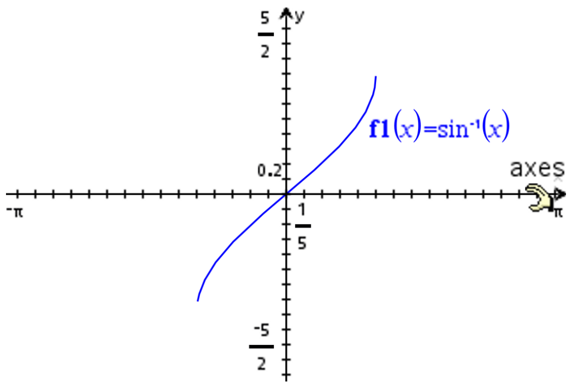
8.2 Inverse Circular Functions

Function	$\sin^{-1}$ or arcsin	$\cos^{-1}$ or arccos	$\tan^{-1}$ or arctan
Domain	$[-1, 1]$	$[-1, 1]$	R
Range	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

arcsin function when $y = \sin(x)$ is restricted to $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ the resulting function is one-to-one and therefore its inverse exists:

$$\arcsin : [-1, 1] \to \mathbb{R}, \arcsin(x) = y \qquad \text{where} \quad \sin(y) = x, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

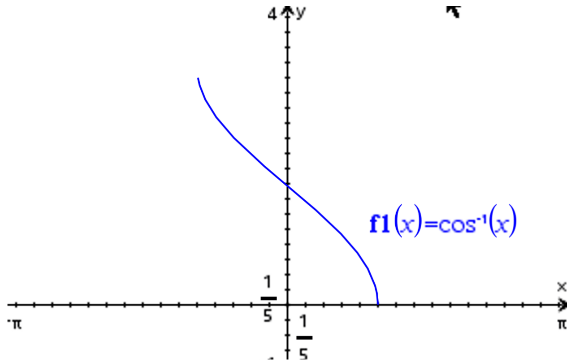
- domain of $\arcsin(x)$: $x \in [-1, 1]$
- range of $\arcsin(x)$: $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\arcsin(\sin(x)) = x \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\sin(\arcsin(x)) = x \forall x \in [-1, 1]$



arccos function when $y = \cos(x)$ is restricted to $x \in [0, \pi]$ the resulting function is one-to-one and therefore the inverse function exists.

$$\arccos : [-1, 1] \to \mathbb{R}, \arccos(x) = y \qquad \text{where} \quad \cos(y) = x, y \in [0, \pi]$$

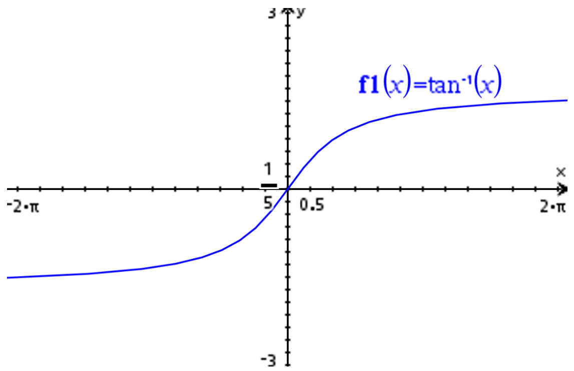
- domain of $\arccos(x)$: $x \in [-1, 1]$
- range of $\arccos(x)$: $y \in [0, \pi]$
- $\arccos(\cos(x)) = x \forall x \in [-1, 1]$
- $\cos(\arccos(x)) = x \forall x \in [0, \pi]$



arctan function when $y = \tan(x)$ is restricted to $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ the resulting function is one-to-one and therefore the inverse function exists.

$$\arctan : \mathbb{R} \to \mathbb{R}, \arctan(x) = y \qquad \text{where} \quad \tan(y) = x, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

- domain of $\arctan(x)$: $x \in \mathbb{R}$
- range of $\arctan(x)$: $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- $\arctan(\tan(x)) = x \forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- $\tan(\arctan(x)) = x \forall x \in \mathbb{R}$



Exam-style Question: 2007 NEAP Spesh E2 A7

> The domain of $f(x) = \sin(2x)$ is restricted so that it is one-to-one. The domain and range of the resulting inverse $f^{-1}(x)$, could be respectively:

The usual domain of $\sin(x)$ for one-to-one is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The range of $\sin(x)$ is $[-1, 1]$. The range is unchanged by the horizontal dilation, so the domain of $f^{-1}(x)$ will be $[-1, 1]$. The domain of $\sin(2x)$ to be one-to-one will halve, so it could be $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$. Of the options, $\left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$ is the only suitable option as it maintains the same period as $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$. Therefore the range of the inverse will be $\left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$. Therefore **C**.

8.3 Symmetry Properties

Sine

$$-\sin(\alpha) = \sin(2\pi - \alpha) \qquad \sin(\pi - \alpha) = \sin(\alpha) \qquad \sin\left(\alpha + \frac{\pi}{2}\right) = \cos(\alpha) \qquad \sin\left(\alpha - \frac{\pi}{2}\right) = -\cos(\alpha)$$

Cosine

$$-\cos(\alpha) = \cos(\pi - \alpha) \qquad \cos(2\pi - \alpha) = \cos(\alpha) \qquad \cos\left(\alpha - \frac{\pi}{2}\right) = \sin(\alpha) \qquad \cos\left(\alpha + \frac{\pi}{2}\right) = -\sin(\alpha)$$

Tangent

$$\tan\left(\alpha + \frac{\pi}{2}\right) = \tan\left(\alpha - \frac{\pi}{2}\right) - \cot(\alpha)$$

8.4 Solutions of Equations with Circular Functions

Sine general solutions For $a \in [-1, 1]$ the general solution of $\sin(x) = a$ is:

$$x = 2n\pi + \arcsin(a) \qquad \text{or} \qquad x = (2n + 1)\pi - \arcsin(a), n \in \mathbb{Z}$$

Cosine general solutions For $a \in [-1, 1]$ the general solution of $\cos(x) = a$ is:

$$x = 2n\pi \pm \arccos(a), n \in \mathbb{Z}$$

Tangent general solutions For $a \in \mathbb{R}$ the general solution of $\tan(x) = a$ is:

$$x = n\pi + \arctan(a), n \in \mathbb{Z}$$

8.5 Product-to-sum identities

$$2\cos(x)\cos(y) = \cos(x-y) + \cos(x+y) \qquad 2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y) \qquad 2\sin(x)\cos(y) = \sin(x+y) + \sin(x-y)$$

8.6 Sum-to-product identities

$$\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \qquad \cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \qquad \sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

9 Stats and Probability :skull:

9.1 Tech

9.1.1 sam_prob\bpd(n,p,lower,upper)

Outputs the probability distribution of a random binomial variable with n trials and probability of success p for a range of successes from lower to upper inclusive. It will also output the total probability of that range of successes.

9.1.2 sam_prob\pdanal(x,px)

Fill in the values from the **discrete** probability distribution where the set x is the row of x values and px is the row of $\Pr(X = x)$ values. It will tell you the mean and variance of that distribution.

9.1.3 scriptedmath\expec(f(x),x)

Find the expected value of a continuous distribution with a density function f(x).

9.1.4 scriptedmath\var(f(x),x)

Find the variance of a continuous distribution with a density function f(x).

9.1.5 scriptedmath\sd(f(x),x)

Find the standard deviation of a continuous distribution with a density function f(x).

9.1.6 scriptedmath\mode(list,freqlist)

Find the mode of list.

9.1.7 scriptedmath\ns(eq,var)

Finds a relation to describe the non-negative integer solutions for the variable var to the equation.

9.1.8 normcdf(lower, upper, μ, σ)

Finds the area under a normal density curve over an interval. For $\Pr(X \geq x)$ set the lower bound to be x and the upper bound to be ∞. For $\Pr(X \leq x)$ set the upper bound to be x and the lower bound to be $-\infty$. Input σ and μ as given in the question.

9.1.9 invnorm(left tail prob, μ, σ)

Used for solving $Pr(X > x_c) = C$. Input invnorm(1-C, μ, σ) to get the value for x_c.

9.1.10 binomcdf(n, p, lower, upper)

Gives the $\Pr(\mathit{lower} < X < \mathit{upper})$ for $X \sim \mathrm{Bi}(n, p)$.

9.1.11 binompdf(n, p, x)

Gives the $\Pr(X = x)$ for $X \sim \mathrm{Bi}(n, p)$.

9.1.12 invbinomn(cumulative prob, prob, numsuccess)

Menu+5+5+D. Inverse binomial with respect to N (number of trials). Given the probability of success on each trial (prob), and the number of successes (numsuccesses), this function returns the minimum number of trials, N, such that for $X \sim \mathrm{Bi}(N, \mathit{prob})$, that $\Pr(X < N)$ is less than cumulative probability.

9.1.13 sam_prob\normsum(x1,ltp1,x2,ltp2)

Find the mean and standard deviation of a normal distribution given the left-tail probability of two values. Put in one of the values for x1 and its left-tail probability for ltp1, and do the same for the second value.

9.1.14 Determine the Mean and Standard Deviation of a Normal Distribution

Suppose we have a normal distribution with mean 2 and unknown standard deviation, and we know that $\Pr(X > 3) = 0.3$. We first standardise to the standard normal random variable $Z \sim N(0, 1^2)$. Using the left tail area (in this case 1-0.3=0.7) to find the percentile on the standard normal distribution, then solve this for $\dfrac{3-\mu}{\sigma}$ substituting appropriately to obtain the unknown standard deviation. The same process can be used for an unknown mean.

Unknown Mean and Unknown Standard Deviation Suppose $X \sim N(\mu, \sigma^2)$, but we know $\Pr(X > 5) = 0.65$ and $\Pr(X < 1) = 0.05$. Observe the following:

$$\Pr(X > 5) = \Pr\left(Z > \frac{5-\mu}{\sigma}\right) = 0.65 \qquad \text{and} \qquad \Pr(X < 1) = \Pr\left(Z < \frac{1-\mu}{\sigma}\right) = 0.05$$

We can then use **invnorm()** to find the z-scores of each equation above, and set up an appropriate system of equations to solve.

$$\text{invnorm}(1 - 0.65, 0, 1) = \frac{5-\mu}{\sigma} \qquad \text{and} \qquad \text{invnorm}(0.05, 0, 1) = \frac{1-\mu}{\sigma}$$

9.1.15 Simulating Sample Means

randNorm(μ, σ, n) will give a list of n values randomly selected in a normal distribution of mean μ and standard deviation σ. To find the sample mean apply mean() to this function.

To do this for m samples, add a **Lists and Spreadsheets** page in the CAS. In column A in the formula box input seq(mean(randNorm(μ
Remember n is the sample size and m is the number of samples. Name the column smeans. Now add a **Data and Statistics** page and add smeans to the x-axis.

9.1.16 Confidence Interval (Spesh)

To quickly find a confidence interval, in a calculator window go to menu+6+6+1. Set input method to stats, then input the necessary values.

9.1.17 Determine a Confidence Interval (Methods)

Menu+6+6+5 inserts the 1-Prop z interval function. Input the number of successes x, the sample size n, and the confidence level C.

Exam-style Question: 2016 VCAA Methods B3g

> A school has a class set of 22 new laptops kept in a recharging trolley. Provided each laptop is correctly plugged into the trolley after use, its battery recharges.
>
> On a particular day, a class of 22 students uses the laptops. All laptop batteries are fully charged at the start of the lesson. Each student uses and returns exactly one laptop. The probability that a student does not correctly plug their laptop into the trolley at the end of the lesson is 10%. The correctness of any student's plugging-in is independent of any other student's correctness.
>
> The laptop supplier finds that, in a particular sample of 100 laptops, six of them have a battery life of less than three hours.
>
> Determine the 95% confidence interval for the supplier's estimate of the proportion of interest. Give the values correct to two decimal places.

menu+6+6+5 then input successes=6, n=100, C=0.95

9.1.18 Hypothesis Testing - One Tail

Menu+6+7+1 $\rightarrow$ zTest. μ_0 is the original population mean. σ is the general standard deviation (do not divide by $\sqrt{n}$). $\bar{x}$ is the sample mean. n is the sample size. Choose the type of alternative hypothesis and it will give you the necessary values.
For two tail-tests: Do the same as for a one tail test, but choose the $\mu \neq \mu_0$ test type.

9.1.19 nSolve for brute force solving

In situations where solve() doesn't give a solution, even in approximate form, you can use nsolve() to brute force solve an equation.

9.1.20 Type II Error Probability

Instances of one-tail tests For a 5% significance one-tail hypothesis test, in order to calculate the probability of type II error use **invnorm(0.05,μ,$\frac{\sigma}{\sqrt{n}}$)** to get the value for the sample mean which determines the boundary between rejection and acceptance of H_0. From this, execute a **normcdf(ans,∞,true mean,$\frac{\sigma}{\sqrt{n}}$)** to find the probability of a Type II error.

Instances of two-tail tests For a 5% significance two-tail hypothesis test, in order to calculate the probability of type II error use **invnorm(0.025,μ,$\frac{\sigma}{\sqrt{n}}$)** to get the lower bound value for the 'acceptance region' and save this in a variable name like l. From this, do $\mu - l$ and save this as d then $\mu + d$ (save this as u). Then execute **normcdf(l,u,true mean,$\frac{\sigma}{\sqrt{n}}$)** to find the probability of a Type II error.

9.2 Linear Functions of Random Variables

Consider a random variable Y which is a linear function of random variable X

$$Y = aX + b \qquad a, b \in \mathbb{R}$$

Linear Functions of Discrete Random Variables If X is a discrete random variable $Y = aX + b$ is also a discrete random variable.

Linear Functions of Continuous Random Variables If X is a continuous random variable, $Y = aX + b, a \neq 0$ is also a continuous random variable. For $a > 0$:

$$\Pr(Y \leq y) = \Pr(aX + b \leq y) = \Pr\left(X \leq \frac{y-b}{a}\right)$$

$$\therefore \Pr(Y \leq y) = \int_{-\infty}^{\frac{y-b}{a}} f(x)dx$$

Mean of Linear Functions of Random Variables

Mean of Discrete Random Variables For D.R.V's, the expected value $\mathrm{E}(X)$ of X gives:

$$\mathrm{E}(X) = \sum_x x \cdot \Pr(X = x)$$

Mean of Continuous Random Variables For C.R.V's, the expected value $\mathrm{E}(X)$ of X gives:

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx$$

For $Y = aX + b$:

$$\mathrm{E}(Y) = \mathrm{E}(aX + b) = a\mathrm{E}(X) + b$$

Variance of Linear Functions of Random Variables Generally, for X is a random variable with variance σ^2, and $Y = aX + b$ for $a, b \in \mathbb{R}$:

$$\mathrm{Var}(Y) = a^2\mathrm{Var}(X) = a^2\sigma^2$$
$$\mathrm{s.d}(Y) = \sqrt{a^2\sigma^2} = |a|\sigma$$

9.3 Independent Events

$$\text{Events } A, B \text{ are independent} \iff \Pr(A \cap B) = \Pr(A) \times \Pr(B) \qquad \Pr(A|B) = \Pr(A)$$

9.4 Mutually Exclusive Events

Two events A and B are mutually exclusive:

$$\Pr(A \cap B) = 0 \qquad \Pr(A \cup B) = \Pr(A) + \Pr(B)$$

9.5 Linear Combination of Random Variables

The idea of independent events also applies to random variables. If two random variables' joint probability function is the product of their individual probability functions, the random variables are said to be independent.

aX vs X+X+X+...+X Consider X representing the number that turns up on a dice roll. Then $2X$ is the result of one dice roll, doubled. $X+X$ is the the equivalent of rolling the dice twice and adding the results.

$$\mathbb{E}(2X) = \mathbb{E}(X+X) = 2\mathbb{E}(X)$$

$$\text{Var}(2X) = 2^2\text{Var}(X) = 4\text{Var}(X) \qquad \text{vs} \qquad \text{Var}(X+X) = 1^2\text{Var}(X) + 1^2\text{Var}(X) = 2\text{Var}(X)$$

Sum of Identically Distributed Independent Random Variables Variables with the same mean and standard deviation and are independent. The events predicted by the random variables are also independent, so therefore we can find the sum of probabilities by multiplying the individual probabilities. Consider X_1 and X_2 being discrete random variables for two dice rolling.

$$\Pr(X_1 + X_2 = 2) = \Pr(X_1 = 1, X_2 = 2) = \Pr(X_1 = 1) \times \Pr(X_2 = 1) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

Mean of Sum of n Identically Distributed Independent Random Variables $\text{E}(X_1 + X_2 + ... + X_n) = \text{E}(X_1) + \text{E}(X_2) + ... + \text{E}(X_n) = n\mu$

Variance of Sum of n Identically Distributed Independent Random Variables $\text{Var}(X_1 + X_2 + ... + X_n) = \text{Var}(X_1) + \text{Var}(X_2) + ... + \text{Var}(X_n) = n\sigma^2$
$\text{sd}(X_1 + X_2 + ... + X_n) = \sqrt{\text{Var}(X_1 + X_2 + ... + X_n)} = \sqrt{n}\sigma$

Linear Combinations of n Independent Random Variables A random variable Y which is a linear combination of random variables $X_1, X_2, ..., X_n$ is given by $Y = a_1X_1 + a_2X_2 + ... + a_nX_n$ for $a_1, a_2, ..., a_n \in \mathbb{R}$.

$$\text{E}(a_1X_1 + a_2X_2 + ...a_nX_n) = a_1\text{E}(X_1) + a_2\text{E}(X_2) + ... + a_n\text{E}(X_n)$$

$$\text{Var}(a_1X_1 + a_2X_2 + ... + a_nX_n) = a_1^2\text{Var}(X_1) + a_2^2\text{Var}(X_2) + ... + a_n^2\text{Var}(X_n)$$

$$\text{sd}(a_1X_1 + a_2X_2 + ... + a_nX_n) = \sqrt{a_1^2\text{Var}(X_1) + a_2^2\text{Var}(X_2) + ... + a_n^2\text{Var}(X_n)}$$

Linear Combinations of Normal Random Variables A linear combination of normally distributed random variables is also a normally distributed random variable.

9.6 Discrete Probability

Exam-style Question: 2021 NEAP Methods MCQ 9

A discrete random variable X has the following probability distribution, where k and m are real constants.

x	0	1	2	3
$\Pr(X = x)$	m	k	$k - \frac{1}{4}$	k^2

The maximum value of m is:

$$0 \leq \Pr(X = 2) \leq 1 \implies 0 \leq k - \frac{1}{4} \leq 1$$

$$\therefore k_{min} = \frac{1}{4}$$

$$m = 1 - \left(\frac{1}{4} + 0 + \frac{1}{4}^2\right)$$

$$= \frac{11}{16}$$

Exam-style Question: 2016 VCAA Methods MCQ 19

Consider the discrete probability distribution with random variable X below:

x	-1	0	b	$2b$	4
$\Pr(X = x)$	a	b	b	$2b$	0.2

The smallest and largest possible values of $\mathbb{E}(X)$ are:

$$\mathbb{E}(X) = -a + b^2 + 4b^2 + 0.8 = 0.8 + 5b^2 - a$$
$$a + 4b + 0.2 = 1 \to a + 4b = 0.8$$
$$\mathbb{E}(X) = 0.8 + 5b^2 - (0.8 - 4b) = 5b^2 + 4b = b(4 + 5b)$$
$$\text{Parabola with intercept } b = 0, b > 0$$
$$\therefore \mathbb{E}_{min}(X) = 0$$
$$0 \leq a \leq 0.8 \to 0 \leq 0.8 - 4b \leq 0.8$$
$$\implies -0.8 \leq -4b \leq 0 \to 0.2 \geq b \geq 0$$
$$\text{Taking maximum value of b: } b = 0.2$$
$$\therefore \mathbb{E}_{max}(X) = 0.2 \cdot (4 + 5 \cdot 0.2) = 1$$

9.7 Normal Probability

$$X \sim N(\mu, \sigma^2)$$

Symmetry Properties of Normally Distributed Probabilities Consider $Z \sim \mathrm{N}(0, 1)$.

- $\Pr(Z > a) = 1 - \Pr(Z \leq a)$
- $\Pr(Z < -a) = \Pr(Z > a)$
- $\Pr(-a < Z < a) = 1 - 2\Pr(Z \geq a) = 1 - 2\Pr(Z \leq -a)$

Exam-style Question: 2010 NEAP Methods E2 B3

A winery produces bottles of wine in two sizes: standard and large. For each size, the content of the bottles are normally distributed according to the following:

Bottle Size	Mean	Standard Deviation
standard	0.760	0.008
large	μ	σ

The probability that a randomly selected standard bottle contains fewer than 0.750 litres of wine is known to be 0.1056. Of the large bottles, 8% contain more than 1.023 litres and 4% contain less that 0.994 litres. Show that the mean and standard deviation of the large bottles can be found by solving the system of equations: $\mu + 1.4051\sigma = 1.023$ and $\mu - 1.7507\sigma = 0.994$.

$X \sim N(\mu, \sigma^2)$ $\quad$ $\Pr(X > 1.023) = 0.08$ $\quad$ $\Pr(X < 0.994) = 0.04$

$Z \sim N(0, 1)$

$\Pr(X > 1.023) = \Pr\left(Z > \dfrac{1.023 - \mu}{\sigma}\right) = 0.08$ $\quad$ **invnorm(0.92,0,1) gives** $\dfrac{1.023 - \mu}{\sigma} = 1.4051$ standard deviations

$\Pr(X < 0.994) = \Pr\left(Z < \dfrac{0.994 - \mu}{\sigma}\right) = 0.04$ $\quad$ **invnorm(0.04,0,1) gives** $\dfrac{0.994 - \mu}{\sigma} = -1.7507$ standard deviations

$\therefore \mu + 1.4051\sigma = 1.023$

$\mu - 1.7507\sigma = 0.994$ as required

Exam-style Question: 2009 Methods E2 B3d

content...

9.8 The Binomial Distribution

Bernoulli Sequence describes a sequence of trials for which there are two possible outcomes (success, failure), the probability of success p is constant for all trials, and the trials are independent. The number of successes in a Bernoulli sequence of n trials is called a Binomial Random Variable denoted $X \sim \text{Bi}(n, p)$

Exam-style Question: Suzanne Cory 2018 SAC3 Part II Q2e/f

> In an isolated incident, Ricardo got his CAS stolen when he left it unattended in a breakout room in the G16-21 area after school one day. He found it later being sold on Gumtree.
> Chelsea, from his homegroup, thinks she saw who walked into the breakout room. However, she was sitting quite far away, and her accusations are flimsy at best. To verify her claim, Mr. Mott tested her eyesight, by assessing her ability to identify a student from the claimed distance away.
> The probability that she would correctly identify a student is 0.8. In the test to identify 8 students from distance, B is the random variable following a binomial distribution for the number of students she identifies correctly.
>
> e) What is the probability the she identified exactly 7 students correctly, given that she correctly identified the first two (give your answer to 4dp).

We know $B \sim \text{Bi}(8, 0.8)$ We know the outcomes of two out of eight trials, both being successes. So we can treat this as a binomial variable $B_1 \sim \text{Bi}(2, 0.8)$, we then treat the remaining 6 trials as a binomial variable $B_2 \sim \text{Bi}(6, 0.8)$.

$$\Pr(B = 7|\text{first two are correct}) = \Pr(B_2 = 5|B_1 = 2) = \frac{\Pr(B_2 = 5) \times \Pr(B_1 = 2)}{\Pr(B_1 = 2)}$$

$$\frac{\Pr(B_2 = 5) \times \Pr(B_1 = 2)}{\Pr(B_1 = 2)} = \Pr(B_2 = 5) = 0.3932 \qquad \textbf{binompdf(6,0.8,5)}$$

> f) What is the minimum number of students in the test to ensure that the probability of Chelsea identifying at least 7 of them is 0.46?

$$\begin{aligned} & X \sim \text{Bi}(n, 0.8) \\ & \Pr(X \geq 7) \geq 0.46 \\ \implies & \Pr(X \leq 6) \leq 0.54 \\ \implies & n = 8 \qquad \textbf{invBinomN(0.54,0.8,6)} \end{aligned}$$

9.8.1 Inverse Binomial

Exam-style Question: 2008 VCAA Methods MCQ 5

> Let X be a discrete random variable with a binomial distribution. The mean of X is 1.2 and the variance of X is 0.72. The values of n (the number of independent trials) and p (the probability of success in one trial) are:

$$\mu = np = 1.2$$
$$\sigma^2 = np(1 - p)$$
$$\text{Solving the system gives } n = 3, p = 0.4$$

Exam-style Question: 2022 VCAA NHT Methods QB4f

> The proportion of custard doughnuts as a proportion of all doughnuts made in the bakery is 0.44. Find the minimum number of doughnuts required in a box to ensure that the probability of having at least 12 custard doughnuts in a box is greater than 90%.

invBinomN(0.1,0.44,11) gives 35, the minimum number of doughnuts. Cumulative probability is 1-0.9=0.1. Probability of success if the proportion, and the number of successes is one less than the threshold (12).

9.9 Sample Means

Populations and samples

- **Population** is the set of all eligible members of the intended group of study.
- **Sample** is a subset of the population which is selected to make inferences about the population.
- The **population mean** μ is the mean of all values of a measure in the entire population.
- The **sample mean** $\bar{x}$ is the mean of the values of measure in a particular sample. Since this varies based on the contents of a random sample, we consider $\bar{x}$ as being values of a random variable $\bar{X}$

Sample Mean of a normal random variable Let X be a normally distributed random variable with mean μ and standard deviation σ. Let $X_1, X_2, ..., X_n$ represent a sample size of n selected from this population. The sample mean is defined as:

$$\bar{X} = \frac{X_1 + X_2 + ... + X_n}{n}$$

The sample mean $\bar{X}$ is normally distributed with $\mathrm{E}(\bar{X}) = \mu$ and $\mathrm{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$.

Example: Edrolo Sample Mean MCQ

> The diameter, in metres, of trees in a certain forest is distributed with the following density function:
>
> $$g(x) = \frac{12}{2401}(x-3)(x-10)^2, 3 \leq x \leq 10$$
>
> Find the probability that the mean diameter of a sample of 40 trees is less than 6m to 2 decimal places.

Let G be the random variable representing the diameter of a randomly selected tree. The sample size is large enough that the distribution of sample means will be approximately normal.

$$\mathbb{E}(G) = \int_3^{10} x \times \frac{12}{2401}(x-3)(x-10)^2 dx = \frac{29}{5}$$

$$\mathrm{Var}(G) = \mathbb{E}(G^2) - [\mathbb{E}(G)]^2 = \int_3^{10} x^2 \times \frac{12}{2401}(x-3)(x-10)^2 dx - (\frac{29}{5})^2 = \frac{49}{25}$$

We use these results to find the distribution of $\bar{G}$.

$$\mathbb{E}(\bar{G}) = \mathbb{E}(G) = \frac{29}{5}$$

$$\mathrm{sd}(\bar{G}) = \frac{\mathrm{sd}(G)}{\sqrt{40}} = \frac{7}{5\sqrt{40}} = \frac{7}{10\sqrt{10}}$$

For $\Pr(\bar{G} < 6)$, we can use **normcdf(-∞,6,$\mathbb{E}(\bar{G})$,sd($\bar{G}$))**, which gives $\Pr(\bar{G} < 6) = 0.82$

9.10 Normal Approximation of Sample Proportions

The distribution of the sample mean $\bar{X}$ is normal with mean $\bar{x} = \mu$ and the standard deviation $s = \frac{\sigma}{\sqrt{n}}$.

$$\Pr\left(\bar{X} > a\right) = \Pr\left(Z > \frac{a - \bar{x}}{s}\right) \qquad Z \sim (0,1)$$

Example: FreeVCENotes Normal Approximation

> Use the normal approximation to the binomial distribution to find the approximate probability, to 4 decimal places, that in the next 600 rolls of a fair die, the proportion of sixes rolled is less than 0.2.

$$\begin{aligned}
&\text{Let } \hat{P} \text{ be the proportion of sixes rolled}\\
\mathbb{E}(\hat{P}) &= p = \frac{1}{6}\\
\mathrm{sd}(\hat{P}) &= \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{\frac{1}{6}\left(\frac{5}{36}\right)}{600}}\\
&= \sqrt{\frac{1}{4320}}\\
\hat{P} &\sim N\left(\frac{1}{6}, \sqrt{\frac{1}{4320}}\right)\\
\Pr(\hat{P} < 0.2) &= \textbf{normcdf}\left(0, 0.2, \frac{1}{6}, \sqrt{\frac{1}{4320}}\right)\\
&= 0.9858
\end{aligned}$$

9.11 Sample Proportions $\hat{P}$ without Normal Approximation (Methods)

Mean, Variance, Standard Deviation Using the mean and variance from the binomial, we can derive the following about the sample proportion $\hat{P}$:

$$\hat{P} = \frac{X}{n}, \quad X \sim \mathrm{Bi}(n, p)$$

$$\mathbb{E}(\hat{P}) = p \qquad \mathrm{Var}(\hat{P}) = \frac{p(1-p)}{n} \qquad \mathrm{sd}(\hat{P}) = \sqrt{\mathrm{Var}(\hat{P})}$$

Exam-style Question: 2016 VCAA Methods A17

> Inside a container there are one million coloured building blocks. It is known that 20% of the blocks are red. A sample of 16 blocks is taken from the container. For samples of 16 blocks, $\hat{P}$ is the random variable of the distribution of sample proportions of red blocks. **(Do not use a normal approximation.)**

Let $X \sim Bi(16, 0.2)$ be a binomial variable representing the number of red blocks in the sample.

$$\hat{P} = \frac{X}{16} \implies X = 16\hat{P}$$

$$\begin{aligned}
\Pr\left(\hat{P} \geq \frac{3}{16}\right) &= \Pr\left(X \geq \frac{3}{16} \times 16\right)\\
&= 0.6482 \qquad\qquad \textbf{binomcdf(16,0.2,3,16)}
\end{aligned}$$

Exam-style Question: 2016 VCAA Methods E2 B3d

> A supplier of laptops decides to take a sample of 100 new laptops from a number of different schools. For samples of size 100 from the population with a mean battery life of 3 hours 10 minutes and a standard deviation of 6 minutes, $\hat{P}$ is the random variable of the distribution of sample proportions of laptops with battery life less than three hours. It is known that the probability that the probability that a single laptop's battery life is less than 3 hours is 0.0478. Find $Pr(\hat{P} \geq 0.06|\hat{P} \geq 0.05)$ without a normal approximation.

Let Y be the number of laptops with battery less than 3 hours

$Y \sim \text{Bi}(100, 0.0478)$

$$\hat{P} = \frac{Y}{100} \implies \Pr\left(\hat{P} \geq 0.06|\hat{P} \geq 0.05\right) = \Pr\left(Y \geq 0.06 \times 100|Y \geq 0.05 \times 100\right)$$

$$= \Pr\left(Y \geq 6|Y \geq 5\right)$$

$$= \frac{\Pr(Y \geq 6)}{\Pr(Y \geq 5)} = 0.658$$

9.12 Distribution of the sample mean

9.12.1 Central Limit Theorem

Let X be any random variable with mean μ and standard deviation σ. Then, provided that the sample size n is large enough, the distribution of $\bar{X}$ is approximately normal with mean $\text{E}(\bar{X}) = \mu$ and $\text{sd}(\bar{X}) = \frac{\sigma}{\sqrt{n}}$.

9.12.2 Normal Approximation to the Binomial Distribution

If X is a binomial random variable with parameters n, p, then the distribution of X is approximately normal with mean $\mu = np$ and standard deviation $\sigma = \sqrt{np(1-p)}$ provided $np > 5$ and $n(1-p) > 5$.

9.13 Confidence Intervals for the Population Mean

Interpretation of a Confidence Interval For a 95% confidence interval, we expect 95% of such intervals to contain the population mean μ. Whether or not a particular confidence interval contains μ is generally unknown.

Point Estimates The value of the sample mean $\bar{x}$ can be used to estimate the population mean μ, as this is a single-valued estimate, it is called a point estimate of μ.

Interval Estimates An interval estimate for the population mean μ is called a confidence interval for μ.

C% Confidence Interval An approximate $C\%$ confidence interval for μ is given by

$$\left(\bar{x} - z\frac{\sigma}{\sqrt{n}}, \bar{x} + z\frac{\sigma}{\sqrt{n}}\right)$$

where $z \ni \Pr(-z < Z < z) = C\%$, $\bar{x}$ is the sample mean, σ is the standard deviation of the population and n is the size of the sample from which $\bar{x}$ was calculated.

The values of z for common confidence intervals:

- $90\% \to z = 1.6449$
- $95\% \to z = 1.9600$
- $99\% \to z = 2.5758$

To find the z score to use for any $C\%$ confidence interval in CAS: **invnorm($\frac{C}{2}$,0,1)** will give you the value for z.

Exam-style Question: VCAA Spesh 2017 Exam 2 A19

> A confidence interval is to be used to estimate the population mean μ based on a sample mean $\bar{x}$. To decrease the width of a confidence interval by 75%, the sample size must be multiplied by a factor of...

To reduce by 75% is the equivalent to making it a quarter the width, i.e. dividing by four:

$$\frac{\sigma}{4\sqrt{n}} = \frac{\sigma}{\sqrt{16n}}$$

Therefore the sample size must be multiplied by a factor of 16.

9.14 Hypothesis Testing for Mean

Null hypothesis The null hypothesis is denoted by H_0 says that the sample is drawn from a population which has the same mean as before (the mean has not changed after taking the sample). Under this hypothesis, any difference between a sample statistic and a population parameter is explained by sample-sample variation.

Alternative hypothesis The alternative hypothesis H_1 says that the population mean has changed.

p-Values The p value is the probability of observing a value of the sample statistic as extreme as or more extreme than the one observed, assuming that the null hypothesis is true.

Statistical Significance The significance level represents the threshold of unlikelihood that a result must have to cast sufficient doubt on the null hypothesis. This is denoted by α.

If the p-value is less than α, we reject the null hypothesis in favour of the alternative hypothesis. If the p-value is greater than α, we do not reject the null hypothesis.

z-test is the name for the hypothesis test for a mean of a sample drawn from a normally distributed population with known standard deviation.

9.14.1 One-tail tests

For directional alternative hypotheses, one-tail tests are used to calculate p-values.
For $H_1 : \mu > x$ the p-value is calculated considering only values in the upper tail of the normal curve.
For $H_1 : \mu < x$ the p-value is calculated considering only values in the lower tail of the normal curve.

9.14.2 Two-tail tests

Non-directional alternative hypothesis p-values are determined with two-tail tests. $H_1 : \mu \neq x$.

$$p = \Pr\left(\left|\bar{X} - \mu\right| \geq |\bar{x} - \mu|\right)$$

p-value (two-tail)$=2\times p$-value (one-tail)

9.14.3 Relation to Confidence Intervals

Consider the fact of the real number line:

$$a \in (b-c, b+c) \iff |a-b| < c \iff b \in (a-c, a+c)$$

Suppose we have:

$$H_0 : \mu = \mu_0 \qquad H_1 : \mu \neq \mu_0$$

Then:

$$\mu_0 \notin \left(\bar{x} - 1.96\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}\right) \iff \bar{x} \notin \left(\mu_0 - 1.96\frac{\sigma}{\sqrt{n}}, \mu_0 + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

Hence the 95% confidence interval does not contain μ_0 if and only if we should reject H_0 at a 5% significance.

9.14.4 Hypothesis Testing Errors

Type I Rejecting the null hypothesis when it is true. Pr(Type I)$=$ Pr(H_0 rejected$|H_0$ true) $=$ level of significance of test

Type II Not rejecting the null hypothesis when it is false. Pr(Type II)$=$ Pr(H_0 not rejected$|H_0$ false)

Exam-style Question: VCAA 2023 Specialist E2 Sample

> The time taken by a machine to make electronic components varies normally with a mean of 20 seconds and a standard deviation of 2 seconds. After the machine is serviced, it is believed that the mean time taken has been reduced to 18.5 seconds with the standard deviation remaining the same. A statistical test is proposed to check whether there is any evidence of a 1.5 second reduction in the mean time taken to make components. The test statistic will be the mean time taken to make a random sample of 16 such components. The type I error for this test will be $\alpha = 5\%$ with a critical sample mean of 19.2 seconds. Find the type II error β for the test.

This is a one-tail test (as it is reduction, not simply changing). Therefore we use **invnorm**$(0.05, 20, \frac{2}{\sqrt{16}})$ to get 19.17757... We then input this into **normcdf**$(19.17757..., \infty, 18.5, \frac{2}{\sqrt{16}})$=0.088 (3dp)

Question 6 (8 marks)

The heights of mature water buffaloes in northern Australia are known to be normally distributed with a standard deviation of 15 cm. It is claimed that the mean height of the water buffaloes is 150 cm.

To decide whether the claim about the mean height is true, rangers selected a random sample of 50 mature water buffaloes. The mean height of this sample was found to be 145 cm.

A one-tailed statistical test is to be carried out to see if the sample mean height of 145 cm differs significantly from the claimed population mean of 150 cm.

Let $\overline{X}$ denote the mean height of a random sample of 50 mature water buffaloes.

a. State suitable hypotheses H_0 and H_1 for the statistical test. 1 mark

$H_0: \mu = 150$

$H_1: \mu < 150$ *one-tail

b. Find the standard deviation of $\overline{X}$. 1 mark

$sd(\bar{X}) = \frac{15}{\sqrt{50}} = \frac{3\sqrt{2}}{2}$

c. Write down an expression for the p value of the statistical test and evaluate your answer correct to four decimal places. 2 marks

$p = Pr(\bar{X} < 145 | \mu = 150)$

$p = 0.0092$ *menu + 6+6+1 see 9.1.17

d. State with a reason whether H_0 should be rejected at the 5% level of significance. 1 mark

H_0 should be rejected as $p = 0.0092$ is less than the 0.05 threshold of significance.

e. What is the smallest value of the sample mean height that could be observed for H_0 to be **not** rejected? Give your answer in centimetres, correct to two decimal places. 1 mark

$Pr(\bar{X} < \bar{h} | \mu = 150) = 0.05$ $\quad Z \sim N(0,1)$

$Pr\left(Z < \frac{\bar{h} - 150}{\frac{3\sqrt{2}}{2}}\right) = 0.05 \Rightarrow \bar{h} = 146.51\text{cm}$

f. If the true mean height of all mature water buffaloes in northern Australia is in fact 145 cm, what is the probability that H_0 will be accepted at the 5% level of significance? Give your answer correct to two decimal places. 1 mark

$\Pr(\bar{X} > 146.51 \mid \mu = 145)$ $\quad Z \sim N(0,1)$

$= \Pr\left(Z > \frac{146.51 - 145}{\frac{3\sqrt{2}}{2}}\right) = 0.24$

g. Using the observed sample mean of 145 cm, find a **99% confidence interval** for the mean height of all mature water buffaloes in northern Australia. Express the values in your confidence interval in centimetres, correct to one decimal place. 1 mark

$CI = (139.5,\ 150.5)$

END OF QUESTION AND ANSWER BOOK

Question 6 (9 marks)

A company produces packets of noodles. It is known from past experience that the mass of a packet of noodles produced by one of the company's machines is normally distributed with a mean of 375 grams and a standard deviation of 15 grams.

To check the operation of the machine after some repairs, the company's quality control employees select two independent random samples of 50 packets and calculate the mean mass of the 50 packets for each random sample.

a. Assume that the machine is working properly. Find the probability that at least one random sample will have a mean mass between 370 grams and 375 grams. Give your answer correct to three decimal places. 2 marks

$M \sim N(375, 15)$

$E(\bar{M}) = 375 \quad sd(\bar{M}) = \frac{15}{\sqrt{50}} = \frac{3\sqrt{2}}{2}$

$\therefore \bar{M} \sim N(375, \frac{3\sqrt{2}}{2})$

$Pr(370 < \bar{M} < 375) = 0.491$

$X \sim Bi(2, 0.491) \quad Pr(X \geq 1) = 0.741$

b. Assume that the machine is working properly. Find the probability that the means of the two random samples differ by less than 2 grams. Give your answer correct to three decimal places. 3 marks

$\bar{M}_1 \sim N(375, \frac{3\sqrt{2}}{2}) \quad \bar{M}_2 \sim N(375, \frac{3\sqrt{2}}{2})$

$E(\bar{M}_1 - \bar{M}_2) = 0 \quad sd(\bar{M}_1 - \bar{M}_2) = 3$

$Pr(-2 < \bar{M}_1 - \bar{M}_2 < 2) = 0.495$

To test whether the machine is working properly after the repairs and is still producing packets with a mean mass of 375 grams, the two random samples are combined and the mean mass of the 100 packets is found to be 372 grams. Assume that the standard deviation of the mass of the packets produced is still 15 grams. A two-tailed test at the 5% level of significance is to be carried out.

c. Write down suitable hypotheses H_0 and H_1 for this test. 1 mark

$H_0: \mu = 375$

$H_1: \mu \neq 375$

d. Find the p value for the test, correct to three decimal places. 1 mark

$p = 0.046$

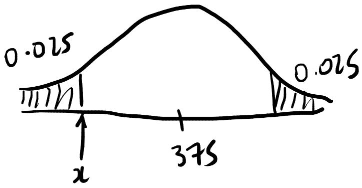
e. Does the mean mass of the sample of 100 packets suggest that the machine is working properly at the 5% level of significance for a two-tailed test? Justify your answer. 1 mark

No, $p\ (=0.046) < 0.05$

$\therefore$ reject $H_0 \Rightarrow$ the machine is not working properly

f. What is the smallest value of the mean mass of the sample of 100 packets for H_0 to be **not** rejected? Give your answer correct to one decimal place. 1 mark

$x = 372.1$ g * invnorm$\left(0.025, 375, \frac{15}{\sqrt{100}}\right)$



END OF QUESTION AND ANSWER BOOK

10 Differential Calculus

10.1 Derivatives of x=f(y)

For $x = f(y)$ where f is a one-to-one function:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}, \frac{dx}{dy} \neq 0$$

10.2 Derivatives of Inverse Circular Functions

10.3 Second Derivatives

10.4 Points of Inflection

10.5 Related Rates

10.6 Rational Functions

11 Integral Calculus

11.1 Rules

11.1.1 Definite Integral

The definite integral on interval $[a, b]$ denotes the signed area enclosed by the graph $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$.

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$

11.2 Inverse Circular Functions

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{a}\right) + c, x \in (-a, a)$$
$$\int \frac{-1}{\sqrt{a^2 - x^2}} dx = \arccos\left(\frac{x}{a}\right) + c, x \in (-a, a)$$
$$\int \frac{a}{a^2 + x^2} dx = \arctan\left(\frac{x}{a}\right) + c, x \in \mathbb{R}$$

11.3 u-Sub

$$\int f(u)\frac{du}{dx} dx = \int f(u)du$$

Linear substitutions Antiderivatives of expressions in the form $f(x)[g(x)]^n$ for a linear function g can be found using linear substitution:

Exam-style Question:

Evaluate $\int_0^1 xe^{-x^2} dx$.

$$\int_0^1 xe^{-x^2} dx = \frac{1}{2}\int_0^1 (2x)e^{-x^2} dx$$
$$u = -x^2 \qquad \frac{du}{dx} = -2x \qquad x = 0 \implies u = 0, x = 1 \implies u = -1$$
$$= \frac{-1}{2}\int_0^{-1} e^u du$$
$$= \frac{-1}{2}\left[e^u\right]_0^{-1}$$
$$= \frac{-1}{2}\left(e^{-1} - 1\right)$$
$$= \frac{1}{2} - \frac{1}{2e}$$

Exam-style Question: 2006 NEAP E2 A11

Using a suitable substitution, $\int_0^{\frac{\pi}{4}} \left(\frac{\cos x}{\sin x + \cos^2 x} \right) dx$ can be expressed as:

Let $u = \sin x$

$$\int \frac{\cos x}{\sin x + (1 - \sin^2 x)} dx = \int \frac{1}{u + 1 - u^2} du$$

Exam-style Question:

Using the substitution $u = \pi - x$,

a) Show that $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx$.

$$u = \pi - x \qquad x = \pi - u \qquad \frac{du}{dx} = -1 \qquad du = -dx$$

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \int_{\pi}^{0} \frac{(\pi - u) \sin(\pi - u)}{1 + \cos^2(\pi - u)} (-du)$$

$$= \int_0^{\pi} \frac{(\pi - u) \sin(\pi - u)}{1 + \cos^2(\pi - u)} du$$

$$= \int_0^{\pi} \frac{(\pi - x) \sin(x)}{1 + \cos^2 x} dx \text{ (Because u is just a dummy variable)}$$

b) Hence deduce that $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{4}$.

$$\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x} dx - \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

$$2 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx \implies \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx$$

$$v = \cos x, dx = \frac{-1}{\sin x} du, x = 0 \to v = 1, x = \pi \to v = -1$$

$$\pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx = -\pi \int_1^{-1} \frac{1}{1 + u^2} du$$

$$= \pi \int_{-1}^{1} \frac{1}{1 + u^2} du = \pi \left[\arctan u\right]_{-1}^{1}$$

$$= \frac{\pi^2}{2}$$

$$\therefore \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{1}{2} \cdot \frac{\pi^2}{2} = \frac{\pi^2}{4}$$

11.4 Integration by Trig Identities

Integrals of form $\int \sin^m(x)\cos^n(x)dx$ can be considered in three cases:

Case 1: Power of sine is odd If m is odd, let $m = 2k+1, k \in \mathbb{Z}$

$$\sin^{2k+1}(x) = \left(\sin^2 x\right)^k \sin x = \left(1-\cos^2 x\right)^k \sin x$$

We can then use the substitution $u = \cos x$ to evaluate integrals.

Case 2: Power of cosine is odd If m is even and n is odd, let $n = 2k+1, k \in \mathbb{Z}$

$$\cos^{2k+1}(x) = \left(\cos^2 x\right)^k \cos x = \left(1-\sin^2 x\right)^k \cos x$$

We can then use the substitution $u = \sin x$ to evaluate integrals.

Case 3: Both powers are even If both m and n are even, we use the identities:

$$\sin^2 x = \frac{1}{2}\left(1-\cos(2x)\right) \qquad \cos^2 x = \frac{1}{2}\left(1+\cos(2x)\right) \qquad \sin(2x) = 2\sin(x)\cos(x)$$

Products to Sums Identities Recall the product to sum identities:

$$2\cos x\cos y = \cos(x-y)+\cos(x+y) \qquad 2\sin x\sin y = \cos(x-y)-\cos(x+y) \qquad 2\sin x\cos y = \sin(x+y)+\sin(x-y)$$

We can also use these for integration in appropriate questions.

11.5 Partial Fractions

Proper Fractions If $g(x)$ and $h(x)$ are polynomials then $f(x) = \frac{g(x)}{h(x)}$ is a rational function. If the degree of g is less than the degree of h then $f(x)$ is a proper fraction.

For **proper fractions**, we resolve into proper fractions with the following rules:

- For every linear factor $ax+b$ in the denominator, there will be a partial fraction of form $\frac{A}{ax+b}$
- For every repeated linear factor $(ax+b)^2$ in the denominator, there will be two partial fractions of form $\frac{A}{ax+b}$ and $\frac{B}{(ax+b)^2}$
- For every irreducible quadratic factor ax^2+bx+c in the denominator, there will be a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$.
- For every repeated irreducible quadratic factor $(ax^2+bx+c)^2$ in the denominator, there will be two partial fractions of form $\frac{Ax+B}{ax^2+bx+c}$ and $\frac{Cx+D}{(ax^2+bx+c)^2}$

For **improper fractions**, those where the degree of the numerator is greater than the denominator, we use polynomial division to first create a proper fraction then resolve as normal.

11.6 Integration by Parts

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$$

Exam-style Question: VCAA 2007 Spesh E2 B2

> Let $f(x) = x\arctan(x)$.
> a) Find $f'(x)$ and calculate the slope of f at $x = 0$.

$$f'(x) = \arctan(x) + x\cdot\frac{1}{x^2+1} = \arctan(x) + \frac{x}{x^2+1}$$

$$f'(0) = 0$$

d) Use the result for $f'(x)$ in part **a** to show that an antiderivative of $\arctan(x)$ is

$$x\arctan(x) - \frac{1}{2}\ln(1+x^2)$$

$$\begin{aligned}
f'(x) &= \arctan(x) + \frac{x}{x^2+1} \\
\int f'(x)dx &= \int \frac{x}{x^2+1}dx + \int \arctan(x)dx \\
\text{Let } u = 1+x^2 \qquad f(x) &= \int \arctan(x)dx + \frac{1}{2}\int \frac{1}{u}du \\
\int \arctan(x)dx &= x\arctan(x) - \frac{1}{2}\ln(u) + c \\
&= x\arctan(x) - \frac{1}{2}\ln(1+x^2) \text{ is an antiderivative as required}
\end{aligned}$$

11.7 Properties of the Definite Integral

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, a < c < b$$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

11.8 Area Between Curves

Let f and g be continuous functions on the interval $[a, b]$ such that $\forall x \in [a, b] f(x) \geq g(x)$. The area of the region bounded between the two curves and the lines $x = a$ and $x = b$ can be found by evaluating:

$$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b f(x) - g(x)dx$$

When the graphs intersect on the interval, you must evaluate an integral for each section.

11.9 Volumes of Solids of Revolution

By rotating a function around an axis you form a solid of revolution.

Revolution around the x-axis If the region being rotated is bounded by $y = f(x)$, $x = a$, $x = b$ and the x-axis, then:

$$V = \pi\int_a^b f(x)^2 dx$$

Revolution around the y-axis If the region being rotated is bounded by $x = f(y)$, $y = a$, $y = b$ and the y-axis, then:

$$V = \pi\int_a^b f(y)^2 dy$$

Regions not bound by an axis If the region to be rotated is bounded by two functions instead of by an axis, use the area between two curves form:

$$V = \pi \int_a^b (f(x))^2 - (g(x))^2 dx$$

11.10 Lengths of Curves in Plane

The length of a curve of $f(x)$ (for f is differentiable and f' is continuous) from $x = a$ to $x = b$ is given by:

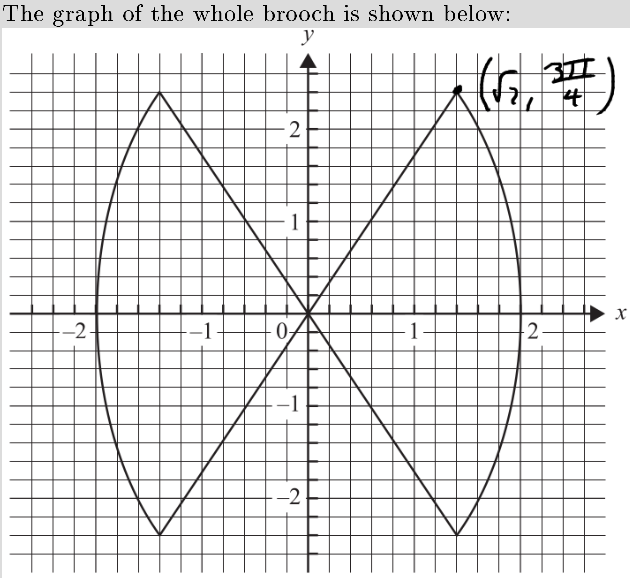
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

Exam-style Question: VCAA Spesh 2017 E2 B3e

A brooch is designed using inverse circular functions. The edges of the brooch in the first and third quadrants are described by these piecewise functions:

$$\text{Q1: } f(x) = \begin{cases} 3\arcsin\left(\frac{x}{2}\right) & 0 \le x \le \sqrt{2} \\ 3\arccos\left(\frac{x}{2}\right) & \sqrt{2} < x \le 2 \end{cases} \qquad \text{Q3: } g(x) = \begin{cases} -3\arccos(\frac{-x}{2}) & -2 \le x < -\sqrt{2} \\ -3\arcsin(\frac{-x}{2}) & -\sqrt{2} \le x \le 0 \end{cases}$$

The graph of the whole brooch is shown below:



The perimeter of the brooch has a gold border. Show that the length of the gold border needed is given by a definite integral of the form $\int_0^2 \sqrt{a + \frac{b}{4 - x^2}} dx$, where $a, b \in \mathbb{R}$. Find the values of a and b.

$$f'(x) = \begin{cases} \frac{3}{\sqrt{4-x^2}}, & 0 < x < \sqrt{2} \\ \frac{-3}{\sqrt{4-x^2}}, & \sqrt{2} < x < 2 \end{cases}$$

$$L = 4\left[\int_0^2 \sqrt{1+\left(\frac{3}{\sqrt{4-x^2}}\right)^2} dx\right] = 4\int_0^2 \sqrt{1+\frac{9}{4-x^2}} dx$$

$$= \int_0^2 \sqrt{16+\frac{144}{4-x^2}} dx$$

$\therefore$ $a = 16$, $b = 144$

11.11 Areas of Surfaces of Revolution

Revolution About x-axis If the region bounded by $y = f(x)$, the x-axis and $x = a$, $x = b$ is rotated about the x-axis, the area of the surface formed is given by:

$$A = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Revolution About y-axis If the region bounded by $x = f(y)$, the y-axis and $y = a$, $y = b$ is rotated about the y-axis, the surface area of the surface formed is given by:

$$A = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Surface Area of Parametrics Consider a curve defined by $x = f(t)$ and $y = g(t)$ for $a \leq t \leq b$. If $P(f(t), g(t))$ traces the curve exactly once from $t = a$ to $t = b$ then the area formed by rotating about the x-axis is given by:

$$A = 2\pi \int_a^b |g(t)|\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Total Surface Area of a Solid

$$\Sigma A = A + \pi\,(f(a))^2 + \pi\,(f(b))^2$$

For A being the area of the surface, and a, b being the limits of the solid of revolution.

11.12 Reduction Formulas

Example: Reduction Formulas Example 1

$$\text{Let } I_n = \int_1^e (ln(t))^n dt\ n \in \mathbb{Z}^+$$

i) Show that $I_n = e - nI_{n-1}$ for $n \in \mathbb{Z}^+$.

$$I_n = \int_1^e (ln(t))^n dt$$

$$u = (ln(t))^n \quad \frac{du}{dt} = \frac{n(ln(t))^{n-1}}{t} \quad \frac{dv}{dt} = 1 \quad v = t \quad \text{(I.B.P)}$$

$$I_n = [t(ln(t))^n]_1^e - \int_1^e n(ln(t))^{n-1} dt$$

$$= e - n\int_1^e n(ln(t))^{n-1} dt$$

$$= e - nI_{n-1} \text{ As required.}$$

ii) Hence, or otherwise, find the exact value of I_3.

$$\begin{aligned} I_3 &= e - 3I_2 \\ &= e - 3(e - 2I_1) = -2e + 6(e - I_0) \\ &= 4e - 6I_0 = 4e - 6\int_1^e (ln(t))^0 dt = 4e - [t]_1^e \\ &= 4e - 6(e-1) = 6 - 2e \end{aligned}$$

12 Differential Equations

12.1 Tech

12.1.1 deSolve(diffeq,iv,dv)

Press menu+4+D to get this function up. In the diffeq, denote derivatives with an apostrophe, iv is the independent variable and dv is the dependent variable.

12.1.2 euler(dydx,iv,dv,{x0,x1},y0,h)

Input the euler method function. dydx is the derivative of the dependent variable (dv) w.r.t the independent variable (iv). x0 is the x value stipulated in the initial condition, x1 is the maximum x value you wish to enumerate to. y0 is the y value in the initial condition and h is the step size.

12.2 Verification of Solutions

We can verify solutions to DiffEqs by substitution.

12.3 Proportionality/Inverse Proportionality

Exam-style Question: VCAA 2007 Spesh E2 A14

> The rate at which a type of bird flu spreads throughout a population of 1000 birds in a certain area is proportional to the product of the number N of infected birds and the number of birds still **not** infected after t days. Initially two birds in the population are found to be infected. A differential equation, the solution of which models the number of infected birds after t days, is

My Blunder Solution:

$$\frac{dN}{dt} = k(N-2)(1000-N)$$

Correct Solution:

$$\frac{dN}{dt} \propto N(1000-N) \therefore \frac{dN}{dt} = kN(1000-N)$$

The intiial value does not matter in this instance, as it is proportional to the product of the **currently** infected and non-infected birds, not the birds infected **since** the initial measurement.

12.4 Logistic DiffEq

$$\frac{dP}{dt} = rP\left(1-\frac{P}{K}\right)$$

Where P is the population at time t, r is the 'growth parameter' (rate), K is the upper limit of the population, $\frac{dP}{dt} \to 0$ as $P \to K$. Maximum $\frac{dP}{dt}$ occurs at $P = \frac{K}{2}$.

General Solution The general solution to a logistic diffeq $\frac{dP}{dt}$:

$$P(t) = \frac{P_0 k}{P_0 + (k-P_0)e^{-rt}} = \frac{P_0 k e^{rt}}{P_0 e^{rt} + (k-P_0)}, \qquad P_0 = P(0)$$

Exam-style Question: VCAA 2023 Spesh Sample Q2

> In a certain region, 500 rare butterflies are released to maintain the species. It is believed that the region can support a maximum of 30 000 such butterflies. The butterfly population P, t years after release can be modelled by the logistic differential equation $\frac{dP}{dt} = rP\left(1-\frac{P}{30000}\right)$ where r is the growth rate of the population.
> a) Use an integration technique and partial fractions to solve the DE above to find P in terms of r and t.
> b) Given that after 10 years there are 1930 butterflies, find r to 2 decimal places.

a)

$$r\frac{dt}{dP} = \frac{1}{P} + \frac{1}{30000-P}$$
$$rt = \ln(P) - \ln(30000-P) + c, c \in \mathbb{R}$$
$$P(0) = 500 \implies c = \ln(59)$$
$$\therefore rt = \ln\left(\frac{59P}{30000-P}\right)$$
$$P = \frac{30000}{1+59e^{-rt}}$$

b)

$$P(10) = 1930$$
$$r = \frac{1}{t}\ln\left(\frac{59P}{30000-}\right) = \ln\left(\frac{59(1930)}{30000-1930}\right) \approx 0.14$$

12.5 Separation of Variables

$$\frac{dy}{dx} = f(x)g(y) \implies \int f(x)dx = \int \frac{1}{g(y)}dy$$

12.5.1 Solving

To solve $\frac{dy}{dx} = f(x)g(y)$:
1) Write expression clearly as a product of a function in x and a function in y.
2) Substitute these expressions into the separation of variables form.
3) Antidifferentiate both expressions with $+c$ included (you only need this on one side).

Exam-style Question: 2018 Exam 2 Question A9

A solution to the differential equation $\frac{dy}{dx} = \frac{2}{\sin(x+y)-\sin(x-y)}$ can be obtained from:

a) $\int 1dx = \int 2\sin(y)dy$
b) $\int \cos(y)dy = \int \csc(x)dx$
c) $\int \cos(x)dx = \int \csc(y)dy$
d) $\int \sec(x)dx = \int \sin(y)dy$
e) $\int \sec(x)dx = \int \csc(y)dy$

$$\frac{dy}{dx} = \frac{1}{\cos(x)\sin(y)}$$
$$\sin(y)\frac{dy}{dx} = \sec(x)$$
$$\int \sin(y)dy = \int \sec(x)dx \qquad \text{(D)}$$

12.6 DiffEqs + Related Rates Crossover Event

Example: Inverted Cone

An inverted cone has height h and radius r. It is being filled at k L per minute. The depth of the water is x at time t minutes. Construct a diffeq for $\frac{dx}{dt}$ and solve given the cone was initially empty.

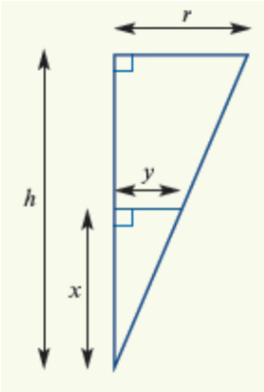
Let V be the volume of water after t minutes

k litres $= 1000k\text{cm}^3$

$\frac{dV}{dt} = 1000k$

$\frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt}$

$V = \frac{1}{3}\pi y^2 x$



By similar triangles: $\frac{y}{r} = \frac{x}{h} \implies y = \frac{xr}{h} \implies \quad V = \frac{1}{3}\pi\frac{r^2x^2}{h^2}x = \frac{\pi r^2}{3h^2}x^3$

$\therefore \frac{dV}{dx} = \frac{\pi r^2}{h^2}x^2 \implies \frac{dx}{dV} = \frac{h^2}{\pi r^2}\frac{1}{x^2} \therefore \frac{dx}{dt} = \frac{h^2}{\pi r^2 x^2} \times 1000k = \frac{1000kh^2}{\pi r^2 x^2}, k > 0$

$t = \int \frac{\pi r^2 x^2}{1000kh^2}dx$

$= \frac{\pi r^2 x^3}{3000kh^2} + c, c \in \mathbb{R}$

Then solve appropriately for c.

12.7 Definite Integration to solve DiffEqs

For differential equations $\frac{dy}{dx} = f(x)$ with $y_0 = f(x_0)$ being known, the values of y at x_1 can be found:

$$y(x_1) = \int_{x_0}^{x_1} f(x)dx + y_0$$

12.8 Newton's Law of Cooling DiffEq

Newton's Law of Cooling involves a DiffEq that describes the rate of change of temperature $\frac{dT}{dt}$ is proportional to the difference between the temperature and the outside. Note that $k \in \mathbb{R}^-$ in the below statement.

$$\frac{dT}{dt} \propto (T - T_{external}) \implies \frac{dT}{dt} = k(T - T_{external})$$

12.9 Applications of DiffEqs - Concentration Problems

Q=concentration

$$\frac{dQ}{dt} = \frac{dQ_{in}}{dt} - \frac{dQ_{out}}{dt}$$

In flow: $\frac{dQ_{in}}{dt} = \frac{dQ_{in}}{dV_{in}} \times \frac{dV_{in}}{dt}$.
In tank: Mass at any time t, volume at any time t, concentration $C = \frac{mass}{volume}$
Out flow: $\frac{dQ_{out}}{dt} = \frac{dQ_{out}}{dV_{out}} \times \frac{dV_{out}}{dt}$. Out flowing concentration=concentration in tank C.

Exam-style Question: 2006 NEAP E1 A21

> A 5L container is initially full of fresh water. Saline solution with a concentration of 10g/L flows into the tank at 1L/min. At the same time, the mixture in the container flows out at 2L/min. If S is the quantity of salt in the container at time t minutes, write a differential equation describing this situation.

Into tank: $\frac{dQ_{in}}{dt} = 10 \times 1 = 10$

In tank: $C = \frac{S}{5 + (1-2)t} = \frac{S}{5 - t}$

Out of tank: $\dfrac{dQ_{out}}{dt} = \dfrac{S}{5-t} \times 2$

Therefore $\frac{dS}{dt} = \frac{dQ_{in}}{dt} - \frac{dQ_{out}}{dt} = 10 - \frac{2S}{5-t}$

12.10 Euler's Method

If $\dfrac{dy}{dx} = g(x)$ with $y = y_0$ when $x = x_0$, then:

$$x_{n+1} = x_n + h \qquad \text{and} \qquad y_{n+1} = y_n + hg(x_n)$$

Example: Euler's Method Basic Case

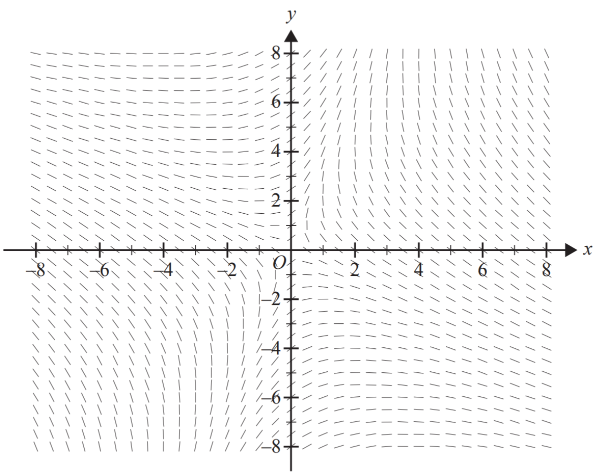
Let $\frac{dy}{dx} = x^2y$ with $y(1) = 4$. Apply Euler's method to find y_3 using steps of 0.2.

$$\begin{aligned}
&g(x,y) = x^2y, \qquad h = 0.2 \\
&0: \quad x_0 = 1 && y_0 = 4 \\
&1: \quad x_1 = 1.2 && y_1 = 4 + 0.2(1)^2(4) = 4.8 \\
&2: \quad x_2 = 1.4 && y_2 = 4.8 + 0.2(1.2)^2(4.8) = 6.18 \\
&3: \quad x_3 = 1.6 && y_3 = 6.18 + 0.2(1.4)^2(6.18) = 8.606
\end{aligned}$$

12.11 Slope Fields

Exam-style Question: 2018 Exam 2 Question A10

The differential equation that best represents the slope field below is



a) $\dfrac{dy}{dx} = \dfrac{2x+y}{y-2x}$ b) $\dfrac{dy}{dx} = \dfrac{x+2y}{2x-y}$ c) $\dfrac{dy}{dx} = \dfrac{2x-y}{x+2y}$ d) $\dfrac{dy}{dx} = \dfrac{x-2y}{y-2x}$ e) $\dfrac{dy}{dx} = \dfrac{2x+y}{2y-x}$

Plot using **Graphs** page, Menu+3+8 to enter a diffeq. **A** has the same shape, so A is the answer.

12.12 Finding Solutions to DiffEqs

13 Pseudocode

13.1 Syntax

Assignment A variable is a string of one or more letters that acts as a placeholder that can be assigned different values. For example $x \leftarrow 3$ reads 'assign the value 3 to the variable x' or 'x takes the value 3'.

Control of Flow of Steps

If-Then Blocks This construct provides a means of making decisions within an algorithm. If the condition is true, then the code between the If-Then block and the EndIf statement will execute.

```
If condition then
    execute these instructions
End If
```

For loops Repeated execution of code blocks. Note the convention that 'from 1 to n' is inclusive.

```
For i From 1 to n
    execute these instructions
End For
```

While loops Indefinite repeated iterations of instructions provided some condition remains true.

```
While condition is true
    execute these instructions
End While
```

Functions Functions are sections of pseudocode that can be used to complete a specific task. Once a function is defined it can be used ('called') within another algorithm.

```
define function_name(args):
    execute these instructions
return output
```

13.2 Example Algorithms

Example: Newton-Raphson Method to tolerance of 0.001

```
define newtonraphson(f(x),x0)
    df(x)← f′(x)
    i← 0
    prevx←x0
    While i<1000 do
        nextx←prevx - f(prevx) ÷ df(prevx)
        If 0.001 < nextx - prevx < 0.001 Then
            Return nextx
        Else
            prevx←nextx
            i←i+1
    EndWhile
```

Example: Numerical Integration with Trapezium Method

```
define f(x)
    return (function rule)

sum←0
a←lower bound
b←upper bound
n←number of trapeziums
h←(b-a)/n
left←a
```

```
right←a+h
for i from 1 to n
    strip← 0.5(f(left)+f(right)) × h
    sum← sum + strip
    left← left + h
    right← right + h
end for
print sum
```

Example: Estimate of Long-Term Average for Number of Dice Rolls for a Six

```
sum←0
for i from 1 to 1000
    outcome←0
    count←0
    while outcome ≠ 6
        outcome←randominteger(1,6)
        count←count + 1
    end while
    sum←sum + count
end for
print sum/1000
```

Example: Bisection Method for x-intercepts

Algorithm:

```
define f(x):
    return (function rule)
a←lower guess
b←upper guess
c←(a+b)/2
t←tolerance
if f(a)× f(b)>0 then
    print "Incorrect initial estimates"
else
    while b-a>2× t
        if f(a)× f(c)<0 then
            b←c
        else
            a←c
        end if
        c←(a+b)/2
        endwhile
print c
end if
```

Comments:

Function f(x) is chosen. For example, $x^2 - 2$. Good choices for a and b are a=1 and b=2 as f(1)<0 and f(2)>0

The **while** loop requires that we continue the iteration until the desired accuracy is reached.
The **if** statement inside the loop ensures that we continue to choose values for which there is a sign difference. For each iteration an average of the new pair is determined.

14 Other Tech

14.1 Functions

14.1.1 scriptedmath\poi(f(x),x)

Gives stationary points, axes intercepts, straight-line asymptotes, and endpoints.

14.1.2 scriptedmath\inv(f(x),pointX)

Finds $f^{-1}(x)$. If the function is not one-to-one it will restrict domain to contain the point where x=pointX.

14.1.3 sam_other\newtonraphson(f,g,e,n)

Uses the Newton-Raphson method to approximate roots. f is the function, the roots of which we are trying to estimate. g is the derivative function. e is the initial estimate, and n is the number of iterations.

14.2 Parametric Functions

14.2.1 scriptedmath\pder(vfunc, parameter)

For a vector function in form $[x(t)y(t)]$ and parameter t. This function will find $\frac{dy}{dx}$.

15 Advanced Maths

15.1 The Derivative of f(x)=4x+1

This could be a tough one. It might be best to take an indirect approach. Let's start with what we know:

$$f(x) = 4x + 1$$

and make the problem simpler by translating it to a coordinate system I'm going to call "Laplace Space"

$$\mathscr{L}\{f(x)\} = \mathscr{L}\{4x + 1\}$$

Next we should refer to our table of Laplace Transforms, to convert our functions in x to functions in s.

$$F(s) = \frac{4}{s^2} + \frac{1}{s} = \frac{s+4}{s^2}$$

Now we're in "Laplace Space". The cool thing about "Laplace Space" is that integration becomes division and differentiation becomes multiplication. To find the derivative, we generally want to multiply everything by s.

$$sF(s) = \frac{s+4}{s}$$

In "Laplace Space", the derivative of a function is $sF(s) - f(0)$.

$$sF(s) - f(0) = \frac{s+4}{s} - (4(0) + 1) = \frac{s+4-s}{s} = \frac{4}{s}$$

The left hand side of our equation is now the version of $f'(x)$ that exists in "Laplace Space". To find what our derivative is actually equal to, we need to convert everything back to "Cartesian Space"

$$\mathscr{L}^{-1}\{sF(s) - f(0)\} = \mathscr{L}^{-1}\{\frac{4}{s}\}$$

$$f'(x) = 4$$

And that's our answer, $f'(x) = 4$. By taking a detour through "Laplace Space", we were able to treat this like an algebra problem and avoid all of that troublesome, limit-based calculus.

15.1.1 Our Table of Laplace Transforms

1	$\frac{1}{s}$
$e^{at}, a \in \mathbb{R}$	$\frac{1}{s-a}$
$t^n, n \in \mathbb{Z}^+$	$\frac{n!}{s^{n+1}}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$
$\sin(at), a \in \mathbb{R}$	$\frac{a}{s^2+a^2}$
$\cos(at), a \in \mathbb{R}$	$\frac{s}{s^2+a^2}$
$t\sin(at), a \in \mathbb{R}$	$\frac{2as}{(s^2+a^2)^2}$
$t\cos(at), a \in \mathbb{R}$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$\sin(at) - at\cos(at), a \in \mathbb{R}$	$\frac{2a^3}{(s^2+a^2)^2}$
$\sin(at) + at\cos(at), a \in \mathbb{R}$	$\frac{2as^2}{(s^2+a^2)^2}$
$t^n e^{at}, a \in \mathbb{R}, n \in \mathbb{Z}^+$	$\frac{n!}{(s-a)^{n+1}}$

$\cos(at) - at\sin(at), a \in \mathbb{R}$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$
$\cos(at) + at\sin(at), a \in \mathbb{R}$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
$\sin(at+b), a, b \in \mathbb{R}$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$
$\cos(at+b), a, b \in \mathbb{R}$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
$\sinh(at), a \in \mathbb{R}$	$\frac{a}{s^2-a^2}$
$\cosh(at), a \in \mathbb{R}$	$\frac{s}{s^2-a^2}$
$e^{at}\sin(bt), a, b \in \mathbb{R}$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt), a, b \in \mathbb{R}$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sinh(bt), a, b \in \mathbb{R}$	$\frac{b}{(s-a)^2-b^2}$
$e^{at}\cosh(bt), a, b \in \mathbb{R}$	$\frac{s-a}{(s-a)^2-b^2}$
$f(ct), c \in \mathbb{R}$	$\frac{1}{c}F(\frac{s}{c})$

Table 1: Laplace Transforms for Equations in t

15.2 Sandwich Theorem

Let $g(x) \leq f(x) \leq h(x)$ for all x near a and

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$$

Then

$$\lim_{x\to a} f(x) = L$$

The function f is 'sandwiched' between g and h.

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