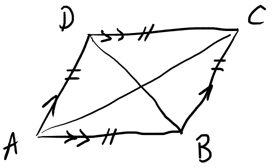
Prove that the diagonals of a rhombus are perpendicular using a vector method.



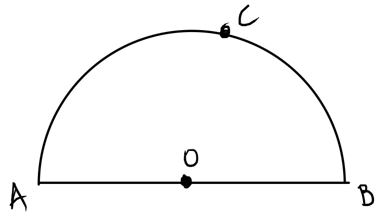
$|\vec{AB}| = |\vec{AD}| = |\vec{DC}| = |\vec{CB}|$

$\vec{AB} = \vec{DC}$ and $\vec{AD} = \vec{BC}$

$$\begin{aligned}\vec{AC}\cdot\vec{BD} &= (\vec{AB}+\vec{BC})\cdot(\vec{BC}+\vec{CD})\\ &= (\vec{AB}+\vec{BC})\cdot(\vec{BC}-\vec{AB})\\ &= |\vec{BC}|^2 - |\vec{AB}|^2\\ &= 0 \quad \text{by definition}\end{aligned}$$

$\vec{AC}\cdot\vec{BD} = 0 \iff \vec{AC} \perp \vec{BD}$ $\blacksquare$

Prove that the angle subtended by a diameter at a point on a circle is a right angle.



Let O be the centre of the circle and let $\vec{AB}$ be the diameter. Let C be a point on the circle's circumference other than A or B.

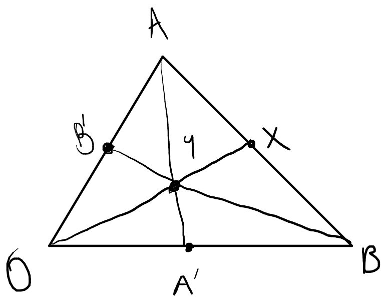
Let $\underset{\sim}{a} = \vec{OA}$, $\underset{\sim}{c} = \vec{OC}$, $\vec{OB} = -\underset{\sim}{a}$ $\Rightarrow$ $|\vec{OA}| = |\vec{OB}| = |\vec{OC}| = r$ where r = radius of the circle.

$\vec{AC} = \vec{AO} + \vec{OC} = \underset{\sim}{c} - \underset{\sim}{a}$ and $\vec{BC} = \vec{BO} + \vec{OC} = \underset{\sim}{a} + \underset{\sim}{c}$

$$\begin{aligned}\vec{AC}\cdot\vec{BC} &= (\underset{\sim}{c}-\underset{\sim}{a})\cdot(\underset{\sim}{a}+\underset{\sim}{c})\\ &= |\underset{\sim}{c}|^2 - |\underset{\sim}{a}|^2 = r^2 - r^2 = 0 \quad \text{by definition}\end{aligned}$$

Hence $\vec{AC} \perp \vec{BC}$ $\Rightarrow$ $\angle ACB$ is a right angle $\blacksquare$

Prove that the medians of a triangle are concurrent.



Let $\underset{\sim}{a} = \vec{OA}$, $\underset{\sim}{b} = \vec{OB}$

Show that $|AY| : |YA'| = |BY| : |YB'| = 2:1$

$\vec{AY} = \lambda AA'$ $\quad \vec{BY} = \mu \vec{BB'}$ for $\lambda, \mu \in \mathbb{R}$

$AA' = \frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a}$ $\quad \vec{BB'} = -\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{a}$

$\Rightarrow \vec{AY} = \lambda(\frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a})$ $\quad \vec{BY} = \mu(\frac{1}{2}\underset{\sim}{a} - \underset{\sim}{b})$

$\vec{BY}$ can also be defined as:

$\vec{BY} = \vec{BA} + \vec{AY} = -\underset{\sim}{b} + \underset{\sim}{a} + \lambda(\frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a})$ $\quad \therefore -\mu\underset{\sim}{b} + \frac{\mu}{2}\underset{\sim}{a} = (1-\lambda)\underset{\sim}{a} + (\frac{\lambda}{2} - 1)\underset{\sim}{b}$

Equating coefficients of independent vectors $\underset{\sim}{a}, \underset{\sim}{b}$:

$\frac{\mu}{2} = 1 - \lambda$ ① $\quad -\mu = \frac{\lambda}{2} - 1$ ②

$2\times$① + ②: $0 = 2 - 2\lambda + \frac{\lambda}{2} - 1$

$1 = \frac{3}{2}\lambda$

$\lambda = \frac{2}{3}$

$\lambda = \frac{2}{3} \rightarrow$ ① gives $\mu = \frac{2}{3}$

$\therefore |\vec{AY}| : |\vec{YA'}| = |\vec{BY}| : |\vec{YB'}| = 2:1$

By symmetry, the intersection of $\vec{AA'}$ and $\vec{OX}$ must also divide $\vec{AA'}$ into a ratio of 2:1, and therefore this intersection is Y.

Hence the three medians are concurrent at centroid Y. $\blacksquare$