VCE Specialist Mathematics

Written examination 2 – End of year

Sample questions

These sample questions are intended to demonstrate how new aspects of Units 3 and 4 of VCE Specialist Mathematics may be examined in written examination 2. They do **not** constitute a full examination paper.

SECTION A – Multiple-choice questions

Question 1

Consider the following statement.

P Q

'For all integers n, if n^2 is even, then n is even.'

Which one of the following is the contrapositive of this statement?

$\neg Q \Rightarrow \neg P$

A. For all integers n, if n^2 is odd, then n is odd.

B. There exists an integer n such that n^2 is even and n is odd.

C. There exists an integer n such that n is even and n^2 is odd.

D. For all integers n, if n is odd, then n^2 is odd.

E. For all integers n, if n is even, then n^2 is even.

Question 2

The procedure below has been written in pseudocode.

```
declare integer n
declare integer f
declare integer t1
declare integer t2
set f to 0
set t1 to 2
set t2 to 3
set n to 3
repeat n times
       f = t1 + 2 × t2
       t2 = f
       print f
end loop
```

$f = 2 + 6 = 8$

$f = 2 + 16 = 18$

$f = 2 + 36 = 38$

The output of the pseudocode is a list of numbers.

The final number in the list is

A. 3
B. 18
C. 38
D. 72
E. 78

Question 3

A vector perpendicular to both of the lines represented by $\underset{\sim}{r}_1 = 2\underset{\sim}{i} + 3\underset{\sim}{j} + t\left(\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$ and $\underset{\sim}{r}_2 = 3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k} + t\left(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}\right)$ is given by

A. $\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 3 & 0 \\ 3 & 1 & -2 \end{vmatrix}$

$\begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix}$

B. $\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 3 & 0 \\ 2 & 1 & -1 \end{vmatrix}$

C. $\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & 1 & -2 \\ 2 & 3 & 0 \end{vmatrix}$

D. $\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & 1 & -2 \\ 1 & 2 & -1 \end{vmatrix}$

E. $\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix}$

Question 4

Consider two points with coordinates (5, −6, 4) and (−3, −1, −10).

Which one of the following is the equation of the straight line that passes through these two points?

A. $\underset{\sim}{r}(t) = -3\underset{\sim}{i} - \underset{\sim}{j} - 10\underset{\sim}{k} + t\left(8\underset{\sim}{i} - 5\underset{\sim}{j} + 14\underset{\sim}{k}\right)$

$8\underset{\sim}{i} - 5\underset{\sim}{j} + 14\underset{\sim}{k}$

B. $\underset{\sim}{r}(t) = 5\underset{\sim}{i} - 6\underset{\sim}{j} + 4\underset{\sim}{k} + t\left(3\underset{\sim}{i} + \underset{\sim}{j} + 10\underset{\sim}{k}\right)$

C. $\underset{\sim}{r}(t) = -3\underset{\sim}{i} - \underset{\sim}{j} - 10\underset{\sim}{k} + t\left(5\underset{\sim}{i} - 6\underset{\sim}{j} + 4\underset{\sim}{k}\right)$

D. $\underset{\sim}{r}(t) = 5\underset{\sim}{i} - 6\underset{\sim}{j} + 4\underset{\sim}{k} + t\left(-3\underset{\sim}{i} - \underset{\sim}{j} - 10\underset{\sim}{k}\right)$

E. $\underset{\sim}{r}(t) = 8\underset{\sim}{i} - 5\underset{\sim}{j} + 14\underset{\sim}{k} + t\left(-3\underset{\sim}{i} - \underset{\sim}{j} - 10\underset{\sim}{k}\right)$

Question 5

A plane is perpendicular to the vector $\underset{\sim}{n} = \underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and passes through the point (3, 2, −4).

The Cartesian equation of this plane is

A. $3x + 2y - 4z = -11$

B. $-x + y - 3z = 11$

C. $-3x - 2y + 4z = -11$

D. $x - y + 3z = 11$

E. $x - y + 3z = 3$

$x - y + 3z = -11$

Question 6

The shortest distance between the planes given by $5x - 4y - 12z = 10$ and $-15x + 12y + 36z = 20$ is

A. 0

B. $\frac{10}{3\sqrt{185}}$

C. $\frac{10}{\sqrt{185}}$

D. $\frac{50}{3\sqrt{185}}$

E. $\frac{50}{\sqrt{185}}$

$\Pi_1: 5x - 4y - 12z = 10$

$\Pi_2: 5x - 4y - 12z = \frac{-20}{3}$

Parallel Planes

$d = \frac{k}{|\underset{\sim}{n}|}$

$d_1 = \frac{2\sqrt{185}}{37}$ $\qquad d_2 = \frac{-4\sqrt{185}}{111}$

$d_1 - d_2 = \frac{10\sqrt{185}}{111} = $ D

Question 7

The time taken by a machine to make electronic components varies normally with a mean of 20 seconds and a standard deviation of 2 seconds. After the machine is serviced, it is believed that the mean time taken has been reduced to 18.5 seconds with the standard deviation remaining the same.

A statistical test is proposed to check whether there is any evidence of a 1.5 second reduction in the mean time taken to make components. The test statistic will be the mean time taken to make a random sample of 16 such components. The type I error for the test will be $\alpha = 5\%$ with a critical sample mean of 19.2 seconds.

The type II error (β) for the test is closest to

A. 8%

B. 34%

C. 36%

D. 46%

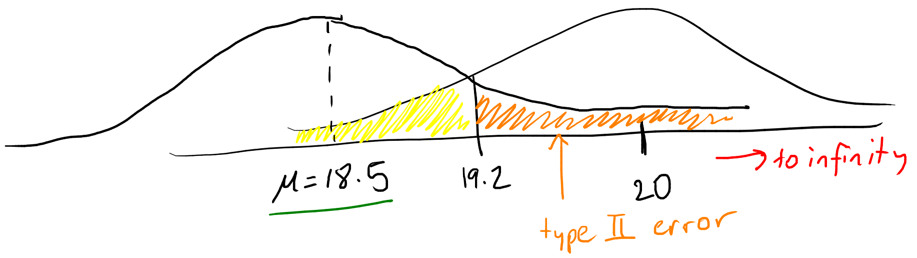
E. 95%

Type II = not rejecting when false

$sd(\bar{X}) = \frac{2}{\sqrt{n}} = \frac{2}{\sqrt{16}}$

$\text{normCdf}\left(19.2, \infty, 18.5, \frac{2}{\sqrt{16}}\right)$

0.080756711166



SECTION B

Question 1 (10 marks)

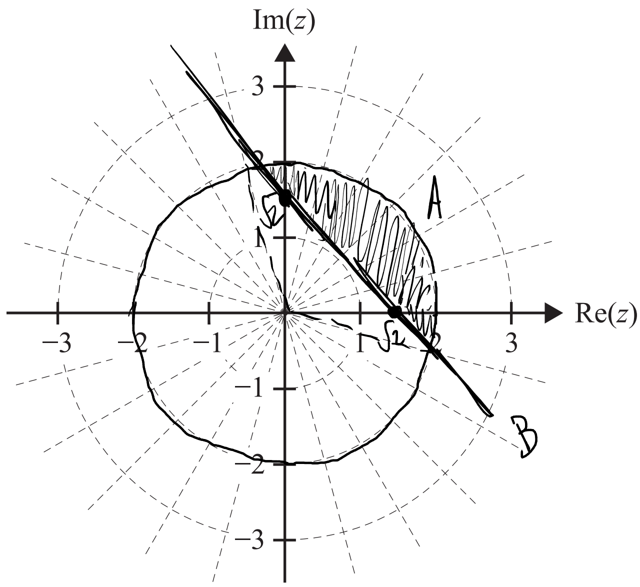
a. Express $\left\{z : |z| = \left|z - 2\operatorname{cis}\left(\frac{\pi}{4}\right)\right|,\ z \in C\right\}$ in the form $y = ax + b$, where $a, b \in R$. 2 marks

Let $z = x + yi$

$\sqrt{x^2+y^2} = \sqrt{(x-\sqrt{2})^2+(y-\sqrt{2})^2}$

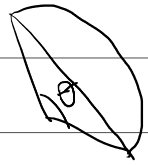
$y = -x + \sqrt{2}$

b. On the Argand diagram below, sketch and label $A = \{z : z\bar{z} = 4,\ z \in C\}$ and sketch and label $B = \left\{z : |z| = \left|z - 2\operatorname{cis}\left(\frac{\pi}{4}\right)\right|,\ z \in C\right\}$. Label the axis intercepts of the graph of B. 3 marks



c. On the Argand diagram in **part b.**, shade the region defined by $\{z : z\bar{z} \le 4,\ z \in C\} \cap \{z : \operatorname{Re}(z) + \operatorname{Im}(z) \ge \sqrt{2},\ z \in C\}$. 1 mark

d. Find the area of the shaded region in **part c.** 2 marks



$\theta = 8 \cdot \frac{\pi}{12} = \frac{2\pi}{3}$

$A = 2 \times \left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right) = 2 \times \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$= \frac{4\pi}{3} - \sqrt{3}$ units^2

e. The elements of $\{z : z\bar{z} \le 4,\ z \in C\} \cap \left\{z : |z| = \left|z - 2\operatorname{cis}\left(\frac{\pi}{4}\right)\right|,\ z \in C\right\}$ provide two of the cube roots of w, where $w \in C$.

Write down all three cube roots of w in the form $r\operatorname{cis}(\theta)$ and find w in the form $a + ib$, where $a, b \in R$. 2 marks

Intersections of circle and line are 2 roots z_1, z_2

$z_1 = 2\operatorname{cis}\left(\frac{-\pi}{12}\right)$ $z_2 = 2\operatorname{cis}\left(\frac{7\pi}{12}\right)$

roots spaced by $\frac{2\pi}{3}$ $\therefore z_3 = 2\operatorname{cis}\left(\frac{-\pi}{12} - \frac{2\pi}{3}\right) = 2\operatorname{cis}\left(\frac{-3\pi}{4}\right)$

$\therefore w = 4\sqrt{2} - 4\sqrt{2}i$

Question 2 (10 marks)

In a certain region, 500 rare butterflies are released to maintain the species.
It is believed that the region can support a maximum of 30 000 such butterflies.
The butterfly population, P, t years after release can be modelled by the logistic differential equation $\frac{dP}{dt} = rP\left(1 - \frac{P}{30\,000}\right)$, where r is the growth rate of the population.

a. Use an integration technique and partial fractions to solve the differential equation above to find P in terms of r and t. 3 marks

$\frac{dP}{dt} = rP\left(\frac{30\,000 - P}{30\,000}\right)$

$\int \frac{30\,000}{P(30\,000 - P)}\,dP = \int r\,dt$

$\int \frac{1}{P} + \frac{1}{30000 - P}\,dP = rt$

$\ln(P) - \ln(30000 - P) + c = rt,\ c \in R$

$t = 0, P = 500$

$-\ln\left(\frac{500}{29\,500}\right) = c = -\ln\left(\frac{1}{59}\right) = \ln(59)$

$\therefore rt = \ln\left(\frac{59P}{30000 - P}\right)$

$\Rightarrow P = \frac{30000e^{rt}}{e^{rt} + 59}$

b. Given that after 10 years there are 1930 butterflies in the population, find the value of r correct to two decimal places. 2 marks

$1930 = \frac{30\,000e^{10r}}{e^{10r} + 59}$

$\Rightarrow r = 0.14$

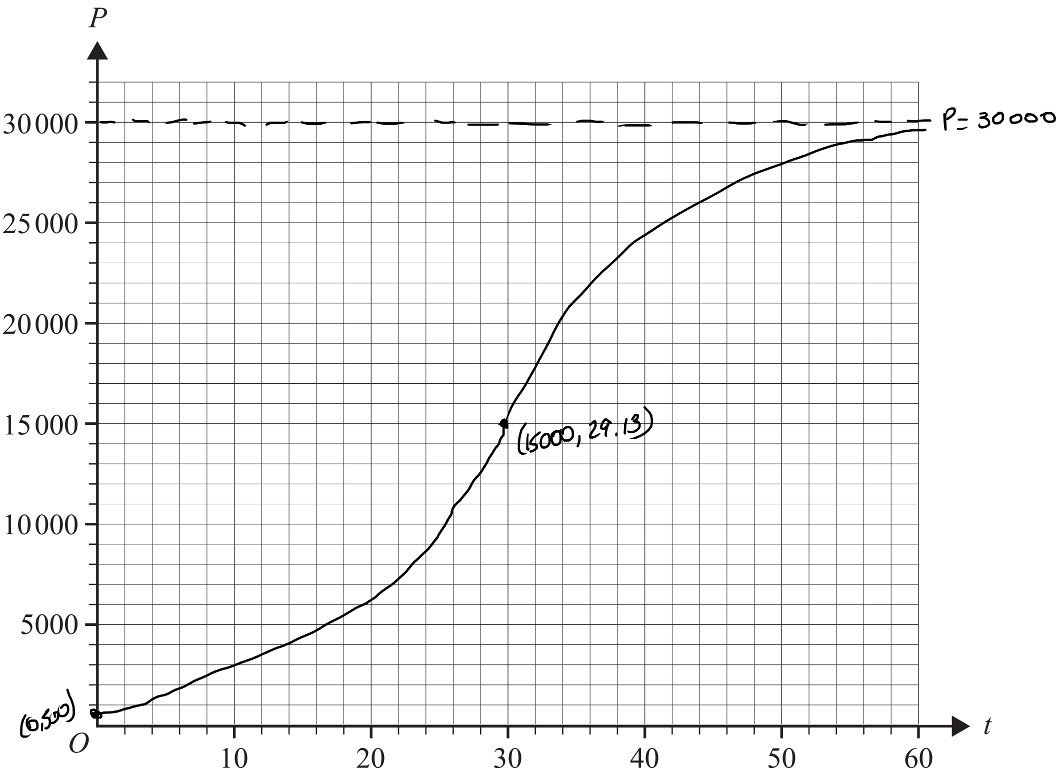
c. What is the initial rate of increase of the population, correct to one decimal place? 1 mark

$\left.\frac{dP}{dt}\right|_{t=0} = 68.9$ butterflies per year

d. After how many years will the population reach 10 000 butterflies? Give your answer correct to one decimal place. 1 mark

$t = \frac{1}{r} \ln\left(\frac{59P}{30000-P}\right) = \frac{1}{0.14} \ln\left(\frac{59 \times 10000}{20000}\right) = 24.2$ years

e. Sketch the graph of P versus t on the axes below, showing the value of the vertical intercept. Label the point of fastest population growth as a coordinate pair (t, P), with t labelled correct to two decimal places, and label the asymptote with its equation. 3 marks



Question 3 (10 marks)

A plane, Π_1, is described by the parametric equations

$$x = 1 + 2s + 3t$$
$$y = -2 - s - 2t$$
$$z = 2 - s + t$$

A second plane, Π_2, contains the point $P(1, 0, 3)$ and is parallel to the plane Π_1.

a. Find a vector equation of the plane Π_1 in the form $\underset{\sim}{r} = \underset{\sim}{a} + s\underset{\sim}{b} + t\underset{\sim}{c}$. 2 marks

Π_1: $\underset{\sim}{r} = \underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k} + s(2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}) + t(3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$, $s, t \in R$

b. Hence, find a Cartesian equation of the plane Π_1. 2 marks

Choose 2 points on Π_1. Let $\underset{\sim}{r}_0 = \underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$

$s=1, t=0$: $\underset{\sim}{r}_1 = 3\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$ $\quad$ $s=0, t=1$: $\underset{\sim}{r}_2 = 4\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}$

$\underset{\sim}{a} = \underset{\sim}{r}_1 - \underset{\sim}{r}_0 = 2\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$, $\quad \underset{\sim}{b} = \underset{\sim}{r}_2 - \underset{\sim}{r}_0 = 3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$

$\underset{\sim}{n} = \underset{\sim}{a} \times \underset{\sim}{b} = -3\underset{\sim}{i} - 5\underset{\sim}{j} - \underset{\sim}{k}$ $\quad \therefore \Pi_1: -3x - 5y - z = 5$

c. Find a Cartesian equation of the plane Π_2. 1 mark

Π_2: $-3x - 5y - z = -6$

d. i. Find the shortest distance between the planes Π_1 and Π_2. 2 marks

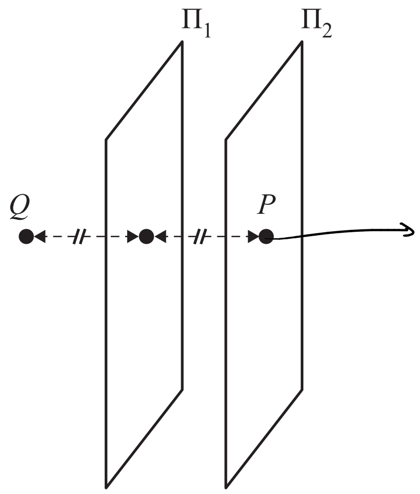
$\Pi_1: 3x+5y+z=-5 \qquad \Pi_2: 3x+5y+z=6$

$D_1 = \frac{-5}{\sqrt{3^2+5^2+1^2}} \qquad D_2 = \frac{6}{\sqrt{3^2+5^2+1^2}}$

$= \frac{-\sqrt{35}}{7} \qquad = \frac{6\sqrt{35}}{35}$

$\therefore$ shortest distance $= \frac{11\sqrt{35}}{35}$ units

ii.



Hence, find the coordinates of point Q, which is the reflection of point P in the plane Π_1, as shown in the diagram above. 3 marks

$P(1,0,3) \qquad \underset{\sim}{n} = 3\underset{\sim}{i} + 5\underset{\sim}{j} + \underset{\sim}{k}$

$\underset{\sim}{r}(t) = (\underset{\sim}{i}+3\underset{\sim}{k}) + t(3\underset{\sim}{i}+5\underset{\sim}{j}+\underset{\sim}{k}) \qquad \Pi_1: 3x+5y+z=-5$

$x=1+3t,\ y=5t,\ z=3+t$

$\therefore 3+9t+25t+3+t=-5$

$t=\frac{-11}{35}$

$\therefore M\left(\frac{2}{35}, \frac{-11}{7}, \frac{94}{35}\right)$ is intersection of line and Π_1

M is midpoint of $\overrightarrow{PQ}$

$\therefore \frac{1}{2}\left((x_1\underset{\sim}{i}+y_1\underset{\sim}{j}+z_1\underset{\sim}{k})+(\underset{\sim}{i}+3\underset{\sim}{k})\right) = \frac{2}{35}\underset{\sim}{i} - \frac{11}{7}\underset{\sim}{j} + \frac{94}{35}\underset{\sim}{k}$

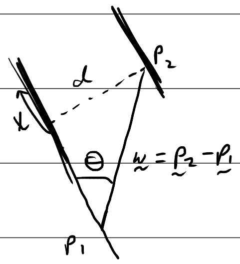
$\Rightarrow x = \frac{-31}{35},\ y=\frac{-22}{7},\ z=\frac{83}{35}$

$\therefore Q\left(\frac{-31}{35}, \frac{-22}{7}, \frac{83}{35}\right)$

Question 4 (10 marks)

a. Find the shortest distance between the two parallel lines given by $\underset{\sim}{r}(t) = 4\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k} + t\left(-\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right)$, where $t \in R$, and $\underset{\sim}{r}(s) = 5\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k} + s\left(-\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right)$, where $s \in R$. 3 marks

$\underset{\sim}{\hat{v}} = \widehat{(-\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k})} = \frac{\sqrt{11}}{11}(-\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k})$



$\underset{\sim}{P_2} = 5\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k}$ $\quad \underset{\sim}{P_1} = 4\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$

$\underset{\sim}{w} = \underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$d = |\underset{\sim}{w}| \sin\theta$

$= |\underset{\sim}{\hat{v}} \times \underset{\sim}{w}| = \frac{3\sqrt{110}}{11}$ units

b. Given that the lines with equations $\underset{\sim}{r}(t) = \underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k} + t\left(3\underset{\sim}{i} + 5\underset{\sim}{j} - a\underset{\sim}{k}\right)$, where $t \in R$, and $\underset{\sim}{r}(s) = -6\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k} + s\left(4\underset{\sim}{i} - 10\underset{\sim}{j} + 6\underset{\sim}{k}\right)$, where $s \in R$, intersect, find the value of a and the point of intersection. 4 marks

$\underset{\sim}{r}(t) = \underset{\sim}{r}(s)$

$(1+3t)\underset{\sim}{i} + (5t-3)\underset{\sim}{j} + (6-at)\underset{\sim}{k} = (4s-6)\underset{\sim}{i} + (2-10s)\underset{\sim}{j} + (1+6s)\underset{\sim}{k}$

$1+3t = 4s-6$ (1) $\quad 5t-3 = 2-10s$ (2) $\quad 6-at = 1+6s$ (3)

solving (1) and (2) simultaneously gives $t=-1$, $s=1$

Sub $t=-1$, $s=1 \rightarrow$ (3) gives $a = -11$

$\therefore \underset{\sim}{r} = -2\underset{\sim}{i} - 8\underset{\sim}{j} + 7\underset{\sim}{k} \implies (-2, -8, 7)$

c. The line with equation $\underset{\sim}{r}(t) = \underset{\sim}{i} + \underset{\sim}{j} - 5\underset{\sim}{k} + t\left(4\underset{\sim}{i} + b\underset{\sim}{j} + 2\underset{\sim}{k}\right)$, where $t, b \in R$, is parallel to the plane with equation $2x - 3y - z = 2$.

Find the value of b and the shortest distance of the line from the plane. 3 marks

$\underset{\sim}{n} = 2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}$

$(2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}) \cdot (4\underset{\sim}{i} + b\underset{\sim}{j} + 2\underset{\sim}{k}) = 0$

$8 - 3b - 2 = 0$

$b = 2$

shortest dist $= \dfrac{|\underset{\sim}{n} \cdot (\underset{\sim}{i} + \underset{\sim}{j} - 5\underset{\sim}{k})|}{|\underset{\sim}{n}|} = \dfrac{2}{\sqrt{14}}$

Question 5 (10 marks)

a. Given the points $A(1, 0, 2)$, $B(2, 3, 0)$ and $C(1, 2, 1)$

i. find the vector $\overrightarrow{AB} \times \overrightarrow{AC}$ 1 mark

$\overrightarrow{AB} \times \overrightarrow{AC} = \underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

ii. show that the Cartesian equation of the plane Π_1, containing the points A, B and C, is $x + y + 2z = 5$. 1 mark

$\underset{\sim}{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

$\underset{\sim}{n} \cdot (x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}) = x + y + 2z = k, \ k \in \mathbb{R}$

$\overrightarrow{OA} = \underset{\sim}{i} + 2\underset{\sim}{k}$

$\underset{\sim}{n} \cdot (\underset{\sim}{i} + 2\underset{\sim}{k}) = k = 5$ $\quad \therefore x + y + 2z = 5$ as required

b. A second plane, Π_2, has the Cartesian equation $x - y - z = 0$.
L is the line of intersection of the planes Π_1 and Π_2.

i. Find the coordinates of the point P, where L crosses the y-z plane. 1 mark

$x - y - z = 0$ ① $\quad x + y + 2z = 5$ ②

①+②: Let $z = \lambda$, $x = \frac{5-\lambda}{2}$, $y = \frac{5-3\lambda}{2}$, $z = \lambda$

$x = 0 \Rightarrow \lambda = 5 \quad \therefore x = 0, y = -5, z = 5 \quad \therefore (0, -5, 5)$

ii. Hence, find the vector equation of the line L. 2 marks

$\Pi_1: \underset{\sim}{n}_1 = \underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k} \qquad \Pi_2: \underset{\sim}{n}_2 = \underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$

Parallel to L: $\underset{\sim}{n}_1 \times \underset{\sim}{n}_2 = \underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$

Point on L: $(0, -5, 5)$

$\therefore \underset{\sim}{r}(\lambda) = (-5\underset{\sim}{j} + 5\underset{\sim}{k}) + \lambda(\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k})$

iii. Find the distance from the point A to the plane Π_2. 2 marks

$P(0, -5, 5)$ on Π_2, $\underset{\sim}{n}_2 = \underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$

$d = |\overrightarrow{AP} \cdot \hat{\underset{\sim}{n}}_2|$

$= \frac{\sqrt{3}}{3}$ units

iv. Find the distance from the point A to the line L. 3 marks

$\underset{\sim}{r}(\lambda) = \lambda\underset{\sim}{i} + (3\lambda - 5)\underset{\sim}{j} + (5 - 2\lambda)\underset{\sim}{k}$

$\underset{\sim}{d} = \underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$

$\overrightarrow{AQ} = \underset{\sim}{r}(\lambda) - \overrightarrow{OA} = (t-1)\underset{\sim}{i} + (3t-5)\underset{\sim}{j} + (3-2t)\underset{\sim}{k}$

$\overrightarrow{AQ} \cdot \underset{\sim}{d} = 0 \qquad |AQ|\Big|_{t=\frac{11}{7}} = \frac{\sqrt{21}}{7}$ units = distance

$14t - 22 = 0$

$\Rightarrow t = \frac{11}{7}$

Question 6 (11 marks)

The position vector $\underset{\sim}{r}_S(t)$, from an origin O, of a sparrow t seconds after being sighted is modelled by $\underset{\sim}{r}_S(t) = 23t\,\underset{\sim}{i} + 5t\,\underset{\sim}{j} + \left(4\sqrt{2}\sin\left(\frac{\pi t}{2}\right) + 4\sqrt{2}\right)\underset{\sim}{k}$, $t \geq 0$, where $\underset{\sim}{i}$ is a unit vector in the forward direction, $\underset{\sim}{j}$ is a unit vector to the left and $\underset{\sim}{k}$ is a unit vector vertically up. Displacement components are measured in centimetres.

a. Find the value of t when the sparrow first lands on the ground. 2 marks

$\underset{\sim}{k}$ component $= 0$: $4\sqrt{2}\sin\left(\frac{\pi}{2}t\right) + 4\sqrt{2} = 0$

$t = 4n - 1,\ n \in \mathbb{Z}$

First landing at $t = 3$ for $t \geqslant 0$

b. Find the distance of the sparrow from O when it first lands. Give your answer correct to one decimal place. 2 marks

first landing at $t=3$

$\underset{\sim}{r}_s(3) = 69\underset{\sim}{i} + 15\underset{\sim}{k}$

$|\underset{\sim}{r}_s(3)| = 3\sqrt{554} = 70.6\text{cm}$

c. Find the maximum flight speed, in centimetres per second, of the sparrow. Give your answer correct to one decimal place. 2 marks

$\dot{\underset{\sim}{r}}(t) = 23\underset{\sim}{i} + 5\underset{\sim}{j} + 2\sqrt{2}\pi\cos\left(\frac{\pi}{2}t\right)\underset{\sim}{k}$

$|\dot{\underset{\sim}{r}}(t)| = \sqrt{2\left(4\pi^2\cos^2\left(\frac{\pi t}{2}\right) + 277\right)}$

Max $|\dot{\underset{\sim}{r}}(t)|$ occurs at $\cos^2\left(\frac{\pi t}{2}\right) = 1$

$\therefore |\dot{\underset{\sim}{r}}(t)|_{max} = 25.2\text{ cm s}^{-1}$

A second bird, a miner, flies such that its velocity vector $\underset{\sim}{v}_M(t)$, relative to the same origin O, is modelled by $\underset{\sim}{v}_M(t) = 6\underset{\sim}{i} + \underset{\sim}{j} + \left(\frac{\pi}{6}\cos\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{k}$, $t \geq 0$, where velocity components are measured in centimetres per second.

d. Given that the miner has an initial position vector of $10\underset{\sim}{i} + 4\underset{\sim}{j} + 4\sqrt{2}\underset{\sim}{k}$, show that its position vector at time t seconds is given by $\underset{\sim}{r}_M(t) = (6t+10)\underset{\sim}{i} + (t+4)\underset{\sim}{j} + \left(\sin\left(\frac{\pi t}{6}\right) + 4\sqrt{2}\right)\underset{\sim}{k}$. 2 marks

$\underset{\sim}{r}_m(t) = \int 6\,dt\,\underset{\sim}{i} + \int dt\,\underset{\sim}{j} + \int \frac{\pi}{6}\cos\left(\frac{\pi}{6}t\right)dt\,\underset{\sim}{k}$

$= 6t\underset{\sim}{i} + t\underset{\sim}{j} + \sin\left(\frac{\pi}{6}t\right)\underset{\sim}{k} + \underset{\sim}{c},\ \underset{\sim}{c} \in \mathbb{R}^3$

$\underset{\sim}{r}_m(0) = \underset{\sim}{c} = 10\underset{\sim}{i} + 4\underset{\sim}{j} + 4\sqrt{2}\underset{\sim}{k}$

$\therefore \underset{\sim}{r}_m(t) = 6t\underset{\sim}{i} + t\underset{\sim}{j} + \sin\left(\frac{\pi}{6}t\right)\underset{\sim}{k} + 10\underset{\sim}{i} + 4\underset{\sim}{j} + 4\sqrt{2}\underset{\sim}{k}$

$= (6t+10)\underset{\sim}{i} + (t+4)\underset{\sim}{j} + \left(\sin\left(\frac{\pi t}{6}\right) + 4\sqrt{2}\right)\underset{\sim}{k}$

e. The sparrow and the miner are at the same position at different times.

Find the coordinates of this position and the times at which each bird is at this position. 3 marks

$\underset{\sim}{r}_s(\lambda) = \underset{\sim}{r}_m(\gamma),\ \lambda, \gamma \in \mathbb{R} \Rightarrow 23\lambda = 6\gamma + 10$ ①, $5\lambda + \gamma + 4$ ②

$\Rightarrow \lambda = 2,\ \gamma = 6$

$\underset{\sim}{r}_s(2) = 46\underset{\sim}{i} + 10\underset{\sim}{j} + 4\sqrt{2}\underset{\sim}{k}$

$\underset{\sim}{r}_m(6) = 46\underset{\sim}{i} + 10\underset{\sim}{j} + 4\sqrt{2}\underset{\sim}{k}$

$\therefore P(46, 10, 4\sqrt{2})$ is the position

Sparrow at P at 2 seconds, Miner at P at 6 seconds

Answers to multiple-choice questions

Question	Answer
1	D
2	C
3	E
4	A
5	B
6	D
7	A