BOOLEAN ALGEBRA

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Contents

- A binary relation R from a set A to B is a subset of $A \times B$. That is, $R = \{(a, b) | a \in A, b \in B\} \subseteq A \times B$. If $(a, b) \in R$, then we say that the element 'a is related to b' and write aRb.
- A binary relation a set A to A is said to be a binary relation on A.

Types of relations:

1. **Reflexive relation:** A binary relation R on a A is said to be a reflexive relation if $(a, a) \in R$ for all $a \in A$.
   Ex: Let A be the set of positive integers and R be the binary relation on A defined by $(a, b) \in R$ if and only if a divides b. Then R is reflexive as every integer divides itself.
2. **Symmetric relation:** A binary relation R on a A is said to be symmetric if $(a, b) \in R \implies (b, a) \in R$ for all $a, b \in A$.
   Ex: The relations "is parallel to "and "is perpendicular to "are symmetric relations on the set of all straight lines.

3. **Antisymmetric relation:** A binary relation R on a set A is said to be antisymmetric if $(a, b) \in R \implies (a, b) \notin R$ unless $a = b$.
Ex: The binary relation R defined by $(a, b) \in R$ if and only if $a \geq b$ is antisymmetric on the set of positive integers.
4. **Transitive relation:** A binary relation R on a A is said to be transitive if $(a, c) \in R$ whenever both $(a, b) \in R$ and $(b, c) \in R$.
Ex: The relation "is parallel to "is transitive, but the relation "is perpendicular to "is not transitive on the set of straight lines.
5. **Equivalence relation:** A binary relation on a set is said to be an equivalence relation if it is reflexive, symmetric and transitive.

3. **Partial ordering relation:** A binary relation on a set is said to be an equivalence relation if it is reflexive, antisymmetric and transitive. A nonempty set A with a partial ordering relation R is a partially ordered set (abbreviated as poset). **For each ordered pair $(a, b) \in R$, we write $a \leq b$ instead of aRb where $\leq$ is a generic symbol and commonly read as "less than or equal to "**. It is often denoted as (A, R) or $\langle A, R \rangle$ or $(A, \leq)$.
Ex: Let A be the set of positive integers and R be the binary relation on A defined by $a \leq b$ if and only if a divides b. Then $(A, \leq)$ is a poset.

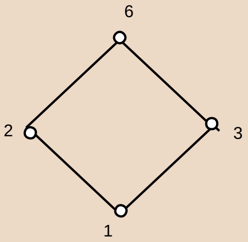
- **Comparable elements:** Let $(A, \leq)$ is a poset. Two elements $a, b \in A$ are said to be comparable if either $a \leq b$ or $b \leq a$.
- **Chain:** Let $(A, \leq)$ is a poset. A subset of A is called a chain if every two elements in the subset are comparable. The number of elements in a chain is known as the length of the chain.
- **Antihain:** Let $(A, \leq)$ is a poset. A subset of A is called an antichain if no two distinct elements in the subset are comparable.
- **Totally ordered set:** A poset $(A, \leq)$ is called a totally ordered set if A is a chain. In this case, the binary relation $\leq$ is called a total ordering relation.
- **Cover of an element:** Let $(A, \leq)$ is a poset. An element $b \in A$ is said to cover an element $a \in A$ if $a \leq b$ and there is no element $c \in A$ such that $a \leq c \leq b$.

Hasse diagram

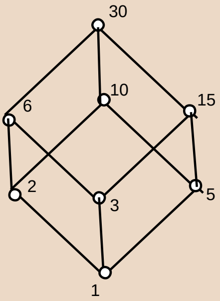
A poset $(A, \leq)$ is graphically represented by Hasse diagram. The following steps are to be followed to draw Hasse diagram corresonding to a given poset $(A, \leq)$.

- Each element of A is represented by a small circle or a dot.
- The circle for $x \in A$ is drawn below the circle for $y \in A$ if $x \leq y$. A line is drawn if y covers x.
- If $x \leq y$ but y doesn't cover x, then x and y are not connected directly by a single line.

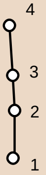
Example 1.1.



The poset (A, |) where A={1,2,3,6}



The poset (D, |) where D={1,2,3,5,6,10,15,30}



The poset (C, $\leq$) where C={1,2,3,4}

Here $|$ is the relation "divides "and $\leq$ is the relation "less than or equal to ".

We note the following terminologies for a given poset $(A, \leq)$.

- **Maximal element:** An element $a \in A$ is said to be a maximal element of A if there is no $b \in A$ such that $a \neq b$ and $a \leq b$. We note that 6,30 and 4 are the maximal elements of $(A, |), (D, |)$ and $(C, \leq)$ respectively.
- **Minimal element:** An element $a \in A$ is said to be a minimal element of A if there is no $b \in A$ such that $a \neq b$ and $b \leq a$.1 is the minimal element of $(A, |), (D, |)$ and $(C, \leq)$.
- **Upper bound:** Let $a, b \in A$. An element $c \in A$ is said to be an upper bound of a and b if $a \leq c$ and $b \leq c$.
- **Lower bound:** An element $c \in A$ is said to be an lower bound of a and b if $c \leq a$ and $c \leq b$.

- **Least upper bound (lub):** An element $c \in A$ is said to be a least upper bound of a and b if c is an upper bound for a and b, and there is no upper bound d of a and b such that $d \leq c$.
  In $(D, |)$ of example 1.1, the element 30 is an upper bound of 2 and 3, but it is not the least upper bound. The lub for 2 and 3 is 6.
- **Greatest lower bound (glb):** An element $c \in A$ is said to be an greatest lower bound of a and b if c is a lower bound for a and b, and there is no lower bound d of a and b such that $c \leq d$.

Lattice:

A partially ordered set is said to be a lattice if every two elements in the set have a unique glb and unique lub. Let $(L, \leq)$ be a lattice. For any two elements a, b, let

$a \vee b$: **lub of** a **and** b and $a \wedge b$: **glb of** a **and** b.

Then $(L, \leq, \vee, \wedge)$ is an algebraic system defined by the lattice $(L, \leq)$.

Example 2.1.

Let $P(S)$ be the power set of a nonempty set S. Then $(P(S), \subseteq)$ is a lattice where $A \vee B = A \cup B$ and $A \wedge B = A \cap B$. This defines the algebraic system $(P(S), \subseteq, \cup, \cap)$.

Example 2.2.

Let N^+ be the set of all positive integers. Then $(N^+, |)$ ($a|b$ if a divides b) is a lattice where $a \vee b = lcm(a, b)$ and $a \wedge b = gcd(a, b)$.

Theorem 2.3.

For any elements a, b *in a lattice* $(A, \leq)$,

- $a \leq a \vee b$ *and* $b \leq a \vee b$
- $a \wedge b \leq a$ *and* $a \wedge b \leq b$

Theorem 2.4.

For any elements a, b, c, d *in a lattice* $(A, \leq)$, *if* $a \leq b$ *and* $c \leq d$

- $a \vee c \leq b \vee d$
- $a \wedge c \leq b \wedge c$

Duality Principle

Let $(A, \leq)$ be a poset. Let $\geq$ be a binary relation on A such that for any a, b in A, $a \geq b$ if and only if $b \leq a$. We note that $(A, \geq)$ is a poset.

- If $(A, \leq)$ is a lattice, then so is $(A, \geq)$
- The join operation of the algebraic system defined by the lattice $(A, \leq)$ is the meet operation of the algebraic system defined by $(A, \geq)$ and vice versa.
- Consequenntly, given any valid statement concerning the general properties of the lattices, we can obtain another valid statement by replacing the relation $\leq$ with $\geq$, the meet operation with the join operation and the join operation with the meet operation. This is known as principle of duality for lattices.
- If the statement remains the same after dualism, then such a statement is called self dual.

Properties of aglebraic systems defined by lattices:

Let $(A, \leq, \vee, \wedge)$ be the algebraic system defined by the lattice $(A, \leq)$. For any elements $a, b, c \in A$,

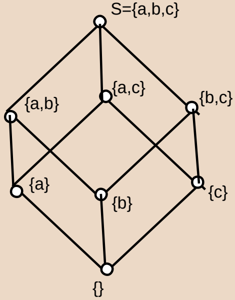
1. Commutative property:
   - $a \vee b = b \vee a$
   - $a \wedge b = b \wedge a$
2. Associative property:
   - $(a \vee b) \vee c = a \vee (b \vee c)$
   - $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
3. Idempotent property:
   - $a \vee a = a$
   - $a \wedge a = a$
4. Absorption property:
   - $a \wedge (a \vee b) = a$
   - $a \vee (a \wedge b) = a$

Distributive lattice: A lattice is said to be a distributive lattice if the meet operation is distributes over the join operation and the join operation s distributes over the meet operation. For any a, b, c

- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Example 3.1.

Let $S = \{a, b, c\}$. Then $(P(S), \subseteq)$ is a distributive lattice.



Theorem 3.2.

If the meet operation is distributive over the join operation in a lattice, then the join operation is also distributive over the meet operation. If the join operation is distributive over the meet operation in a lattice, then the meet operation is also distributive over the join operation.

Proof.

Given that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) - - - - - - - - - - - - - - (1)$

To prove $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,

$$\begin{aligned}
\text{Consider } (a \vee b) \wedge (a \vee c) &= [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c] && \text{from (1)} \\
&= a \vee [(a \vee b) \wedge c] && \text{(absorption law)} \\
&= a \vee [c \wedge (a \vee b)] && \text{(commutative law)} \\
&= a \vee [(c \wedge a) \vee (c \wedge b)] && \text{from (1)} \\
&= [a \vee (c \wedge a)] \vee (c \wedge b) && \text{(associative law)} \\
&= a \vee (c \wedge b) && \text{(absorption law)} \\
&= a \vee (b \wedge c) && \text{(commutative law)}
\end{aligned}$$

Second part follows from the principle of duality. □

Problems:

Q1. Let a and b be two elements in a lattice $(A, \leq)$. Show that $a \wedge b = b$ if and only if $a \vee b = a$.

Sol.

Let

$a \wedge b = b$ $\quad ----------------------(2)$

Consider $a \vee (a \wedge b) = a$ (absorption law)

$a \vee b = a$ from (2)

Conversely, let $a \vee b = a$ $\quad --------------------(3)$

Consider $b \wedge (a \vee b) = b$ (absorption law)

$a \wedge b = b$ $\square$

Q2. Let a, b, c be elements in a lattice $(A, \leq)$. Show that

- i. $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$
- ii. $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$

Sol.

i. $a \leq a \vee b$ and $a \leq a \vee c \implies a \leq (a \vee b) \wedge (a \vee c) - - - - - - - - - (4)$
$b \leq a \vee b$ and $c \leq a \vee c \implies b \wedge c \leq (a \vee b) \wedge (a \vee c) - - - - - - - (5)$
From (4) and (5), $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$. (By Theorem 2.4)
ii. $(a \wedge b) \leq a$ and $(a \wedge c) \leq a \implies (a \wedge b) \vee (a \wedge c) \leq a - - - - - - (6)$
$(a \wedge b) \leq b$ and $(a \wedge c) \leq c \implies (a \wedge b) \vee (a \wedge c) \leq (b \vee c) - - - - - - (7)$
From (6) and (7), $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$. (By Theorem 2.4)

□

Q3. Let a, b, c be elements in a lattice $(A, \leq)$. Show that if $a \leq b$, then $a \vee (b \wedge c) \leq b \wedge (a \vee c)$.
Q4. Let $(A, \leq, \vee, \wedge)$ be an algebraic system where $\vee, \wedge$ are binary operations satisying absorption law. Show that $\vee$ and $\wedge$ also satisfy idempotent law.

Q5. Let $(A, \vee, \wedge)$ be an algebraic system where $\vee, \wedge$ are binary operations satisying commutative, associative and absorption laws. Define a binary operation $\leq$ as follows: for all $a, b \in A, a \leq b$ if and only if $a \wedge b = a$. Show that $\leq$ is a poset. Also show that $a \vee b$ is lub of a and b and $a \wedge b$ is glb of a and b in $(A, \leq)$.

Q6.(Cancellation laws) Let $(A, \leq)$ be a distributive lattice. Show that if $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ for some a, then $x = y$.

Q7. Show that a lattice is distributive if and only if for any elements a, b, c in lattice $(a \vee b) \wedge c \leq a \vee (b \wedge c)$.

Universal lower and upper bounds: An element a in a lattice $(A, \leq)$ is called a universal lower bound if for every element $b \in A$, $a \leq b$. We use '0'to denote universal lower bound. An element a in a lattice $(A, \leq)$ is called a universal upper bound if for every element $b \in A$, $b \leq a$. We use '1' to denote universal upper bound. If a lattice has a universal lower (upper) bound, then it is unique. In the lattice $(P(S), \subseteq)$, the nullset ϕ and the set S are the universal lower and upper bounds respectively.

Theorem 3.3.

Let $(A, \leq)$ be a lattice with universal upper and lower bounds 1 and 0. For any elements $a \in A$

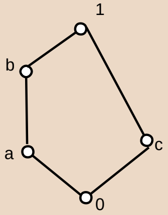
- $a \vee 1 = 1$
- $a \wedge 1 = a$
- $a \vee 0 = a$
- $a \wedge 0 = 0$

Complement of an element: Let $(A, \leq)$ be a lattice with universal upper and lower bounds 1 and 0. For any element $a \in A$, an element b is said to be a complement of a if $a \vee b = 1$ and $a \wedge b = 0$

An element in a lattice may have more than one complement. Not all the elements in a lattice have complements. It's evident that '0'is the unque complement of '0'and vice versa.

Complemented lattice: A lattice is said to be a complemented lattice if every element in the lattice has a complement. Clearly, a complemented lattice has a universal lower and upper bounds.

Example 3.4.



Complement of a and b is c. Complement of c are a, b.

Theorem 3.5.

In a distributive lattice, if an element has a complement, it is unique.

Proof.

Suppose an element a has two complements b and c. i.e.
$a \vee b = a \vee c = 1$ and $a \wedge b = a \wedge c = 0$.
Consider
$$\begin{aligned} b &= b \wedge 1 \\ &= b \wedge (a \vee c) \\ &= (b \wedge a) \vee (b \vee c) \\ &= 0 \vee (b \vee c) \\ &= (a \wedge c) \vee (b \vee c) \\ &= c \wedge (a \vee b) \\ &= c \wedge 1 \\ &= c \end{aligned}$$
Thus $b = c$ □

Boolean lattice: A complemented and distributive lattice is called a boolean lattice.

Example 4.1.

$(P(S), \subseteq)$ is a boolean lattice.

Let $(A, \leq)$ be a boolean lattice. Since every element a has a unique complement $\bar{a}$, we have another unary operation known as complementation and denoted by $^{-}$. Thus we can say that the lattice $(A, \leq)$ defines an algebraic system $(A, \leq, \vee, \wedge)$ where $\vee$ and $\wedge$ are the join and meet operations respectively. The algebraic system defined by a boolean lattice is known as **boolean algebra.**

Theorem 4.2.

DeMorgan's laws: For any elements a, b in a boolean algebra $(A, \leq, \vee, \wedge)$,

- $\overline{a \vee b} = \bar{a} \wedge \bar{b}$
- $\overline{a \wedge b} = \bar{a} \vee \bar{b}$

Proof.

We have to prove that $(a \vee b) \vee (\bar{a} \wedge \bar{b}) = 1$ and $(a \vee b) \wedge (\bar{a} \wedge \bar{b}) = 0$.

Consider
$$\begin{aligned}(a \vee b) \vee (\bar{a} \wedge \bar{b}) &= [(a \vee b) \vee \bar{a}] \wedge [(a \vee b) \vee \bar{b}] \text{ (distributive law)}\\ &= [\bar{a} \vee (a \vee b)] \wedge [a \vee (b \vee \bar{b})] \text{ (associative law)}\\ &= [(\bar{a} \vee a) \vee b] \wedge [a \vee 1] \text{ (associative law)}\\ &= [1 \vee b] \wedge [a \vee 1]\\ &= 1 \wedge 1 = 1\end{aligned}$$

Similarly,
$$\begin{aligned}(a \vee b) \wedge (\bar{a} \wedge \bar{b}) &= (\bar{a} \wedge \bar{b}) \wedge (a \vee b) \text{ (commutative law)}\\ &= [(\bar{a} \wedge \bar{b}) \wedge a] \vee [(\bar{a} \wedge \bar{b}) \wedge b] \text{ (distributive law)}\\ &= [a \wedge (\bar{a} \wedge \bar{b})] \vee [(\bar{a} \vee \bar{b}) \wedge b] \text{ (commutative law)}\\ &= [(a \wedge \bar{a}) \wedge \bar{b}] \vee [\bar{a} \vee (\bar{b} \wedge b)] \text{ (associative law)}\\ &= [0 \wedge \bar{b}] \vee [(\bar{a} \wedge 0]\\ &= 0 \vee 0 = 0\end{aligned}$$

The second part follows from principle of duality. □

Uniqueness of finite boolean algebras: We show that a finite boolean algebra has 2^n elements for some n. An element of a boolean algebra is called an atom if it covers 0.

Lemma 4.3.

In a distributive lattice, if $b \wedge \bar{c} = 0$, then $b \leq c$.

Proof.

We know that $0 \vee c = c$

$(b \wedge \bar{c}) \vee c = c$ (given)

$c \vee (b \wedge \bar{c}) = c$ (commutative law)

$(c \vee b) \wedge (c \vee \bar{c}) = c$ (distributive law)

$(c \vee b) \wedge 1 = c$

$(c \vee b) = c$

$(b \vee c) = c$

Thus $b \leq b \vee c \implies b \leq c$. □

Lemma 4.4.

Let $(A, \leq, \vee, \wedge, ^{-})$ be a finite boolean algebra. Let b be any nonzero element in A and $a_1, a_2, \ldots, a_k$ be all the atoms of A such that $a_i \leq b$. Then $b = a_1 \vee a_2 \vee \ldots a_k$.

Proof.

Since $a_1 \leq b, a_2 \leq b, \ldots, a_k \leq b$, it follows that
$a_1 \vee a_2 \vee \ldots a_k \leq b$ ———————————————————————(8)
For notational convinience, let $c = a_1 \vee a_2 \vee \ldots a_k$. Suppose $b \wedge \bar{c} \neq 0$, then there exists an atom a such that $a \leq b \wedge \bar{c}$.
Now $a \leq b \wedge \bar{c}$ and $b \wedge \bar{c} \leq b \implies a \leq b$ ———————————————(9)
$a \leq b \wedge \bar{c}$ and $b \wedge \bar{c} \leq \bar{c} \implies a \leq \bar{c}$
From (9), a is equal to one of the atoms $a_1, a_2, \ldots, a_k$. Also $a \leq c$.
Combining $a \leq c$ and $a \leq \bar{c}$, we get $a \leq c \wedge c \implies a \leq o$, which is impossible. Thus $b \wedge c = 0 \implies b \leq c$. That is
$b \leq a_1 \vee a_2 \vee \ldots a_k$. ————————————————————————(10)
Form (8) and (10) and antisymmetric property, $a_1 \vee a_2 \vee \ldots a_k = b$. □

Lemma 4.5.

Let $(A, \leq, \vee, \wedge, ^{-})$ be a finite boolean algebra. Let b be any nonzero element in A and $a_1, a_2, \ldots, a_k$ be all the atoms of A such that $a_i \leq b$. Then $b = a_1 \vee a_2 \vee \ldots a_k$ is the unique way to represent b as a join of atoms.

Proof.

Suppose that we have alternatinve representation $b = a_{j1} \vee a_{j2} \vee \ldots a_{jt}$.
Since b is the lub of $a_{j1}, a_{j2}, \ldots a_{jt}$, it is true that
$a_{j1} \leq b, a_{j2} \leq b, \ldots, a_{jt} \leq b$. Consider an atom a_{ju} $(1 \leq u \leq t)$. Since
$a_{ju} \leq b$.
we have $a_j \wedge b = a_{ju}$
$a_{ju} \wedge (a_1 \vee a_2 \vee \ldots a_k = a_{ju})$
$(a_{ju} \wedge a_1) \vee (a_{ju} \wedge a_2) \vee \ldots (a_{ju} \wedge a_k) = a_{ju}$
Then for some a_i $(1 \leq i \leq k)$, $a_{ju} \wedge a_i \neq 0$.
Since a_{ju} and a_i are atoms, we must have $a_{ju} = a_1$. Thus each atom in the alternative representation is an atom in the original one, and the lemma follows. □

From the above lemmas, it is clear that there is one to one correspondence between the elements of a boolean lattice and subset of atoms. As a matter of fact, there is one to one correspondence from $(A, \leq)$ to $(P(S), \subseteq)$, where S is the set of all atoms.

Theorem 4.6.

Let $(A, \vee, \wedge, ^-)$ be a finite boolean algebra. Let S be the set of all atoms. Then $(A, \vee, \wedge, ^-)$ is isomorphic to the algebraic system defined by the lattice $(P(S), \subseteq)$.

It follows from the above lemmas that **there exists a finite boolean algebra of 2^n elements for any $n > 0$.**

Boolean expression: Let Let $(A, \vee, \wedge, ^{-})$ be a finite boolean algebra. A boolean expression over $(A, \vee, \wedge, ^{-})$ is defined as follows:

- An element of A is a boolean expression.
- Any variable name is a boolean expression.
- If e_1 and e_2 are boolean expressions, then $\overline{e_1}, e_1 \vee e_2$ and $e_1 \wedge e_2$ are boolean expressions.

Example 5.1.

$0 \vee x$, $(x_1 \vee x_2) \wedge \overline{(2 \vee 3)}$

Let $E(x_1, x_2, \ldots, x_n)$ be a boolean expression of n variables over a boolean algebra $(A, \vee, \wedge, ^{-})$. By assignment of values to the variables $x_1, x_2, \ldots, x_n$, we mean an assignment of elements of A to be the values of the variables. For an assignment of values to the variables, we can evaluate $E(x_1, x_2, \ldots, x_n)$ by substituting the variables in the expression by their values.

Two boolean expressions of n variables are said to be equivalent if thet assume the same vales for every assignment of values to the n variables. If $E_1(x_1, x_2, \ldots, x_n)$ and $E_2(x_1, x_2, \ldots, x_n)$ are equivalent, then we write $E_1(x_1, x_2, \ldots, x_n) = E_2(x_1, x_2, \ldots, x_n)$.

Example 5.2.

$(x_1 \wedge x_2) \vee (x_1 \wedge \bar{x_3})$ is equivalent to $x_1 \wedge (x_2 \vee \bar{x_3})$.

Boolean function: A function $f : A^n \to A$ is said to be a boolean functionnif it can be specified by a boolean expression of n variables.

Minterm: A booleam expression of n variables $x_1, x_2, \ldots, x_n$ is said to be a minterm if it is of the form $\tilde{x_1} \wedge \tilde{x_2} \wedge \cdots \wedge \tilde{x_n}$ where $\tilde{x_i}$ is either x_i or $\bar{x_i}$.
Disjunctive normal form (DNF): A boolean expression over $(\{0,1\}, \wedge, \vee, ^{-})$ is said to be in disjunctive normal form if it is join of minterms.

Example 5.3.

$(x_1 \wedge \bar{x_2} \wedge x_3) \vee (\bar{x_1} \wedge \bar{x_2} \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3)$

Maxterm: A booleam expression of n variables $x_1, x_2, \ldots, x_n$ is said to be a maxterm if it is of the form $\tilde{x_1} \vee \tilde{x_2} \vee \cdots \vee \tilde{x_n}$ where $\tilde{x_i}$ is either x_i or $\bar{x_i}$.
Conjunctive normal form (CNF): A boolean expression over $(\{0,1\}, \wedge, \vee, ^{-})$ is said to be in conjunctive normal form if it is meet of maxterms.

Example 5.4.

$(x_1 \vee \bar{x_2} \vee x_3) \wedge (\bar{x_1} \vee \bar{x_2} \vee x_3)$

DNF: Given a function $\{0,1\}^n \to \{0,1\}$, we can obtain a boolean expression in DNF corresponding to this function by having a minterm corresponding to each ordered n tuple of $0's$ and $1's$ for which the value of the function is 1. For each n tuple with the functional value is 1, we have the minterm $\tilde{x_1} \vee \tilde{x_2} \vee \cdots \vee \tilde{x_n}$ where $\tilde{x_i} = \begin{cases} x_i & if \;\; i^{th} \; componet \; is \; 1 \\ \bar{x}_i & if \;\; i^{th} \; componet \; is \; 0 \end{cases}$.

CNF: We can obtain a boolean expression in CNF corresponding to this function by having a maxterm corresponding to each ordered n tuple of $0's$ and $1's$ for which the value of the function is 1. For each n tuple with the functional value is 0, we have the minterm $\tilde{x_1} \vee \tilde{x_2} \vee \cdots \vee \tilde{x_n}$ where

$$\tilde{x_i} = \begin{cases} x_i & if \;\; i^{th} \; componet \; is \; 0 \\ \bar{x}_i & if \;\; i^{th} \; componet \; is \; 1 \end{cases}$$

Q8. Let $E(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (\bar{x_2} \wedge x_3)$ be a boolean expression over $(\{0,1\}, \wedge, \vee, ^-)$. Write the boolean expression in both DNF and CNF.

Sol:

	$(x_1 \wedge x_2)$	$(x_1 \wedge x_3)$	$(\bar{x_2} \wedge x_3)$	f
000	0	0	0	0
001	0	0	1	1
010	0	0	0	0
011	0	0	0	0
100	0	0	0	0
101	0	1	1	1
110	1	0	0	1
111	1	1	0	1

DNF : $(\bar{x_1} \wedge \bar{x_2} \wedge x_3) \vee (x_1 \wedge \bar{x_2} \wedge x_3) \vee (x_1 \wedge x_2 \wedge \bar{x_3}) \vee (x_1 \wedge x_2 \wedge x_3)$
CNF :$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x_2} \vee x_3) \wedge (x_1 \vee \bar{x_2} \vee \bar{x_3}) \wedge (\bar{x_1} \vee x_2 \vee x_3)$ □

References

[1] Liu C L, Elements of discrete mathematics,2^{nd} *edition*, *McGraw Hill Book Company, New Dehli*, (2007).

THANK YOU