Math for Engineers, vol. I

Signals & Noise

Foundations for understanding

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Contents

Chapter 1

Metric spaces

(Introductory anecdote.)

A vast swath of mathematics is built off of **sets**, which are abstract tools for conceptualizing collections of mathematical objects. In notation, we often will build a set in our minds using a **comprehension**; for example, constructing the set

$$P = \{p \in \mathbb{N} \mid \nexists n \in \mathbb{N}, n \neq p \,:\, n \text{ divides } p\}$$

allows us to reason collectively about the prime numbers, as a set.

The study of sets in and of themselves is an incredibly rich field of inquiry. As it happens, we cannot go off willy-nilly, naïvely constructing sets as our minds see fit. An immediate example was shown by Russell in the early 20th century, who asked one to conceive of the set of all sets that don't contain themselves:

$$S = \{A \text{ a set} \mid A \notin A\}$$

The problem arises when one asks the reasonable question: does S contain itself? Here we see that the definition of S leads to a contradiction: if S does not contain itself, then S must be in S; and, similarly, if S is in S, then it cannot contain itself.

The problem, Russell and his contemporaries discovered, was in the implicit axiom of "naïve" set theory that any collection that could be reasoned about could be instantiated as a mathematical set. It turns out that certain notions are "too ugly" to reason about—at least, without the help of some more recent, esoteric objects that grant us a little more leeway. Today, as a result of the work of these set theorists, we have a much more solid foundation on which to build our mathematical understanding; even so, as we will see when we begin to develop our theory of measure, seemingly benign axioms of set theory will again lead us to a very uncomfortable logical place.

For now, we begin our journey with sets—and with structures placed upon them, which will eventually yield an unending landscape of objects for us to study.

1.1 Sets

Before

1.1.1 Objects and functions

Before

1.1.2 The naturals and the rationals

1.2 Real numbers

Our goal in this chapter is to develop the theory of a set X augmented with a structure imbued by a *metric*,

$$d : X \times X \to [0, \infty)$$

which intuitively measures the "distance" between two points. As you can imagine—given that the central structure of these objects is specified by a mapping to the reals—this theory is intimately linked with the behavior of the real numbers.

The history of the real numbers ...

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1.2.1 Construction

One of the most common ways to build the reals is to, in a sense, "fill in" the missing numbers between the rationals. This construction, due to Dedekind, has a certain physical intuition to it that makes it quite fun. Once we have some of the machinery of metric spaces, we will explore a "slicker" construction.

We're going to leverage the fact that $\mathbb{Q}$ is *totally ordered*, meaning that we could, in theory, take all of the rationals and line them up such that

$$\cdots q_{n-1} \leq q_n \leq q_{n+1} \cdots$$

(with, of course, the caveat that the line would go off infinitely in either direction). But, because we have this lovely ordering, we can speak meaningfully of, for example, sets of the form

$$A_r = \{x \in \mathbb{Q} \mid x < r\}$$

for some $r \in \mathbb{Q}$. We can easily also build the complement of A_r—it is simply $B_r = \{x \in \mathbb{Q} \mid x \geq r\}$. We have, in a sense, "split' $\mathbb{Q}$ into two halves. Importantly, however, observe that A_r has no greatest element; indeed, for any $q = r - \epsilon \in A_r$, $\epsilon \in \mathbb{Q}$, note that $q + \epsilon/2 < r$ is also in A_r.

We can do this construction for any $r \in \mathbb{Q}$; hence, we can, in a sense, *identify* the rationals with these cuts. But let's not stop here: instead of splitting at a particular rational r—which will only net us splits identified with the rationals—let's take *any* splitting of $\mathbb{Q}$ into two sets, A and $B = A^c$, such that

1. Neither A nor B is empty.
2. A is *closed downwards*: for $x, y \in \mathbb{Q}$ with $x < y$, then $y \in A \Rightarrow x \in A$.
3. A has no greatest element.

The first condition prevents us from adding elements representing $\pm\infty$. It's the second and third conditions that are the "secret sauce" for expanding on our earlier construction: they generalize the notion of taking $x < r$ by "adding in" splits that aren't of that form (with rational r) by using the total order on $\mathbb{Q}$ (ideas like "closed downwards" and "no greatest element").

To see that we can in fact find more such splits, consider the following sets:

$$A_{\sqrt{2}} = \left\{x \in \mathbb{Q} \mid x < 0 \text{ or } x^2 < 2\right\} \tag{1.1}$$
$$B_{\sqrt{2}} = \left\{x \in \mathbb{Q} \mid x > 0 \text{ and } x^2 > 2\right\} \tag{1.2}$$

Note that the equality must be strict in both cases, as there is no rational r such that $r^2 = 2$. Because of this, it isn't possible to place this partition in the form above; however, it is a valid partition according to our rules (in fact, it is the one corresponding to $\sqrt{2}$!).

1.2.2 Sequences

1.3 Metricc spaces

Chapter 2

Linearity and translation-invariance

A monophonic record contains information about sound encoded in a groove etched into the vinyl, with the sound's amplitude corresponding to the height of the groove. This physical geometry of the record has to be transformed into sound that reaches your ear: on today's systems, the groove is first transduced into an electric current through a needle, and then this electric current is transformed into sound through vibrating elements in speakers.

At each stage, the information content is transformed into a new form; however, at least to a first approximation, each transformation posesses two fundamentally important qualities. First, the transformations are **linear**: that is, a doubling of the height of the groove results in a doubling of the transduced electric current, and hence a doubling of the resultant sound amplitude out of the speakers; and, similarly, were we to add together the groove heights generated by two different sounds on our platter, the resultant output at the speaker would be the superposition of those two sounds. Second, the transformations are **translation-invariant** in time: if you were to plaly the record tomorrow, it would sound exactly the same as it would if you played it today. Our goal in this section is to characterize transformations of information that retain these two properties. As we will see, despite their apparent simplicity, linearity and translation-invariance actually provide immense constraints on the behavior of general transformations.

It's important to note that in real world apparatuses, these properties only hold approximately. In our example, linearity breaks down for a number of reasons: the record is only of finite thickness, and hence there is a breakdown of the transduction process when the groove height becomes too large;

what's more, due to practical limitations on operational amplifier circuits, the process that drives the speakers from transduced electrical currents suffers from nonlinear distortion, especially at high frequencies, resulting in a characteristic "crunchy" sound. Translation invariance is clearly limited in our example as well: if we translated far enough *back* in time, we would be attempting to play the record before the record player was made; and, if we tried to play our record millions of years in the future, our turntable would clearly be degraded to the point of no longer functioning. However, many practical apparatuses have an operational regime in which their behavior is very approximately linear and/or translation-invariant; and, as we will see, the beautiful mathematics imparted by these properties provides numerous practical benefits for analysis and understanding.

2.1 Signals and systems

Let's begin by defining our objects of study. First, we would like to formally characterize "information content", the object being acted upon by our transformations of interest; for example, this information may be electrical voltage fluctuations over time, or patterns of light passing through a 2D aperture, or the pressure of air induced by a speaker at every point in a room. Formally, we call such information content a **signal**, which we define as a "nice" function

$$u : G \to \mathbb{R}$$

where we impose on G the structure of a *topological group*, meaning that G posesses a topological structure (open sets) and a group structure (a binary operation $\_ + \_$ and an inverse $-\_$) that are "compatible" with one another, in the sense that both of the functions $\_ + \_$ and $-\_$ are continuous in G's topology. We denote the set of all such signals on G by $S(G)$; for the time being, we will leave it nebulous as to what functions are "allowed" signals, and which are too "ugly" to be dealt with, a point which we will return to once we have a few more tools at our disposal.

Although we won't use much of this formalism *per se*, it can be helpful to think of this topological group structure as being the "minimal" amount of structure we need to impose on the domain of our signal in order for translation to "make sense". The group operation *is*, in a manner of speaking, the translation: for example, when G is $\mathbb{R}$ (as in, for example, the case where we have signals that vary over *time*), then the group operations on $\mathbb{R}$ (*i.e.*, $+$ and $-$) anchor the behavior of translation forward and backward *in time* to the addition and subtraction *of timestamps* in $\mathbb{R}$; more specifically, for any $\tau \in \mathbb{R}$, we can construct a transformation $T_\tau : S(\mathbb{R}) \to S(\mathbb{R})$ defined by

$$T_\tau (f(t)) = f(t - \tau), \forall t$$

This transformation is a **translation by** τ, and is evidently fundamentally linked to the subtraction (*i.e.*, addition and negation) on $\mathbb{R}$.

Although it is possibble to develop a theory of signals for arbitrary topological groups G, as we've begun to explore, the most practical examples come from the cases $\mathbb{R}^n$ (called **continuous signals**) and $\mathbb{Z}^n$ (called **discrete signals**), with addition (equipped with negation) as the group operation. Note, however, that no matter what G is, we will always be interested in selecting our set of "allowed" signals $S(G)$ to be a vector space over $\mathbb{R}$, using pointwise scalar multiplication and addition as the vector space operations; this structure will become indispensable.

As stated at the beginning, however, our direct interest is not necessarily with the signals themselves, but with the way in which one signal is transformed into another—for example, the transduction of mechanical motion of the turntable needle into an electrical signal. We call such a transformation of signals a **system**,

$$A : S(F) \to S(G)$$

(noting that the case $F = G$ is of particular interest for our study). Systems in general have very little governing mathematical structure: there is a wide gamut of different systems having vastly different timbres. However, as noted in the introduction, there are a few classes of systems that are remarkably well-behaved.

First, we note that, as in the case of metric spaces, the most immediately interesting mappings between objects are those that preserve those objects' underlying mathematical structure. The most readily apparent structure on $S(G)$, as we noted above, is its *vector space* structure over $\mathbb{R}$; hence, most immediately, it would seem that we would first want to consider those systems that are vector space homomorphisms (that is, linear operators) between signal spaces.

Definition 1 (Linear system)**.** A system $A : S(F) \to S(G)$ is **linear** if and only if

1. A preserves scalar multiplication; that is, for any $f \in S(F)$, $\alpha \in \mathbb{R}$,

$$A(\alpha f) = \alpha A(f)$$

2. A preserves vector addition; that is, for any $f, g \in S(F)$,

$$A(f + g) = A(f) + A(g)$$

Linear systems are a rich field of investigation, and are the general objects of study in linear algebra. ...

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$S(G)$ has more structure than that of a vector space, however: we baked in the fact that G is a topological group! Because of this, it also seems reasonable for us to consider systems that, in a sense, preserve this structure. First, we generalize our construction of a transformation that "shifts" a signal from earlier: given the signal space $S(G)$, where G's group operation is $\_ + \_$ and inverse is $-\_$, as well as an $x \in G$, the **translation by** x system $T_x : S(G) \to S(G)$ is defined, for all $f \in S(G)$, by

$$T_x(f(g)) = f(g + (-x)), \forall g \in G$$

Note that translation systems are actually quite well behaved:

Claim 1. For any G and $x \in G$, the translation system $T_x : S(G) \to S(G)$ is a linear operator.

Proof. (Left as exercise.) □

Now that we have these systems induced by G's group structure, we can easily find systems that "respect" the induced shifts:

Definition 2 (Translation-invariant system)**.** A system $A : S(G) \to S(G)$ is **translation-invariant** if and only if, for any $f \in S(G)$ and $x \in G$:

$$(A \circ T_x)(f) = (T_x \circ A)(f)$$

That is, for a translation-invariant system, shifting the input gives an identical output, shifted by the same amount; put another way, translation-invariant systems are precisely those that commute with the translation operators.

Example 1. As an exercise, consider the following closely-related problems:

1. Consider the linear difference equation

   $$y[n] = \alpha y[n-1] + u[n]$$

   (The notation $f[n]$ is common to distinguish discrete signals from their continuous counterparts.) Consider also the additional constraint of zero initial conditions for y; *i.e.*,

   $$\lim_{n \to -\infty} y[n] = 0$$

Prove that the system $A : S(\mathbb{Z}) \to S(\mathbb{Z})$ induced by letting $A(u)$ be the solution y of this difference equation is both linear and time-invariant.

2. Further, consider the forced linear differential equation

$$\dot{y}(t) = \alpha y(t) + u(t)$$

with the constraint of zero initial conditions for y; *i.e.*,

$$\lim_{t \to -\infty} y(t) = 0$$

Prove that the system $A : S(\mathbb{R}) \to S(\mathbb{R})$ induced by letting $A(u)$ be the solution y of this differential equation is both linear and time-invariant.

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For brevity, we denote systems that posess both linearity and translation-invariance **LTI systems**; in some texts, these are denoted **linear shift-invariant (LSI) systems**.

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2.2 Discrete systems in one dimension

Let's focus our attention for now on an example case: one-dimensional discrete signals, $S : \mathbb{Z} \to$, and LTI systems between them.

We begin by developing a representation of an arbitrary discrete signal $u[n]$ as a sum of translated copies of simpler functions; the hope is that, if we can define the behavior of an LTI system on the simpler functions, we can extend the result to the full function by leveraging linearity and translation-invariance. One such simple function is the **unit impulse** (also known as the **Dirac delta**):

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

(The importance of this particular choice of simple function will build as we develop our theory!)

With the unit impulse in hand, note that we can "build up" any discrete signal $u[n]$ by placing a scaled and shifted impulse at each time point:

$$u[n] = \sum_{k=-\infty}^{\infty} u_k \, \delta[n-k]$$

where the scaling coefficients $u_k = u[k]$ now encode u's information content. Now, consider an LTI system A's action on u: letting $\delta_k[n] = (T_k\,\delta)[n] = \delta[n-k]$ be the impulse shifted by k, since

$$u = \sum_k u_k\,\delta_k$$

by linearity we must have

$$A\,u = \sum_k u_k\;(A\,\delta_k)$$

Hence, A's behavior is entirely determined by how it transforms translated impulses. But wait, there's more! By translation-invariance, we must also have

$$(A\,\delta_k)[n] = ((A \circ T_k)\;\delta)[n] = ((T_k \circ A)\;\delta)[n] = (A\,\delta)[n-k]$$

That is, A's action on translated impulses is completely determined by its action on the standard impulse.

Putting it all together, we have (in "vector" form):

$$A\,u = \sum_k u_k\,T_k\;(A\,\delta)$$

Amazing! A's behavior is fully specified by $A\,\delta$, its action on the unit impulse; by scaling and translating this response to the impulse, we can obtain A's output for any arbitrary input signal u. This special output signal is denoted the **impulse response** of A:

$$h_A = A\,\delta$$

Recalling that u_k is really just $u[k]$ in disguise, we can write the equation above in "coordinate" form:

$$\begin{aligned}(A\,u)[n] &= \sum_k u[k]\;(T_k\,h_A)[n] \\ &= \sum_k u[k]\;h_A[n-k]\end{aligned}$$

And so we get a glimpse at what A is doing: A's output for u is a superposition of scaled and shifted copies of h_A, A's impulse response, where each copy (indexed by k) is scaled by the corresponding value of u for that shift.

The operation performed by A between u and the impulse resposnse h_A is of its own importance:

Definition 3 (Convolution (discrete))**.** Given two discrete signals $x, y \in S(\mathbb{Z})$, the **convolution** signal, $x * y \in S(\mathbb{Z})$, is defined by

$$(x * y)[n] = \sum_{k=-\infty}^{\infty} x[k]\ y[n-k]$$

With this definition in hand, we see immediately that the action of an LTI system is to convolve input signals with the system's impulse response:

$$A\,u = u * h_A$$

How delightfully parsimonious!

Convoution brings some algebraic structure of its own, being a binary operation on signals. (As we will see in the continuous case, this structure can be highly nontrivial!) For example:

Claim 2. Convolution is commutative; that is, $\forall x, y \in S(\mathbb{Z})$,

$$x * y = y * x$$

Proof. (Left as exercise.) □

As you may have already pieced together, the unit impulse is the identity for discrete convolution:

$$(\delta * x)[n] = (x * \delta)[n] = \sum_k x[k]\ \delta[n-k] = x[n]$$

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2.2.1 The z transform and the discrete Fourier transform

We've now seen that the behavior of LTI systems is extremely well-constrained as compared to the general case: LTI systems perform one function, and that is convolution with an impulse response function. But, let's not forget that these are *linear operators*; hence, we can characterize their behavior from the perspective of vector spaces as well, not just from the perspective of the convolution algebra we introduced above. So, then, the first question is this: what is the spectrum of an LTI system? What are its eigenvectors? In the general case of linear operators, this question can get quite gnarly; however, as we will see, LTI systems are exceptionally well-behaved in this regard.

Consider a discrete signal given by the exponential

$$u[n] = z^n$$

for some $z \in \mathbb{R}$. Then, for an LTI system A, we have

$$A\,u = u * h_A = h_A * u$$

which, in coordinates, becomes

$$\begin{aligned}(A\,u)[n] &= \sum_k h_A[k]\; u[n-k] \\ &= \sum_k h_A[k]\; z^{n-k} \\ &= \left(\sum_k h_A[k]\, z^{-k}\right) z^n \\ &= H_A(z)\; u[n]\end{aligned}$$

And so we see that z^n is an eigenfunction of A, with $H_A(z)$ as the corresponding eigenvalue! (One must, of course, consider carefully the convergence of the sum that reduces to H_A; we will cover this point in more detail as we progress.)

We can, however, go even further: why not consider values of z that are in $\mathbb{C}$ as well? We can still evaluate our system's convolution in the same way, and when we do, we still arrive at the same result: z^n for complex z is also an eigenfunction of A in the same way. (Why?) With this extended definition, we have extracted another crucial object describing our LTI system A. The *eigenvalue* of A corresponding to the eigenfunction $z^n, z \in \mathbb{C}$ in the expression above yields a map $H_A : \mathbb{C} \to \mathbb{C}$ defined by

$$H_A(z) := \sum_k h_A[k]\, z^{-k}$$

This function is known as the **z-transform** of A's impulse response (and hence, the z-transform corresponding to A). We will see that, just like the impulse response, the z-transform encodes all of the information about A's action on inputs; however, the form of that information is much more conducive to analysis.

Example 2. The translation by 1 system, also called the **unit delay**,

$$(T_1\,x)[n] = x[n-1]$$

is LTI, with impulse response

$$h_{T_1}[n] = \delta[n-1]$$

The corresponding z-transform is

$$H_{T_1}[z] = z^{-1}$$

(which is generally used in engineering texts as the symbol for a unit delay).

Proof. (Exercise.) $\square$

2.3 Higher dimensions

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2.4 Continuous systems

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2.5 General considerations and Pontryagin duality

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Chapter 3

Randomness

Imagine rolling a standard six-sided die. From a physical standpoint, the mechanics are quite clear: the die has an initial position, orientation, and linear and angular momentum; and, once released, the die is subjected to the expected forces—the downward pull of gravity, collisions and friction with the ground plane. Eventually, the die will most likely come to rest, with one of its faces "facing up" the most, a notion which can be easily formalized using the inner product between the normals of the die's faces and the ground plane.

Though it's difficult to imagine, we can take the space of initial states of the die S_i—12 dimensions, encompassing linear and angular position and momentum—and "color" each point of it according to what face of the die ended up face up when the die came to rest; that is, we can construct a map from initial state to final roll:

$$f_d : S_i \to \{1, 2, \ldots, 6\}$$

There are a few features of this "die map" that are worth commenting on. The first is that, owing to the chaotic nature of the underlying dynamics, the geometry of this "coloring" of state space is magnificently, confoundingly complicated. To be sure, there are symmetries to be exploited within the problem; for example, we immediately know that the solution must be translationally invariant in the axes parallel to the ground plane. However, many questions remain: how large are contiguous "tiles" of the mapping, connected regions of state space that all map to the same value?

The second feature is that, as constructed above, the die map is most certainly not defined on all of S_i; for example, while our physical intuition is that such a scenario would never happen, mathematically, if the die were dropped with no initial velocity perfectly edge-on, the die would come to rest balanced perfectly on its edge, with no face "winning out". (We could similarly construct an initial state that lands the die exactly on one corner!) But that lends a reasonable question: why don't we get such results in practice? If we were to adjoin an element $\perp$ to the output of f_d indicating no clear winning face, then what does the set

$$f_d^{-1}(\perp)$$

look like? How "big" is it? Does it take up "volume" in state space?

And of course, the lingering question through all of this: why is it that we can reasonably approximate this setup by saying that the results are "uniformly distributed" on $\{1, 2, \ldots, 6\}$?

Set theorists and analysts alike began experimenting with ways of measuring the "size" of sets near the turn of the 20th century. The critical insight came from Henri Lebesgue who, in 1902, recognized that the problem of measure was of fundamental importance to the problem of *integration*, which at the time was suffering from severe theoretical deficiencies. Many mathematicians were hard at work researching methods of function approximation—for example, Joseph Fourier carried out fundamental work on approximating functions as infinite series, each term a trigonometric function. However, while Fourier and others spoke of the convergence of such series (f_n):

$$f(x) = \sum_{n=0}^{\infty} f_n(x)$$

it remained devilishly difficult to ascertain whether the convergence of this series implied that

$$\int_{[a,b]} \sum_{n=0}^{\infty} f_n(x) \stackrel{?}{=} \sum_{n=0}^{\infty} \int_{[a,b]} f_n(x)$$

(It does not immediately follow that this sum is even integrable!) Having an integral that satisfied this property—namely, respectiing sequences in function space—was a highly desirable goal; and, while Riemann's formulation from several decades earlier seemed to reluctant to yield this result, Lebesgue's measure-theoretic formulation made this, and many more properties of analytic value, manifest, while also drastically expanding the space of functions that one could do calculus with.

Enter Andrey Kolmogorov.

When you think about it, probability theory is largely concerned with "mesuring" sets, namely, sets of *outcomes* of some process, or *states* of some system: ideally, if a set of states E is "large", then the outcome of our experiment, or the state of our system, is more likely to be one of those contained in E. This fundamental link between probability theory and measure theory—providing a sort of geometry for randomness—was made by Kolmogorov in 1933. As we will see, Kolmogorov's formulation provides us a wealth of tools to play with in developing our understanding of "randomness".

3.1 Measure

What does it mean to identify the "size" of a set? We clearly need some kind of "ruler"! Let's try to identify the properties we want out of this set "ruler", which measures the subsets of some given set, X. Right away, it's clear that our domain is the powerset of X, $\mathcal{P}(X)$. We'll also take "size" to be a nonnegative real (and, possibly infinite) quantity, although it's worth noting that there are theories that are more expansive in their notion of what "size" is. For now, our "ruler" is, ideally, a function

$$m : \mathcal{P}(X) \to [0, \infty]$$

We will develop two theories of measure here: first, the theory of *Lebesgue*, which has its roots in the geometry of Euclidean space; and second, the theroy of *Borel*, which utilizes a topology on X. As it turns out, these two notions are linked in a beautiful way for Euclidean spaces.

3.1.1 The problem of measure

To start out, we'll say that we are trying to measure a subset of $\mathbb{R}$; that is, our "ruler"'s ideal domain is $\mathcal{P}(\mathbb{R})$.

What else do we need? Well, we need proper **normalization** of the measure, in order for numerical results to be uniquely specified. In the case of $\mathbb{R}$, this normalization can be achieved by simply taking

$$m([0, 1]) = 1$$

(In the cases of other measures, we may take a slightly different normalization.) At the very least, we would like to make sure that

$$m(\varnothing) = 0$$

In the case of $\mathbb{R}$, we also note that our space has some geometry built in by its vector space structure; hence, again citing geometric intuition, we might also desire that our "ruler" is **translation-invariant**; letting $A+h = \{x+h \mid x \in A\}$ denote the set of translates of A by h, we ask that

$$\forall h \in \mathbb{R} \,:\, m(A+h) = m(A)$$

(Built into this, we might also add the requirement that our "ruler" assigns equal sizes to geometrically *congruent* sets—sets that are the "same" up to translation and rotation.)

Now comes the real meat of our "ruler", the property that gives it real mathematical teeth. Suppose you have two separate objects, A and B, each with its own size that you can measure; and, suppose further, that you make a new object AB by gluing the two objects together without intersecting. Intuitively, you would like your notion of size to be **additive**, so that the size of AB is the sum of the sizes of A and B. That is, for our measure, we would like *disjoint* sets to *sum*:

$$\forall A, B, A \cap B = \varnothing \,:\, m(A \cup B) = m(A) + m(B)$$

(This, of course, would imply that we could carry out this procedulre with any finite collection (A_n).)

Here is where things start to get hairy. For example, assuming the Axiom of Choice (which we do want!), the following result throws a monkey wrench into the works:

Claim 3 (Banach-Tarski Paradox). Let A and B be bounded subsets of $\mathbb{R}^n$, $n \geq 3$, both with nonempty interior. Then, there is an integer k, and partitions $A = \bigcup_{n=1}^{k} A_n$, $B = \bigcup_{n=1}^{k} B_n$ into disjoint subsets, such that for each $i \in \{1, 2, \ldots, k\}$, A_i is geometrically congruent to B_i. (See proof in Appendix.)

Although we've set out just to work with $\mathbb{R}$, this result has sobering implications—in particular, that there is no way for us to have a nontrivial finitely additive "ruler" defined for all subsets of $\mathbb{R}^n$ for $n \geq 3$; indeed, we should have, based on finite additivity:

$$m(A) = m\left(\bigcup_{n=1}^{k} A_n\right) = \sum_{n=1}^{k} m(A_n) = \sum_{n=1}^{k} m(B_n) = m\left(\bigcup_{n=1}^{k} B_n\right) = m(B)$$

where the middle equality was based on the geometric congruence of the A_n and B_n. However, the choices of A and B were completely arbitrary! We could have decided to let A be a kernel of corn and B be Jupiter; obviously, something went wrong. Somehow, the geometry of sets (facilitated by the Axiom of Choice) has allowed us to carve up A and B into pieces that are so profoundly ugly that they break our notion of measure when we try to reassemble them. (Extra neat!)

What's more, finite additivity doesn't provide us enough structure to fully solve one of the conundrums we set out to investigate, the eixistence of integrals of infinite series of functions. As we will see, to build an integral that has all of the properties we desire, we will need to expand our axiom, to **countable additivity**; that is, for *any* sequence of sets (A_n), with $A_i \cap A_j = \varnothing$ when $i \neq j$ (that is, *pairwise disjoint*), we have

$$m\left(\bigcup_n A_n\right) = \sum_n m\left(A_n\right)$$

(We note that going even further, to allow *uncountable* unions, is actually mathematically rather boring; indeed, if A is an uncountable set, and (r_α) an uncountable collection of real numbers indexed by A, then $\sum_{\alpha \in A} r_\alpha$ diverges if (r_α) has an uncountable number of nonzero entries. That is, the theory of uncountable sums essentially reduces to the original theory of sums of countable nonzero terms (which might converge), plus a theory of sums of uncountable nonzero terms (which all diverge). And so, we are content stopping at countable.)

Unfortunately, countably additive measures get us into even deeper trouble. While at first we were safe in 1 or 2 dimensions from Banach-Tarski's wrath, with *countable* additivity, we become subject to a corollary with similar counterintuitive conclusions about sets' size. Far uglier, however, is the fact that we can now explicitly construct sets for which we have no hope of assigning a consistent measure.

Theorem 1 (Vitali Existence Theorem)**.** *Let $\mathcal{A} \subset \mathcal{P}([0,1))$, and let $\mu : \mathcal{A} \to [0,\infty]$ satisfy $\mu([0,1)) = 1$, translation invariance, and countable additivity. Then $\mathcal{A} \subsetneq \mathcal{P}([0,1))$; that is, there exist nonmeasurable sets.*

Proof. Begin by identifying $[0,1)$ with the circle by gluing the ends. Note that we can extend addition (and hence subtraction) to this setting by performing the operation modulo 1:

$$x + y \,(\mathrm{mod}\ 1) = \begin{cases} x + y, & x + y < 1 \\ x + y - 1, & x + y \geq 1 \end{cases}$$

We can similarly extend this definition to set translation in the expected way; and, further, we can show that the translation invariance of μ implies translation invariance with these new circular operations. (How?)

Vitali observed that we can construct a curious equivalence relation on this structure, owing to the algebraic properties of $\mathbb{R}$ and $\mathbb{Q}$; that is, we say two elements $x, y \in [0, 1)$ are *equivalent* if they are separated by a rational; that is,

$$x \sim y \Leftrightarrow x - y \in \mathbb{Q}$$

Going one step further, we can partition $[0, 1)$ into *equivalence classes*, or sets of equivalent elements, using this relation:

$$[x] = \{y \in [0, 1) \mid x - y \in \mathbb{Q}\}$$

Observe, however, that these equivalence classes are essentially "copies" of $\mathbb{Q} \cap [0, 1)$, formed by taking the difference between x and every available rational number; that is,

$$[x] = \mathbb{Q} + x \pmod 1$$

Now, we can use the Axiom of Choice to construct a new set, V, by choosing exactly one element from each of the equivalence classes constructed above. (Note immediately that, since each of the equivalence classes is countable, but the circle is uncountable, V must necessarily be uncountable.) Now, consider the translates of V by rational numbers r, $V_r = V + r \pmod 1$. Our goal is to show that these translates of V actually partition $[0, 1)$, so that we may apply our beloved countable additivity.

First, let us show that

$$[0, 1) = \bigcup_{r \in \mathbb{Q} \cap [0,1)} V_r = \mathcal{F}$$

where we denute the union by $\mathcal{F}$ for brevity. We note that, by construction, for any $x \in [0, 1)$, there is precisely one $y \in V$ such that $x \sim y$ (since we chose this y from the respective equivalence class); hence, $x - y \in \mathbb{Q}$, and so $x \in V_{x-y}$. Since we can repeat this for any such x, it follows that $[0, 1) \subset \mathcal{F}$. The reverse inclusion comes easily, as each $V_r \subset [0, 1)$ by construction.

To see that the individual V_r are disjoint, note that $x \in V_r \cap V_s$ would imply that x is a translate by both r and s of elements in V; that is,

$$x = \begin{cases} y + r & \pmod 1 \\ z + s & \pmod 1 \end{cases}$$

for some $y, z \in V$; hence, $y+r = z+s \pmod 1$. But since r and s are rational, this means that $y - z \in \mathbb{Q}$, and hence $y \sim z$. But y and z were chosen, by construction, to be *unique* representatives from their equivalence classes; that is, $y \sim z \Rightarrow y = z$. This implies that $r = s$. Since $x \in V_r \cap V_s \Rightarrow r = s$, it follows that $V_r \cap V_s = \varnothing$ whenever $r \neq s$.

Now that we've established that the V_r partition $[0, 1)$, we can proceed to "sizing-up" our sets. Suppose that $\mu(V)$ exists. Since μ is translation invariant, then $\mu(V_r) = \mu(V + r \pmod 1) = \mu(V)$ for any r. Hence, by countable additivity, we have

$$1 = \mu([0,1)) = \mu\left(\bigcup_{r\in\mathbb{Q}\cap[0,1)} V_r\right) = \sum_{r\in\mathbb{Q}\cap[0,1)} \mu(V_r) = \sum_{r\in\mathbb{Q}\cap[0,1)} \mu(V)$$

This immediately implies that there is no consistent way to assign a value to $\mu(V)$: if it is zero, then the final sum above is zero, inconsistent with our assignment of a value of 1 to $\mu([0,1))$; on the other hand, if it is nonzero, then the sum must necessarily diverge, again yielding an inconsistency. It follows that V is **nonmeasurable**. $\square$

The takeaway message here is that the Axiom of Choice—while extremely useful for allowing to construct the rest of our mathematics—has devilish consequences when it comes to measure theory. The plan of action, however, is not to flee in despair! Instead, we must recognize that, in order to build a usable theory, we must acknowledge the inherent limitations in our quest. To that end, we must ask ourselves: what is the best we can do, as far as identifying where our "ruler" is defined?

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Definition 4 (σ-algebra)**.** Given a set X, a **σ-algebra** is a set $\Sigma \subset \mathcal{P}(X)$ such that

1. Σ contains the "universal set": $X \in \Sigma$
2. Σ is closed under *complements*: if $A \in \Sigma$ then $X \setminus A \in \Sigma$
3. Σ is closed under *countable unions*: if $(A_n) \subset \Sigma$ then $\bigcup_n A_n \in \Sigma$

...

So, where does that leave us? For a *general* "ruler"-like object, the primary elements are normalization and countable additivity, with the extra bits sprinkled in for flavor when necessary. We'll call such a "ruler" a **measure**.

Definition 5 (Measure)**.** Let X be a set, and Σ a σ-algebra on X. Then a function

$$\mu : \Sigma \to [0, \infty]$$

is a **measure** if

1. $\mu(\varnothing) = 0$
2. For a sequence (E_n) of disjoint sets in Σ,

$$\mu\left(\bigcup_n E_n\right) = \sum_n \mu(E_n)$$

3.1.2 Lebesgue measure

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(Caratheodoroy's theorem)

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3.1.3 Borel measure

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3.2 Probability spaces

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