Constant Coefficients

In the case of constant coefficients,

$$\mathcal{L}_x[y] = \frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f(x),$$

we first solve for the homogeneous solution,

$$\frac{d^2y_h}{dx^2} + p\frac{dy_h}{dx} + q\,y_h = 0.$$

This is done by noting that $Ae^{i\lambda x}$ is an eigenstate of the derivative operator:

$$\frac{d}{dx}\left(Ae^{i\lambda x}\right) = i\lambda\left(Ae^{i\lambda x}\right), \quad \frac{d^2}{dx^2}\left(Ae^{i\lambda x}\right) = -\lambda^2\left(Ae^{i\lambda x}\right)$$

Thus,

$$\mathcal{L}_x[e^{i\lambda x}] = [-\lambda^2 + ip\lambda + q]\,Ae^{i\lambda x}$$

The characteristic polynomial in this case is:

$$P(\lambda) = -\lambda^2 + ip\lambda + q$$

For $Ae^{\lambda t}$ to be a solution

$$\mathcal{L}_x[Ae^{i\lambda x}] = 0$$

$$\Rightarrow \quad [-\lambda^2 + ip\lambda + q]\,Ae^{i\lambda x} = 0$$

$$\Rightarrow \quad -\lambda^2 + ip\lambda + q = 0 \quad \text{just like before.}$$

Then: $\lambda = \frac{1}{2}\left[ip \pm \sqrt{-p^2+4q}\right]$

Take:

$$\lambda_1 = \frac{1}{2}\left[ip + \sqrt{4q-p^2}\right], \quad \lambda_2 = \frac{1}{2}\left[ip - \sqrt{4q-p^2}\right]$$

The most general solution of the homogeneous equation is then a linear combination of these two eigenvectors

$$\boxed{y_h(x) = A_1 e^{i\lambda_1 x} + A_2 e^{i\lambda_2 x}}$$

since:

$$\mathcal{L}_x[y_h] = \mathcal{L}_x[A_1 e^{\lambda_1 x} + A_2 e^{\lambda_2 x}] = \mathcal{L}_x[A_1 e^{\lambda_1 x}] + \mathcal{L}_x[A_2 e^{\lambda_2 x}]$$

$$\mathcal{L}_x[y_h] = 0.$$

Notice, however, that if $p^2 = 4q$, $\lambda_1 = \lambda_2$, and it would seem that there is only one solution to the second order differential equation. This is the degenerate eigenvalue case, and it means that we have to work harder to find the second solution.

Notice that

$$\mathcal{L}_x[y] = \frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy.$$

$$= \left[\left(\frac{d}{dx} + \frac{p}{2}\right)^2 + q - \frac{p^2}{4}\right]y$$

after completing the square. When $\lambda_1 = \lambda_2 = \frac{ip}{2}$

$$\mathcal{L}[y_h] = \left(\frac{d}{dx} + \frac{p}{2}\right)^2 y = 0$$

or:

$$\left(\frac{d}{dx} + \frac{p}{2}\right)\left[\frac{dy_h}{dx} + \frac{p}{2}y_h\right] = 0.$$

Since $\left(\frac{d}{dx} + \frac{p}{2}\right) e^{-\frac{p}{2}x} = 0$,

$$\frac{dy_h}{dx} + \frac{p}{2}y_h = A_2 e^{-\frac{p}{2}x}.$$

which is a first order ODE that we can solve using integration factors:

$$e^{-\frac{p}{2}x}\frac{d}{dx}\left(e^{\frac{p}{2}x} y_h\right) = A_2 e^{-\frac{p}{2}x}$$

$$\frac{d}{dx}\left(e^{\frac{p}{2}x} y_h\right) = A_2$$

$$e^{\frac{p}{2}x} y = A_2 x + A_1$$

$$\Rightarrow \boxed{y_h = A_1 e^{-\frac{p}{2}x} + A_2 x e^{-\frac{p}{2}x}}$$

in the special case where $4q = p^2$.

Thus:

$$\boxed{\begin{aligned} y_h &= A_1 e^{i\lambda_1 x} + A_2 e^{i\lambda_2 x} && \text{for } 4q^2 \neq p^2 \\ y_h &= (A_1 + x A_2) e^{-\frac{p}{2}x} && \text{for } 4q^2 = p^2 \end{aligned}}$$

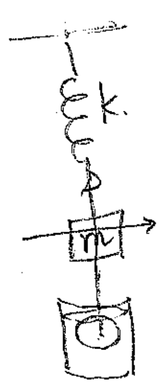
While the above is a perfectly fine form of the solution of the homogeneous equation, we often find it more convenient to write it using:

$$\begin{aligned} e^{i\lambda x} &= e^{i\left[\frac{ip}{2} \pm \sqrt{4q - p^2}\right]x} \\ &= e^{-\frac{p}{2}x} e^{\pm i\sqrt{4q-p^2}\,x} \\ &= e^{-\frac{p}{2}x}\left[\cos\left[\sqrt{4q-p^2}\,x\right] \pm i \sin\left[\sqrt{4q-p^2}\,x\right]\right] \end{aligned}$$

Then:

$$\boxed{y_h(x) = e^{-\frac{p}{2}x}\left[A\cos\left[\sqrt{4q-p^2}\,x\right] + B\sin\left[\sqrt{4q-p^2}\,x\right]\right]}$$

Ex <u>Damped Harmonic Motion</u>



$$\sum \vec{F}_y = -ky - bv = m\frac{d^2y}{dt^2}$$

$$\frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m}y = 0$$

$p = \frac{b}{m}$, $q = \frac{k}{\omega} = \omega_0^2$.

Then $i\lambda_1 = -\frac{b}{2m} \pm i\sqrt{q - \frac{p^2}{4}} = -\frac{b}{2m} \pm i\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$

<u>If $\omega_0 > \frac{b}{2m}$:</u> Then $\sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2}$ is a real number, and we use the solution of the form,

$$y_h(t) = e^{-\frac{b}{2m}t}\left[A\cos\left[\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}\, t\right] + B\sin\left[\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}\, t\right]\right]$$

The mass oscillates with time. This is the lightly damped solution.

<u>If $\omega_0 < \frac{b}{2m}$:</u> Then $\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$ is imaginary, and:

$$\sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} = i\sqrt{\left(\frac{b}{2m}\right)^2 - \omega_0^2}.$$

The convenient form of the solution is:

$$y_h(t) = \underbrace{A_1 e^{-\left[\frac{b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \omega_0^2}\right]t}}_{\text{Dies to zero fastest}} + A_2 e^{-\left[\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \omega_0^2}\right]t}.$$

The system does not oscillate; it only dies off to zero exponentially fast.

If $\omega_0 = \frac{b}{2m}$: Then:

$$y_h(t) = (A_1 + t A_2)\, e^{-\frac{b}{2m}t}.$$

This is the critically damped system

Notice that there are two different solutions of the linear, 2^{nd} order ODE, just like there was only one solution for a linear, 1^{st} order ODE. This turns out to be true in general. Namely,

"There are precisely n, linearly independent solutions of an n^{th} order, linear, ordinary differential equation."

Notice also that y_h for a 2^{nd} order ODE has two arbitrary constants, while y_h for a 1^{st} order ODE has one. This is true in general. Namely, the homogeneous solution for a n^{th} order, linear ODE has n constants (or, is unique up to n arbitrary constants). Thus, to obtain a unique solution to the ODE we will have to specify n, linearly independent boundary conditions. For 2nd order ODE, they are often taken to be:

$$y(x_0) = y_0, \quad \left.\frac{dy}{dx}\right|_{x_0} = y'_0.$$

Inhomogeneous Solutions

For the inhomogeneous, 2nd order ODE:

$$\mathcal{L}_x[y] = \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x),$$

there are a variety of ways to find the solution to this inhomogeneous equation. We first note that since the derivative is a linear operator:

$$\mathcal{L}_x(y_h + y_p) = \mathcal{L}_x[y_p]$$

where y_h is the solution of the homogeneous equation,

$$\mathcal{L}_x[y_h] = 0$$

Thus, $y = y_h + y_p$, and we only have to find the particular solution, y_p.

There are a number of ways to do so. The one we will focus on here is the method of undetermined coefficients. This is because we can obtain a closed form solution of a general 2nd order (or higher!) linear ODE, and because it is directly related to the Green's function. We will talk about two other approaches later.

In the method of undetermined coefficients we look
for solutions of the particular solution by using the
solution of the homogeneous solution. Because
the equation is 2<u>nd</u> order we know that there
are two linearly independent solutions of the homogeneous
equation. Call them $y_1(x)$ and $y_2(x)$. Then:

$$\mathcal{L}_x[y_1(x)] = 0, \quad \mathcal{L}_x[y_2(x)] = 0$$

We use these solutions to construct a solution
of the inhomogeneous equation by taking

$$y_p(x) = a_1(x) y_1(x) + a_2(x) y_2(x)$$

where $a_1(x)$ and $a_2(x)$ are chosen so that:

$$\mathcal{L}_x[y_p] = f(x)$$

Notice, however, that the inhomogeneous ODE
just one equation, while <u>two</u> unknown functions
are introduced. We have to impose one more
condition on a_1 and a_2 if we hope to determine
them. This we shall do the following way.

We note that:

$$\frac{dy_P}{dx} = \frac{da_1}{dx} y_1 + \frac{da_2}{dx} y_2 + a_1 \frac{dy_1}{dx} + a_2 \frac{dy_2}{dx}.$$

We then require that:

$$\boxed{0 = \frac{da_1}{dx} y_1 + \frac{da_2}{dx} y_2}$$

for all x. Then:

$$\frac{d^2y_P}{dx^2} = \frac{da_1}{dx}\frac{dy_1}{dx} + \frac{da_2}{dx}\frac{dy_2}{dx} + a_1 \frac{d^2y_1}{dx^2} + a_2 \frac{d^2y_2}{dy^2},$$

so that:

$$\mathcal{L}[y_P] = a_1 \frac{d^2y_1}{dx^2} + a_2 \frac{d^2y_2}{dy^2} + \frac{da_1}{dx}\frac{dy_1}{dx} + \frac{da_2}{dx}\frac{dy_2}{dx} + p(x)\left[a_1 \frac{dy_1}{dx} + a_2 \frac{dy_2}{dx}\right] + q(x)\left[a_1 y_1 + a_2 y_2\right]$$

$$= a_1 \mathcal{L}_x[y_1] + a_2 \mathcal{L}_x[y_2] + \frac{da_1}{dx}\frac{dy_1}{dx} + \frac{da_2}{dx}\frac{dy_2}{dx}$$

Thus:

$$\frac{da_1}{dx}\frac{dy_1}{dx} + \frac{da_2}{dx}\frac{dy_2}{dx} = f(x)$$

and the imposed condition

$$\frac{da_1}{dx} y_1 + \frac{da_2}{dx} y_2 = 0$$

We first solve for $\frac{da_1}{dx}$:

$$\frac{da_1}{dx} y_2 \frac{dy_1}{dx} + \frac{da_2}{dx} y_2 \frac{dy_2}{dx} = y_2 f(x)$$

$$\frac{da_1}{dx} y_1 \frac{dy_2}{dx} + \frac{da_2}{dx} y_2 \frac{dy_2}{dx} = 0$$

$$- \left[y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx} \right] \frac{da_1}{dx} = y_2 f(x)$$

The combination:

$$W[y_1, y_2] = y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx}$$

is called the Wronskian. If y_1 and y_2 are linearly independant, $W[y_1, y_2] \neq 0$, and:

$$\boxed{\frac{da_1}{dx} = - \frac{y_2(x) f(x)}{W[y_1, y_2]}}$$

Similarly:

$$\boxed{\frac{da_2}{dx} = \frac{y_1(x) f(x)}{W[y_1, y_2]}}$$

and we are left with two simple integrals to do. Notice, and this would seem to involve two constants.

however, that the total solution is:

$$y(x) = y_h(x) + y_p(x)$$

$$= \underbrace{A_1 y_1(x) + A_2 y_2(x)}_{y_h(x)} + \underbrace{a_1(x) y_1(x) + a_2(x) y_2(x)}_{y_p(x)},$$

$$\boxed{y(x) = [A_1 + a_1(x)]y_1(x) + [A_2 + a_2(x)]y_2(x).}$$

So, different choices of the constants coming from the integral of $\frac{da_1}{dx}$ and $\frac{da_2}{dx}$ absorbed in A_1 and A_2. Thus, ultimately a_1 and a_2 are determined based on the boundary conditions for $y(x)$. Indeed:

$$\frac{dy}{dx} = [A_1 + a_1(x)]\frac{dy_1}{dx} + [A_2 + a_2(x)]\frac{dy_2}{dx} +$$

$$\frac{da_1}{dx} y_1 + \frac{da_2}{dx} y_2 = 0$$

$$\boxed{\frac{dy}{dx} = [A_1 + a_1(x)]\frac{dy_1}{dx} + [A_2 + a_2(x)]\frac{dy_2}{dx}.}$$

If x is restricted to be between $x_1 \leq x \leq x_2$, then we can choose $a_1(x_1)=0$, $a_2(x_2)=0$, so that

$$a_1(x) = -\int_{x_1}^{x} \frac{y_2(\tilde{x}) f(\tilde{x})}{W(\tilde{x})}\, d\tilde{x}, \quad \text{while}$$

$$a_2(x) = +\int_{x_2}^{x} \frac{y_1(\tilde{x}) f(\tilde{x})}{W(\tilde{x})}\, d\tilde{x} = -\int_{x}^{x_2} \frac{y_1(\tilde{x}) f(\tilde{x})}{W(\tilde{x})}\, d\tilde{x}.$$

Then:

$$y_p(x) = a_1(x)\, y_1(x) + a_2(x)\, y_2(x).$$

$$= -y_1(x)\int_{x_1}^{x} \frac{y_2(\tilde{x}) f(\tilde{x})\, d\tilde{x}}{W[\tilde{x}]} - y_2(x)\int_{x}^{x_2} \frac{y_1(\tilde{x}) f(\tilde{x})\, d\tilde{x}}{W[\tilde{x}]}$$

$$= -\int_{x_1}^{x} \underbrace{\left(\frac{y_1(x)\, y_2(\tilde{x})}{W(\tilde{x})}\right)}_{x \geq \tilde{x}} f(\tilde{x})\, d\tilde{x} - \int_{x}^{x_2} \underbrace{\frac{y_2(x)\, y_1(\tilde{x}) f(\tilde{x})\, d\tilde{x}}{W[\tilde{x}]}}_{x \leq \tilde{x}}$$

This can be written in a surprisingly simple form using the heaviside function since:

$$\int_{x_1}^{x_2} (\cdot)\, \theta(x-\tilde{x})\, d\tilde{x} = \int_{x_1}^{x} \underbrace{(\cdot)\, \theta(x-\tilde{x})\, d\tilde{x}}_{x \geq \tilde{x}_1} + \int_{x}^{x_2} \underbrace{\cancel{(\cdot)\, \theta(x-\tilde{x})}}_{x \leq \tilde{x}}^{\to 0} d\tilde{x}$$

$$= \int_{x_1}^{x} (\cdot)\, d\tilde{x}$$

where $(\cdot)$ means any function of $\tilde{x}, x$, while

$$\int_{x_1}^{x_2} (\cdot)\,\theta(\tilde{x}-x)\,d\tilde{x} = \int_{x_1}^{x} (\cdot)\underbrace{\theta(\tilde{x}-x)}_{=0}\,d\tilde{x} + \int_{x}^{x_2} (\cdot)\,\theta(\tilde{x}-x)\,d\tilde{x}$$

$x > \tilde{x}$

$$= \int_{x}^{x_2} (\cdot)\,\theta(\tilde{x}-x)\,d\tilde{x}$$

Thus:

$$y_p(x) = -\int_{x_1}^{x_2} \left[\frac{y_1(x)\,y_2(\tilde{x})}{W(\tilde{x})}\right] f(\tilde{x})\,\theta(x-\tilde{x})\,d\tilde{x} - \int_{x_1} \left[\frac{y_2(x)\,y_1(\tilde{x})}{W(\tilde{x})}\right] f(\tilde{x})\,\theta(\tilde{x}-x)\,d\tilde{x}$$

$$= \int_{x_1}^{x_2} \left\{ -\left(\frac{y_1(x)\,y_2(\tilde{x})\,\theta(x-\tilde{x}) + y_2(x)\,y_1(\tilde{x})\,\theta(\tilde{x}-x)}{W(\tilde{x})}\right)\right\} f(\tilde{x})\,d\tilde{x}$$

$$G(x,\tilde{x}) = -\left[\frac{y_1(x)\,y_2(\tilde{x})\,\theta(x-\tilde{x}) + y_2(x)\,y_1(\tilde{x})\,\theta(\tilde{x}-x)}{W[\tilde{x}]}\right]$$

is called the Green's function. With G, we can write:

$$\boxed{y_p(x) = \int_{x_1}^{x_2} G(x,\tilde{x})\,f(\tilde{x})\,d\tilde{x}}$$

and:

$$y(x) = A_1 y_1(x) + A_2 y_2(x) + \int_{x_1}^{x_2} G(x,\tilde{x})\,f(\tilde{x})\,d\tilde{x}.$$

To implement the boundary conditions, note that:

$$y(x) = y_h(x) + y_p(x) = A_1 y_1(x) + A_2 y_2(x) + y_p(x)$$

$$y(0) = A_1 y_1(0) + A_2 y_2(0) + y_p(0)$$

Since $y_1 = \cos(k_0 x) \Rightarrow y_1(0) = 1$, $y_2(x) = \sin(k_0 x) \Rightarrow y_2(0) = 0$

$\Rightarrow y_1(0) = A_1 + y_p(0)$. Since $y_1(0) = 0 \Rightarrow 0 = A_1 + y_p(0)$.

$$\Rightarrow A_1 = -y_p(0)$$

Next,

$$\frac{dy}{dx} = A_1 \frac{dy_1}{dx} + A_2 \frac{dy_2}{dx} + \frac{dy_p}{dx}$$

$\frac{dy_1}{dx} = -k_0 \sin(k_0 x) \Rightarrow \left.\frac{dy}{dx}\right|_L = 0$. $\frac{dy_2}{dx} = k_0 \cos(k_0 x) \Rightarrow \frac{dy_2}{dx} = k_0$

$$\Rightarrow \left.\frac{dy}{dx}\right|_L = k_0 A_2 + \left.\frac{dy_p}{dx}\right|_L \Rightarrow 0 = k_0 A_2 + \left.\frac{dy_p}{dx}\right|_L \Rightarrow A_2 = -\frac{1}{k_2} \left.\frac{dy_p}{dx}\right|_L$$

Next,

$$y_p(x) = \int_0^L G(x,\tilde{x}) f(\tilde{x})\, d\tilde{x}, \quad G(x,\tilde{x}) = -\frac{[y_1(\tilde{x}) y_2(\tilde{x}) \theta(x-\tilde{x}) + y_1(\tilde{x}) y_2(x) \theta(\tilde{x}-$$

$$\Rightarrow G(0,\tilde{x}) = -\frac{[y_1(0) y_2(\tilde{x}) \underbrace{\theta(-\tilde{x})}_{=0} + y_1(\tilde{x}) \underbrace{y_2(0)}_{=0} \theta(\tilde{x})]}{W[y_1, y_2](0)}$$ since $\theta(-\tilde{x}) =$

$$\Rightarrow y_p(0) = 0 \quad \Rightarrow \boxed{A_1 = 0}$$

Notice that $G(x,\bar{x})$ depends explicitly on x_1 and x_2, and thus depends explicitly on this choice. Such a G is often called a two-sided Green's function.

This choice may not be appropriate for some problems, and in fact the choice of x_1 + x_2 to determine G is intimately related to the boundary conditions for $y(x)$ itself. If, for example, x_1 is not restricted to be between x_1 and x_2, but rather $x \geq x_0$, then we could choose $a_1(x_0) = 0$, $a_2(x_0) = 0$. This would result in the one-sided (retarded or advanced) Green's function.

Ex: Consider the differential equation

$$f(t) = \frac{d^2y}{dx^2} + \left(\frac{2\pi}{L}\right)^2 y$$

with $f(t) = f_0 \cos\left[\frac{4\pi x}{L}\right] \pm f_0 \cos[kx]$ and $0 \leq x \leq L$. Take the boundary condition to be: $y(0) = 0$, $\left.\frac{dy}{dx}\right|_{x=L} = 0$.

What is $y(x)$?

This is precisely the situation we have above with $x_1 = 0$, $x_2 = L$. The two homogeneous equations are:

$$y_1(x) = \cos\left(\frac{2\pi}{L}x\right) = \cos(k_0 x)$$

$$y_2(x) = \sin(k_0 x)$$

$$y_p(x) = -y_1(x)\int_0^x \frac{y_2(\tilde{x}) f(\tilde{x})\, d\tilde{x}}{W[y_1, y_2](\tilde{x})} - y_2(x)\int_x^L \frac{y_1(\tilde{x}) f(\tilde{x})}{W[y_1, y_2](\tilde{x})}\, d\tilde{x}$$

$$\frac{dy_p}{dx} = -\frac{dy_1}{dx}\int_0^x \frac{y_2(\tilde{x}) f(\tilde{x})\, d\tilde{x}}{W[y_1, y_2](\tilde{x})} - \frac{dy_2}{dx}\int_x^L \frac{y_1(\tilde{x}) f(\tilde{x})}{W[y_1, y_2](\tilde{x})}\, d\tilde{x} - \cancel{\frac{y_1(x) y_2(x) f(x)}{W[y_1, y_2]}}$$

$$+ \cancel{\frac{y_2(x)\, y_1(x)\, f(x)}{W[y_1, y_2]}}$$

$$\Rightarrow \left.\frac{dy_p}{dx}\right|_L = -\left.\frac{dy_1}{dx}\right|_L \cancel{\int_0^L \frac{y_2(\tilde{x}) f(\tilde{x})\, d\tilde{x}}{W[y_1, y_2](\tilde{x})}} - \left.\frac{dy_2}{dx}\right|_L \cancel{\int_L^L \frac{y_1(\tilde{x}) f(\tilde{x})\, d\tilde{x}}{W[y_1, y_2]}} = 0$$

$$\Rightarrow \quad A_2 = -\frac{1}{k_0}\left.\frac{dy_p}{dx}\right|_L \quad \Rightarrow \quad \boxed{A_2 = 0}$$

Then:

$$y(x) = y_p(x) = \int_0^L G(x, \tilde{x}) f(\tilde{x})\, d\tilde{x}$$

$$W[y_1, y_2] = \cos(k_0 x)\frac{d}{dx}(\sin k_0 x) - \sin(k_0 x)\frac{d}{dx}(\cos(k_0 x))$$

$$= k_0 \cos^2(k_0 x) + \sin^2(k_0 x)\, k_0 = k_0 \qquad k = 2k_0$$

$$y(x) = -\frac{f_0}{k_0}\int_0^L \left[\cos(k_0 x)\sin(k_0 \tilde{x})\,\theta(x - \tilde{x}) + \sin(k_0 x)\cos(k_0 \tilde{x})\,\theta(\tilde{x} - x)\right]\cos(k\tilde{x})\, d\tilde{x}$$

$$= -\frac{f_0}{k_0}\left\{\cos(k_0 x)\int_0^x \sin(k_0 \tilde{x})\cos(2k_0 \tilde{x})\, d\tilde{x} + \sin(k_0 x)\int_x^L \cos(k_0 \tilde{x})\cos(2k_0 \tilde{x})\, d\tilde{x}\right.$$

$$\sin(ax)\cos(bx) = \frac{1}{2}\left\{\sin[(a+b)x] + \sin[(a-b)x]\right\}$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\left\{\cos[(a-b)x] + \cos[(a+b)x]\right\}$$

$$y(x) = -\frac{f_0}{2k_0}\left\{\cos(k_0 x)\int_0^x \left[\sin(3k_0 \tilde{x}) + \sin(-k_0)\tilde{x}\right] d\tilde{x} +\right.$$

$$\left.\sin(k_0 x)\int_x^L \left[\cos(-k_0 \tilde{x}) + \cos(3k_0 \tilde{x})\right] d\tilde{x}\right\}$$

$$y(x) = -\frac{f_0}{2k_0}\left\{ \cos(k_0x)\left[-\frac{1}{3k_0}\cos(3k_0\tilde{x}) + \frac{\cos(k_0\tilde{x})}{k_0}\right]\Bigg|_0^x \right.$$

$$\left. - \sin(k_0x)\left[\frac{\sin(k_0\tilde{x})}{k_0} + \frac{\sin(3k_0\tilde{x})}{3k_0}\right]\Bigg|_x^L \right\}$$

$$= -\frac{f_0}{2k_0}\left\{ \cos(k_0x)\left[-\frac{1}{3k_0}\cos(3k_0x) + \frac{1}{3k_0} + \frac{\cos(k_0x)}{k_0} - 1/k_0\right]\right.$$

$$\left. - \sin(k_0x)\left[\cancel{\frac{\sin(2\pi)}{k_0}} - \frac{\sin(k_0x)}{k_0} + \cancel{\frac{\sin(6\pi)}{3k_0}} - \frac{\sin(3k_0x)}{3k_0}\right]\right\}$$

$$= -\frac{f_0}{2k_0^2}\left\{ \frac{1}{3}\left(1-\cos(3k_0x)\right)\cos(k_0x) - \cos(k_0x) + \cos^2(k_0x) \right.$$

$$\left. + \sin^2(k_0x) + \frac{\sin(k_0x)\sin(3k_0x)}{3}\right\}$$

$$= -\frac{f_0}{2k_0^2}\left\{ \frac{1}{3} + 1 - \frac{1}{3}\left[\cos(3k_0x)\cos(k_0x) - \sin(3k_0x)\sin(k_0x)\right] \right.$$

$$\left. - \cos(k_0x)\right\}$$

$$\boxed{y(x) = -\frac{f_0}{6k_0^2}\left\{4 - 3\cos(k_0x) - \cos(4k_0x)\right\}}$$

Check:

$$y(0) = 0$$

$$\frac{dy}{dx} = -\frac{f_0}{6k_0^2}\left\{3k_0\sin(k_0x) + 4\sin(4k_0x)\right\}$$

$$\left.\frac{dy}{dx}\right|_L = -\frac{f_0}{6k_0^2}\left\{3k_0\sin(2\pi) + 4\sin(8\pi)\right\} = 0$$

yes!