Ordinary Differential Equations

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The need to solve ordinary differential equations is pervasive in all fields of physics ranging from Newtonian dynamics

$$m\frac{d^2\vec{r}}{dt^2} = \sum \vec{F}\left(\vec{r}, \frac{d\vec{r}}{dt}, t\right)$$

to electrodynamics:

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{V}{R}$$

to thermodynamics:

$$dE = TdS - pdV$$

to quantum mechanics:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi.$$

For the next few weeks we will focus on the solution of ordinary differential equations

$$F\left(y, \frac{dy}{dx}, \ldots, \frac{d^m y}{dx^m}, x\right) = 0$$

in which we wish to determine y as a function of a single variable x. This cannot be done in general, and in fact there isn't even a methodology to solving these equations that will for all ordinary differential equations. With the exception of a few special cases, we are left with solving each on a case-by-case basis.

We start with terminology.

The order of the differential equation is the highest order derivative of y that appears in F.

The differential equation is linear if F is a linear function of $y, \frac{dy}{dx}, \ldots, \frac{d^{(n)}y}{dx^{(n)}}$; otherwise, it is nonlinear.

The differential equation is homogeneous if $F(0, \ldots, 0, x) = 0$; otherwise, it is inhomogeneous.

First order ODEs

First order differential equations have the form,

$$\frac{dy}{dx} = M(y, x).$$

It is not hard to think of a 1st order ODE that cannot be solved analytically; such as:

$$\frac{dy}{dx} = \sin(xy^2).$$

If, however, $M(y,x)$ is separable into a product of two functions:

$$M(y,x) = Y(y)\,X(x),$$

then:

$$\frac{dy}{dx} = Y(y)\,X(x),$$

so that:

$$\frac{1}{Y}\,dy = X(x)\,dx$$

Then:

$$\int_{y_0}^{y} \frac{d\tilde{y}}{Y(\tilde{y})} = \int_{x_0}^{x} X(\tilde{x})\,d\tilde{x}.$$

and we can see that a unique solution of y is obtained if the boundary condition:

$$y(x_0) = y_0$$

is given. What is left is to do the integral — often not possible in closed form — and then solve for y in terms of x, which is also not necessarily possible.

Ex:

$$\frac{dy}{dx} = y^{\frac{1}{2}(1-p)}\sqrt{1-y^{1+p}}$$

with $y(0)=0$. Then:

$$\frac{y^{\frac{p-1}{2}}\,dy}{\sqrt{1-y^{1+p}}} = dx$$

$$x = \int_0^y \frac{s^{\frac{1}{2}(p-1)}\,ds}{\sqrt{1-s^{p+1}}},$$

Let: $\sin u = s^{\frac{1}{2}(p+1)}$

$$\cos u\,du = \tfrac{1}{2}(p+1)\,s^{\frac{1}{2}(p-1)}\,ds$$

$$x = \frac{2}{p+1}\int \frac{\cos u\,du}{\sqrt{1-\sin^2 u}} = \frac{2}{p+1}u$$

$$\tfrac{1}{2}(p+1)x = u \Rightarrow \sin\left[\tfrac{1}{2}(p+1)x\right] = \sin u$$

$$\sin\left[\tfrac{1}{2}(p+1)x\right] = y^{\frac{1}{2}(p+1)} \Rightarrow \boxed{y = \left[\sin\left[\tfrac{1}{2}(p+1)x\right]\right]^{\frac{2}{p+1}}}$$

Ex:

Energy conservation:

$$E = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + V(x)$$

$$\Rightarrow \quad \frac{dx}{dt} = \pm \sqrt{\frac{2}{m}(E - V(x))}$$

$$\boxed{t - t_0 = \pm \int_{x_0}^{x} \frac{d\tilde{x}}{\sqrt{\frac{2}{m}(E - V(\tilde{x}))}}}$$

$$V(x) = \frac{1}{2} m\omega^2 x^2 \qquad x(0) = 0$$

$$t = \pm \int_0^x \frac{d\tilde{x}}{\sqrt{\frac{2E}{m} - \omega^2 \tilde{x}^2}} = \pm \sqrt{\frac{m}{2E}} \int_0^x \frac{d\tilde{x}}{\sqrt{1 - \frac{m\omega^2}{2E}\tilde{x}^2}}$$

$$u = \sqrt{\frac{m}{2E}}\, \omega \tilde{x} \qquad du = \sqrt{\frac{m}{2E}}\, \omega\, d\tilde{x}$$

$$t = \pm \sqrt{\frac{m}{2E}} \sqrt{\frac{2E}{m}} \frac{1}{\omega} \int_0^{\sqrt{\frac{m}{2E}}\omega x} \frac{du}{\sqrt{1 - u^2}}$$

$$\pm \omega t = \sin^{-1} u \Big|_0^{\sqrt{\frac{m}{2E}}\omega x} = \sin^{-1}\left(\sqrt{\frac{m}{2E}}\, \omega x\right)$$

$$x = \pm \sqrt{\frac{2E}{m}} \frac{1}{\omega} \sin(\omega t)$$

Linear 1st order ODE

For linear ordinary differential equations we can also find a general solution.

The most general linear ODE has the form

$$p(x)\frac{dy}{dx} + q(x)y = f(x)$$

but since we can divide out by $p(x)$,

$$\frac{dy}{dx} + Q(x)y = H(x).$$

To solve this equation, we use an integration factor.

Namely, we note that:

$$\frac{d}{dx}(e^{g}y) = e^{g}\frac{dy}{dx} + e^{g}\frac{dg}{dx}y$$

$$\Rightarrow \quad e^{-g}\frac{d}{dx}(e^{g}y) = \frac{dy}{dx} + \frac{dg}{dx}y$$

Thus, by choosing:

$$\boxed{\frac{dg}{dx} = Q(x)},$$

the ODE reduces to:

$$e^{-g}\frac{d}{dx}(e^{g}y) = H(x)$$

which is separable. Then:

$$\frac{d}{dx}\left(e^{g}y\right) = e^{g}H(x)$$

so that
$$e^{g}y = k + \int_{x_0}^{x} e^{g(\tilde{x})}H(\tilde{x})d\tilde{x},$$

and k is an integration constant. Then:

$$y(x) = k\,e^{-g(x)} + e^{-g(x)}\int_{x_0}^{x} e^{g(\tilde{x})}H(\tilde{x})d\tilde{x}$$

Now, we take as out boundary condition:

$$y(x_0) = y_0.$$

Then:

$$y_0 = k\,e^{-g(x_0)}$$

so that:
$$\boxed{y(x) = y_0\,e^{-[g(x)-g(x_0)]} + \int_{x_0}^{x} e^{-[g(x)-g(\tilde{x})]}H(\tilde{x})d\tilde{x}}$$

g(x) is determined through the equation:

$$\frac{dg}{dx} = Q(x)$$

or:
$$g(x) - g(x_0) = \int_{x_0}^{x} Q(s)\,ds$$

And note that:

$$g(x) - g(\tilde{x}) = \int_{x_0}^{x} Q(s)\,ds + \int_{x_0}^{\tilde{x}} Q(s)\,ds = \int_{\tilde{x}}^{x} Q(s)\,ds$$

Thus, once the boundary condition $y(x_0) = y_0$ is imposed, it does not matter what we choose $g(x_0)$ to be.

The solution can then be written as:

$$\boxed{y(x) = y_0 \exp\left[-\int_{x_0}^{x} Q(s)\,ds\right] + \int_{x_0}^{x} H(\tilde{x}) \exp\left[-\int_{\tilde{x}}^{x} Q(s)\,ds\right] d\tilde{x}}$$

Comments

- When $H(x) = 0$, the ODE reduces to:

$$\frac{dy}{dx} + Q(x)y = 0$$

which is a homogeneous ODE. The corresponding solution:

$$y_h(x) = y_0 \exp\left[-\int_{x_0}^{x} Q(s)\,ds\right]$$

is called the homogeneous solution. The H-dependent piece of y:

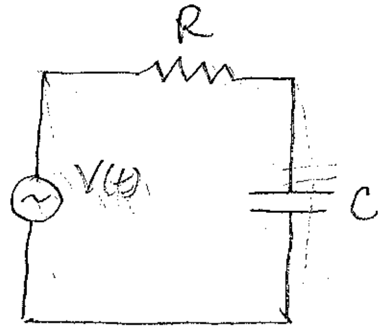
$$y_p(x) = \int_{x_0}^{x} H(\tilde{x}) \exp\left[-\int_{\tilde{x}}^{x} Q(s)\,ds\right] d\tilde{x}$$

is independent of the boundary condition x_0. This is called the inhomogeneous solution.

in physics. Then:

$$\boxed{y(x) = y_0 \exp\left[-\int_{x_0}^{x} Q(s)ds\right] + \int_{x_0}^{\infty} G(x,\tilde{x})H(\tilde{x})d\tilde{x}}$$

Ex



$V(t) = V_0 \sin(\omega t)$

with the boundary condition. (or initial condition)

$$q(0) = q_0$$

then:

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{V_0}{R}\sin(\omega t) \qquad \tau \equiv RC$$

and:

$$q(t) = q_0 \exp\left[-\int_0^t \frac{ds}{\tau}\right] + \int_0^{\infty} G(t,\tilde{t})V(\tilde{t})d\tilde{t}$$

$$G(t,\tilde{t}) = \exp\left[-\int_{\tilde{t}}^{t} \frac{s}{\tau}\right]\theta(t-\tilde{t}) = e^{-\left(\frac{t-\tilde{t}}{\tau}\right)}\theta(t-\tilde{t})$$

$$q_h(t) = q_0\, e^{-t/\tau}$$

$$q_p(t) = \int_0^{\infty} \frac{V(\tilde{t})}{R} e^{-\left(\frac{t-\tilde{t}}{\tau}\right)}\theta(t-\tilde{t})\,d\tilde{t}$$

$$= \frac{V_0}{R}\int_0^t \sin(\omega\tilde{t})\, e^{-\left(\frac{t-\tilde{t}}{\tau}\right)}d\tilde{t}$$

$$= \frac{V_0}{R}e^{-t/\tau}\int_0^t e^{\tilde{t}/\tau}\sin(\omega\tilde{t})\,d\tilde{t}$$

and $y(x) = y_h(x) + y_p(x)$. This is true in general for linear ODEs. The solution can always be written as a superposition of a homogeneous and an inhomogeneous solution. In terms of physics, the inhomogeneous solution depends on $H(x)$, which can often be identified as an external, driving term, an when $x = \sqrt{t}$, y_p is identified as the steady state solution. Then y_h is the transient solution.

- By making use of the heaviside function

$$\theta(x) = \begin{cases} 1 & \text{if } x \geq 1 \\ 0 & \text{if } x < 0. \end{cases},$$

we can write:

$$y_p(x) = \int_{x_0}^{\infty} H(\tilde{x}) \exp\left[-\int_{\tilde{x}}^{x} Q(s)ds\right] \theta(x-\tilde{x})\, d\tilde{x}$$

since:

$$y_p(x) = \underbrace{\int_{x_0}^{x} H(\tilde{x}) \exp\left[-\int_{\tilde{x}}^{x} Q(s)ds\right] \theta(x-\tilde{x})\, d\tilde{x}}_{x > \tilde{x}} + \underbrace{\int_{x}^{\infty} H(\tilde{x}) \exp\left[-\int_{\tilde{x}}^{x} Q(s)ds\right] \theta(x-\tilde{x}) d\tilde{x}}_{x < \tilde{x}} \rightarrow = 0$$

The function:

$$G(x,\tilde{x}) \equiv \exp\left[-\int_{\tilde{x}}^{x} Q(s)ds\right] \theta(x-\tilde{x})$$

is called a Green's function, which plays an essential

$$q_p(t) = \frac{V_0}{R} e^{-t/\tau} \int_0^t \frac{1}{2i}\left(e^{i\omega\tilde{t}} - e^{-i\omega\tilde{t}}\right) e^{\tilde{t}/\tau}\, d\tilde{t}$$

$$= \frac{V_0 e^{-t/\tau}}{R} \operatorname{Re}\left(\int_0^t \frac{1}{i} e^{(i\omega + \frac{1}{\tau})\tilde{t}}\, d\tilde{t}\right)$$

$$= \frac{V_0}{R} e^{-t/\tau} \operatorname{Re}\left[\frac{1}{i}\frac{1}{(i\omega + 1/\tau)} e^{(i\omega + 1/\tau)t}\Big|_0^t\right]$$

$$= \frac{V_0\, e^{-t/\tau}}{R} \operatorname{Re}\left[\frac{1}{-\omega + i/\tau}\left[e^{(i\omega + 1/\tau)t} - 1\right]\right]$$

$$= \frac{V_0\, e^{-t/\tau}}{R}\left[\frac{1}{-\omega + i/\tau}\left[e^{(i\omega + 1/\tau)t} - 1\right] + \frac{1}{-\omega - i/\tau}\left[e^{(-i\omega + 1/\tau)t} - 1\right]\right]$$

$$= \frac{V_0 e^{-t/\tau}}{R}\left[\frac{1}{-\omega + i/\tau} e^{(i\omega + 1/\tau)t} - \frac{1}{\omega + i/\tau} e^{(-i\omega + 1/\tau)t} - \frac{1}{-\omega + i/\tau} + \frac{1}{\omega + i/\tau}\right]$$

$$= \frac{V_0 e^{-t/\tau}}{R}\left[\frac{(\omega + i/\tau) e^{(i\omega + 1/\tau)t} - (-\omega + i/\tau) e^{(-i\omega + 1/\tau)t}}{-\omega^2 - 1/\tau^2} - \frac{(\omega + i/\tau) + (-\omega + i/\tau)}{-\omega^2 - 1/\tau^2}\right]$$

$$= -\frac{V_0 e^{-t/\tau}}{R}\frac{1}{\omega^2 + 1/\tau^2}\left[\omega e^{t/\tau}\left[e^{i\omega t} + e^{-i\omega t}\right] + i/\tau\left(e^{i\omega t} - e^{-i\omega t}\right)e^{t/\tau} + 2\omega\right]$$

$$= \frac{V_0 e^{-t/\tau}}{R(\omega^2 + 1/\tau^2)}\left[\omega - \omega\cos(\omega t) e^{t/\tau} + \frac{1}{\tau}\sin(\omega t) e^{t/\tau}\right]$$

$$\boxed{q_p(t) = \frac{\tau^2}{1+(\omega\tau)^2}\left[\omega\left[e^{-t/\tau} - \cos(\omega t)\right] + \frac{1}{\tau}\sin(\omega t)\right]\frac{V_0}{R}}$$

Notice that $q_p(0) = 0$ as expected and $q_h(0) = q_0$.
Notice also that as $t \gg \tau$, $q_h(t) \approx 0$, while

$$q_p(t) \simeq \frac{\tau}{1+(\omega\tau)^2}\frac{V_0}{R}\left[\sin(\omega t) - \omega\tau\cos(\omega t)\right]$$

Thus, $q_h(t)$ is the transient solution — which depends on the initial conditions — while $q_p(t)$ is called the driven solution since it depends on $V(t)$, does not die off w/ t, and does not depend on the initial conditions.

Non-linear First Order ODE's

General, non-linear ODE's are solved in a case-by-case basis.

Bernoulli's Eqn

Bernoulli's equation is the simplest generalization of a 1st order linear ODE:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Define $\quad z = y^{1-n}$

$$\frac{dz}{dx} = (1-n)\, y^{-n} \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{1-n} y^n \frac{dz}{dx}$$

so that

$$\frac{1}{1-n} y^n \frac{dz}{dx} + P(x)\, y = Q(x)\, y^n$$

$$\frac{dz}{dx} + (1-n)\, P(x)\, y^{1-n} = Q(x)$$

or

$$\frac{dz}{dx} + (1-n)\, P(x)\, z = Q(x)$$

and we are back to a linear ODE, where the boundary condition is now

$$z(x_0) = y_0^{1-n}, \quad y(x_0) = y_0$$

Exact Differentials

Consider the two dimensional vector field

$$\vec{v}(x,y) = v_1(x,y)\,\hat{e}_1 + v_2(x,y)\,\hat{e}_2$$

that is irrotational, or closed, if

$$\vec{\nabla} \times \vec{v} = 0$$

or

$$\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0$$

$\vec{v}$ is exact if there exists a function F such that

$$\vec{v} = \vec{\nabla} F \quad \text{or} \quad v_1 = \frac{\partial F}{\partial x}, \; v_2 = \frac{\partial F}{\partial y}$$

Equivalently:

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = v_1 dx + v_2 dy.$$

The only exact vector field on the plane is when:

$$F(x,y) = \text{constant}.$$

$\Rightarrow \quad dF = 0$

$\Rightarrow \quad v_1 dx + v_2 dy = 0$

or $\quad \frac{dy}{dx} = -\frac{v_1}{v_2}$, which gives the path that F depends on.

Ex:

$$dE = TdS - pdV$$

is the first law. In a process for which E = constant (the internal energy remains constant),

$$0 = TdS - pdV$$

$$\Rightarrow \quad \frac{dS}{dV} = \frac{P}{T}$$

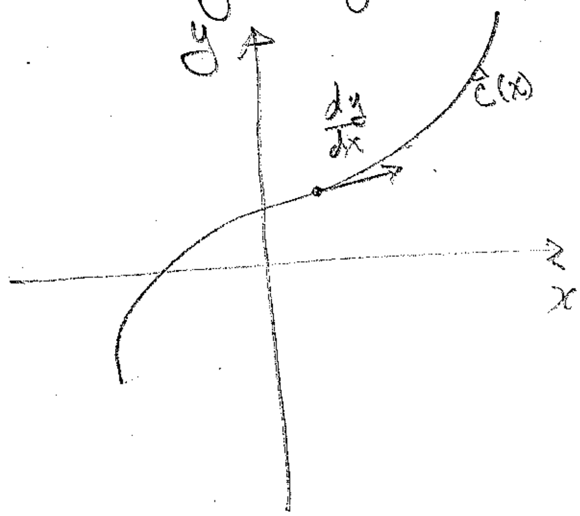
For an ideal gas, $\quad pV = Nk_BT \Rightarrow \frac{P}{T} = \frac{Nk_B}{V}$

$$\frac{dS}{dV} = \frac{Nk_B}{V}$$

$$\boxed{S = Nk_B \log\left(\frac{V}{V_0}\right) + S_0}$$

Graphical Methods

Remember that if we have a function $y(x)$, its graph as a function of x gives a curve:



At each point, $\frac{dy}{dx}$ = [illegible] gives the slope of y wrt x. This observative gives a potential means to graphically solve nonlinear, 1st order ODEs.

For many systems, this is the only way of solving the ODE.

A general 1st order ODE has the form:

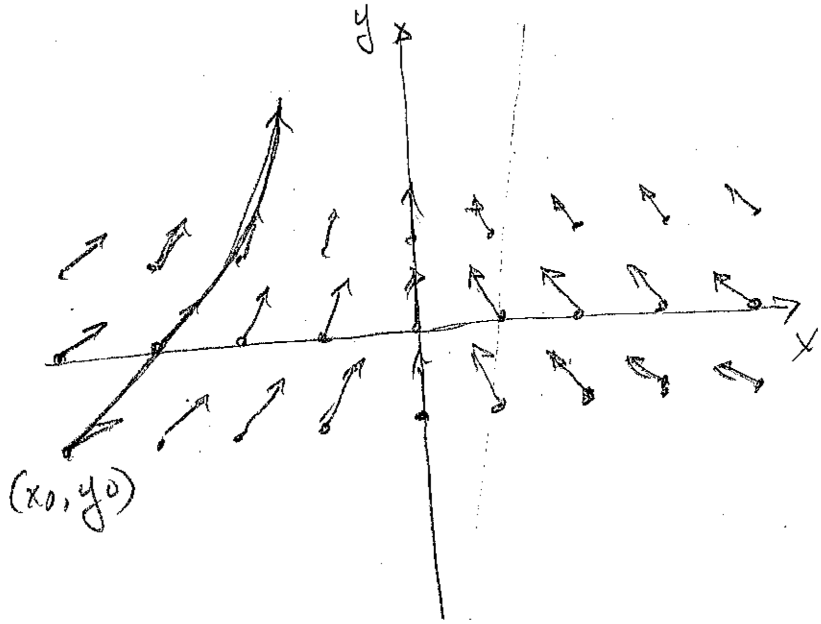
$$\frac{dy}{dx} = M(y, x)$$

Since $M(y,x)$ is known, we know $\frac{dy}{dx}$ at each point in the x-y-plane, and thus we know $\frac{dy}{dx}$ everywhere.

Remembering that $\frac{dy}{dx}$ is a slope, consider the trajectory $\vec{c} = (x, y(x))$ of a particle that satisfies the ODE. Then:

$$\frac{d\vec{c}}{dx} = \left(1, \frac{dy}{dx}\right).$$

and we can graph the slope of $\vec{c}$ everywhere



Which path is taken depends on the boundary condition

$$y(x_0) = y_0$$

or

$$\vec{c}\,(x_0) = (x_0, y_0)$$

Starting at this point draw a trajectory so that it is tangent to $\frac{d\vec{c}}{dx}$ at each point.

This will be the graphical solution of the ODE.

Linear 2nd Order ODE's

We will mostly be concerned with linear, 2nd order ODE's, the most general of which has the form:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = f(x).$$

The homogeneous 2nd order ODE is often written as

$$\frac{d}{dx}\left(P(x)\frac{dy}{dx}\right) + Q(x)y = -\lambda w(x) y,$$

when it is called a Strum-Louiville equation.

Notice that because these are linear equations, we can write them as

$$\mathcal{L}_x[y] = f(x)$$

where:

$$\mathcal{L}_x = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$$

is a linear operator. The homogeneous equation

$$\mathcal{L}_x[y] = 0$$

then means that y lies in the kernal of this operator.