For $x > \hat{x}$

Once again $\delta(x-\hat{x}) = 0$. Thus we have:

$$\frac{d^2 G_>}{dx^2} + p(x)\frac{dG_>}{dx} + q(x) G_> = 0$$

and the subscript > denotes that this is the solution for $x > \hat{x}$. Then by the same argument as for the $x < \hat{x}$ case,

$$G_>(x,\hat{x}) = a_1^>(\hat{x}) y_1(x) + a_2^>(\hat{x}) y_2(\hat{x}).$$

There is, however, only one function $G(x,\hat{x})$, and we need to find a way to join the two solutions. To do that, we will need to look at the behaviour of $G(x,\hat{x})$ at $x = \hat{x}$. Namely, we will have to go back to the ODE:

$$\frac{d^2 G}{dx^2} + p(x)\frac{dG}{dx} + q(x) G = \delta(x-\hat{x})$$

Integrating about $x = \hat{x}$,

$$\int_{\hat{x}-\epsilon}^{\hat{x}+\epsilon} \frac{d^2 G}{dx^2}\, dx + \int_{\hat{x}-\epsilon}^{\hat{x}+\epsilon} p(x)\frac{dG}{dx}\, dx + \int_{\hat{x}-\epsilon}^{\hat{x}+\epsilon} q(x) G\, dx = \int_{\hat{x}-\epsilon}^{\hat{x}+\epsilon} \delta(x-\hat{x})\, d\hat{x}$$

or since:

$$\int_{\hat{x}-\epsilon}^{\hat{x}+\epsilon} p(x)\frac{dG}{dx}\, dx = \int_{\hat{x}-\epsilon}^{\hat{x}+\epsilon} \frac{d}{dx}[pG]\, dx - \int_{\hat{x}-\epsilon}^{\hat{x}+\epsilon} \frac{dp}{dx} G\, dx$$

$$= p(\hat{x}+\epsilon) G(\hat{x}+\epsilon,\hat{x}) - p(\hat{x}+\epsilon) G(\hat{x}-\epsilon,\hat{x}) - \int_{\hat{x}-\epsilon}^{\hat{x}+\epsilon} \frac{dp}{dx} G(x)\, dx$$

and:

$$\int_{\tilde{x}-\epsilon}^{\tilde{x}+\epsilon} \frac{d^2G}{dx^2}\,dx = \frac{dG}{dx}\Bigg|_{\tilde{x}-\epsilon}^{\tilde{x}+\epsilon}$$

we get:

$$1 = \frac{dG}{dx}\Bigg|_{\tilde{x}+\epsilon} - \frac{dG}{dx}\Bigg|_{\tilde{x}-\epsilon} + p(\tilde{x}+\epsilon,\tilde{x})G(\tilde{x}+\epsilon,\tilde{x}) - p(\tilde{x}-\epsilon)G(\tilde{x}-\epsilon,\tilde{x}) + \int_{\tilde{x}-\epsilon}^{\tilde{x}+\epsilon}\left[q - \frac{dp}{dx}\right]G(x,\tilde{x})\,dx$$

Now take $\epsilon \to 0$. We will require that $G(x,\tilde{x})$ be continuous everywhere. Thus:

$$\lim_{\epsilon\to 0}\left[G(\underbrace{\tilde{x}+\epsilon}_{x>\tilde{x}},\tilde{x}) - G(\underbrace{\tilde{x}-\epsilon}_{x<\tilde{x}},\tilde{x})\right] = 0.$$

$$\boxed{G_>(\tilde{x},\tilde{x}) = G_<(\tilde{x},\tilde{x})} \quad \forall\, \tilde{x}.$$

The pG terms vanish while

$$\lim_{\epsilon\to 0}\int_{\tilde{x}-\epsilon}^{\tilde{x}+\epsilon}\left[q - \frac{dp}{dx}\right]G(x,\tilde{x})\,dx = 0$$

Thus:

$$\boxed{1 = \frac{dG_>}{dx}\Bigg|_{\tilde{x}} - \frac{dG_<}{dx}\Bigg|_{\tilde{x}}}$$

and the derivative of G has a jump discontinuity.

For a 2^{nd} order ODE

$$\mathcal{L}_x[G(x,\tilde{x})] = \frac{d^2G}{dx^2} + p(x)\frac{dG}{dx} + g(x)G = \delta(x-\tilde{x})$$

To solve this equation, we divide x into two regions.

For $x < \tilde{x}$:

Here, $\delta(x-\tilde{x}) = 0$, so that:

$$\frac{d^2G_<}{dx^2} + p(x)\frac{dG_<}{dx} + g(x)\,G_< = 0.$$

where we denote the solution in this region by a subscript $<$. Then $G_<$ is a solution of the homogeneous equation in this region:

$$G_<(x,\tilde{x}) = a_1^< y_1(x) + a_2^< y_2(x).$$

Here, $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous equation:

$$\mathcal{L}_x[y_1] = 0, \quad \mathcal{L}_x[y_2] = 0$$

$A_1^<$ and $A_2^<$ are constants that are independent of x, but they may depend on $\tilde{x}$. To emphasize this, we write

$$\boxed{G_<(x,\tilde{x}) = a_1^<(\tilde{x})\,y_1(x) + a_2^<(\tilde{x})\,y_2(x)}$$

There are four unknowns, $a_1^<(\hat{x})$, $a_2^<(\hat{x})$, $a_1^>(\hat{x})$, $a_2^>(\hat{x})$, and only two equations: (5)

$$\boxed{\begin{aligned} 0 &= G_>(\hat{x},\hat{x}) - G_<(\hat{x},\hat{x}) \\ 1 &= \left.\frac{dG_>}{dx}\right|_{\hat{x}} - \left.\frac{dG_<}{dx}\right|_{\hat{x}} \end{aligned}}$$

Two of these unknows cannot be determined, and in the end, will be set by the boundary condition of $y(x)$. In practice, the nature of the problem will determine what we choose the additional condition on $a_1^<$, $a_2^<$, $a_1^>$, and $a_2^>$ to be. Indeed, we see that the two above equations can be written as:

$$0 = \left[a_1^>(\hat{x}) - a_1^<(\hat{x})\right] y_1(\hat{x}) + \left[a_2^>(\hat{x}) - a_2^<(\hat{x})\right] y_2(\hat{x})$$

$$1 = \left[a_1^>(\hat{x}) - a_1^<(\hat{x})\right] \left.\frac{dy_1}{dx}\right|_{\hat{x}} + \left[a_2^>(\hat{x}) - a_2^<(\hat{x})\right] \left.\frac{dy_2}{dx}\right|_{\hat{x}}$$

and only the differences between the a's matter!

Defining:
$$\Delta a_1 = a_1^> - a_1^<$$
$$\Delta a_2 = a_2^> - a_2^<,$$

$$0 = \Delta a_1 y_1(\hat{x}) + \Delta a_2 y_2(\hat{x})$$
$$1 = \Delta a_1 \left.\frac{dy_1}{dx}\right|_{\hat{x}} + \Delta a_2 \left.\frac{dy_2}{dx}\right|_{\hat{x}}$$

so that

$$\boxed{\Delta a_1 = \frac{-y_2(\tilde{x})}{W[y_1, y_2](\tilde{x})}, \qquad \Delta a_2 = \frac{y_1(\tilde{x})}{W[y_1, y_2](\tilde{x})}}$$

We will need two additional conditions to determine $G(x,\tilde{x})$ exactly. It is not hard to show that different choices of these conditions will result in shifts of the homogeneous solution, and thus will be determined by the boundary conditions on $y(x)$. In practice, the nature of the differential equation and what it describes physically will determine the two additional condition on $G(x,\tilde{x})$.

Ex Suppose we have once again the situation that the ODE holds for $x_1 \le x \le x_1$. Then we choose:

$$G(x_1, \tilde{x}) = 0,$$
$$G(x_2, \tilde{x}) = 0.$$

Since $x_1 \le \tilde{x}$ for any $\tilde{x}$, the first condition gives:

$$G_<(x_1, \tilde{x}) = 0.$$

Since $\tilde{x} \le x_2$ for any $\tilde{x}$, the second condition gives

$$G_>(x_2, \tilde{x}) = 0.$$

Equivalently:

$$0 = a_1^<(\tilde{x})y_1(x_1) + a_2^<(\tilde{x})y_2(x_1)$$

and

$$0 = a_1^>(\tilde{x})y_1(x_2) + a_2^>(\tilde{x})y_2(x_2).$$

This gives the two additional equations that will determine $G(x,\tilde{x})$ precisely. To do so, write the first equation as

$$0 = [a_1^<(\tilde{x}) - a_1^>(\tilde{x}) + a_1^>(\tilde{x})]y_1(x_1) + [a_2^<(\tilde{x}) - a_2^>(\tilde{x}) + a_2^>(\tilde{x})]y_2(x_1)$$

or:

$$\Delta a_1 y_1(x_1) + \Delta a_2 y_2(x_1) = a_1^>(\tilde{x})y_1(x_1) + a_2^>(\tilde{x})y_2(x_1)$$

$$0 = a_1^>(\tilde{x})y_1(x_2) + a_2^>(\tilde{x})y_2(x_2).$$

$$\Rightarrow \boxed{a_1^>(\tilde{x}) = \frac{[\Delta a_1(\tilde{x})y_1(x_1) + \Delta a_2(\tilde{x})y_2(x_1)]y_2(x_2)}{[y_1(x_1)y_2(x_2) - y_1(x_2)y_2(x_1)]}}$$

while

$$\boxed{a_2^>(\tilde{x}) = -\left[\frac{\Delta a_1(\tilde{x})y_1(x_1) + \Delta a_2(\tilde{x})y_2(x_1)}{y_1(x_1)y_2(x_2) - y_1(x_2)y_2(x_1)}\right]y_1(x_2)}$$

Then $a_1^<(\tilde{x})$ and $a_2^<(\tilde{x})$ is determined through

$$a_1^<(\tilde{x}) = a_1^>(\tilde{x}) - \Delta a_1(\tilde{x})$$

$$a_2^<(\tilde{x}) = a_2^>(\tilde{x}) - \Delta a_2(\tilde{x})$$

The Green's function is then:

$$G(x,\tilde{x}) = G_>(x,\tilde{x})\theta(x-\tilde{x}) + G_<(x,\tilde{x})\theta(\tilde{x}-x)$$

Laplace Transformation

Remember that in solving the homogeneous, linear 2nd order ODE

$$\frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta y = 0$$

we made use of the fact that $e^{i\lambda x}$ is an eigenvector of the derivative operator:

$$\frac{d}{dx}(e^{i\lambda x}) = i\lambda e^{i\lambda x}.$$

How, though, should we solve the inhomogeneous equation? One method is to use Green's functions. Another method is to use Laplace transforms, which is particularly useful for time-dependent systems with given initial conditions.

Let $f(t)$ be a function of time that decreases sufficiently fast as $t \mapsto \infty$. Then the Laplace transform of $f(t)$ is:

$$L[f](p) = \int_0^\infty f(t)e^{-pt}dt.$$

p can be a complex number. This is an integral operator, or equivalently, an integral transform. It is certainly linear:

$$L[af+bg](p) = aL[f](p) + bL[g](p).$$

Notice that $L[f](p)$ only depends on the value of

$f(t)$ for $t \geq 0$. For convenience, we take $f(t)=0$ for $t<0$. [5?]

There are a number of properties of the Laplace transform.

$$\mathcal{L}\left[\frac{df}{dt}\right](p) = \int_0^\infty \frac{df}{dt} e^{-pt} dt$$

$$= \int_0^\infty \left\{ \frac{d}{dt}(f e^{-pt}) - f(t)\frac{de^{-pt}}{dt} \right\} dt$$

$$= -f(0) + p\int_0^\infty f(t) e^{-pt} dt.$$

or:

$$\boxed{\mathcal{L}\left[\frac{df}{dt}\right](p) = -f(0) + p\,\mathcal{L}[f](p).}$$

Indeed,

$$\boxed{\mathcal{L}\left[\frac{d^{(n)}f}{dt^{(n)}}\right](p) = -\left.\frac{d^{(n-1)}f}{dx^{(n-1)}}\right|_0 + p\,\mathcal{L}\left[\frac{d^{(n-1)}f}{dx^{(n-1)}}\right](p)}$$

Then:

$$\mathcal{L}[t^n](p) = \int_0^\infty t^n e^{-pt} dt = \int_0^\infty \left(\frac{-1}{p}\right) t^n \frac{de^{-pt}}{dt} dt$$

$$= -\frac{1}{p}\int_0^\infty \left\{ \frac{d}{dt}(t^n e^{-pt}) - n t^{n-1} e^{-pt} \right\} dt$$

$$= +\frac{n}{p}\int_0^\infty t^{n-1} e^{-pt} dt.$$

$$\boxed{\mathcal{L}[t^n](p) = \frac{n}{p}\,\mathcal{L}[t^{n-1}](p)}$$

Next:

$$L[\theta(t-t_0)](p) = \int_0^\infty \theta(t-t_0)\, e^{-pt}\,dt = \int_0^{t_0} \cancel{\theta(t-t_0)\, e^{-pt}\,dt}_{\,t<t_0} + \int_{t_0}^\infty \theta(t-t_0)\, e^{-pt}\,dt_{\,t>t_0}$$

$$= \int_{t_0}^\infty e^{-pt}\,dt$$

$$\boxed{L[\theta(t-t_0)](p) = \frac{e^{-pt_0}}{p}}$$

And:

$$L[e^{izt}](p) = \int_0^\infty e^{izt}\, e^{-pt}\,dt = \int_0^\infty e^{(iz-p)t}\,dt$$

$$= \frac{e^{(iz-p)t}}{iz-p}\Bigg|_0^\infty$$

which converges if $p > -\mathrm{Im}(z)$.

$$= \frac{-1}{iz-p}$$

Thus:

$$L[\sin(zt)] = L\left[\frac{1}{2i}\left[e^{izt} - e^{-izt}\right]\right] = \frac{-1}{2i}\left[\frac{1}{iz-p} - \frac{1}{-iz-p}\right]$$

$$= \frac{-1}{2i}\left[\frac{-iz-p-(iz-p)}{z^2+p^2}\right]$$

$$L[\sin(zt)] = \frac{z}{z^2+p^2}$$

$$L[\cos(zt)] = \frac{p}{z^2+p^2}$$

and so on.

There is a table of the Laplace transform of known functions. The strength of the Laplace transformation comes in solving linear, 2nd order ODE's. w/ inhomogeneous terms.

Given the 2nd order ODE:

$$\frac{d^2y}{dt^2} + \alpha \frac{dy}{dt} + \beta y = f(t)$$

with the initial condition,

$$y(0) = y_0, \quad \left.\frac{dy}{dt}\right|_0 = \dot{y}_0,$$

we solve this equation by using Laplace transformations.

First we define:

$$Y(p) = L[y](p).$$

Next, we take the Laplace transform of both sides of the ODE,

$$L\left[\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y\right](p) = L[f](p)$$

$$L\left[\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y\right](p) = L\left[\frac{d^2y}{dt^2}\right](p) + \alpha L\left[\frac{dy}{dt}\right](p) + \beta Y$$

$$= -\left.\frac{dy}{dt}\right|_0 + pL\left[\frac{dy}{dt}\right] + \alpha L\left[\frac{dy}{dt}\right] + \beta Y$$

$$= -\dot{y}_0 + (\alpha + p)\left[-y_0 + pY\right] + \beta Y$$

or $\quad -\dot{y}_0 - (\alpha + p) y_0 + [p^2 + \alpha p + \beta] Y = F(p)$

where $\quad F(p) = \mathcal{L}[f](p)$.

Then:

$$\boxed{Y(p) = \frac{F(p)}{p^2 + \alpha p + \beta} + \frac{\dot{y}_0 + (\alpha + p) y_0}{p^2 + \alpha p + \beta}}$$

Once F(p) is know, it would seem that we would then take the "inverse" Laplace transform of Y(p) to find y(t). Doing so would require knowledge of contour integration, which we do not cover. Instead, the tactic we will employ is the look-up method. Manipulate the right hand side of the above until it has a form we can find in the table of Laplace transform.

For example, the homogeneous term is:

$$Y_h(p) = \frac{\dot{y} + (\alpha + p) y_0}{(p^2 + \alpha p + \beta)}$$

Completing the square,

$$p^2 + \alpha p + \beta = \left(p + \frac{\alpha}{2}\right)^2 + \beta - \frac{\alpha^2}{4}$$

$$Y_h(p) = \frac{\dot{y} + (\alpha + p) y_0}{\left(p + \frac{\alpha}{2}\right)^2 + \beta - \frac{\alpha^2}{4}}$$

Next, we note that

$$\mathcal{L}\left[e^{-at}\sin bt\right] = \frac{b}{(p+a)^2+b^2},$$

$$\mathcal{L}\left[e^{-at}\cos bt\right] = \frac{p+a}{(p+a)^2+b^2}.$$

Thus,

$$Y_h(p) = \frac{\dot{y}_0 + \left[p+\frac{\alpha}{2}+\frac{\alpha}{2}\right]y_0}{(p+\frac{1}{2}\alpha)^2+\beta-\frac{\alpha^2}{4}}$$

$$= \frac{\dot{y}_0+\frac{\alpha}{2}y_0}{(p+\frac{1}{2}\alpha)^2+\beta-\frac{\alpha^2}{4}} + \frac{(p+\frac{\alpha}{2})y_0}{(p+\frac{1}{2}\alpha)^2+\beta-\frac{\alpha^2}{4}}$$

$$= \frac{\dot{y}+\frac{\alpha}{2}y_0}{\sqrt{\beta-\frac{\alpha^2}{4}}}\left[\frac{\sqrt{\beta-\frac{\alpha^2}{4}}}{(p+\frac{1}{2}\alpha)^2+\beta-\frac{\alpha^2}{4}}\right] + y_0\left[\frac{p+\frac{\alpha}{2}}{(p+\frac{\alpha}{2})^2+\beta-\frac{\alpha^2}{4}}\right].$$

Then:

$$y_h(t) = \frac{(\dot{y}+\frac{\alpha}{2}y_0)}{\sqrt{\beta-\frac{\alpha^2}{4}}}\cdot e^{-\frac{\alpha}{2}t}\sin\left[\sqrt{\beta-\frac{\alpha^2}{4}}\,t\right] +$$

$$y_0\, e^{-\frac{\alpha}{2}t}\cos\left(\sqrt{\beta-\frac{\alpha^2}{4}}\,t\right).$$

$$\boxed{y_h(t) = \frac{e^{-\frac{\alpha}{2}t}\sin\left[\sqrt{\beta-\frac{\alpha^2}{4}}\,t\right]}{\sqrt{\beta-\frac{\alpha^2}{4}}} + \left[\frac{\frac{\alpha}{2}}{\sqrt{\beta-\frac{\alpha^2}{4}}}\, e^{-\frac{\alpha}{2}t}\sin\left[\sqrt{\beta-\frac{\alpha^2}{4}}\,t\right] + e^{-\frac{\alpha}{2}t}\cos\left[\sqrt{\beta-\frac{\alpha^2}{2}}\,t\right]\right]y_0}$$

The inhomogeneous solution will, however, depend on $F(p)$.

Convolution

There are times when the look-up method does not work, or when $F(p)$ is to difficult to calculate. In these cases we resort to convolutions.

Remember that,

$$Y_p(p) = \frac{F(p)}{p^2 + \alpha p + \beta} = T(p)F(p)$$

$T(p)$ is called the transfer function. It depends solely on the form of the original differential equation.

The question then becomes, what must $y_p(t)$ if $Y_p(p) = T(p)F(p)$?

Consider:

$$L[g(t)h(t)](p) = \int_0^\infty g(t)h(t)e^{-pt}dt \neq L[g(t)](p)\,L[f(t)](p)$$

Namely, the Laplace tranform of two functions is not the product of the Laplace transforms. It has to be more complicated.

Let $G(p) = L[g](p) = \int_0^{\infty} g(\sigma) e^{-p\sigma} d\sigma$

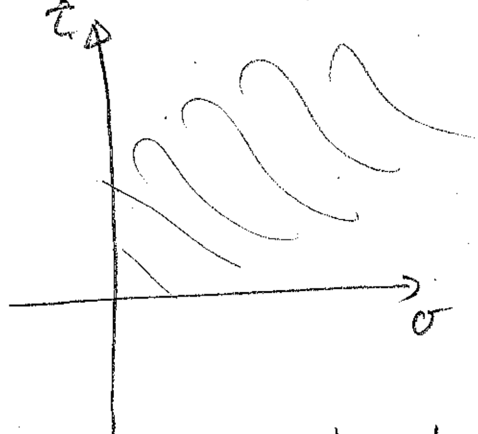
$$H(p) = L[h](p) = \int_0^{\infty} f(\tau) e^{-p\tau} d\tau.$$

Then:

$$G(p)H(p) = \int_0^{\infty} g(\sigma) e^{-p\sigma} d\sigma \int_0^{\infty} f(\tau) e^{-p\tau} d\tau$$

$$= \int_0^{\infty}\int_0^{\infty} g(\sigma) f(\tau) e^{-p(\sigma+\tau)} d\sigma\, d\tau.$$

and the integral is over the 1st quadrant of the σ-τ plane:



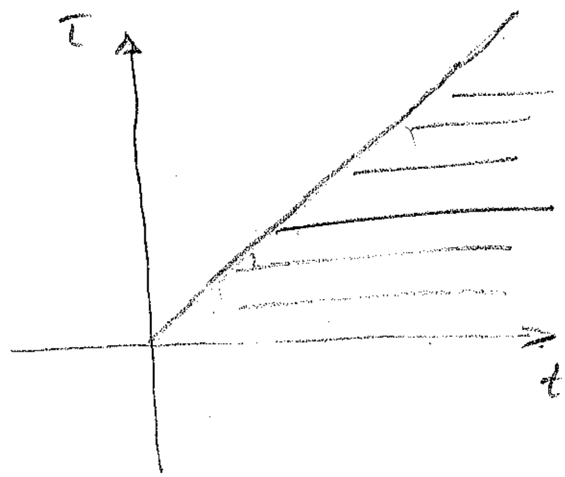
We would like to express

$$G[p]H[p] = \int_0^{\infty} c(t) e^{-pt} dt$$

as the Laplace transform of a specific function. To do so we first define $\sigma + \tau = t$, and we consider this to be a change in variable for σ; thus, τ is fixed. Then $\sigma = t - \tau$, and the integral over t ranges from $t = \tau$ to infinity.

$$G[p]H(p) = \int_0^\infty \left(\int_\tau^\infty g(t-\tau) f(\tau) e^{-pt}\, dt \right) d\tau.$$

and the domain of integration is:



The problem is that the exponential term is stuck w/ in an integral that goes from τ to infinity. It does not have the form we want.

However, as long as we integrate over the <u>same</u> domain in τ-t space, the value of the integrals will not change. The above integral integrates t first then τ. We can just as easily integrate τ first and then t. In doing so, we would first integrate τ from 0 to t, and t from 0 to ∞. Thus

$$G[p]H[p] = \int_0^\infty dt \int_0^t g(t-\tau) f(\tau)\, e^{-pt}\, d\tau.$$

$$= \int_0^\infty \left(\int_0^t g(t-\tau) f(\tau)\, d\tau \right) e^{-pt}\, dt.$$

$$\Rightarrow \quad c(t) = \int_0^t g(t-\tau) f(\tau)\, d\tau$$

which is called the <u>convolution</u>. It is often denoted as:

$$\boxed{g * f = \int_0^t g(t-\tau) f(\tau)\, d\tau}$$

In our case, this means that:

$$y_p(t) = \int_0^t T(t-\tau) f(\tau)\, d\tau.$$

where:

$$T(p) = \mathcal{L}[T(t)] = \int_0^\infty T(t) e^{-pt}\, dt.$$

$$T(p) = \frac{1}{(p-p_+)(p-p_-)} \quad \text{where } p_+ = \frac{1}{2}\left[-\alpha + \sqrt{\alpha^2 - 4\beta}\right]$$

$$= -\frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}$$

$$p_- = -\frac{\alpha}{2} - \sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}$$

$$T(t) = \frac{e^{\left[-\frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}\right]t} - e^{\left[-\frac{\alpha}{2} - \sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}\right]t}}{2\sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}} \quad \text{from LT}$$

$$= \frac{e^{\frac{\alpha}{2}t}\left[e^{-\sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}\, t} - e^{\sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}\, t}\right]}{2\sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}}$$

In the special case where

$$f(t) = \delta(t-\hat{\tau}),$$

then:

$$\frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta y = \delta(t-\hat{\tau})$$

$\Rightarrow$ y is a Green's function

$$y_p(t) = \int_0^t T(t-\tau)\, f(\tau)\, d\tau$$

$$= \int_0^t T(t-\tau)\, \delta(\tau-\hat{\tau})\, d\tau$$

If $t < \hat{\tau}$, then $y_p(t) = 0$ since $\delta(\tau-\hat{\tau})$ is alway zero.

If $t > \hat{\tau}$, then

$$y_p(t) = T(t-\hat{\tau})$$

Thus,

$$y_p(t) = \frac{e^{\alpha/2(t-\hat{\tau})}\left[e^{-\sqrt{(\frac{\alpha}{2})^2-\beta}\,(t-\hat{\tau})} - e^{\sqrt{(\frac{\alpha}{2})^2-\beta}\,(t-\hat{\tau})}\right]}{2\sqrt{(\frac{\alpha}{2})^2-\beta}}\,\theta(t-\hat{\tau})$$

which gives the Green's function once $y_h(t)$ is included.