The exact form of $\mathcal{L}$ is quite arbitrary, and that is because we can do the following:

- change variables:

$$x \longmapsto z(x).$$

- Use an integration factor by defining

$$y = e^{g}\mu.$$

- A combination of both.

In the first

$$\frac{d}{dx} = \frac{dz}{dx}\frac{d}{dz}$$

$$\frac{d^2}{dx^2} = \frac{d^2z}{dx^2}\frac{d}{dz} + \left(\frac{dz}{dx}\right)\frac{d}{dx}\left(\frac{d}{dz}\right)$$

$$= \frac{d^2z}{dx^2}\frac{d}{dz} + \left(\frac{dz}{dx}\right)^2\frac{d^2}{dz^2}$$

Then:

$$\mathcal{L}_x[y] = \frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + by$$

$$= \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2} + \frac{d^2z}{dx^2}\frac{dy}{dz} + p(z)\frac{dz}{dx}\frac{dy}{dz} + q(z)y$$

$$= \left(\frac{dz}{dx}\right)^2\left[\frac{d^2y}{dz^2} + \left[\frac{\frac{d^2z}{dx^2} + p(z)\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\right]\frac{dy}{dz} + \frac{q(z)y}{\left(\frac{dz}{dx}\right)^2}\right]$$

$$\mathcal{L}_x[y] = \left(\frac{dz}{dx}\right)^2\mathcal{L}_z[y]$$

while in the second

$$\frac{dy}{dx} = e^g \left[\frac{dg}{dx}\mu + \frac{d\mu}{dx} \right].$$

$$\frac{d^2y}{dx^2} = e^g \frac{dg}{dx}\left[\frac{dg}{dx}\mu + \frac{d\mu}{dx} \right] + e^g\left[\frac{d^2g}{dx^2}\mu + \frac{dg}{dx}\frac{d\mu}{dx} + \frac{d^2\mu}{dx^2} \right]$$

$$= e^g\left[\frac{d^2\mu}{dx^2} + 2\frac{dg}{dx}\frac{d\mu}{dx} + \left(\frac{d^2g}{dx^2} + \left(\frac{dg}{dx}\right)^2 \right)\mu \right]$$

Then,

$$\mathcal{L}_x[y] = e^g\left[\frac{d^2\mu}{dx^2} + 2\frac{dg}{dx}\frac{d\mu}{dx} + \left(\frac{d^2g}{dx^2} + \left(\frac{dg}{dx}\right)^2 \right)\mu + p(x)\left[\frac{dg}{dx}\mu + \frac{d\mu}{dx} \right] + q(x)\mu \right]$$

$$= e^g\left[\frac{d^2\mu}{dx^2} + \left(2\frac{dg}{dx} + p(x) \right)\frac{d\mu}{dx} + \left[q(x) + p(x)\frac{dg}{dx} + \frac{d^2g}{dx^2} + \left(\frac{dg}{dx}\right)^2 \right]\mu \right]$$

Both transformations allow us to change the form of a given form of the differential equation to one that matches a differential equation for whom the solutions are will known.

Dirac Delta Function

We have, up to now two Green's functions.

For the first order ODE:

$$\frac{dy}{dx} + Q(x)\, y = H(x)$$

for which:

$$G_{(1)}(x,\tilde{x}) = \exp\left[-\int_{\tilde{x}}^{x} Q(s)\,ds\right] \theta(x-\tilde{x})$$

while for the second order ODE:

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f(x)$$

$$G_{(2)}(x,\tilde{x}) = -\left[\frac{y_1(x)\,y_2(\tilde{x})\,\theta(x-\tilde{x}) + y_2(x)\,y_1(\tilde{x})\,\theta(\tilde{x}-x)}{W[y_1,y_2](\tilde{x})}\right].$$

Notice that for the first order ODE,

$$\frac{dG_{(1)}}{dx} = -Q(x)\exp\left[-\int_{\tilde{x}}^{x} Q(s)\,ds\right]\theta(x-\tilde{x}) + \exp\left[-\int_{\tilde{x}}^{x} Q(s)\,ds\right]\frac{d\theta(x-\tilde{x})}{dx}$$

$$= -Q(x)\,G_{(1)} + \exp\left[-\int_{\tilde{x}}^{x} Q(s)\,ds\right]\frac{d\theta(x-\tilde{x})}{dx}$$

$$\Rightarrow \qquad \frac{dG_{(1)}}{dx} + Q(x)\,G_{(1)}(x) = \exp\left[-\int_{\tilde{x}}^{x} Q(s)\,ds\right]\frac{d\theta(x-\tilde{x})}{dx}$$

Now, $\frac{d\theta(x-\tilde{x})}{dx}$ cannot be zero everywhere. If it was, then $G_{(1)}$ would be a solution a solution of the homogeneous equation, (which is unique) and it is not. We denote this function by,

$$\delta(x-\tilde{x}) = \frac{d\theta(x-\tilde{x})}{dx},$$

It is call the Dirac δ-function, but do call it a function is a slight misnomer, as we will see. First, we note the following properties of this function.

Note that,

$$\delta(x) = \frac{d\theta(x)}{dx}$$

so that for $a<0$

$$\int_a^y \delta(x)\,dx = \theta(y) - \theta(a) = \theta(x)$$

If $y<0$, $\quad \int_a^y \delta(x)\,dx = 0$

If $y>0$, $\quad \int_a^y \delta(x)\,dx = 1$

irrespective of the choice of a. In particular:

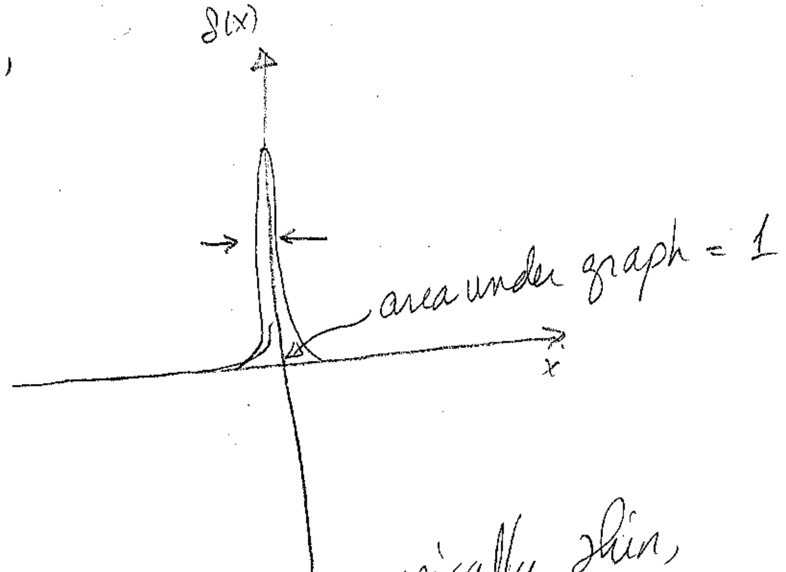
$$\lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} \delta(s)\,ds = 1.$$

Thus, the area under the $\delta(x)$ is always 1, and the width of the function is infinitely thin. Thus:

$$\delta(x) = \begin{cases} \infty & \text{if } x=0 \\ 0 & \text{if } x\neq 0. \end{cases}$$

Moreover, $\delta(-x) = \delta(x)$; the δ-function is an even function.

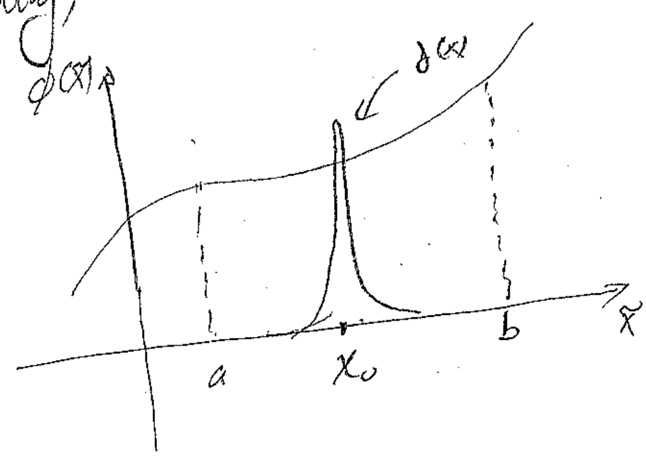
Or graphically,



Because the δ-function is surgically thin,

$$\int_{a}^{b} \phi(\tilde{x})\,\delta(\tilde{x}-x)\,d\tilde{x} = \begin{cases} \phi(x) & \text{if } a<\tilde{x}<b \\ 0 & \text{otherwise} \end{cases}$$

Graphically,



Since

$$u = \tilde{x} - x$$

$$\int_a^b \phi(\tilde{x})\,\delta(\tilde{x}-x)\,d\tilde{x} = \lim_{\epsilon\to 0}\int_{x-\epsilon}^{x+\epsilon} \phi(\tilde{x})\,\delta(\tilde{x}-x)\,d\tilde{x}$$

$$= \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} \phi(u+x)\,\delta(u)\,du =$$

Since u is small,

$$\phi(u+x) \approx \phi(x) + \left.\frac{d\phi}{dx}\right|_x u + \cdots$$

after expanding in a Taylor series. Thus:

$$\int_a^b \phi(\tilde{x})\,\delta(\tilde{x}-x)\,d\tilde{x} = \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\left[\phi(x) + \left.\frac{d\varphi}{dx}\right|_x u\right]\delta(u)\,du.$$

$$= \phi(x) + \frac{d\varphi}{dx}\overbrace{\int_{-\epsilon}^{\epsilon} u\,\delta(u)\,du}^{=0} \quad \text{since } u \text{ is odd}$$

$$\Rightarrow \quad \int_a^b \phi(\tilde{x})\,\delta(\tilde{x}-x)\,d\tilde{x} = \phi(x).$$

A few more properties:

$$\int_{-a}^{a} \tilde{x}^n\,\delta(\tilde{x})\,d\tilde{x} = 0,$$

while:

$$\int_a^b \phi(\tilde{x})\,\frac{d\delta(\tilde{x}-x)}{d\tilde{x}}\,d\tilde{x} = \int_a^b\left\{d\overbrace{\left[\phi(\tilde{x})\,\delta(\tilde{x}-x)\right]}^{=0} - \frac{d\phi}{d\tilde{x}}\,\delta(\tilde{x}-x)\,d\tilde{x}\right\}$$

$$= -\int_a^b \frac{d\phi}{d\tilde{x}}\,\delta(\tilde{x}-x)\,d\tilde{x}$$

$$= \begin{cases} -\dfrac{d\varphi}{dx} & \text{if } a<x<b \\ 0 & \text{otherwise} \end{cases}$$

Thus:

$$\int_a^b \phi(\tilde{x})\,\frac{d^{(n)}\delta(\tilde{x}-x)}{d\tilde{x}^{(n)}}\,d\tilde{x} = \begin{cases} (-1)^n \dfrac{d^{(n)}\phi}{dx^{(n)}} & \text{if } a<x<b. \\ 0 & \text{otherwise.} \end{cases}$$

Next,

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\delta(a\tilde{x})\,d\tilde{x} = \lim_{\epsilon\to 0}\int_{-\epsilon/|a|}^{\epsilon/|a|}\frac{1}{|a|}\delta(u)\,du$$

$u = |a|\tilde{x}$
since $\delta(x) = \delta(-x)$

$$= \lim_{\tilde{\epsilon}\to 0}\int_{-\tilde{\epsilon}}^{\tilde{\epsilon}}\frac{\delta(\tilde{x})\,d\tilde{x}}{|a|}$$

$\tilde{\epsilon} = \epsilon/a$, $u = \tilde{x}$

$$\Rightarrow \quad \delta(ax) = \frac{\delta(x)}{|a|}$$

Finally, let $f(x)$ be a function with a simple pole at x_1 so that $f(x_1) = 0$.

Then:

$$\int_a^b \delta[f(\tilde{x})]\,d\tilde{x} = \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\delta[f(\tilde{x})]\,d\tilde{x}$$

since $f(x) \neq 0$ everywhere else.

Expanding in a Taylor series about $x = x_1$:

$$f(x) = f(x_1) + \left.\frac{df}{dx}\right|_{x_1}(x - x_1) = \left.\frac{df}{dx}\right|_{x_1}(x - x_1)$$

$$\int_a^b \delta[f(\tilde{x})]\,d\tilde{x} = \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\delta\left[\left.\frac{df}{dx}\right|_{x_1}(\tilde{x} - x_1)\right]d\tilde{x}$$

so that

$$\delta[f(x)] = \frac{\delta(x - x_1)}{\left|\left.\frac{df}{dx}\right|_{x_1}\right|}$$

If $f(x)$ has simple zeros at $x_1, \ldots, x_n$, then it is not hard to see that:

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x-x_i)}{\left|\frac{df}{dx}\Big|_{x_i}\right|}$$

If the δ-function does not seem like a standard function, it isn't. It is at times called a generalized function or a distribution, and the functions on which it acts are called test functions; they are functions that are extremely well-behaved at infinity. Indeed, there isn't just <u>one</u> δ-function; there are <u>representations</u> of δ functions.

<u>For example:</u>

$$\delta(x) = \lim_{\epsilon \to 0} \frac{\sin(x/\epsilon)}{\pi x} \quad \Rightarrow \quad \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk$$

$$\delta(x) = \frac{e^{-x^2/\epsilon}}{\sqrt{\pi\epsilon}}$$

$$\delta(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\epsilon}{x^2+\epsilon}$$

$$\delta(x) = \frac{d\theta}{dx}$$

and so on

δ-functions in 3D

If we work in cartesian coordinates, the generalization of the δ-function to 3D is straightforward.

Since:

$$\int_{-\infty}^{\infty} \delta(x-x_0)\,dx = 1.$$

Then:

$$\int_{-\infty}^{\infty} \delta(x-x_0)dx \int_{-\infty}^{\infty} \delta(y-y_0)dy \int_{-\infty}^{\infty} \delta(z-z_0)dz = 1$$

or:

$$\int_{R^3} \delta^3(\vec{r}-\vec{r}_0)\, d^3r = 1$$

where: $\vec{r} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3$, $\vec{r}_0 = x_0\hat{e}_1 + y_0\hat{e}_2 + z_0\hat{e}_3$, and:

$$\delta^3(\vec{r}-\vec{r}_0) \equiv \delta(x-x_0)\,\delta(y-x_0)\,\delta(z-z_0).$$

Indeed,

$$\int_D \delta^3(\vec{r}-\vec{r}_0)d^3r = 1 \quad \text{for } D \subset R^3 \text{ and } \{\vec{0}\} \in D$$

Then:

$$\int_D \varphi(\vec{r}')\,\delta^3(\vec{r}'-\vec{r})\,d^3\vec{r}' = \varphi(\vec{r}) \quad \text{for } \vec{r} \in D.$$

In spherical coordinates, we also know what $\delta^3(\vec{r})$ is, and we have known it since 7B. Remember Gauss's Law:

$$\oint_{S'} \vec{E}\cdot d\vec{A} = \frac{q_{encl.}}{\epsilon_0}$$

Remember also the Divergence Theorem:

$$\oint_{S'} \vec{E}\cdot d\vec{A} = \int_D \nabla\cdot\vec{E}\, d^3\vec{r}$$

as well as $\vec{E} = -\vec{\nabla} V$. Putting this altogether,

$$-\int_D \nabla^2 V\, d^3\vec{r} = \frac{q_{encl}}{\epsilon_0}$$

Lets apply this to a point charge. Then:

$$V = \frac{q}{4\pi\epsilon_0 r}$$

$$-\int \frac{q}{4\pi\epsilon_0} \nabla^2\left(\frac{1}{r}\right) d^3\vec{r} = \frac{q}{\epsilon_0}$$

or

$$\int \left[-\frac{1}{4\pi}\vec{\nabla}^2\left(\frac{1}{r}\right)\right] d^3\vec{r} = 1.$$

But:

$$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\vec{\nabla} r$$

$$= -\frac{\vec{r}}{r^3}$$

$$\vec{\nabla} r = \vec{\nabla}\sqrt{x^2+y^2+z^2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r},\quad \frac{\partial r}{\partial y} = \frac{y}{r},\quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\vec{\nabla} r = \frac{\vec{r}}{r}$$

$$\vec{\nabla}^2\left(\frac{1}{r}\right) = -\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^3}\right) = -\left[\frac{\vec{\nabla}\cdot\vec{r}}{r^3} - 3\frac{\vec{r}\cdot\vec{\nabla}r}{r^4}\right]$$

$$\vec{\nabla}\cdot\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$ Then

$$\vec{\nabla}^2\left(\frac{1}{r}\right) = -\left[\frac{3}{r^3} - \frac{3\vec{r}}{r^4}\cdot\left(\frac{\vec{r}}{r}\right)\right] = 0.$$

Thus, $-\frac{1}{4\pi}\vec{\nabla}^2\left(\frac{1}{r}\right)$ vanishes when $r \neq 0$, but the integral is unity. This is a δ-function:

$$\boxed{\delta^3(\vec{r}) = -\frac{1}{4\pi}\vec{\nabla}^2\left(\frac{1}{r}\right)}$$ which is spherically symmetric.

Remembering in spherical coordinates that

$$\int \delta^3(\vec{r})\,d^3\vec{r} = \int \delta^3(\vec{r})\cdot 4\pi r^2\,dr$$

since $\delta^3(\vec{r})$ is spherically symmetric. Thus:

$$\int \delta^3(\vec{r})\,4\pi r^2\,dr \equiv \int \delta(r)\,dr$$

$$\Rightarrow \quad \boxed{\delta^3(\vec{r}) = \frac{\delta(r)}{4\pi r^2}}$$

is another representation.

Greens Function

Remember that for the 1st order, linear ODE:

$$\frac{dG_{(1)}}{dx} + Q(x)G_{(1)}(x,\tilde{x}) = e^{-\int_{\tilde{x}}^{x} Q(s)ds} \frac{d\theta(x-\tilde{x})}{dx}.$$

But $\frac{d\theta(x-\tilde{x})}{dx} = \delta(x-\tilde{x})$

And $e^{-\int_{\tilde{x}}^{x} Q(s)ds}\delta(x-\tilde{x}) = 1\cdot\delta(x-\tilde{x})$

Thus: $G_{(1)}(x,\tilde{x})$ is the solution of the inhomogeneous differential equation:

$$\frac{dG_{(1)}}{dx} + Q(x)G_{(1)}(x,\tilde{x}) = \delta(x-\tilde{x})$$

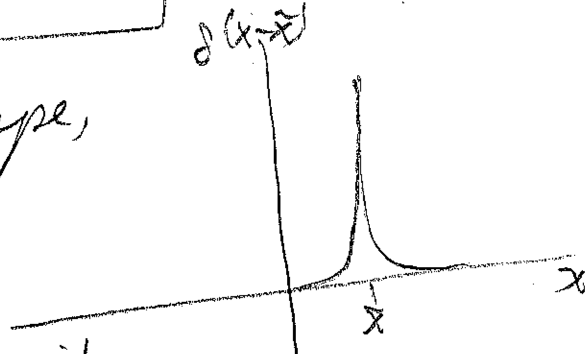
This is in fact the diffinition of Green's function.

Given a <u>linear</u>, ordinary differential operator, $\mathcal{L}_x$, the corresponding Green's function is the solution of of the inhomogeneous ODE:

Notice sign!

$$\boxed{\mathcal{L}_x[G(x,\tilde{x})] = \delta(x-\tilde{x})}$$

Because $\delta(x-\tilde{x})$ has the shape,



$\delta(x-\tilde{x})$ is sometimes called a <u>unit impulse</u>.

Of course, the precise form of $G(x, x')$ will depend on the choice of boundary conditions, and that in turn will be guided by the boundary conditions for the problem at hand. But to understand the physical meaning behind the green's function — and thus its importance, consider the following linear ODE:

$$\mathcal{L}_x[y] = f_1(x) + f_2(x),$$

which is driven by two inhomogeneous terms (sources).

Then y has the form:

$$y(x) = y_h(x) + y_{p_1}(x) + y_{p_2}(x)$$

where $y_h(x)$ is the solution of the homogeneous equation,

$$\mathcal{L}_x[y_h] = 0,$$

$y_{p_1}(x)$ is the solution of the particular equation driven by $f_1(x)$ only:

$$\mathcal{L}_x[y_{p_1}] = f_1(x),$$

and $y_{p_2}(x)$ is the solution of the particular equation driven by $f_2(x)$ only:

$$\mathcal{L}_x[y_{p_2}(x)] = f_2(x)$$

This follows from linearity:

$$\mathcal{L}_x[y_h + y_{p_1} + y_{p_2}] = \mathcal{L}_x[y_h] + \mathcal{L}_x[y_{p_1}] + \mathcal{L}_x[y_{p_2}]$$
$$= 0 + f_1(x) + f_2(x).$$

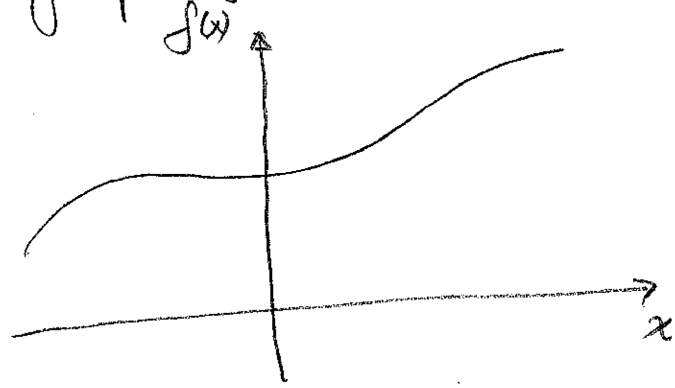
Thus, the particular solution to a sum of two inhomogeneous term is the sum of particular solutions to each inhomogeneous term acting on the system separately. In physics, this would be a direct consequence of the superposition principle.

This is a simple but important observation that can be applied to <u>any</u> inhomogeneous term.

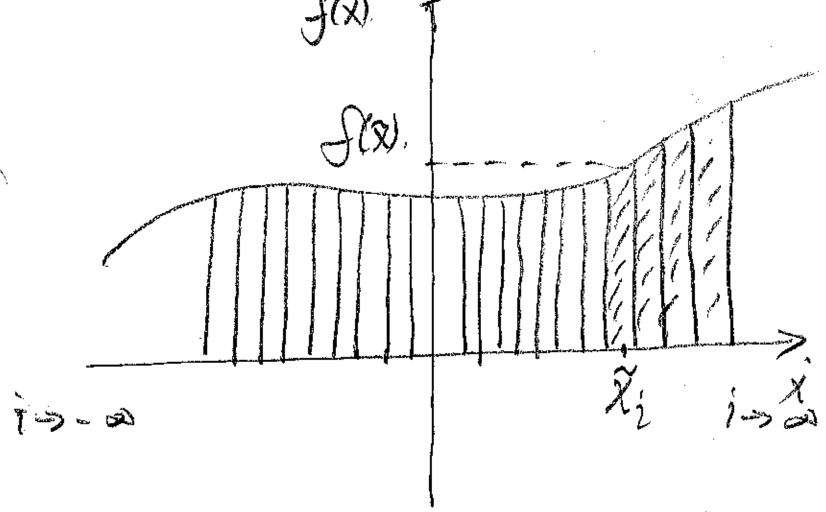
Consider the linear ODE:

$$\mathcal{L}_x[y] = f(x).$$

and graph $f(x)$ as a function of x:



Solve this problem by dividing $f(x)$ into pieces.



Then we can approximate:

$$f(x) \cong \sum_{i=-\infty}^{\infty} f(\tilde{x}_i)\, s(x-\tilde{x}_i)\, \Delta\tilde{x}$$

where: $s(x-x_i) = \begin{cases} \frac{1}{\Delta\tilde{x}} & \text{if } \tilde{x}_i < x \le \tilde{x}_i + \Delta\tilde{x} \\ 0 & \text{otherwise} \end{cases}$

By doing so, we have divided $f(x)$ into small slivers, and we know that the solution to

$$\mathcal{L}_x[y] = f(x) \approx \sum_{i=-\infty}^{\infty} f(\tilde{x}_i)\, s(x-\tilde{x}_i)\, \Delta\tilde{x}$$

is given by:

$$y(x) = y_h(x) + \sum_{i=-\infty}^{\infty} y_p(x, \tilde{x}_i)\, \Delta\tilde{x}$$

where y_p is the solution of the ODE:

$$\mathcal{L}_x[y_p(x,\tilde{x})] = f(\tilde{x}_i)\, s(x-\tilde{x}_i)$$

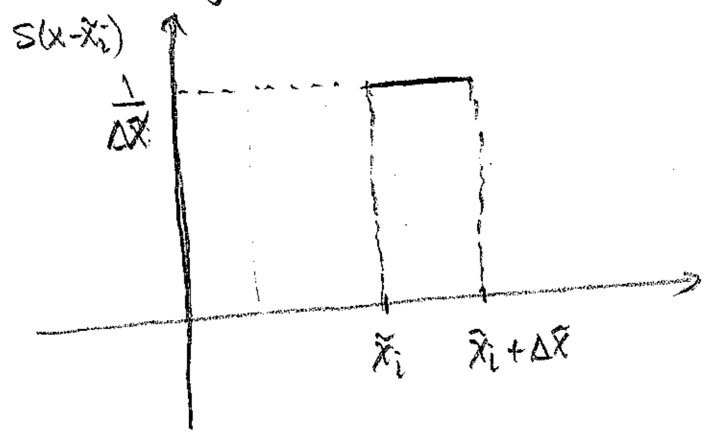
Importantly, $f(\hat{x}_i)$ is independent of x, and thus:

$$\frac{\mathcal{L}_x\,[y_p(x,\hat{x}_i)]}{f(\hat{x}_i)} = S(x-\hat{x}_i)$$

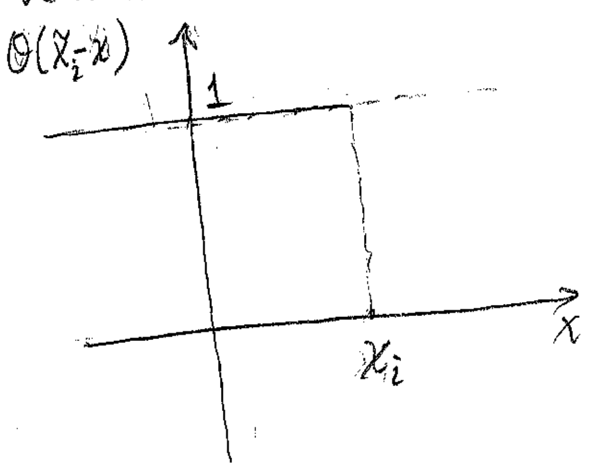
$\Downarrow$

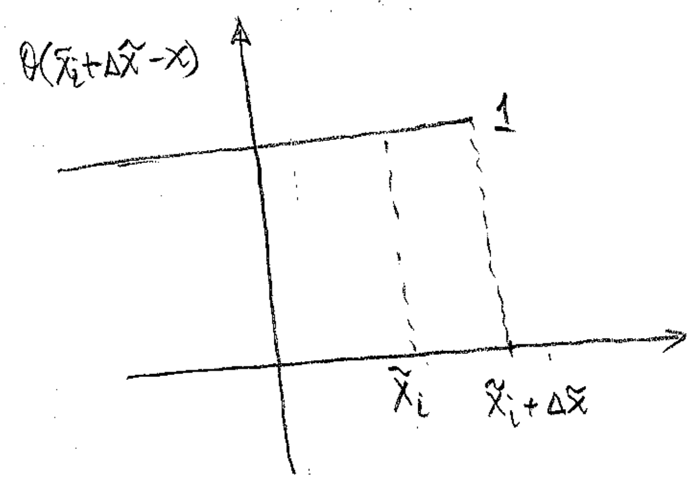
$$\mathcal{L}_x\left[\frac{y_p(x,\hat{x}_i)}{f(\hat{x}_i)}\right] = S(x-\hat{x}_i)$$

Now, if we graph $S(x-\hat{x}_i)$,



While:





Then:

$$S(x-\tilde{x}_i) = \frac{\theta(\tilde{x}_i+\Delta\tilde{x}-x) - \theta(\hat{x}_i - x)}{\Delta\tilde{x}}$$

This approximation becomes more accurate as $\Delta\hat{x} \mapsto 0$. In this limit,

$$\lim_{\Delta\hat{x}\mapsto 0} \frac{\theta(\tilde{x}_i+\Delta\tilde{x}-x)-\theta(\tilde{x}_i-x)}{\Delta\tilde{x}} = \frac{d\theta}{d\tilde{x}} = \delta(\hat{x}-x) = \delta(x-\hat{x}_i)$$

Then:

$$f(x) = \int_{-\infty}^{\infty} f(\hat{x})\,\delta(x-\hat{x})\,d\hat{x},$$

and:

$$\mathcal{L}_x\left[\frac{y_p(x,\hat{x})}{f(\hat{x})}\right] = \delta(x-\hat{x})$$

$$G(x,\hat{x}) \equiv \frac{y_p(x,\hat{x})}{f(\hat{x})}$$ so that

$$\mathcal{L}_x\left[G(x,\hat{x})\right] = \delta(x-\hat{x}).$$

and:

$$\boxed{y(x) = y_h(x) + \int_{-\infty}^{\infty} G(x,\hat{x})\,f(\hat{x})\,d\hat{x}}$$