Ordinary Differential Equations

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The need to solve ordinary differential equations is pervasive in all fields of physics ranging from Newtonian dynamics

$$m\frac{d^2\vec{r}}{dt^2} = \sum \vec{F}\left(\vec{r}, \frac{d\vec{r}}{dt}, t\right)$$

to electrodynamics:

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{V}{R}$$

to thermodynamics:

$$dE = TdS - pdV$$

to quantum mechanics:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi.$$

For the next few weeks we will focus on the solution of ordinary differential equations.

$$F\left(y, \frac{dy}{dx}, \ldots, \frac{d^m y}{dx^{(m)}}, x\right) = 0$$

in which we wish to determine y as a function of a single variable x. This cannot be done in general, and in fact there isn't even a methodology to solving these equations that will for all ordinary differential equations. With the exception of a few special cases, we are left with solving each on a case-by-case basis.

We start with terminology.

The order of the differential equation is the highest order derivative of y that appears in F.

The differential equation is linear if F is a linear function of $y, \frac{dy}{dx}, \dots \frac{d^{(n)}y}{dx^{(n)}}$; otherwise, it is nonlinear.

The differential equation is homogeneous if $F(0,\dots,0,x)=0$; otherwise, it is inhomogeneous.

First order ODEs

First order differential equations have the form,

$$\frac{dy}{dx} = M(y, x).$$

It is not hard to think of a 1st order ODE that cannot be solved analytically; such as:

$$\frac{dy}{dx} = \sin(xy^2).$$

If, however, $M(y, x)$ is separable into a product of two functions:

$$M(y, x) = Y(y)\,X(x),$$

then:

$$\frac{dy}{dx} = Y(y)\,X(x),$$

so that

$$\frac{1}{Y}\,dy = X(x)\,dx$$

Then:

$$\int_{y_0}^{y} \frac{d\tilde{y}}{Y(\tilde{y})} = \int_{x_0}^{x} X(\tilde{x})\,d\tilde{x}.$$

and we can see that a unique solution of y is obtained if the boundary condition:

$$y(x_0) = y_0$$

is given. What is left is to do the integral – often not possible in closed form – and then solve for y in terms of x, which is also not necessarily possible.

Ex:

$$\frac{dy}{dx} = y^{\frac{1}{2}(1-p)} \sqrt{1-y^{1+p}}$$

with $y(0)=0$. Then:

$$\frac{y^{\frac{p-1}{2}}\,dy}{\sqrt{1-y^{1+p}}} = dx$$

$$x = \int_0^y \frac{s^{\frac{1}{2}(p-1)}\,ds}{\sqrt{1-s^{p+1}}},$$

Let:
$$\sin u = s^{\frac{1}{2}(p+1)}$$
$$\cos u\,du = \tfrac{1}{2}(p+1)\, s^{\frac{1}{2}(p-1)}\,ds$$

$$x = \frac{2}{p+1}\int \frac{\cos u\,du}{\sqrt{1-\sin^2 u}} = \frac{2}{p+1}\,u$$

$$\tfrac{1}{2}(p+1)x = u \Rightarrow \sin\left[\tfrac{1}{2}(p+1)x\right] = \sin u$$

$$\sin\left[\tfrac{1}{2}(p+1)x\right] = y^{\frac{1}{2}(p+1)} \Rightarrow \boxed{y = \left[\sin\left[\tfrac{1}{2}(p+1)x\right]\right]^{\frac{2}{p+1}}}$$

Ex:

Energy conservation:

$$E = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + V(x)$$

$$\Rightarrow \quad \frac{dx}{dt} = \pm \sqrt{\frac{2}{m}(E - V(x))}$$

$$\boxed{t - t_0 = \pm \int_{x_0}^{x} \frac{d\tilde{x}}{\sqrt{\frac{2}{m}(E - V(\tilde{x}))}}}$$

$$V(x) = \frac{1}{2} m\omega^2 x^2 \qquad x(0) = 0$$

$$t = \pm \int_0^x \frac{d\tilde{x}}{\sqrt{\frac{2E}{m} - \omega^2 \tilde{x}^2}} = \pm \sqrt{\frac{m}{2E}} \int_0^x \frac{d\tilde{x}}{\sqrt{1 - \frac{m\omega^2}{2E}\tilde{x}^2}}$$

$$u = \sqrt{\frac{m}{2E}}\, \omega \tilde{x} \qquad du = \sqrt{\frac{m}{2E}}\, \omega\, d\tilde{x}$$

$$t = \pm \sqrt{\frac{m}{2E}} \sqrt{\frac{2E}{m}} \frac{1}{\omega} \int_0^{\sqrt{\frac{m}{2E}}\omega x} \frac{du}{\sqrt{1 - u^2}}$$

$$\pm \omega t = \sin^{-1} u \Big|_0^{\sqrt{\frac{m}{2E}}\omega x} = \sin^{-1}\left(\sqrt{\frac{m}{2E}}\, \omega x\right)$$

$$x = \pm \sqrt{\frac{2E}{m}} \frac{1}{\omega} \sin(\omega t)$$