This class will be about giving examples to the theorems and proofs that were proved last class in Verification 8. i.e. Theorem 1.

Hintikka Formulas

The Hintikka Formula ϕ_S^n is parametrized by the number of rounds one has to survive the game and S the structure over which we want to prove the n-equivalence.

We showed the Hintikka Formula in a recursive manner with the base case i.e. n=0 meaning no quantifiers are allowed:

base case n=0:

$$\phi_S^{n=0} = \bigwedge_{\substack{\alpha \text{ are atoms} \\ \text{over appr. signature} \\ S \models \alpha}} \alpha \vee \bigwedge_{\substack{\alpha \text{ are atoms} \\ S \not\models \alpha}} \neg\alpha$$

Example:

Definintion structure:
$S = (\underbrace{1,2,3,4}_{\text{Universe } U^S}, \underbrace{<}_{ordering}, \underbrace{A}_{\{0\}}, \underbrace{B}_{\{1,2,3\}}$

<,A,B are relationship symbols

This is a respresentation of a finite string containing the two letters A and B.
S in particular is the string "a,b,b,b" as A = {0} i.e. a is at the zeroth postion and the rest of the string are b's because of B ={1,2,3}

Question: What is the Hintikka Formula for this structure?
Lets start with the Hintikka Formula level 0 i.e. ϕ_S^0
This Structure does not allow any interpretation of free variables therefore we are not allowed to use <,A,B as they are requiring variables.
Therefore we are not able to write anything besides:
true: the atom that does not use any variables
false: the negated atom that are not true in S

i.e. : $\phi_S^0 = \text{true} \vee \neg\text{false}$
Now the Hintikka Formula level 1 i.e. ϕ_S^1:

But lets first go more general how is the Hintikka Formula on level n defined:

$$\phi_S^n = \bigwedge_{u \in U^S} \exists x \phi_{S[x=u]}^{n-1} \vee \forall x \bigvee_{u \in U^S} \phi_{S[x=u]}^{n-1}$$

$u \in U^S$: all objects of my universe

What is $\phi_{S[x=0]}^0$?
$\phi_{S[x=0]}^0 = \text{true} \wedge (x = x) \wedge A(x) \wedge \neg\text{false} \wedge \neg(x < x) \wedge \neg B(x)$

This are all formulas that are possible with the interpretation of $x = 0$

How does it look when we have $x=1$?
$\phi^0_{S[x=1]} = \text{true} \wedge (x=x) \wedge B(x) \wedge \neg\text{false} \wedge \neg(x<x) \wedge \neg A(x)$

note: $A(x)$ turns to $B(x)$ in $x=1$ as the second position of $'abbb'$ is b instead of a where $x=0$

Question: how does the Hintikka Formula for $x=2$ look like? Answer it looks the same as $x=1$ and $x=3$ because the positions from 1 till 3 are all b ($'abbb'$)

So summarizing how does the Hintikka Formula look like?

$$\phi^0_S = \exists x \phi^0_{S[x=0]} \wedge \exists x \phi^0_{S[x=1]} \wedge \exists x \phi^0_{S[x=2]} \wedge \exists x \phi^0_{S[x=3]}$$

Now that we have all Hintikka Formulas for each element of our universe we go the next step creating the Hintikka Formula on quantifier level 1 i.e. $\phi^1_{S[x=1]}$.

$$\phi^1_S = (\exists x \phi^0_{S[x=0]}) \wedge (\exists x \phi^0_{S[x=1]}) \ldots \wedge \ldots \wedge \forall x (\phi^0_{S[x=0]} \vee \phi^0_{S[x=1]} \ldots \vee \ldots)$$

When writing the statement out what does the Hintikka Formula mean?
There is x labeled a ($A(x) = true$), there is x labeled by b ($B(x) = true$), and for every x, x is either labeled by a or by b ($\forall x A(x) \vee B(x)$)

> **Note**
>
> take $S' = (\{0,1\}, <, \underbrace{A}_{\{1\}}, \underbrace{B}_{\{0\}})$="ba".
> This structure is actually different from the structure before but when the Hintikka Formulas of level one of S and S' are logically equivalent i.e. $\phi^1_S \equiv \phi^1_{S'}$. Note: syntactically the formulas are not the same but they are still logically equivalent

Now lets look at all Hintikka Formulas of string structures with 'a' and 'b'. i.e.

$string\_structures = \{\epsilon, 'a', 'b', 'aa', 'bb', 'ab', 'ba. \ldots\}$

We know that 'ba' and 'abbb' have the same Hintikka Formula. Might it be that we can put the strings into classes where all strings have the same Hintikka Formula? Yes we can.

1. All strings where one position is labeled by 'a' and one position labeled by 'b' e.g. 'abbb'
2. all positions are only labeled 'a'
3. all positions are only labeled 'b'
4. ϵ i.e. empty string

There is only a limited number of partitions possible when having a low level of the Hintikka Formula the higher the level the more classes can we create.

Lets now look at the Hintikka Formula n=2...

$$S = 'abbb'$$

The definintion of the Hintikka Formula n=2 is:

$$\phi_S^2 = \bigwedge_{u \in U^S} \exists x \quad \phi^1_{S[x=u]} \wedge \forall x \quad \bigvee_{u \in U^S} \phi^1_{S[x=u]}$$

We note that ϕ^1_S which is the Hintikka Formula of level 1 is different from $\phi^1_{S[x=u]}$ as it is an expanded structure i.e. our variables are assigned a value from the universe U.

Lets look what the individual $\phi^1_{S[x=u]}$ are?

- $\phi^1_{S[x=0]}$

> $\phi^1_{S[x=0]} = \bigwedge_{v \in U^S} \exists y \phi^0_{S[x=0][y=v]} \quad \wedge \quad \forall y \bigvee_{v \in U^S} \phi^0_{S[x=0][y=v]}$
> How does it look when $[y = 2]$
>
> $$\phi^0_{S[x=0][y=2]} = \text{true} \wedge (x = x) \wedge (y = y) \wedge (x < y) \wedge A(x) \wedge B(y) \wedge \ldots$$
> $$\ldots \wedge \neg \text{false} \wedge \neg(x < x) \wedge \neg(y < y) \wedge \neg A(y) \wedge \neg B(x)$$
>
> What are the important things in this formula stating something about the structure? Its this three things: $A(x), B(y)$ and $(x < y)$ the rest is obvious.
> How does it look when $\phi^0_{S[x=0][y=0]}$
>
> $$\phi^0_{S[x=0][y=0]} = \text{true} \wedge (x = x) \wedge (y = y) \wedge (x = y) \wedge A(x) \wedge A(y) \wedge \ldots$$
> $$\ldots \wedge \neg \text{false} \wedge \neg(x < x) \wedge \neg(y < y) \wedge \neg(y < x) \wedge \neg B(x) \wedge \neg B(y)$$

How does the complete formula look like?

$$\phi_S^2 = (\exists x \exists y (x < y) \wedge A(x) \wedge B(y) \wedge \ldots) \wedge$$
$$.. \wedge (\exists x \exists y (x < y) \wedge B(x) \wedge A(y) \wedge \ldots) \wedge \ldots$$

Now that we can assume how the higher order Hintikka Formulas look like lets look again at the individual classes of Hintikka Formulas.
Are the Hintikka Formulas for $S = \text{'abbb'}$ and $S = \text{'abbbbb'}$ logically eqivalent? Yes they are.
How about $S = \text{'abbb'}$? They are different as we are missing the part of the Hintikka Formula where both elements are 'b' i.e $\phi^2_S = \ldots \exists x \exists y \quad (x < y) \wedge B(x) \wedge B(y) \ldots$

The higher the level is the more different strings can we distinguish in the Hintikka Formulas.

While on order 0 we could only we only had three classes $\{\epsilon, a, b\}$ at order one we could already distinguish more classes i.e. ${

Lets Recall:
Property P containing is not definable in FO if for all n $\exists S, S'$ while S, S' are n-equivalent while $S \in P$ and $S \notin P$. Lemma 10.

We can rewrite the property S, S' are n-equivalent by saying their Hintikka Formula are logically equivalent i.e $\phi^n_S \equiv \phi^n_{S'}$

A third way to describe S, S' are n-equivalent is by stating Theorem 1 S and S' are n-equivalent when the Duplicator survives n-rounds in the Ehrenfeucht-Fraise game $G_{S,S'}$

Break

Automata theory

Book for reference: Automata theory, Languages and computation 3rd edition Hopcroft, motwani, Udine

Finite State automata (over finite words) and regular expressions

We have:
Symbols: e.g. a,b,c... or 0,1,2...
Words: e.g. abbca
variables used for words are w, u, v e.g. $u = abbca$
length of a word: e.g $|a|$= number of Symbols of word
e.g. $u = abbca$ i.e. $|u| = 5$

A special word ϵ is the empty word $\epsilon =''''$ and $|\epsilon| = 0$

Basic operation Concatenation denoted by $(\cdot)$
given two words u, w the concatenation of u and w is denoted by $u \cdot w$ important: $u \cdot w \neq w \cdot u$.
Sometimes we can denote a concatenation as $uw = u \cdot w$

Sometimes we want to concatenate two sets of words A and B where the Concatenation is
$A \cdot B = \{u \cdot w : u \in A \wedge w \in B\}$

special cases
$w \cdot \epsilon = w$
$A \cdot \{\epsilon\} = A$
but
$A \cdot \{\emptyset\} = \emptyset$
The difference is that ϵ is an empty word $\emptyset$ is an empty set

iterative concatination
For all $n \geq 0$ we define, $A^n = A \cdot A \cdot \ldots$ for n times.
Notable mentions:

$$A^0 = \{\epsilon\}$$

$$A^{n+1} = A \cdot A^n$$

Kleene-closure

Kleene-closure of a set A denoted by A^*, is defined as:

$$A^* = \bigcup_{u \geq 0} A^n = \{\epsilon\} \cup A \cup A \cdot A \cup A \cdot A \cdot A \cup \ldots$$

i.e. all finite words that can be obtained by Concatenation of A with itself.
In all important cases the result of a Kleene-closure is a infinite set but consist of finite words.

Example:
$A = \{a\}$
$A^* = \{\epsilon\} \cup \{a\} \cup \{aa\} \ldots$

What is A^+
$A^+ = A^* - \{\epsilon\}$
Example:

$A = \{a\}$
$A^+ = \{a\} \cup \{aa\} \dots$

What is a Language?

Given an Alphabet A (Finite set of Symbols) a Word over A is just an element of A^*

A Language L over A, is a subset of $A^*(L \leq A^*)$ i.e. a language is a subset of words of A^*

Automata

Automata stands for Deterministic Finite State Automata

Definintion: Deterministic Finite State Automata

A DFA \mathcal{A} is a tuple (Q, A, δ, q_0, F) where

- Q is a finite set of states
- A is a finite set of Symbols i.e. Alphabet
- δ is a Transition function $\delta : Q \times A \to Q$
- q_0 is the initial state and is $q_0 \in Q$
- F is a subset of Q i.e $F \subseteq Q$ containing all final states

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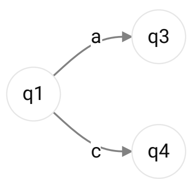
How do we paint a Automata?
example graph:



What means Deterministic?
Deterministic means that if you have a state $s1$ and Symbol a. Then you can only reach one other state $s2$ when the Automata is in state $s1$ and gets the input a.

We can generalize δ the transition function from Symbols to Words
We have:

$$\hat{\delta} : Q \times A^* \to Q$$

is defined as follows:

What happens if we concatenate the empty word ϵ to q?

- $\hat{\delta}(q, \epsilon) = q$

How does the state change if we apply a word and then concatenate a symbol a to the start?e.g. wa

- $\hat{\delta}(q, wa) = \hat{\delta}(\delta(q, w), a)$

Acceptance condition: Defines how a word w needs to be formed so that a Automata $\mathcal{A}$ (starting from a initial state q_0) after processing ends in a state which is part of the set of final states F.
i.e $\hat{\delta}(q_0, w) \in F$

The language $L(\mathcal{A})$ accepted by $\mathcal{A}$ consists of all and only those words in $A*$ which are accepted by the Automata (as of the definition of the Acceptance condition) that means the language will be a subset of $A*$ i.e. $L(\mathcal{A}) \subseteq A^*$

Regular Languages

A language $L \subseteq A^*$ is Regular if it is accepted by a DFA $\mathcal{A}$ that is $L = L(\mathcal{A})$

The difference between δ and $\hat{\delta}$ is that one only has symbols (δ)as inputs the other has words ($\hat{\delta}$) as inputs. From now on we will use δ for both automata with words and Automata that have symbols as input indiscriminantely.

Non Deterministic Finite State Atomata (NFA)

Definition:
An NFA $\mathcal{A}$ is a tuple (Q, A, Δ, q_0, F) where Q, A, q_0 and F are defined as in DFA except for the transition function δ where:

$$\delta : Q \times A \rightarrow 2^Q$$

That means that from one state q one can reach more than one state when it gets a symbol as input. Depending on a random variable for example)

Notationally a δ of a NFA can be also defined as $\delta \subseteq Q \times A \times Q$ We denote a the transition function of a NFA as capital delta i.e. Δ.

Summary

- When we use δ we are in the Deterministic world
- When we use Δ we are in the non-Deterministic world.

As before we generalize δ from symbols to words.
One problem that we face is that after applying one word the state can differ i.e. the state when we apply the second symbol might be multiple symbols. Therfore we express the current state now as a set of possible states.

$\hat{\delta} = \{q\}$
For all words w and all symbols a
$\hat{\delta}(q, wa) = \bigcup_{p \in \delta(\hat{q}, w)} \delta(p, a)]$

This means that we need to create a set out of all possible outcomes of the NFA when applying the word w to it. Then we apply to each state of the machine in this set the input a to get our resulting set of states.

The Non-Deterministic Acceptance condition

A word w is accepted by a NFA $\mathcal{A}$ if (and only if) $\hat{\delta}(q_0, w) \bigcap F \neq \emptyset$.

This means that when a word is applied to an Automata there needs to be a chance for it to finish in a state which is part of the final states. i.e. one of the states in the set of all states that the Automata can be in after applying a word needs to be in the set of final states F.

The Language L accepted by an NFA $\mathcal{A}$ consists of all words (and only those) Words over the alphabet A accepted by $\mathcal{A}$.

As before we will use δ for $\hat{\delta}$ indiscriminantely.

Theorem 2:
For all NFA $\mathcal{A}$ there exists a DFA $\mathcal{A}'$ such that $L(A) = L(A')$ and vice versa.

Proof: NFA $\leftarrow$ DFA is trivial as a DFA is a NFA with all probabilities being 1.
NFA $\rightarrow$ DFA
Can be proved by the subset construction and is proven in Verification 10

generalization Delta can be in a set of states not only in one state like in DFA
$\delta : 2^Q \times A^* \rightarrow 2^Q$
$\delta(\underbrace{\underline{P}}_{\text{set of states}}, w) = \bigcup_{p \in \underline{P}} \delta(p.w)$