DFA and NFA
Are the non Deterministic versions of Automata.

We also stated Theorem 2 2 which states that DFA≡ NFA.

We have not proofed it but now follows the proof!

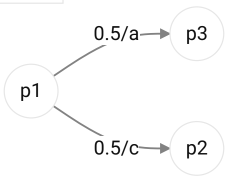
## Proof Theorem 2

First we proof DFA $\leftarrow$ NFA by showing that NFAs are just a special case of DFAs.
let $\mathcal{A} = \{Q, A, \Delta, q_0, F\}$ be a NFA thats accepts the Language $L$. The corresponding DFA would be the following:
$\mathcal{A} = \{Q', A, \delta', q_0', F'\}$
We have a equivalent alphabet $A$, but different set of states $Q'$, a different Transition function $\delta'$, a different initial state $q_0$ and a different set of end states $F$.

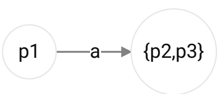
Where $Q' = 2^Q$ which is the Power set of $Q$. This means that $Q'$ consists of all subsets of $Q$ and $Q$ itself.
$q_0' = \{q_0\}$ is a Singleton only consisting of the initial state of $Q$

Now comes the definition of the transition relation $\delta'$:
For all $p \in Q'$ where p is a set of states of $Q$ (see definition of $Q'$) and all symbols $a \in A$:
$\delta'(\underbrace{p}_{Q'}, a) = \Delta(\underbrace{p}_{Q}, a)$
What does this mean?
The state that the DFA can reach from state $p \in Q'$ by reading $a$ is equal to the set of states that the NFA can reach in state $p \in Q$ when reading in $a$.
Probably like this: (own interpretation)

NDA




NFA




Now we have to define the set of finite states of our DFA $F'$
Let's remember that the acceptance condition for a NFA is existential so only one of the possible states to reach needs to be an end state.
$F' = \{ \underbrace{p}_{\text{states of } \mathcal{A}'} \in Q' : \underbrace{p}_{\text{set of states of} \mathcal{A}} \cap F \neq \emptyset\}$
$F'$ contains all sets of $Q'$ that contain a final state of $Q$!

What remains of the proof? exercise

- We proof by induction on the length of the word $w$ that $\delta'(q_0', w) = \Delta(q_0, w)$

> The base case is $|w| = 0$
> For the inductive step we need to proof that the assumption holds for all words with $|w| \leq n$ and proof it for the word $|wa|$ with the length n+1

- We need to also proof that when the computations of the NFA are successful also the computation of the DFA is successful.

Ask Sasha if he knows the proof of this

Now we can define using Theorem 2 that Regular Languages are those languages $L \subset A^*$ that are recognized by the NFA.

This has the advantage that we can switch between the two classes.

For example when we have a Automaton which accepts the language $L$ and we want to find a Automaton that accepts the complemetary language.

> **Note**
>
> title what is the Complement of a Language?
> If we have a language $L \subseteq A^*$ then the Complement of a Language $\overline{L}$ consists of all all the words of the that are not part of $L$ i.e. $\overline{L} = A^* - L$

This is important because it heavily used in model checking.

It is easy to find the Complement of a Language by exploiting Determinism of DFA.

Important: NFA and DFA are not always logically equivalent but when they are it makes our life easier.

### NFA with $\epsilon$-moves

It is very similar to the normal NFA but has $\epsilon$-moves

> **Definition: NFA with epsilon-move**
>
> We have an Automaton $(Q, A, \Delta, q_0, F)$ where $Q, A, q_0$ and $F$ have definitions equla to the normal NFA.
> The only difference is $\Delta$
> $\Delta : Q \times (A \cup \{\epsilon\}) \rightarrow 2^Q$
>
> This means that independent of the state one is in it is always possible to read the empty word $\epsilon$ as input. This is simmilar to a idle move.
>
> How do we define $\hat{\Delta}$ the transition function accepting words?
>
> $\hat{\Delta}(q, \epsilon) = \epsilon\text{-closure}(q)$

Now lets consider:
$q \in Q$ and $p \in 2^Q$
An $\epsilon$-move is a transition of the form $\Delta(q, \epsilon)$.
What is an epsilon-closure:

> **epsilon-closure**
>
> Is the set of states that you can reach within a NFA with epsilon-move from a state $q$ by repeatedly applying only the empty word $\epsilon$.
> This set of states is denoted as $\epsilon$-closure(q)

important: NFAs are not a special case of NFA with epsilon-move without the epsilon move. It $\epsilon$-moves do not increase expressiveness.

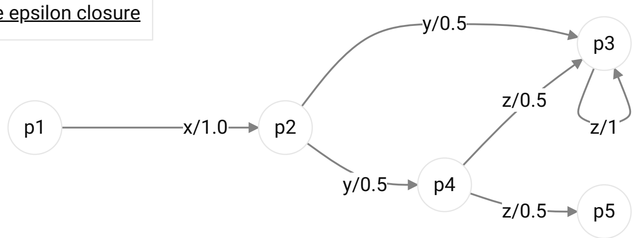
We can not only define the epsilon-closure for single states but also for a set of states:
for all $q \in 2^Q$ we can define the epsilon-closure of the set $(\underline{q})$=$\bigcup_{q \in \underline{q}} \epsilon\text{-closure}(q)$

How do we generalize it to entire words $w$ preceeded by a symbol $a$?
$\hat{\Delta}(q, wa) = \epsilon = \text{closure}(\bigcup_{p \in \hat{\Delta}(q,w)} \Delta(p, a)$

What does this mean? It is the $\epsilon$-closure of all states that one can reach by first applying first the individual symbols of the word $w$ followed by a symbol $a$. After every symbol applied a $\epsilon$-closure needs to be applied. Example:

w=xy
a=z

Example epsilon closure




How does our $\epsilon$-closure($\Delta, wa$) look like?

1. when we apply x the first symbol of the word $w$ we add to our expression:
   $\epsilon\text{-closure}(p_2)$
2. when we read in y we have two states that we can possibly reach ($p_3, p_4$)so we add them both
   $\epsilon\text{-closure}(p_2) \cup \epsilon\text{-closure}(p_3) \cup \epsilon\text{-closure}(p_4)$
3. Now that we process $a = z$ we have two states that we can be in, first: $p3$ which adds itself and from state, and second:$p_4$ we can reach $p_3$ and $p5$.
   $\epsilon\text{-closure}(p_2) \cup \epsilon\text{-closure}(p_3) \cup \epsilon\text{-closure}(p_4) \cup \epsilon\text{-closure}(p3) \cup \epsilon\text{-closure}(p_5) \cup \epsilon\text{-closure}(p3)$
4. Now summarizing we delete the double-mentions and denote from which symbol each $\epsilon$-closure comes from
   $\underbrace{\epsilon\text{-closure}(p_2)}_{x} \cup \underbrace{\epsilon\text{-closure}(p_3)}_{y\&z\&z} \cup \underbrace{\epsilon\text{-closure}(p_4)}_{y} \cup \underbrace{\epsilon\text{-closure}(p_5)}_{z}$

## Theorem 3

> ✎ **Theorem 3**
>
> For all NFA with epsilon-move $\mathcal{A}$ there exists a NFA without $\epsilon$-move $\mathcal{A}'$ such that the languages L are accepted by both Automaton i.e. $L(\mathcal{A}) = L(\mathcal{A}'')$ and vice versa.

To summarize:
DFA$\equiv$NFA$\equiv$NFA with epsilon-move

What is a good illustration of the epsilon move:
Sometimes we have a limited set of symbols (like a limited resource). The epsilon move enables that one can change the Automaton state without using any symbol but just do a $epsilon$-move.

## Regular Expressions

Let $A$ be a finite Alphabet.
We define 3 Classes:

1. Restricted Regular Expressions

   > They are built from a finite set of words over $A$ (made of symbols of $A$) by using the operations:
   >
   > 1. $\cup$ (Union)
   > 2. $\cdot$ (Concatenation)
   > 3. $*$ (Kleene-closure also called Kleene star)

2. General Regular Expressions

   > They are built from a finite set of words over $A$ (made of symbols of $A$) by using the operations:
   >
   > 1. $\cup$ (Union)
   > 2. $\cdot$ (Concatenation)
   > 3. $*$ (Kleene-closure also called Kleene star)
   > 4. $\cap$ (Intersection)
   > 5. $\neg$(Complementation)

3. Star-free Regular Expressions
   Like General Regular Expressions without the Kleene star:

   > 1. $\cup$ (Union)
   > 2. $\cdot$ (Concatenation)
   > 3. $\cap$ (Intersection)
   > 4. $\neg$(Complementation)

Restricted Regular Expressions and General Regular Expressions are equally expressive. They are second order logic.

[Star-free Regular Expressions](#) are strictly less expressive than [Restricted Regular Expression](#)s and [General Regular Expressions](#). The absence of the [Kleene star](#) makes it only first order logic.

---

**break**

## [Theorem 4](#)

> ### [Theorem 4](#)
>
> For every [Finite State Automata](#), there exists a [Restricted Regular Expression](#) that defines the same language and vice versa.

---

Proof:

**First direction [FA](#) $\rightarrow$ [Restricted Regular Expression](#)**

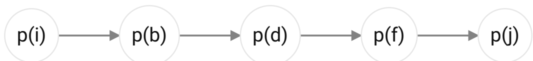
Goal: we want to turn a [DFA](#) to a [Restricted Regular Expression](#).

Let $\mathcal{A} = (\{q_1, q_2 \dots q_n\}, A, \delta, \underbrace{q_1}_{\text{initial state}}, F)$ be a [DFA](#).

And we give the states a ordering relation i.e $q_1 < q_2 < q_3 \dots$.

Let $R^k_{i,j}$ be the set of words $w$ such that $\delta(q_i, w) = q_j$ and for all proper prefix $v$ of $w$ with $v \neq w$ and $v \neq \epsilon$. If one applies the prefix $y$ on any state i.e. $\delta(q, v)$ then the resulting state $q_l$ needs to be less than when we apply the full word on the state i.e. $\delta(q, w) = q_k$ i.e. $q_l \leq q_k$

visual:



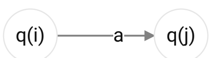

And all this states that we need to step though (inbetween) to get to $p_j$ need to be less or equal $k$ i.e. $(a \leq k) \land (b \leq k) \land (c \leq k) \land (d \leq k) \land (e \leq k) \land (f \leq k)$

The sets $R^k_{i,j}$ can be defined recursively.

- $R^0_{i,j} \rightarrow$ is a way to forbid any intermediate steps as the index of the states starts at

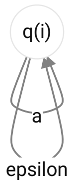
> $$R^0_{i,j} = \begin{cases} \{a : \quad \delta(q_i, a) = q_j & \text{if } i \neq j \\ \{a : \quad \delta(q_i, a) = q_j \cup \{\epsilon\} & \text{if } i = j \end{cases}$$
> How does this look visually.
> In the first case we change state but do not have any states in between.

case i != j



> In the second case we do not change state. That means that we have to cover all symbols that lead back to our initial state as well as the $\epsilon$-move.

case i = j




- Now we define $R_{i,j}^{k}$ by splitting it up in multiple parts

1. The paths from $i$ to $j$ where the intermediate steps are strictly smaller than k i.e. $< k$. This can be expressed as follows:

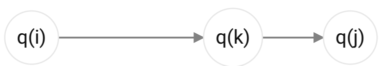
$$R_{i,j}^{k-1}$$

**strictly less than k**


2. The path from $i$ to $j$ where we pass through $q_k$ exactly once. This can be expressed as follows

$$R_{i,k}^{k-1} \cup R_{k,j}^{k-1}$$

**pass through q_k once**




3. And lastly where we pass through $q_k$ multiple times.

> A path from $q_k$ to $q_k$ is part of the class $R_{k,k}^{k-1}$. But we can go from $q_k$ to $q_k$ through different ways again and again passing through other $q_k$. Therefore we create a bigger set of all possible ways to go from $q_k$ to $q_k$ using the [Kleene star]() i.e. $(R_{k,k}^{k-1})^*$
> To explain more: what does $R_{k,k}^{k-1}$ contain?
> It contains all ways from $q_k$ to $q_k$ i.e. $R_{k,k}^{k-1} = \{R_{k,k}^{k-1})(1), R_{k,k}^{k-1}(2), \dots, R_{k,k}^{k-1}(n)\}$
> By using the [Kleene star]() we create a set of set of all combinations of going from $q_k$ to $q_k$ i.e.
> $(R_{k,k}^{k-1})* = \{\{R_{k,k}^{k-1})(1)\}, \{R_{k,k}^{k-1})(1), R_{k,k}^{k-1})(2)\}, \{R_{k,k}^{k-1})(2), R_{k,k}^{k-1})(1)\}\dots\}$

**pass through q_k multiple times**

To summarize

$$R^k_{i,j} = R^{k-1}_{i,j} \cup R^{k-1}_{i,k} \cdot (R^{k-1}_{k,k})^{\star} \cdot R^{k-1}_{k,j}$$

note: We are currently using Restricted Regular Expressions as we only use $\cup, \cdot$ and the Kleene star $\star$

From what we showed we can conclude that for all possible choices for $i, j, k$ there exists a Restricted Regular Expression that defines $R^k_{i,j}$.

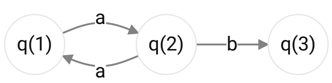
How can we exploit this sets to define a Language $L(\mathcal{A})$?

$$L(\mathcal{A}) = \underbrace{\bigcup_{q_j \in F} R^n_{1,j}}_{\text{Restricted Regular expression defining} L(\mathcal{A})}$$

**Now we prove from FA $\leftarrow$ Restricted Regular Expression**

Goal: we want to turn a Restricted Regular Expression to a NFA with epsilon-move.

Further restriction: We want the resulting Automaton have at most one final state without exiting transition . What is a final state without exiting transition? It is a final state from which one can not leave anymore visually speaking $q_3$ would be such a state.




The proof is by induction on the structural complexity of the Restricted Regular Expression $R$.

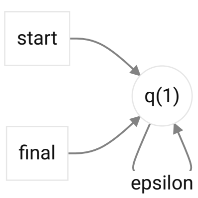
Base cases:

important: we denote the final and the initial state by a box pointing towards the initial state.

$R = \{\epsilon\}, \emptyset, \{a\}$

- How does the Automaton look like for a language $\{\epsilon\}$?

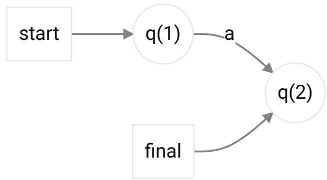
> It has one state, $q_1$, which is also the final state, accepts the empty word, and when it reads the empty word one starts at $q_1$ and ends at $q_1$.




- How does the Automaton look like for the emptyset $\emptyset$?

> It does not have a final state as we never can change state as we can not read in a word or $\epsilon$

- How does the Automaton look like for a Language with a single Symbol i.e. $\{a\}$




**inductive step**

$R$ are Regular Expressions.
$R = R_1 \cup R_2$
By Hypothesis there exists two NFA with epsilon-moves:

- $\mathcal{A}_1 = Q_1, A, \delta_1, q_1, \{f_1\}$
- $\mathcal{A}_2 = Q_2, A, \delta_2, q_2, \{f_2\}$

We assume for that there exist only one final state $f$ for each of the Automata without exiting transition.
We assume that $\mathcal{A}_1$ recognizes the Regular Expression $R_1$, $\mathcal{A}_2$ recognizes Regular Expression $R_2$.
Without the loss of generality we assume that $Q_1 \cap Q_2 = \emptyset$ i.e. the two Automata have different states.

It is easy to show that the following Non Deterministic Finite State Atomata $\mathcal{A}$ recognizes the languages $L(\mathcal{A}_1) \cup L(\mathcal{A}_2)$. $\mathcal{A}$ is the following:

$$\mathcal{A} = \{Q_1 \cup Q_2 \cup \{q_0, f_0\}, A, \Delta, q_0, \{f_0\}\}$$

What does $Q_1 \cup Q_2 \cup \{q_0, f_0\}$ mean?

> We join the states of $Q_1$ and $Q_2$ but we add a new initial state $q_0$ and a new final state $f_0$. Which are also the initial- and final-state of the Automaton $\mathcal{A}$.

Furthermore lets define the Transition function $\Delta$ where:
$\Delta(q_0, \epsilon) = \{q_1, q_2\}$

> What does this mean? Instead of starting at $q_1$ or $q_2$ we use the new state $q_0$ as start state. Then the next state is, by using the magic of NFA, either the previous start state of $\mathcal{A}_1$ $q_1$, or of $\mathcal{A}_2$ $q_2$.

$\Delta(q, a) = \Delta_1(q, a)$

> for $(q \in Q_1 - \{f_1\}) \wedge (a \in A \cup \{\epsilon\})$

$\Delta(q, a) = \Delta_2(q, a)$

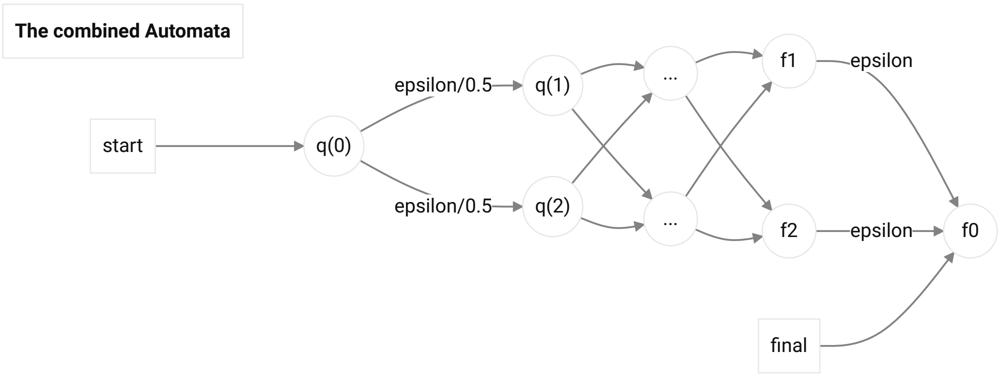
> for $(q \in Q_2 - \{f_2\}) \wedge (a \in A \cup \{\epsilon\})$

> what does this mean? If we are in a state that originally belonged to $\mathcal{A}_1$ then we use the transition function of $\mathcal{A}_1$. If we are in a state that originally belonged to $\mathcal{A}_2$ we use the transition function of $\mathcal{A}_2$. With only one exception: if we are in one of the previous endstates $f_1$ or $f_2$ the now following rule applies.

$\Delta(f1, \epsilon) = \Delta(f_2, \epsilon) = \{f_0\}$

> What does this mean? if we are in one of the previous endstates $f_1$ or $f_2$ and we get an empty word we jump to the new final state $f_0$

How does this visually look?




The next cases
$R = R_1 \cdot R_2$ and $R = R_1^\star$ will be shown in Verification 11