SWARTHMORE

STAT021 Statistical Methods II

Lecture 8 Two-Way ANOVA Model

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Review - One-Way ANOVA Model

One-Way Analysis of Variance Model

The **ANOVA model** for a quantitative response variable and one categorical explanatory variable with K values is

$$\begin{array}{ccccccccc} \text{Data} & = & \text{Grand Mean} & + & \text{Group Effect} & + & \text{Error} \\ Y & = & \mu & + & \alpha_k & + & \epsilon \end{array}$$

where k refers to the specific category of the explanatory variable and $k = 1, 2, \cdots, K$, and $\epsilon \overset{iid}{\sim} N(0, \sigma)$.

The null and alternative hypotheses for the ANOVA model are

$H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_K = 0$;

H_a: at least one $\alpha_k \neq 0$.

Review - One-Way ANOVA Table

The **One-Way ANOVA table** is

	Degree of Freedom	**Sum of Squares**	**Mean Square**	**F statistic**	**P-value**
Model	$K-1$	SSG	MSG	$F = \frac{MSG}{MSE}$	$P(F_{K-1,n-K} > F)$
Error	$n-K$	SSE	MSE		
Total	$n-1$	SST			

If the proper conditions hold, the P-value is calculated using the upper tail of an F distribution with $K-1$ and $n-K$ degrees of freedom.

The fraction of variability explained by the model is measured by

$$R^2 = \frac{SSGroup}{SSTotal} = 1 - \frac{SSE}{SSTotal}$$

Outline

- Example 1: Fruit fly life span
- Two-way ANOVA model
  - Model without interaction: the additive model
  - Model with interaction
  - Interaction plot
- Example 2: Velantine's Day love level
- Example 3: Not all smiles are created equal

Example 1: Fruit fly life span

In the previous lectures, we evaluated the relationship between the life span of male fruit flies (*Longevity*) and the amount of sexual activity (*Treatment*), where *Treatment* is a categorical variable with 5 categories.

- **8virgin**: Each male fruit fly was assigned to live with 8 virgin female fruit flies.
- **1virgin**: Each male fruit fly was assigned to live with 1 virgin female fruit fly.
- **8pregnant**: Each male fruit fly was assigned to live with 8 pregnant fruit flies.
- **1pregnant**: Each male fruit fly was assigned to live with 1 pregnant fruit fly.
- **None**: Each male fruit fly lived alone.

In fact, *Treatment* is a combination of two variables: number of female fruit flies (*Partners*) and whether they are pregnant (*Type*).

Today, we will evaluate the relationship between *Longevity* and two categorical variables *Partners* and *Type*; each has two categories (to make things simple, let's ignore the *None* group).

Example 1: Fruit fly life span

```r
# 8 out of 100 observations from the fruit fly data sample
fly[c(1,2,26,27,51,52,76,77), ]
```

```
##    Partners     Type Longevity
## 1     Eight Pregnant        35
## 2     Eight Pregnant        37
## 26      One Pregnant        46
## 27      One Pregnant        42
## 51      One   Virgin        21
## 52      One   Virgin        40
## 76    Eight   Virgin        16
## 77    Eight   Virgin        19
```

- Response variable: *Longevity*, quantitative
- Explanatory variables:
  - *Partners*, categorical; *One* or *Eight*.
  - *Type*, categorical; *Pregnant* or *Virgin*

- Relationship between *Longevity* and *Partners* (effect of *Partners* on *Longevity*)
- Relationship between *Longevity* and *Type* (effect of *Type* on *Longevity*)
- Effect of *Type* on *Longevity* and how it is affected by *Partners*; or effect of *Partners* on *Longevity* and how it is affected by *Type*.

Example 1: Fruit fly life span

```r
t.test(Longevity ~ Partners, data=fly, var.equal = T)
```

```
##
##  Two Sample t-test
##
## data:  Longevity by Partners
## t = 2.8702, df = 98, p-value = 0.005027
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
##   3.005628 16.474372
## sample estimates:
##   mean in group One mean in group Eight
##               60.78               51.04
```

```r
summary(aov(Longevity ~ Partners, data=fly))
```

```
##             Df Sum Sq Mean Sq F value  Pr(>F)
## Partners     1   2372  2371.7   8.238 0.00503 **
## Residuals   98  28214   287.9
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

$$R^2 = \frac{2372}{2372+28214} = 0.078$$

Example 1: Fruit fly life span

```r
t.test(Longevity ~ Type, data=fly, var.equal = T)
```

```
##
##  Two Sample t-test
##
## data:  Longevity by Type
## t = 5.2304, df = 98, p-value = 9.598e-07
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
##  10.14042 22.53958
## sample estimates:
## mean in group Pregnant   mean in group Virgin
##                  64.08                  47.74
```

```r
summary(aov(Longevity ~ Type, data=fly))
```

```
##             Df Sum Sq Mean Sq F value  Pr(>F)
## Type         1   6675    6675   27.36 9.6e-07 ***
## Residuals   98  23911     244
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

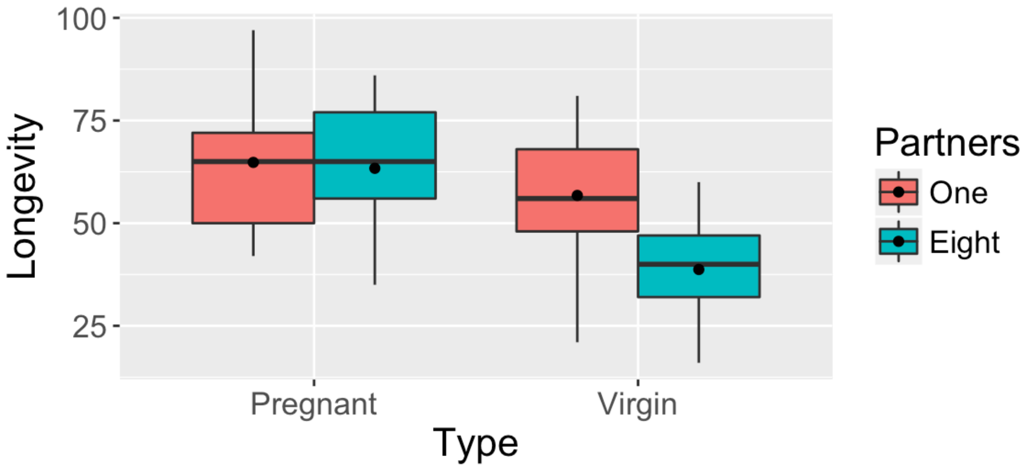
$$R^2 = \frac{6675}{6675+23911} = 0.218$$

ANOVA F test and two-sample t test

- When there are only two categories, the ANOVA F test is equivalent to a **pooled** two-sample t test
  - ANOVA assumes equal variance across groups, so does the **pooled** two-sample t test.
  - $F = t^2$
  - P-values are equal
- t test tells the direction of the relationship. t can be positive or negative; $F = t^2$ is always positive.
- ANOVA model gives the R^2 value.

Example 1: Fruit fly life span

```r
ggplot(fly,aes(x=Type,y=Longevity,fill=Partners))+geom_boxplot()+
  stat_summary(fun.y=mean,geom="point",position=position_dodge(.75))
```



- Effect of *Partners* on *Longevity*?
- Effect of *Type* on *Longevity*?
- Effect of *Partners* on *Longevity* by *Type*?

```r
aggregate(Longevity ~ Partners + Type, data=fly, FUN=mysummary)
```

```
##   Partners     Type Longevity.n Longevity.mean Longevity.sd
## 1      One Pregnant    25.00000       64.80000     15.65248
## 2    Eight Pregnant    25.00000       63.36000     14.53983
## 3      One   Virgin    25.00000       56.76000     14.92838
## 4    Eight   Virgin    25.00000       38.72000     12.10207
```

Two-Way ANOVA Additive Model

The **additive model** for a quantitative response Y based on the effects for two categorical explanatory variables A and B is

$$\begin{array}{ccccccccc} \text{Data} & = & \text{Grand Mean} & + & \text{A Effect} & + & \text{B Effect} & + & \text{Error} \\ Y & = & \mu & + & \alpha_k & + & \beta_j & + & \epsilon \end{array}$$

where $k = 1, 2, \cdots, K, j = 1, 2, \cdots, J, \epsilon \overset{iid}{\sim} N(0, \sigma)$.

The two hypotheses being tested are

A **effect** $H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_K = 0$ versus H_a : at least one $\alpha_k \neq 0$.

B **effect** $H_0 : \beta_1 = \beta_2 = \cdots = \beta_J = 0$ versus H_a : at least one $\beta_j \neq 0$.

Two-Way ANOVA Additive Model

For a data set with two explanatory variables A (K levels) and B (J levels), the ANOVA table for the two-way additive model is

	Degree of Freedom	Sum of Squares	Mean Square	F statistic	P-value
Factor A	$K-1$	SSA	MSA	$F_A = \frac{MSA}{MSE}$	$P(F_{K-1,n-K-J+1} > F_A)$
Factor B	$J-1$	SSB	MSB	$F_B = \frac{MSB}{MSE}$	$P(F_{J-1,n-K-J+1} > F_B)$
Error	$n-K-J+1$	SSE	MSE		
Total	$n-1$	SST			

$$R^2 = \frac{SSA + SSB}{SST} = 1 - \frac{SSE}{SST}$$

Example 1: Fruit fly life span

Two-way ANOVA additive model

```
summary(flymodel1 <- aov(Longevity ~ Partners + Type, data=fly))
```

```
##             Df Sum Sq Mean Sq F value   Pr(>F)    
## Partners     1   2372    2372   10.68   0.0015 ** 
## Type         1   6675    6675   30.06 3.32e-07 ***
## Residuals   97  21540     222                     
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- For *Partners*, $F_{Partners} = MS_{Partners}/MSE = 2372/222 = 10.68$, $P = 0.0015$. We reject the null hypothesis. Male fruit flies with different numbers of partners have significantly different life spans.
- For *Type*, $F_{Type} = MS_{Type}/MSE = 6675/222 = 30.06$, $P = 3.3 \times 10^{-7}$. We reject the null hypothesis. The type of the female fruit flies (prenant or virgin) has significant effect on the life spans of male fruit flies.

Example 1: Fruit fly life span

Two-way ANOVA additive model

```
summary(flymodel1 <- aov(Longevity ~ Partners + Type, data=fly))
```

```
##             Df Sum Sq Mean Sq F value   Pr(>F)    
## Partners     1   2372    2372   10.68   0.0015 ** 
## Type         1   6675    6675   30.06 3.32e-07 ***
## Residuals   97  21540     222                     
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

$$R^2 = 1 - \frac{SSE}{SST} = 1 - \frac{21540}{2372 + 6675 + 21540} = 0.296$$

~30% of the variability in *Longevity* is explained by the model with *Partners* and *Type*.

Example 1: Fruit fly life span

Two-way ANOVA additive model

```
summary(flymodel1 <- aov(Longevity ~ Partners + Type, data=fly))
```

```
##             Df Sum Sq Mean Sq F value   Pr(>F)    
## Partners     1   2372    2372   10.68   0.0015 ** 
## Type         1   6675    6675   30.06 3.32e-07 ***
## Residuals   97  21540     222                     
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- Both *Partners* and *Type* have significant effect on *Longevity*. Whose effect is more significant?
- *Type*. It has larger F statistic and smaller P-value.
- Can we tell whether the effect of *Partners* on *Longevity* is affected by *Type* from the model? Or whether the effect of *Type* on *Longevity* is affected by *Partners*?
- From this model, we can't.

Two-Way ANOVA Model with Interaction

For two categorical factors A and B and a quantitative response Y, the ANOVA model with both main effects and the interaction between the two factors is

$$\begin{array}{ccccccccccc} \text{Data} & = & \text{Grand Mean} & + & \text{A Effect} & + & \text{B Effect} & + & \text{Interaction} & + & \text{Error} \\ Y & = & \mu & + & \alpha_k & + & \beta_j & + & \gamma_{kj} & + & \epsilon \end{array}$$

where $k = 1, 2, \cdots, K, j = 1, 2, \cdots, J, \epsilon \overset{iid}{\sim} N(0, \sigma)$.

The three hypotheses being tested are

Main effect of A $H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_K = 0$ versus H_a : at least one $\alpha_k \neq 0$.

Main effect of B $H_0 : \beta_1 = \beta_2 = \cdots = \beta_J = 0$ versus H_a : at least one $\beta_j \neq 0$.

Interaction effect of A & B $H_0 : \gamma_{11} = \cdots = \gamma_{KJ} = 0$ versus H_a : at least one $\gamma_{kj} \neq 0$.

Two-Way ANOVA Model with Interaction

For a data set with two explanatory variables A (K levels) and B (J levels), the ANOVA table for the two-way ANOVA model with interaction is

	Degree of Freedom	Sum of Squares	Mean Square	F statistic	P-value
Factor A	$K-1$	SSA	MSA	$F_A = \frac{MSA}{MSE}$	$P(F_{K-1,n-KJ} > F_A)$
Factor B	$J-1$	SSB	MSB	$F_B = \frac{MSB}{MSE}$	$P(F_{J-1,n-KJ} > F_B)$
$A \times B$	$(K-1)(J-1)$	$SSAB$	$MSAB$	$F_{AB} = \frac{MSAB}{MSE}$	$P(F_{(K-1)(J-1),n-KJ} > F_{AB})$
Error	$n-KJ$	SSE	MSE		
Total	$n-1$	SST			

$$R^2 = \frac{SSA + SSB + SSAB}{SST} = 1 - \frac{SSE}{SST}$$

Example 1: Fruit fly life span

Two-way ANOVA model with interaction

```
summary(flymodel2 <- aov(Longevity ~ Partners * Type, data=fly))
```

```
##                Df Sum Sq Mean Sq F value  Pr(>F)
## Partners        1   2372    2372  11.489 0.00102 **
## Type            1   6675    6675  32.335 1.4e-07 ***
## Partners:Type   1   1722    1722   8.343 0.00479 **
## Residuals      96  19817     206
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- Both *Partners* and *Type* have significant **main effect** on *Longevity*.
- For the interaction between *Partners* and *Type*, $F_{Interaction} = MS_{Interaction}/MSE = 1722/206 = 8.34$, $P = 0.0048$. We reject the null hypothesis. The effect of *Partners* on *Longevity* is significantly different for different levels of *Type*. Similarly, the effect of *Type* on *Longevity* is significantly affected by *Partners*.

Example 1: Fruit fly life span

Two-way ANOVA model with interaction

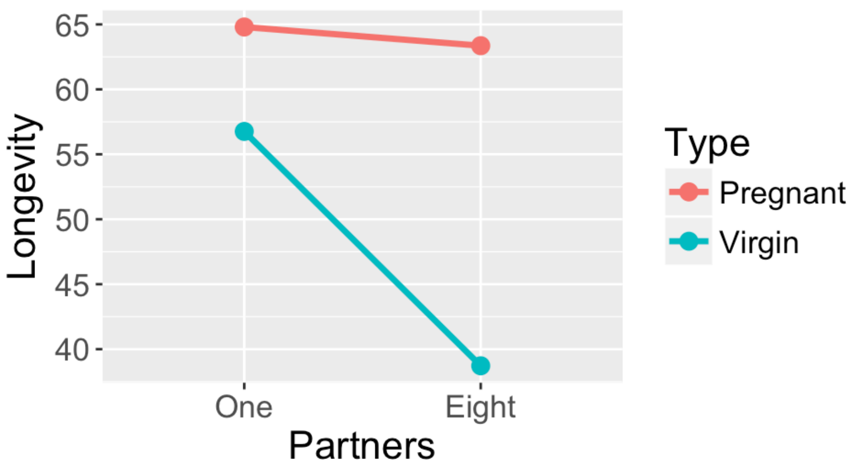
```
summary(flymodel2 <- aov(Longevity ~ Partners * Type, data=fly))
```

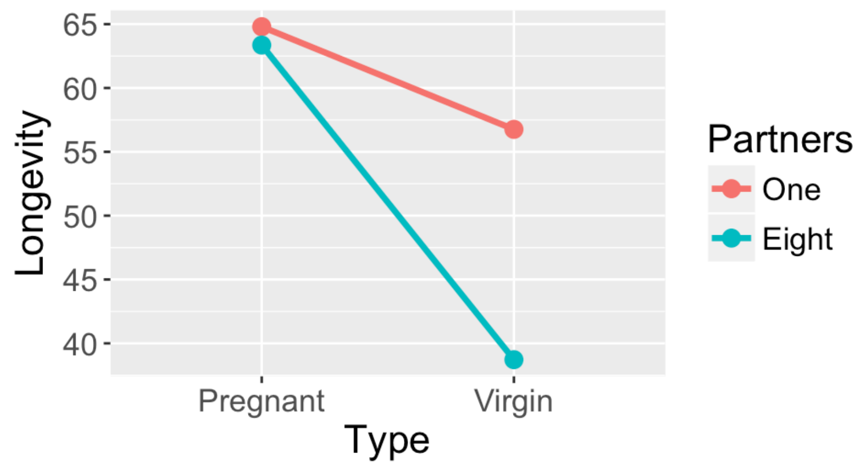
```
##                Df Sum Sq Mean Sq F value  Pr(>F)    
## Partners        1   2372    2372  11.489 0.00102 ** 
## Type            1   6675    6675  32.335 1.4e-07 ***
## Partners:Type   1   1722    1722   8.343 0.00479 ** 
## Residuals      96  19817     206                    
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

$$R^2 = 1 - \frac{SSE}{SST} = 1 - \frac{19817}{2372 + 6675 + 1722 + 19817} = 0.352$$

35.2% of the variability in *Longevity* is explained by the model with *Partners* and *Type* and their interaction.

Interaction plot



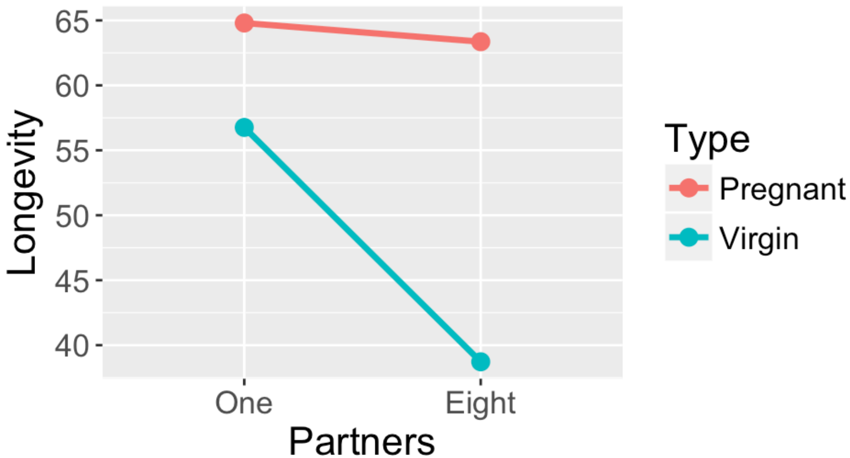


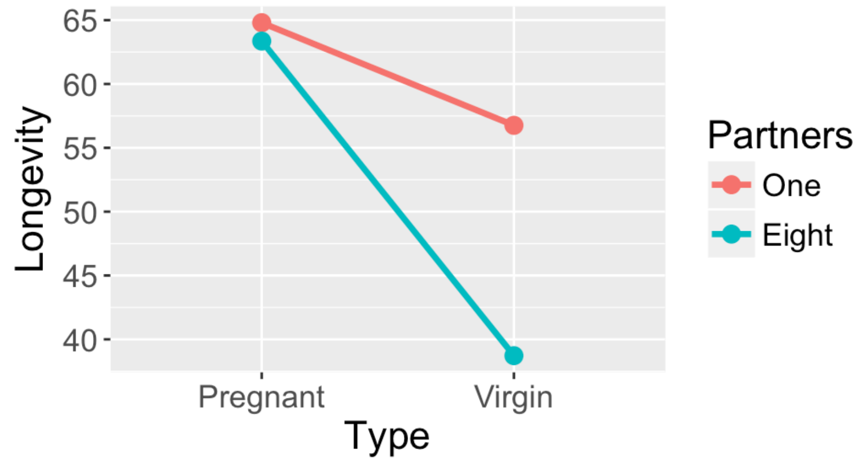
- The interaction plot displays the **mean** *Longevity* of each group.
- Left: effect of *Partners* on *Longevity* for different levels of *Type*

  ∗ When the female fruit flies are pregnant, number of female fruit flies does not affect the life spans of male fruit flies much.

  ∗ When the female fruit flies are virgin, however, more female fruit flies are significantly associated with shorter life spans of male fruit flies.
- How would you comment on the interaction plot on the right?

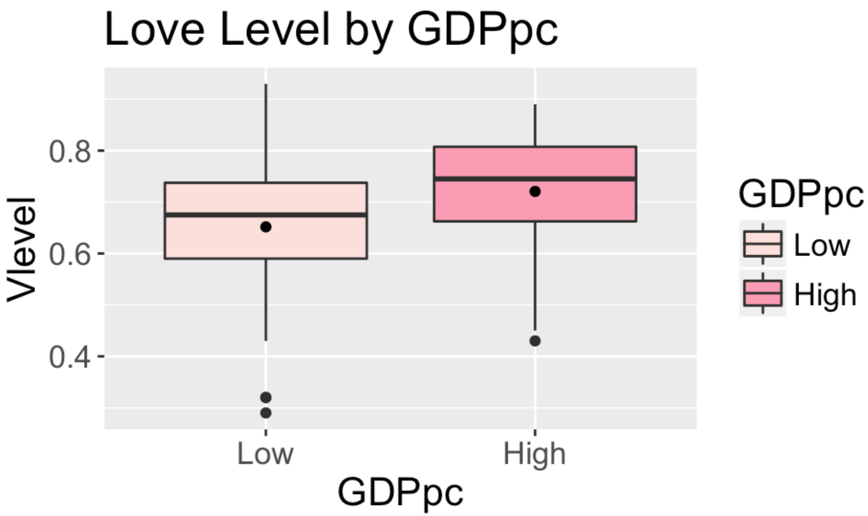
Interaction plot

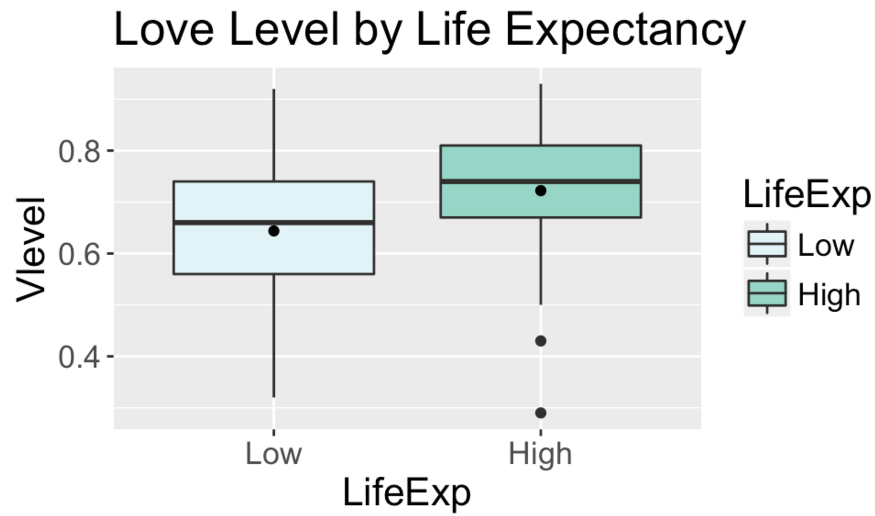




- The interaction in an ANOVA model measures the **difference of differences**.
- How should an interaction plot shows NO interaction between the two explanatory variables look like?
- It should have two or more (if more than two categories) almost **parallel** lines.

Example 2: Valentine's Day love level





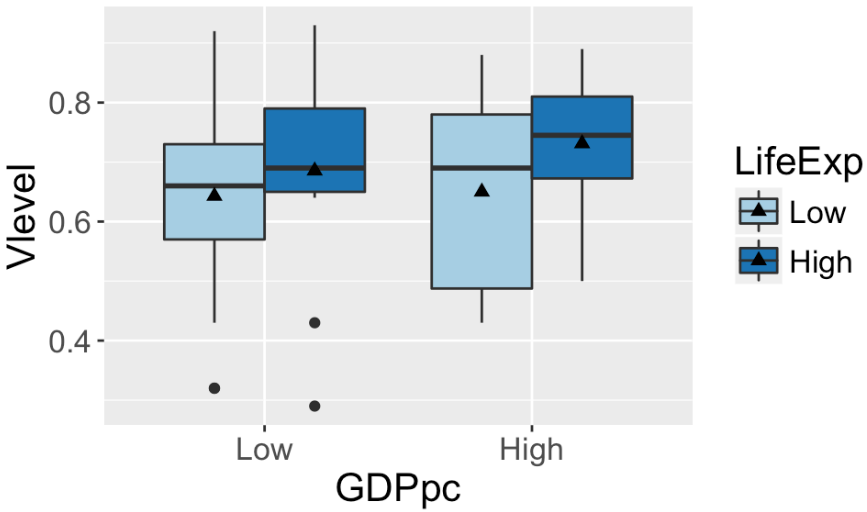

```
##   GDPpc Vlevel.n Vlevel.mean Vlevel.sd
## 1   Low    62.00        0.65      0.14
## 2  High    62.00        0.72      0.11
```

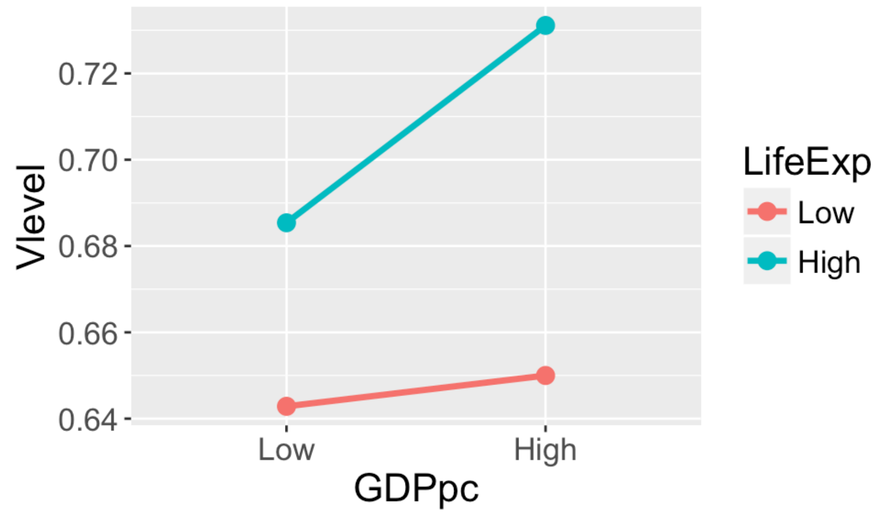
```
##   LifeExp Vlevel.n Vlevel.mean Vlevel.sd
## 1     Low    57.00        0.64      0.14
## 2    High    67.00        0.72      0.12
```

- $Vlevel \sim GDPpc$:
  $F = 4.06, P = 0.0086, R^2 = 0.087$
- $Vlevel \sim LifeExp$:
  $F = 11.95, P = 0.0008, R^2 = 0.089$

$LifeExp = Low$ for life expectancy $<= 70$ and $High$ for life expectancy > 70.

Example 2: Valentine's Day love level






```
##   GDPpc LifeExp Vlevel.n Vlevel.mean Vlevel.sd
## 1   Low     Low   49.000       0.643     0.133
## 2  High     Low    8.000       0.650     0.173
## 3   Low    High   13.000       0.685     0.171
## 4  High    High   54.000       0.731     0.097
```

- Response: *Vlevel*, quantitative
- Explanatory:
  - *GDPpc*: categorical; *Low* or *High*
  - *LifeExp*: categorical; *Low* or *High*

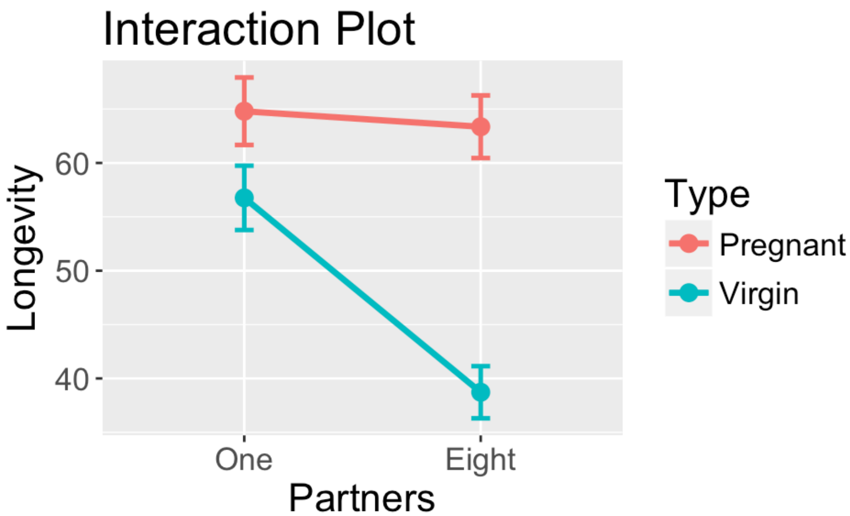
Example 2: Valentine's Day love level

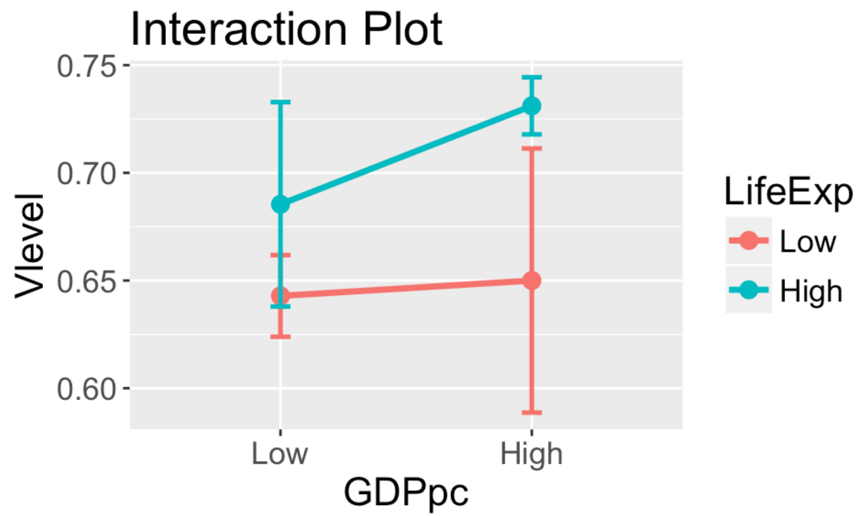
```r
summary(aov(Vlevel ~ GDPpc*LifeExp, data=VD))
```

```
##                Df Sum Sq Mean Sq F value Pr(>F)
## GDPpc           1 0.1470 0.14704   9.243 0.0029 **
## LifeExp         1 0.0582 0.05824   3.661 0.0581 .
## GDPpc:LifeExp   1 0.0062 0.00618   0.389 0.5343
## Residuals     120 1.9091 0.01591
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

- The interaction term is not significant ($P = 0.53 > 0.05$).
- The effect of *GDPpc* on *Vlevel* does not depend on *LifeExp*.
- This interaction plot looks similar to the one of the fruit fly example, why is this one not significant?

Example 1 vs. Example 2





- The error bars are computed by $\bar{y}_{kj} \pm \sqrt{\frac{s_{kj}^2}{n_{kj}}}$
- Whether the interaction term is significant depends on how different the effects are and the variability of the data.

Codes for interaction plot with error bars

```
flysummary <- aggregate(Longevity ~ Partners + Type, data=fly, FUN=mysummary)
flysummary <- data.frame(flysummary, flysummary$Longevity)[-3]
flysummary
```

```
##   Partners     Type  n  mean       sd
## 1      One Pregnant 25 64.80 15.65248
## 2    Eight Pregnant 25 63.36 14.53983
## 3      One   Virgin 25 56.76 14.92838
## 4    Eight   Virgin 25 38.72 12.10207
```

```
ggplot(data=flysummary, aes(x=Partners, y=mean, colour=Type))+
  geom_point(size=3)+ # Add the points for the mean
  geom_line(aes(group=Type), size=1.2)+ # Add the lines
  geom_errorbar(aes(ymax = mean+sd/sqrt(n), ymin = mean-sd/sqrt(n)),
                size=0.9, width=0.08)+ # Add the error bars
  ylab("Longevity") + ggtitle("Interaction Plot") # Set the y-axis label and title
```

Example 3: Not all smiles are created equal

Not all smiles are created equal: Investigating the effects of display authenticity and service relationship on customer tipping behavior

Type: Research paper

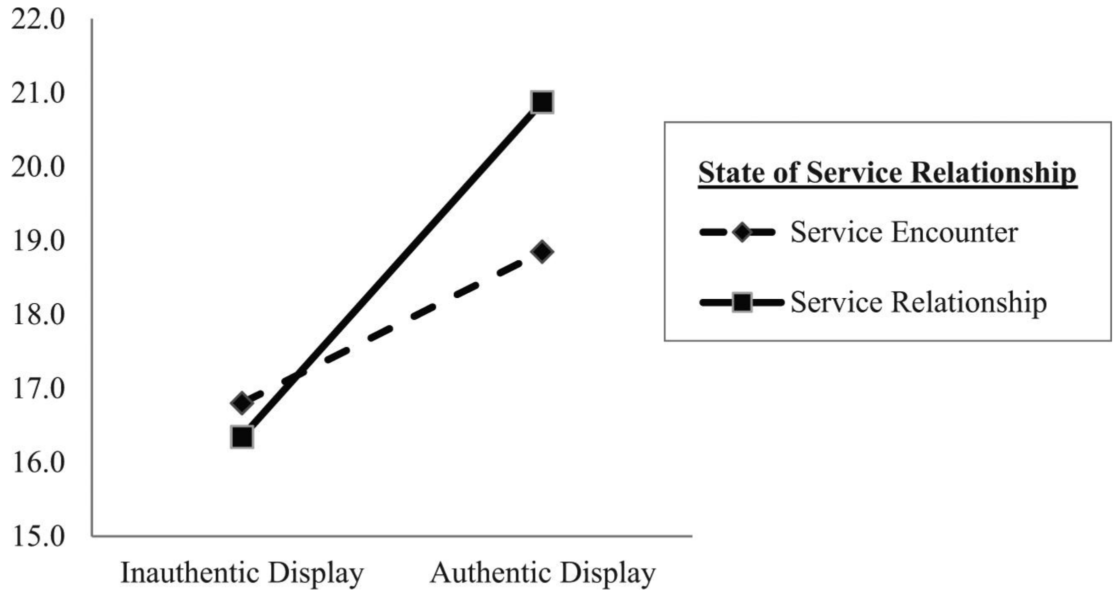
Milos Bujisic , Luorong (Laurie) Wu , Anna Mattila , Anil Bilgihan

International Journal of Contemporary Hospitality Management, Volume: 26 Issue: 2, 2014

- Authenticity: authentic smiles or inauthentic smiles
- Relationship
  - Service relationship: the server and the customer share a history of service interactions, both expect to have repeated contact in the future…
  - Service encounter: one-time deal in which neither party expects to interact with each other in the future.

Example 3: Not all smiles are created equal

Source	Type III SS	Df	MS	*F*	Sig.
Corrected model	844.383	3	281.461	11.813	0.000
Intercept	84244.042	1	84244.042	3535.629	0.000
Authenticity	687.652	1	687.652	28.860	0.000
Relationship	39.004	1	39.004	1.637	0.202
Authenticity * Relationship	97.912	1	97.912	4.109	0.044
Error	5980.621	251	23.827		
Total	90917.299	255			
Corrected total	6825.004	254			



Summary

- Example 1: Fruit fly life span
- Two-way ANOVA model
  - Model without interaction: the additive model $Y = \mu + \alpha_k + \beta_j + \epsilon$
  - Model with interaction $Y = \mu + \alpha_k + \beta_j + \gamma_{kj} + \epsilon$
  - Interaction plot
- Example 2: Valentine's Day love level
- Example 3: Not all smiles are created equal