Math 251 - Exercise Set 2 Zehra's Solutions

March 17, 2024

1 Theorems and Definitions that are used in the solutions

- $p \in \partial S. \iff$ for every $r > 0$, $B(p,r) \cap S \neq \emptyset$ and $B(p,r) \cap (\mathbb{R}^m \setminus S) \neq \emptyset$.

2 Solutions

1. Let $\boldsymbol{A} \subseteq \mathbb{R}^{\boldsymbol{n}}$ and $\boldsymbol{p} \in \mathbb{R}^{\boldsymbol{n}}$. Show that $\boldsymbol{p} \in \bar{\boldsymbol{A}} \Leftrightarrow \boldsymbol{B(p,\varepsilon)} \cap \boldsymbol{A} \neq \emptyset$ for all $\boldsymbol{\varepsilon > 0}$.

2. (a) Show that $\mathbb{R}^{\boldsymbol{n}}$ is an open set by using the definition of open sets.

   Proof:
   Let $\boldsymbol{p} \in \mathbb{R}^{\boldsymbol{n}}$.
   Then for every $\boldsymbol{r > 0}$, $\boldsymbol{B(p,r)} \subseteq \mathbb{R}^{\boldsymbol{n}}$.
   Thus $\mathbb{R}^{\boldsymbol{n}}$ is an open set.

   (b) Show that $\mathbb{R}^{\boldsymbol{n}}$ is a closed subset of $\mathbb{R}^{\boldsymbol{n}}$ by using problem 1.

   Proof:
   Let $\boldsymbol{p} \in \boldsymbol{\partial}\mathbb{R}^{\boldsymbol{n}}$.
   Then for every $\boldsymbol{r > 0}$, $\boldsymbol{B(p,r)} \cap \mathbb{R}^{\boldsymbol{n}} \neq \emptyset$ and $\boldsymbol{B(p,r)} \cap (\mathbb{R}^{\boldsymbol{n}} \setminus \mathbb{R}^{\boldsymbol{n}}) \neq \emptyset$.
   Then $\boldsymbol{B(p,r)} \cap \mathbb{R}^{\boldsymbol{n}} \neq \emptyset$ and $\boldsymbol{B(p,r)} \cap \emptyset \neq \emptyset$.
   Thus $\boldsymbol{p} \in \mathbb{R}^{\boldsymbol{n}}$.
   Thus $\mathbb{R}^{\boldsymbol{n}}$ is a closed subset of $\mathbb{R}^{\boldsymbol{n}}$.

3. Show that the empty set $\emptyset$ is both an open and a closed subset of $\mathbb{R}^{\boldsymbol{n}}$.

   Proof:
   Let $\boldsymbol{p} \in \emptyset$.
   Then for every $\boldsymbol{r > 0}$, $\boldsymbol{B(p,r)} \cap \emptyset = \emptyset$.
   Thus $\emptyset$ is an open subset of $\mathbb{R}^{\boldsymbol{n}}$.
   Let $\boldsymbol{p} \in \boldsymbol{\partial}\emptyset$.
   Then for every $\boldsymbol{r > 0}$, $\boldsymbol{B(p,r)} \cap \emptyset \neq \emptyset$ and $\boldsymbol{B(p,r)} \cap (\mathbb{R}^{\boldsymbol{n}} \setminus \emptyset) \neq \emptyset$.
   Then $\boldsymbol{B(p,r)} \cap \emptyset \neq \emptyset$ and $\boldsymbol{B(p,r)} \cap \mathbb{R}^{\boldsymbol{n}} \neq \emptyset$.
   Thus $\boldsymbol{p} \in \emptyset$.
   Thus $\emptyset$ is a closed subset of $\mathbb{R}^{\boldsymbol{n}}$.
   (**Note:** As we can see from problems 2 and 3, $\emptyset$ and $\mathbb{R}^{\boldsymbol{n}}$ are both open and closed subsets of $\mathbb{R}^{\boldsymbol{n}}$.)

4. (a) Let A and B be subsets of $\mathbb{R}^{\boldsymbol{n}}$. If A and B are closed then show that both $\boldsymbol{A} \cap \boldsymbol{B}$ and $\boldsymbol{A} \cup \boldsymbol{B}$ are closed as well.

   Proof:
   Let $\boldsymbol{p} \in \boldsymbol{\partial A} \cap \boldsymbol{\partial B}$.
   Then for every $\boldsymbol{r > 0}$, $\boldsymbol{B(p,r)} \cap \boldsymbol{A} \neq \emptyset$ and $\boldsymbol{B(p,r)} \cap (\mathbb{R}^{\boldsymbol{n}} \setminus \boldsymbol{A}) \neq \emptyset$ and $\boldsymbol{B(p,r)} \cap \boldsymbol{B} \neq \emptyset$ and $\boldsymbol{B(p,r)} \cap (\mathbb{R}^{\boldsymbol{n}} \setminus \boldsymbol{B}) \neq \emptyset$.
   Then $\boldsymbol{B(p,r)} \cap (\boldsymbol{A} \cap \boldsymbol{B}) \neq \emptyset$ and $\boldsymbol{B(p,r)} \cap (\mathbb{R}^{\boldsymbol{n}} \setminus (\boldsymbol{A} \cap \boldsymbol{B})) \neq \emptyset$.
   Thus $\boldsymbol{p} \in \boldsymbol{\partial}(\boldsymbol{A} \cap \boldsymbol{B})$.
   Thus $\boldsymbol{A} \cap \boldsymbol{B}$ is a closed subset of $\mathbb{R}^{\boldsymbol{n}}$.

(b) If $\{A_i\}_{i \in I}$ is a collection of closed subsets of $\mathbb{R}^n$ then show that $\bigcap_{i \in I} A_i$ is closed, too.

Proof:
Let $p \in \bigcap_{i \in I} \partial A_i$.
Then for every $r > 0$, $B(p, r) \cap A_i \neq \emptyset$ and $B(p, r) \cap (\mathbb{R}^n \setminus A_i) \neq \emptyset$ for all $i \in I$.
Then $B(p, r) \cap \bigcap_{i \in I} A_i \neq \emptyset$ and $B(p, r) \cap (\mathbb{R}^n \setminus \bigcap_{i \in I} A_i) \neq \emptyset$.
Thus $p \in \partial \bigcap_{i \in I} A_i$.
Thus $\bigcap_{i \in I} A_i$ is a closed subset of $\mathbb{R}^n$.

5. Let $A, B \subseteq \mathbb{R}^n$. Prove the following:

(a) int(int(A)) = int(A),

Proof:
Let $p \in \text{int}(\text{int}(A))$.
Then there exists $r > 0$ such that $B(p, r) \subseteq \text{int}(A)$.
Thus $p \in \text{int}(A)$.
Thus $\text{int}(\text{int}(A)) \subseteq \text{int}(A)$.
Let $p \in \text{int}(A)$.
Then there exists $r > 0$ such that $B(p, r) \subseteq A$.
Because $B(p, r) \subseteq A$, $B(p, r) \subseteq \text{int}(A)$.
Thus $p \in \text{int}(\text{int}(A))$.

(b) $\overline{(\overline{A})} = \overline{A}$,
Proof:
Let $p \in \overline{\overline{A}}$.
Then for every $r > 0$, $B(p, r) \cap \overline{A} \neq \emptyset$.
Then for every $r > 0$, $B(p, r) \cap A \neq \emptyset$.
Then $p \in \overline{A}$.
Thus $\overline{\overline{A}} \subseteq \overline{A}$.
Let $p \in \overline{A}$.
Then for every $r > 0$, $B(p, r) \cap A \neq \emptyset$.
Then for every $r > 0$, $B(p, r) \cap \overline{A} \neq \emptyset$.
Then $p \in \overline{\overline{A}}$.
Thus $\overline{A} \subseteq \overline{\overline{A}}$.

(c) $(A \cap B)^\circ = A^\circ \cap B^\circ$,
Proof:
Let $p \in (A \cap B)^\circ$.
Then there exists $r > 0$ such that $B(p, r) \subseteq A \cap B$.
Then there exists $s > 0$ such that $B(p, s) \subseteq A$ and $B(p, s) \subseteq B$.
Then $p \in A^\circ$ and $p \in B^\circ$.
Thus $p \in A^\circ \cap B^\circ$.
Thus $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$.

(d) $R^n = A^\circ \cup \partial A \cup \text{ext}(A)$, where $A^\circ \cap \partial A = \emptyset$, $A^\circ \cap \text{ext}(A) = \emptyset$,$\partial A \cap \text{ext}(A) = \emptyset$,
Proof:
Let $p \in \mathbb{R}^n$.
Then $p \in A^\circ$ or $p \in \partial A$ or $p \in \text{ext}(A)$.
Thus $\mathbb{R}^n = A^\circ \cup \partial A \cup \text{ext}(A)$.
Let $p \in A^\circ \cap \partial A$.
Then there exists $r > 0$ such that $B(p, r) \subseteq A$ and $B(p, r) \cap (\mathbb{R}^n \setminus A) \neq \emptyset$.
Then $p \in \text{ext}(A)$.
Thus $A^\circ \cap \partial A = \emptyset$.

(e) $\overline{A \cup B} = \overline{A} \cup \overline{B}$,
Proof:
Let $p \in \overline{A \cup B}$.
Then for every $r > 0$, $B(p, r) \cap (A \cup B) \neq \emptyset$.
Then for every $r > 0$, $B(p, r) \cap A \neq \emptyset$ or $B(p, r) \cap B \neq \emptyset$.

Then $\boldsymbol{p} \in \overline{\boldsymbol{A}}$ or $\boldsymbol{p} \in \overline{\boldsymbol{B}}$.
Thus $\boldsymbol{p} \in \overline{\boldsymbol{A}} \cup \overline{\boldsymbol{B}}$.
Thus $\overline{\boldsymbol{A} \cup \boldsymbol{B}} \subseteq \overline{\boldsymbol{A}} \cup \overline{\boldsymbol{B}}$.

(f) $\overline{\boldsymbol{A}} = \bigcap\{\boldsymbol{K} \subseteq \mathbb{R}^{\boldsymbol{n}} | \boldsymbol{K} \text{ is closed and } \boldsymbol{A} \subseteq \boldsymbol{K}\}$

Proof:
Let $\boldsymbol{p} \in \overline{\boldsymbol{A}}$.
Then for every $\boldsymbol{r} > \mathbf{0}$, $\boldsymbol{B}(\boldsymbol{p}, \boldsymbol{r}) \cap \boldsymbol{A} \neq \emptyset$.
Let $\boldsymbol{K}$ be a closed subset of $\mathbb{R}^{\boldsymbol{n}}$ such that $\boldsymbol{A} \subseteq \boldsymbol{K}$.
Then for every $\boldsymbol{r} > \mathbf{0}$, $\boldsymbol{B}(\boldsymbol{p}, \boldsymbol{r}) \cap \boldsymbol{K} \neq \emptyset$.
Then $\boldsymbol{p} \in \boldsymbol{K}$.
Thus $\overline{\boldsymbol{A}} \subseteq \boldsymbol{K}$.
Thus $\overline{\boldsymbol{A}} \subseteq \bigcap\{\boldsymbol{K} \subseteq \mathbb{R}^{\boldsymbol{n}} | \boldsymbol{K} \text{ is closed and } \boldsymbol{A} \subseteq \boldsymbol{K}\}$.
Let $\boldsymbol{p} \in \bigcap\{\boldsymbol{K} \subseteq \mathbb{R}^{\boldsymbol{n}} | \boldsymbol{K} \text{ is closed and } \boldsymbol{A} \subseteq \boldsymbol{K}\}$.
Then for every $\boldsymbol{r} > \mathbf{0}$, $\boldsymbol{B}(\boldsymbol{p}, \boldsymbol{r}) \cap \boldsymbol{K} \neq \emptyset$ for all $\boldsymbol{K} \subseteq \mathbb{R}^{\boldsymbol{n}}$ such that $\boldsymbol{K}$ is closed and $\boldsymbol{A} \subseteq \boldsymbol{K}$.
Then for every $\boldsymbol{r} > \mathbf{0}$, $\boldsymbol{B}(\boldsymbol{p}, \boldsymbol{r}) \cap \boldsymbol{A} \neq \emptyset$.
Thus $\boldsymbol{p} \in \overline{\boldsymbol{A}}$.
Thus $\bigcap\{\boldsymbol{K} \subseteq \mathbb{R}^{\boldsymbol{n}} | \boldsymbol{K} \text{ is closed and } \boldsymbol{A} \subseteq \boldsymbol{K}\} \subseteq \overline{\boldsymbol{A}}$.

(g) A is open $\Leftrightarrow$ A $\cap$ $\boldsymbol{\partial}$A $= \emptyset$.

Proof:
$\Rightarrow$ Suppose A is open.
Let $\boldsymbol{p} \in \boldsymbol{A} \cap \boldsymbol{\partial A}$.
Then for every $\boldsymbol{r} > \mathbf{0}$, $\boldsymbol{B}(\boldsymbol{p}, \boldsymbol{r}) \cap \boldsymbol{A} \neq \emptyset$ and $\boldsymbol{B}(\boldsymbol{p}, \boldsymbol{r}) \cap (\mathbb{R}^{\boldsymbol{n}} \setminus \boldsymbol{A}) \neq \emptyset$.
Then $\boldsymbol{p} \in \boldsymbol{\partial A}$.
Thus $\boldsymbol{A} \cap \boldsymbol{\partial A} = \emptyset$.
$\Leftarrow$ Suppose A $\cap$ $\boldsymbol{\partial}$A $= \emptyset$.
Let $\boldsymbol{p} \in \boldsymbol{A}$.
Then for every $\boldsymbol{r} > \mathbf{0}$, $\boldsymbol{B}(\boldsymbol{p}, \boldsymbol{r}) \subseteq \boldsymbol{A}$.
Thus A is open.

6. Let $\boldsymbol{A}, \boldsymbol{B} \subseteq \mathbb{R}^{\boldsymbol{n}}$. Determine if each of the following holds. If it does prove it. Otherwise give a counterexample.

(a) $(\boldsymbol{A} \cup \boldsymbol{B})^\circ \subseteq \boldsymbol{A}^\circ \cup \boldsymbol{B}^\circ$,

Proof:
Let $\boldsymbol{p} \in (\boldsymbol{A} \cup \boldsymbol{B})^\circ$.
Then there exists $\boldsymbol{r} > \mathbf{0}$ such that $\boldsymbol{B}(\boldsymbol{p}, \boldsymbol{r}) \subseteq \boldsymbol{A} \cup \boldsymbol{B}$.
Then there exists $\boldsymbol{s} > \mathbf{0}$ such that $\boldsymbol{B}(\boldsymbol{p}, \boldsymbol{s}) \subseteq \boldsymbol{A}$ or $\boldsymbol{B}(\boldsymbol{p}, \boldsymbol{s}) \subseteq \boldsymbol{B}$.
Then $\boldsymbol{p} \in \boldsymbol{A}^\circ$ or $\boldsymbol{p} \in \boldsymbol{B}^\circ$.
Thus $(\boldsymbol{A} \cup \boldsymbol{B})^\circ \subseteq \boldsymbol{A}^\circ \cup \boldsymbol{B}^\circ$.

(b) $\boldsymbol{A}^\circ \cup \boldsymbol{B}^\circ \subseteq (\boldsymbol{A} \cup \boldsymbol{B})^\circ$,

Proof:
Let $\boldsymbol{p} \in \boldsymbol{A}^\circ \cup \boldsymbol{B}^\circ$.
Then $\boldsymbol{p} \in \boldsymbol{A}^\circ$ or $\boldsymbol{p} \in \boldsymbol{B}^\circ$.
Then there exists $\boldsymbol{r} > \mathbf{0}$ such that $\boldsymbol{B}(\boldsymbol{p}, \boldsymbol{r}) \subseteq \boldsymbol{A}$ or $\boldsymbol{B}(\boldsymbol{p}, \boldsymbol{r}) \subseteq \boldsymbol{B}$.
Then $\boldsymbol{B}(\boldsymbol{p}, \boldsymbol{r}) \subseteq \boldsymbol{A} \cup \boldsymbol{B}$.
Thus $\boldsymbol{A}^\circ \cup \boldsymbol{B}^\circ \subseteq (\boldsymbol{A} \cup \boldsymbol{B})^\circ$.

(c) $\boldsymbol{A} \cap \boldsymbol{B} \subseteq (\boldsymbol{A} \cap \boldsymbol{B})$,

Proof:
Let $\boldsymbol{p} \in \boldsymbol{A} \cap \boldsymbol{B}$.
Then $\boldsymbol{p} \in \boldsymbol{A}$ and $\boldsymbol{p} \in \boldsymbol{B}$.
Thus $\boldsymbol{p} \in \boldsymbol{A} \cap \boldsymbol{B}$.
Thus $\boldsymbol{A} \cap \boldsymbol{B} \subseteq (\boldsymbol{A} \cap \boldsymbol{B})$.

(d) $(\boldsymbol{A} \cap \boldsymbol{B}) \subseteq \boldsymbol{A} \cap \boldsymbol{B}$

Proof:
Let $\boldsymbol{p} \in (\boldsymbol{A} \cap \boldsymbol{B})$.
Then $\boldsymbol{p} \in \boldsymbol{A}$ and $\boldsymbol{p} \in \boldsymbol{B}$.
Thus $\boldsymbol{p} \in \boldsymbol{A} \cap \boldsymbol{B}$.
Thus $(\boldsymbol{A} \cap \boldsymbol{B}) \subseteq \boldsymbol{A} \cap \boldsymbol{B}$.

(e) If $\boldsymbol{A} \subseteq \boldsymbol{B}$ then $\boldsymbol{\partial A} \subseteq \boldsymbol{\partial B}$,

Proof:
$\Rightarrow$ Suppose $\boldsymbol{A} \subseteq \boldsymbol{B}$.
Let $\boldsymbol{p} \in \boldsymbol{\partial A}$.
Then for every $\boldsymbol{r} > \boldsymbol{0}$, $\boldsymbol{B(p,r)} \cap \boldsymbol{A} \neq \emptyset$ and $\boldsymbol{B(p,r)} \cap (\mathbb{R}^{\boldsymbol{n}} \setminus \boldsymbol{A}) \neq \emptyset$.
Then for every $\boldsymbol{r} > \boldsymbol{0}$, $\boldsymbol{B(p,r)} \cap \boldsymbol{B} \neq \emptyset$ and $\boldsymbol{B(p,r)} \cap (\mathbb{R}^{\boldsymbol{n}} \setminus \boldsymbol{B}) \neq \emptyset$.
Then $\boldsymbol{p} \in \boldsymbol{\partial B}$.
Thus $\boldsymbol{\partial A} \subseteq \boldsymbol{\partial B}$.
$\Leftarrow$ Suppose $\boldsymbol{\partial A} \subseteq \boldsymbol{\partial B}$.
Let $\boldsymbol{p} \in \boldsymbol{\partial A}$.
Then for every $\boldsymbol{r} > \boldsymbol{0}$, $\boldsymbol{B(p,r)} \cap \boldsymbol{A} \neq \emptyset$ and $\boldsymbol{B(p,r)} \cap (\mathbb{R}^{\boldsymbol{n}} \setminus \boldsymbol{A}) \neq \emptyset$.
Then for every $\boldsymbol{r} > \boldsymbol{0}$, $\boldsymbol{B(p,r)} \cap \boldsymbol{B} \neq \emptyset$ and $\boldsymbol{B(p,r)} \cap (\mathbb{R}^{\boldsymbol{n}} \setminus \boldsymbol{B}) \neq \emptyset$.
Then $\boldsymbol{p} \in \boldsymbol{\partial B}$.
Thus $\boldsymbol{\partial A} \subseteq \boldsymbol{\partial B}$.

(f) If $\boldsymbol{A} \subseteq \boldsymbol{B}$ then $\mathrm{ext}(\boldsymbol{A}) \subseteq \mathrm{ext}(\boldsymbol{B})$,

Proof:
$\Rightarrow$ Suppose $\boldsymbol{A} \subseteq \boldsymbol{B}$.
Let $\boldsymbol{p} \in \mathrm{ext}(\boldsymbol{A})$.
Then for every $\boldsymbol{r} > \boldsymbol{0}$, $\boldsymbol{B(p,r)} \cap \boldsymbol{A} \neq \emptyset$ and $\boldsymbol{B(p,r)} \cap (\mathbb{R}^{\boldsymbol{n}} \setminus \boldsymbol{A}) \neq \emptyset$.
Then for every $\boldsymbol{r} > \boldsymbol{0}$, $\boldsymbol{B(p,r)} \cap \boldsymbol{B} \neq \emptyset$ and $\boldsymbol{B(p,r)} \cap (\mathbb{R}^{\boldsymbol{n}} \setminus \boldsymbol{B}) \neq \emptyset$.
Then $\boldsymbol{p} \in \mathrm{ext}(\boldsymbol{B})$.
Thus $\mathrm{ext}(\boldsymbol{A}) \subseteq \mathrm{ext}(\boldsymbol{B})$.
$\Leftarrow$ Suppose $\mathrm{ext}(\boldsymbol{A}) \subseteq \mathrm{ext}(\boldsymbol{B})$.
Let $\boldsymbol{p} \in \mathrm{ext}(\boldsymbol{A})$.
Then for every $\boldsymbol{r} > \boldsymbol{0}$, $\boldsymbol{B(p,r)} \cap \boldsymbol{A} \neq \emptyset$ and $\boldsymbol{B(p,r)} \cap (\mathbb{R}^{\boldsymbol{n}} \setminus \boldsymbol{A}) \neq \emptyset$.
Then for every $\boldsymbol{r} > \boldsymbol{0}$, $\boldsymbol{B(p,r)} \cap \boldsymbol{B} \neq \emptyset$ and $\boldsymbol{B(p,r)} \cap (\mathbb{R}^{\boldsymbol{n}} \setminus \boldsymbol{B}) \neq \emptyset$.
Then $\boldsymbol{p} \in \mathrm{ext}(\boldsymbol{B})$.
Thus $\mathrm{ext}(\boldsymbol{A}) \subseteq \mathrm{ext}(\boldsymbol{B})$.

(g) A is closed $\Leftrightarrow$ $\boldsymbol{\partial A} \subseteq \boldsymbol{A}$,

Proof:
$\Rightarrow$ Suppose A is closed.
Let $\boldsymbol{p} \in \boldsymbol{\partial A}$.
Then for every $\boldsymbol{r} > \boldsymbol{0}$, $\boldsymbol{B(p,r)} \cap \boldsymbol{A} \neq \emptyset$ and $\boldsymbol{B(p,r)} \cap (\mathbb{R}^{\boldsymbol{n}} \setminus \boldsymbol{A}) \neq \emptyset$.
Then $\boldsymbol{p} \in \boldsymbol{A}$.
Thus $\boldsymbol{\partial A} \subseteq \boldsymbol{A}$.
$\Leftarrow$ Suppose $\boldsymbol{\partial A} \subseteq \boldsymbol{A}$.
Let $\boldsymbol{p} \in \boldsymbol{\partial A}$.
Then for every $\boldsymbol{r} > \boldsymbol{0}$, $\boldsymbol{B(p,r)} \cap \boldsymbol{A} \neq \emptyset$ and $\boldsymbol{B(p,r)} \cap (\mathbb{R}^{\boldsymbol{n}} \setminus \boldsymbol{A}) \neq \emptyset$.
Then $\boldsymbol{p} \in \boldsymbol{A}$.
Thus $\boldsymbol{\partial A} \subseteq \boldsymbol{A}$.

(h) If $\boldsymbol{A} \subseteq \boldsymbol{B}$ then $\boldsymbol{A'} \subseteq \boldsymbol{B'}$,

Proof:
$\Rightarrow$ Suppose $\boldsymbol{A} \subseteq \boldsymbol{B}$.
Let $\boldsymbol{p} \in \boldsymbol{A'}$.
Then for every $\boldsymbol{r} > \boldsymbol{0}$, $\boldsymbol{B(p,r)} \cap \boldsymbol{A} \setminus \boldsymbol{\{p\}} \neq \emptyset$.
Then for every $\boldsymbol{r} > \boldsymbol{0}$, $\boldsymbol{B(p,r)} \cap \boldsymbol{B} \setminus \boldsymbol{\{p\}} \neq \emptyset$.
Then $\boldsymbol{p} \in \boldsymbol{B'}$.
Thus $\boldsymbol{A'} \subseteq \boldsymbol{B'}$.
$\Leftarrow$ Suppose $\boldsymbol{A'} \subseteq \boldsymbol{B'}$.
Let $\boldsymbol{p} \in \boldsymbol{A'}$.

Then for every $r > 0$, $B(p, r) \cap A \setminus \{p\} \neq \emptyset$.
Then for every $r > 0$, $B(p, r) \cap B \setminus \{p\} \neq \emptyset$.
Then $p \in B'$.
Thus $A' \subseteq B'$.

(i) $int(A) = A^\circ$,

Proof:
Let $p \in \text{int}(A)$.
Then there exists $r > 0$ such that $B(p, r) \subseteq A$.
Then $p \in A^\circ$.
Thus $\text{int}(A) = A^\circ$.

(j) $\partial A \subseteq A'$.

Proof:
Let $p \in \partial A$.
Then for every $r > 0$, $B(p, r) \cap A \neq \emptyset$ and $B(p, r) \cap (\mathbb{R}^n \setminus A) \neq \emptyset$.
Then for every $r > 0$, $B(p, r) \cap A \setminus \{p\} \neq \emptyset$.
Then $p \in A'$.
Thus $\partial A \subseteq A'$.

(k) $\partial(A) \subseteq \partial A$ (Hint: If $B \subseteq \mathbb{R}^n$, then $\partial B = B \cap \mathbb{R}^n \setminus B$).

Proof:
Let $p \in \partial(A)$.
Then for every $r > 0$, $B(p, r) \cap A \neq \emptyset$ and $B(p, r) \cap (\mathbb{R}^n \setminus A) \neq \emptyset$.
Then for every $r > 0$, $B(p, r) \cap A \neq \emptyset$ and $B(p, r) \cap (\mathbb{R}^n \setminus A) \neq \emptyset$.
Then $p \in \partial A$.

7. Find the interior, exterior, closure, boundary, derived set and the set of isolated points for the following sets :

(a) $\emptyset$, $\mathbb{R}^n$

Solution:
$\text{int}(\emptyset) = \emptyset$, $\text{ext}(\emptyset) = \mathbb{R}^n$, $\overline{\emptyset} = \emptyset$, $\partial\emptyset = \emptyset$, $I(\emptyset) = \emptyset$, $\emptyset' = \emptyset$.

(b) Any finite subset A of Euclidean n-space.

Solution:
$\text{int}(A) = \emptyset$, $\text{ext}(A) = \mathbb{R}^n$, $\overline{A} = A$, $\partial A = A$, $I(A) = A$, $A' = A$.

(c) $(-\mathbf{1}, 5) \subseteq$ R,

Solution:
$\text{int}((-1, 5)) = (-1, 5)$, $\text{ext}((-1, 5)) = \mathbb{R} \setminus [-1, 5]$, $\overline{(-1, 5)} = [-1, 5]$, $\partial(-1, 5) = \{-1, 5\}$, $I((-1, 5)) = \emptyset$, $(-1, 5)' = \{-1, 5\}$.

(d) $S = \{(x, y) \mid -1 < x < 5, y = 0\}$ in $\mathbb{R}^2$, i.e. the interval $(-1, 5)$ on the x-axis in the plane. the plane,

Solution:
$\text{int}(S) = \emptyset$, $\text{ext}(S) = \mathbb{R}^2 \setminus [-1, 5]$, $\overline{S} = S \cup \{(-1, 0), (5, 0)\}$, $\partial S = \{(-1, 0), (5, 0)\}$, $I(S) = \emptyset$, $S' = \{(-1, 0), (5, 0)\}$.

(e) $R^2 \setminus \{(0, 0)\}$,

Solution:
$\text{int}(R^2 \setminus \{(0, 0)\}) = R^2 \setminus \{(0, 0)\}$, $\text{ext}(R^2 \setminus \{(0, 0)\}) = \{(0, 0)\}$, $\overline{R^2 \setminus \{(0, 0)\}} = R^2$, $\partial(R^2 \setminus \{(0, 0)\}) = \{(0, 0)\}$, $I(R^2 \setminus \{(0, 0)\}) = \emptyset$, $(R^2 \setminus \{(0, 0)\})' = \{(0, 0)\}$.

(f) A $= \{(x,y,z) \in \mathbb{R}^3 | 0 \leq x < 1, y^2+z^2 \leq 1\}$.

Solution:
$\text{int}(A) = \{(x,y,z) \in \mathbb{R}^3 | 0 \leq x < 1, y^2+z^2 < 1\}$, $\text{ext}(A) = \{(x,y,z) \in \mathbb{R}^3 | x < 0 \text{ or } x \geq 1 \text{ or } y^2+z^2 > 1\}$, $\overline{A} = \{(x,y,z) \in \mathbb{R}^3 | 0 \leq x \leq 1, y^2+z^2 \leq 1\}$, $\partial A = \{(x,y,z) \in \mathbb{R}^3 | x = 0 \text{ or } x = 1, y^2+z^2 = 1\}$, $I(A) = \emptyset$, $A' = \{(x,y,z) \in \mathbb{R}^3 | x = 0 \text{ or } x = 1, y^2+z^2 = 1\}$.

(g) For $a \in \mathbb{R}$, $A = \{(x,y) \in \mathbb{R}^2 \mid x = a, a < y < 1\}$.

Solution:
$\text{int}(A) = \emptyset$, $\text{ext}(A) = \mathbb{R}^2 \setminus \{(a,y) \in \mathbb{R}^2 \mid a < y < 1\}$, $\overline{A} = \{(x,y) \in \mathbb{R}^2 \mid x = a, a \leq y \leq 1\}$, $\partial A = \{(a,y) \in \mathbb{R}^2 \mid a = y \text{ or } y = 1\}$, $I(A) = \emptyset$, $A' = \{(a,y) \in \mathbb{R}^2 \mid a = y \text{ or } y = 1\}$.

8. Let

$$A = \{(x,y) | -2 \leq x \leq 2 \text{ with } x \neq 0, 0 < y \leq 4\} \cup \{(0,y) | 0 < y < 1 \text{ or } 3 < y \leq 4\} \cup \{(3,0),(4,0)\}.$$

(a) Find the interior A° of A. Is A an open set? Why?

Solution:
$A^\circ = \{(x,y) \in \mathbb{R}^2 | -2 < x < 2 \text{ with } x \neq 0, 0 < y < 4\} \cup \{(0,y) | 0 < y < 1 \text{ or } 3 < y < 4\} \cup \{(3,0),(4,0)\}$.
A is not an open set because A contains some of its boundary points.

(b) Write $\mathbb{R}^2 \setminus A$.

Solution:
$\mathbb{R}^2 \setminus A = \{(x,y) \in \mathbb{R}^2 | x < -2 \text{ or } x > 2 \text{ or } x = 0 \text{ and } 0 < y \leq 1 \text{ or } 3 < y < 4 \text{ or } x = 3 \text{ or } x = 4 \text{ and } y = 0\}$.

(c) Write the formula for the exterior of a set and use it to find ext(A).

Solution:
$\text{ext}(A) = \mathbb{R}^2 \setminus \overline{A} = \mathbb{R}^2 \setminus A$.

(d) Write the formula for the boundary ∂A of A and use it to find ∂A. Is A a closed set? Why?

Solution:
$\partial A = \overline{A} \cap \overline{(\mathbb{R}^2 \setminus A)} = \overline{A} \cap \overline{A} = \overline{A}$.
A is a closed set because A contains all of its boundary points.

(e) Write a formula for the closure of A and use it to find A.

Solution:
$\overline{A} = A$.

(f) Find the set $I(A)$ of all isolated points of A.

Solution:
$I(A) = \emptyset$.

(g) Find the derived set A' of A.

Solution:
$A' = \emptyset$.
since $A^\circ = A$ and $A' = \emptyset$, A is not a perfect set.