Math 251 - Exercise Set 2 Zehra's Solutions

March 16, 2024

1 Theorems and Definitions that are used in the solutions

- $p \in \partial S. \iff$ for every $r > 0$, $B(p, r) \cap S \neq \emptyset$ and $B(p, r) \cap (\mathbb{R}^m \setminus S) \neq \emptyset$.

2 Solutions

1. Let

2. Let A(1,4) and B(5,7) be two points in the plane. Find two open sets U and V s.t. $\boldsymbol{A \in U}$, $\boldsymbol{B \in V}$ and $\boldsymbol{U \cap V = \emptyset}$

   Solution:

   Let $\boldsymbol{U = B(A, 1)}$ and $\boldsymbol{V = B(B, 1)}$.
   Then $\boldsymbol{A \in U}$, $\boldsymbol{B \in V}$ and $\boldsymbol{U \cap V = \emptyset}$.

3. Let $\boldsymbol{S} \subseteq \mathbb{R}^{\boldsymbol{m}}$.

   (a) Show that $\boldsymbol{p \in \partial S}$ iff there exists sequences $\boldsymbol{\{p_n\}}$ in $\boldsymbol{S}$ with $\boldsymbol{\lim_{n\to\infty} p_n = p}$ and $\boldsymbol{\{q_n\}}$ in $\mathbb{R}^{\boldsymbol{m}} \setminus \boldsymbol{S}$ with $\boldsymbol{\lim_{n\to\infty} q_n = p}$.

   Proof:

   $\Rightarrow$ Suppose $\boldsymbol{p \in \partial S}$.
   Then for every $\boldsymbol{r > 0}$, $\boldsymbol{B(p, r) \cap S \neq \emptyset}$ and $\boldsymbol{B(p, r)} \cap (\mathbb{R}^{\boldsymbol{m}} \setminus \boldsymbol{S}) \neq \emptyset$.
   Let $\boldsymbol{n} \in \mathbb{N}$.
   Then there exists $\boldsymbol{p_n \in B(p, \frac{1}{n}) \cap S}$ and $\boldsymbol{q_n \in B(p, \frac{1}{n})} \cap (\mathbb{R}^{\boldsymbol{m}} \setminus \boldsymbol{S})$.
   Then $\boldsymbol{\lim_{n\to\infty} p_n = p}$ and $\boldsymbol{\lim_{n\to\infty} q_n = p}$.

$\Leftarrow$ Suppose there exists sequences $\{p_n\}$ in S with $\lim_{n\to\infty} p_n = p$ and $\{q_n\}$ in $\mathbb{R}^m \setminus S$ with $\lim_{n\to\infty} q_n = p$.
Then for every $r > 0$, there exists $N \in \mathbb{N}$ such that $p_n \in B(p,r)$ for all $n \geq N$ and $q_n \in B(p,r)$ for all $n \geq N$.
Then $B(p,r) \cap S \neq \emptyset$ and $B(p,r) \cap (\mathbb{R}^m \setminus S) \neq \emptyset$.
Thus $p \in \partial S$.

(b) Show that S is closed iff the limit of every convergent sequence $\{p_n\}$ in S belongs to S.

Proof:

$\Rightarrow$ Suppose S is closed.
Let $\{p_n\}$ be a convergent sequence in S with $\lim_{n\to\infty} p_n = p$.
Then for every $r > 0$, there exists $N \in \mathbb{N}$ such that $p_n \in B(p,r)$ for all $n \geq N$.
Then $B(p,r) \cap S \neq \emptyset$.
Thus $p \in \overline{S} = S$.

$\Leftarrow$ Suppose the limit of every convergent sequence $\{p_n\}$ in S belongs to S.
Let $p \in \overline{S}$.
Then there exists a sequence $\{p_n\}$ in S with $\lim_{n\to\infty} p_n = p$.
Then $p \in S$ because the limit of every convergent sequence $\{p_n\}$ in S belongs to S.
Thus $\overline{S} = S$.

4. Find $\limsup x_n$ and $\liminf x_n$ if $\{x_n\}$ is

(a) $\{2, 4, 2, 4, 6, 2, 4, 8, 2, 4, 10, \ldots\}$,

Solution:
$\limsup x_n = 6$ and $\liminf x_n = 2$.

(b) $\{1, 0, -1, 1, 0, -1, 1, 0, -1, \ldots\}$,

Solution:
$\limsup x_n = 1$ and $\liminf x_n = -1$.

(c) $\{-2, 1, 1, -2, 1, 1/2, -2, 1, 1/3, -2, 1, 1/4, -2, 1, 1/5, \ldots\}$,

Solution:
$\limsup x_n = 1$ and $\liminf x_n = -2$.

(d) $\{\frac{3n-1}{2n}\}_{n=1}$,

Solution:
$\limsup x_n = \frac{3}{2}$ and $\liminf x_n = \frac{1}{2}$.

(e) $\{0, 1, 0, 3, 0, 5, 0, 7, 0, \dots\}$,

Solution:
$\limsup x_n = 7$ and $\liminf x_n = 0$.

(f) $x_n = -3n + 2$ for $n \geq 1$,

Solution:
$\limsup x_n = 2$ and $\liminf x_n = -\infty$.

5. Find a sequence $\{a_n\}$ with $\limsup a_n = 3$ and $\liminf a_n = -2$.

Solution:
Let $a_n = (-1)^n n$ for $n \geq 1$.
Then $\limsup a_n = 3$ and $\liminf a_n = -2$.

6. Let $S = \{x \in \mathbb{R} | x^3 + 3x^2 + x + 3 < 0\}$. Find $\sup(S)$. Is S bounded below?

Solution:
$\sup(\mathbf{S})$ = 0. Since $x^3 + 3x^2 + x + 3 = (x+1)(x^2 + 2x + 3) = (x+1)(x+1+i\sqrt{2})(x+1-i\sqrt{2})$, $x^3 + 3x^2 + x + 3 < 0$ if and only if $x \in (-1, 0)$.
S is bounded below. Because -1 is a lower bound of S.
????

7. Is the sequence $x_n = \{\frac{\sin(n)}{n^3}\}$ a Cauchy sequence?

Solution:
Yes, the sequence $x_n = \{\frac{\sin(n)}{n^3}\}$ is a Cauchy sequence.

8. Let $A \subseteq \mathbb{R}^n$ and $p \in \mathbb{R}^n$. Show that $p \in \bar{A} \Leftrightarrow B(p, \varepsilon) \cap A \neq \emptyset$ for all $\varepsilon > 0$.

Proof:
$\Rightarrow$ Suppose $p \in \bar{A}$.
Then for every $\varepsilon > 0$, $B(p, \varepsilon) \cap A \neq \emptyset$.