MATH 262 Linear Algebra Lecture Notes of Özgür Kişisel Week 2

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Contents

1 Introduction

Proposition 1.1: Suppose that $\{v_1, v_2, ..., v_n\}$ is a basis for a vector space V so that $dim(V) = n$. Then $\{v_i v_j\}_{1 \leq i \leq j \leq n}$ is a basis for $Sym^2(V)$.

In particular, $dim(Sym^2(V)) = \frac{n(n+1)}{2}$.

Proof:

2 Higher Symmetric Powers

Definition 2.1: The k^{th} symmetric power of a vector space V is the quotient vector space $Sym^k(V) = V^{\otimes k}/\mathcal{U}$ where $\mathcal{U}$ is the subspace spanned by all vectors of the form $v^1 \otimes v^2 \otimes ... \otimes (v \otimes w - w \otimes v) \otimes ... \otimes v^k$ for all $v, w \in V$.

Theorem 2.1: Let $\{v_1, v_2, ..., v_n\}$ be a basis for V. Then $B = \{v_{i_1} v_{i_2} ... v_{i_k}\}_{1 \leq i_1 < i_2 < ... < i_k \leq n}$ is a basis for $Sym^k(V)$.

3 Exterior Powers / Alternating Powers

Assume $char(F) \neq 2$.

Definition 3.1: Let V be a vector space over F. The second exterior (altenating) power of V is $\Lambda^2 V = V^{\otimes k}/\mathcal{U}$ where $\mathcal{U}$ is the subspace spanned by all vectors of the form $v \otimes w + w \otimes v$.
The equivalence class of $v \otimes w$ in $\Lambda^k V$ will be denoted by $v \wedge w$.

$$\overline{w \otimes v} = -\overline{v \otimes w} \implies w \wedge v = -v \wedge w$$

for any $v, w \in V$, $v \wedge v = -v \wedge v \implies 2(v \wedge v) = 0, v \wedge v = 0$.

3.0.1 Universal Property of $\Lambda^2 V$

Let V, Z be a vector space over F. Suppose that and $\psi : V \times V \to Z$ is skew - symmetric bilinear map (altenating bilinear map) ($\psi(v, w) = -\psi(w, v)$). Then there is a unique linear transformation such that $T_\psi : \Lambda^2 V \to Z$ such that $\psi = T_\psi \circ \phi$.
where $\phi(v, w) = v \wedge w$.

Theorem 3.1: Let $\{v_1, v_2, ..., v_n\}$ be a basis for V. Then $B = \{v_i \wedge v_j\}_{1 \leq i < j \leq n}$ is a basis for $\Lambda^2 V$.
In particular, $dim(\Lambda^2 V) = \frac{n(n-1)}{2} = \binom{n}{2}$.

3.1 Higher Exterior Powers

Assume $char(F) \neq 2$.

Definition 3.1.1: Let V be a vector space over F. Say $k \geq 1$ is an integer. The k^{th} exterior power of V is $\Lambda^k V = V^{\otimes k}/\mathcal{U}$ where $\mathcal{U}$ is the subspace spanned by all vectors of the form $v^1 \otimes v^2 \otimes ... \otimes (v \otimes w + w \otimes v) \otimes ... \otimes v^k$.

The equivalence class of $v^1 \otimes v^2 \otimes ... \otimes v^k$ in $\Lambda^k V$ will be denoted by $v^1 \wedge v^2 \wedge ... \wedge v^k$.

4 Digression on Permutations

Definition 4.1: Let n be a positive integer. A permutation of n letters is a bijection. $\sigma : \{1, 2, ..., n\} \rightarrow \{1, 2, ..., n\}$.

Definition 4.2 (Transposition): A permutation T such that $T(i) = j, T(j) = i$ and $T(k) = k$ for all $k \neq i, j$ is called a transposition.
If $j = i \mp 1$, then T is called an adjacent transposition.

Definition 4.3: $sgn(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ can be written as a product of an } \textbf{even} \text{ number of transpositions} \\ -1 & \text{if } \sigma \text{ can be written as a product of an } \textbf{odd} \text{ number of transpositions} \end{cases}$

5 Determinants

5.1 Determinants of Elementary Matrices

5.2 Effects of Elementary Row Operations on Determinants