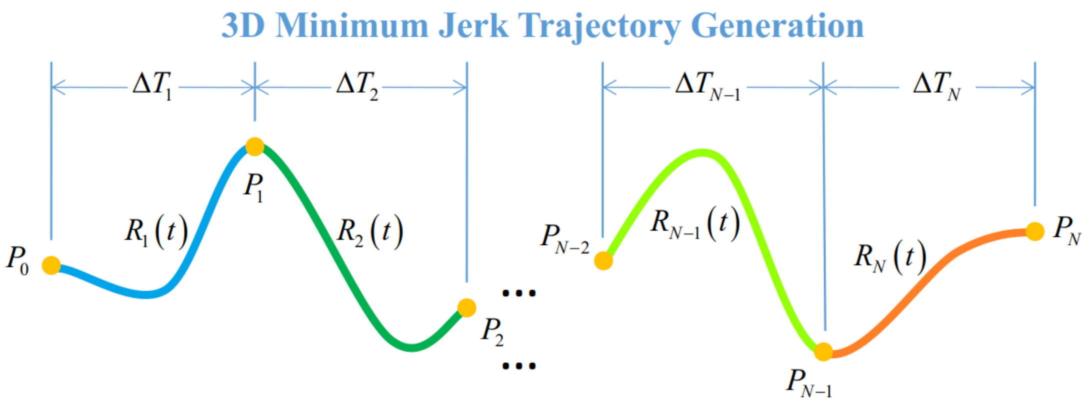
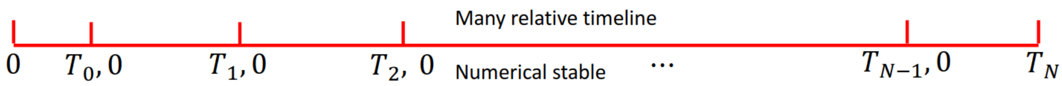
- Unconstrained case




- Use relative timeline




- Optimization Equation

$$\min_{z(t)} \int_{t_0}^{t_M} v(t)^{\mathrm{T}} \mathbf{W} v(t) \mathrm{d}t,$$

$$s.\ t.\ \ z^{(s)}(t) = v(t),\ \forall t \in [t_0, t_M],$$

Boundary value: $z^{[s-1]}(t_0) = \bar{z}_o,\ z^{[s-1]}(t_M) = \bar{z}_f,$

Intermediate value: $z^{[d_i-1]}(t_i) = \bar{z}_i,\ 1 \leq i < M,$

$$t_{i-1} < t_i,\ 1 \leq i \leq M.$$

- Theory

**Theorem (Optimality Conditions).** *A trajectory, denoted by $z^*(t) : [t_0, t_M] \mapsto \mathbb{R}^m$, is optimal, if and only if the following conditions are satisfied:*

- *The map $z^*(t) : [t_{i-1}, t_i] \mapsto \mathbb{R}^m$ is parameterized as a $2s-1$ degree polynomial for any $1 \leq i \leq M$;*
- *The boundary and intermediate conditions all hold;*
- *$z^*(t)$ is $2s - d_i - 1$ times continuously differentiable at $t_i$ for any $1 \leq i < M$.*

*Moreover, a unique trajectory exists for these conditions.*

- Problem become as linear equation system

$$\min_{z(t)} \int_{t_0}^{t_M} v(t)^{\mathrm{T}} \mathbf{W} v(t) \mathrm{d}t,$$
$$s.\ t.\ \ z^{(s)}(t) = v(t),\ \forall t \in [t_0, t_M],$$
$$z^{[s-1]}(t_0) = \bar{z}_o,\ z^{[s-1]}(t_M) = \bar{z}_f,$$
$$z^{[d_i-1]}(t_i) = \bar{z}_i,\ 1 \leq i < M,$$
$$t_{i-1} < t_i,\ 1 \leq i \leq M.$$

$$\Rightarrow \mathbf{Mc} = \mathbf{b}$$

$$\mathbf{M} = \begin{pmatrix} \mathbf{F}_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{E}_1 & \mathbf{F}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_2 & \mathbf{F}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{F}_{M-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{E}_M \end{pmatrix}$$

$$\mathbf{b} = (\mathbf{D}_0^{\mathrm{T}}, \mathbf{D}_1^{\mathrm{T}}, \mathbf{0}_{m \times \bar{d}_1}, \ldots, \mathbf{D}_{M-1}^{\mathrm{T}}, \mathbf{0}_{m \times \bar{d}_{M-1}}, \mathbf{D}_M^{\mathrm{T}})^{\mathrm{T}}$$

- **Deriving the matrix M**

**Derivative constraints P0:**

**We have 3 derivative constrains at initial point**

$$D_0^T = \begin{bmatrix} Initpos \\ Initvel \\ Initacc \end{bmatrix} F_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \end{bmatrix} C_1 = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \end{bmatrix}^T$$

**Derivative constraints PN:**

**We have 3 derivative constrains at terminal point**

$$D_M^T = \begin{bmatrix} Terminalpos \\ Terminalvel \\ Terminalacc \end{bmatrix}$$

$$E_M = \begin{bmatrix} 1 & t_m & t_m^2 & t_m^3 & t_m^4 & t_m^5 \\ 0 & 1 & 2t_m & 3t_m^2 & 4t_m^3 & 5t_m^4 \\ 0 & 0 & 2 & 6t_m & 12t_m^2 & 20t_m^3 \end{bmatrix}$$

$$C_M = \begin{bmatrix} c_0' & c_1' & c_2' & c_3' & c_4' & c_5' \end{bmatrix}^T$$

**Continuity constraints :**

**We have 6 Continuity constraints at each intermediate Position**

$$\begin{cases} \boldsymbol{f_i(t_i) = p_i} \\ \boldsymbol{f_i(t_i) = f_{i+1}(0)} \\ \boldsymbol{f_i(t_i)^1 = f_{i+1}(0)^1} \\ \boldsymbol{f_i(t_i)^2 = f_{i+1}(0)^2} \\ \boldsymbol{f_i(t_i)^3 = f_{i+1}(0)^3} \\ \boldsymbol{f_i(t_i)^4 = f_{i+1}(0)^4} \end{cases}$$

$$E_1 = \begin{bmatrix} 1 & t_1 & t_1^2 & t_1^3 & t_1^4 & t_1^5 \\ 1 & t_1 & t_1^2 & t_1^3 & t_1^4 & t_1^5 \\ 0 & 1 & 2t_1 & 3t_1^2 & 4t_1^3 & 5t_1^4 \\ 0 & 0 & 2 & 6t_1 & 12t_1^2 & 20t_1^3 \\ 0 & 0 & 0 & 6 & 24t_1 & 60t_1^2 \\ 0 & 0 & 0 & 0 & 24 & 120t_1 \end{bmatrix} \quad F_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -24 & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} c_0' & c_1' & c_2' & c_3' & c_4' & c_5' \end{bmatrix}^T$$

- **Solve the equation**

$$c = M^{-1}b$$

- **Result**