Connected Graphs and Spanning Trees

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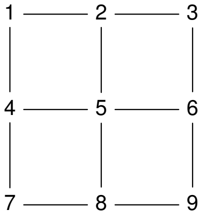
Describing the problem I

$G = (V, E)$ - graph

1. V - set of vertices
2. E - (multi)set of edges

Example:



$V = \{1, \ldots, 9\}$
$E = \{< 1, 2 >; < 1, 4 >; < 2, 3 >; < 2, 5 >; < 3, 6 >; < 4, 5 >; < 4, 7 >; < 5, 6 >; < 5, 8 >;$
$< 6, 9 >; < 7, 8 >; < 8, 9 >\}$

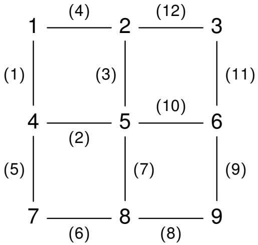
Describing the problem II

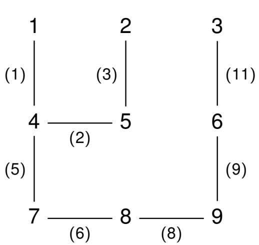
$G = (V, E)$ connected;

$T = (V, E')$ spanning tree of G when $\begin{cases} T \text{ is a tree} \\ E' \subseteq E \end{cases}$

Theorem

Every connected graph has a spanning tree.





Towards formalization

$$(\forall G : \texttt{Graph})\texttt{connected}(G) \Rightarrow (\exists G' : \texttt{Graph}) \subseteq G.\texttt{tree}(G')$$

We need a witness, $\texttt{mktree} : \texttt{Graph} \to \texttt{Graph}$

$$(\forall G : \texttt{Graph})\texttt{connected}(G) \Rightarrow \texttt{connected}(\texttt{mktree}(G)) \wedge \texttt{nocycle}(\texttt{mktree}(G))$$

To do :

1. data representations for mathematical objects ($\texttt{Graph}$)
2. define $\begin{cases} \texttt{connected} : \texttt{Graph} \to \texttt{Bool} \\ \texttt{nocycle} : \texttt{Graph} \to \texttt{Bool} \\ \texttt{mktree} : \texttt{Graph} \to \texttt{Graph} \end{cases}$

Spanning forests I

In the attempt of proving the desired properties we realized is much easier to prove a more general result:

Every graph has a spanning forest!

Definition

A **spanning forest** of a graph is a subgraph that consists of a set of spanning trees, one for each maximal connected component of the initial graph.

$\mathtt{mktree} : \mathtt{Graph} \rightarrow \mathtt{Graph}, \mathtt{mktree}(G) = \text{spanning forest of } G$

Spanning forests II

Example

`G=({1,2,3,4,5,6,7,8}, <1,2>;<1,6>;<1,5>;<3,4>;<3,7>;<4,7>;<5,6>)`

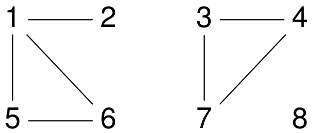
`mktree(G)=({1,2,3,4,5,6,7,8},<1,2>;<1,6>;<1,5>;<3,4>;<3,7>)`

Remark

The value is relative to the order chosen for the edges.

Functions on graphs

$G = (\{1, 2, 3, 4, 5, 6, 7, 8\}, < 1, 2 >; < 1, 6 >; < 1, 5 >; < 3, 4 >; < 3, 7 >; < 4, 7 >; < 5, 6 >)$



- $\begin{cases} \mathtt{mcc} : \mathtt{Vertex\ Graph} \rightarrow \mathtt{VtxSet} \\ \mathtt{mcc}(A, G) = \text{max. connected comp. of } A \text{ in G} \\ \mathtt{mcc}(6, G) = \{1, 2, 5, 6\} \end{cases}$
- $\begin{cases} \mathtt{nomcc} : \mathtt{Graph} \rightarrow \mathtt{Nat} \\ \mathtt{ncc}(G) = \text{ no. of max. connected components} \\ \mathtt{ncc}(G) = 3 \end{cases}$

Properties to prove

Assuming $\begin{pmatrix} \texttt{mcc} : \texttt{Vertex Graph} \to \texttt{VtxSet} \\ \texttt{nomcc} : \texttt{Graph} \to \texttt{Nat} \\ \texttt{nocycle} : \texttt{Graph} \to \texttt{Bool} \\ \texttt{mktree} : \texttt{Graph} \to \texttt{Graph} \end{pmatrix}$ we need to prove

1. $(\forall G : \texttt{Graph}, A : \texttt{Vertex})\texttt{mcc}(A, \texttt{mktree}(G)) = \texttt{mcc}(A, G)$
2. $(\forall G : \texttt{Graph})\texttt{nomcc}(\texttt{mktree}(G)) = \texttt{nomcc}(G)$
3. $(\forall G : \texttt{Graph})\texttt{nocycle}(\texttt{mktree}(G)) = \texttt{true}$

Remark

If $\texttt{nomcc}(G) = 1$ then
$\texttt{connected}(G) \Rightarrow \texttt{connected}(\texttt{mktree}(G)) \wedge \texttt{nocycle}(\texttt{mktree}(G))$