Model Theory

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Who and what

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- Course structure:
  - One two-hour lecture a week: Thursday 16:00–18:00 JST.
  - Three assignments, each worth 20%, due in weeks 4, 8, 12.
  - Final report: two sections, A and B, each worth 20%.
  - Final mark computed as:
    Assgn 1 + Assgn 2 + Assgn 3 + Sect A + Sect B.
  - Section B will be noticeably more difficult than Section A.
  - In practice, this means that you will be able to get up to 80% based on the assignments and Section A alone. For more than 80% you will need Section B.

Course materials

- For all participants

  `https://imi.kyushu-u.ac.jp/~daniel/model-theory/model-theory.html`

- Kyushu University students:

  `https://moodle.s.kyushu-u.ac.jp/course/view.php?id=21801`

  Some quite incomplete course notes are there already.

  Slides from each lecture will be there, and in due course, assignments.

- Reference books:

[1] R. Diaconescu, *Institution-independent model theory*, Birkhäuser 2008
[2] D. Marker, *Model Theory: An Introduction*, Springer 2002
[3] W. Hodges, *A shorter model theory*, Cambridge University Press, 1997
[4] H.-D. Ebbinghaus, J. Flum, W. Thomas, *Mathematical Logic: Second Edition*, 1994
[5] C.C. Chang, H. Jerome Keisler, *Model Theory: Third Edition*, Elsevier, 1990

“ Model theory $=$ universal algebra $+$ logic”

C.C. Chang, H. Jerome Keisler, *Model Theory*

“Model theory is about the classification of mathematical structures by means of logical formulas (...) large part of model theory is directly about constructions (of models) and only indirectly about classification”
“Model theory $=$ algebraic geometry - fields”

W. Hodges, *Shorter Model Theory*

”Model theory $=$ toolbox for the formal specification and verification of software, in which models are regarded as representations of reality”

Formal specification viewpoint

Many-Sorted First-Order Model Theory

Many-sorted sets

Definition 1 (Many-sorted set)

Let S be a set of elements called *sorts* (or *indices*). An S-sorted set is an S-indexed family of sets denoted $\{A_s\}_{s \in S}$ or $\{A_s \mid s \in S\}$, which is

- *empty* if $A_s = \emptyset$ for all $s \in S$;
- *finite* if (a) A_s is finite for all $s \in S$, and
  (b) $\{s \in S \mid A_s \neq \emptyset\}$ is finite.

Let $A = \{A_s\}_{s \in S}$ and $B = \{B_s\}_{s \in S}$.

- $A \cup B = \{A_s \cup B_s\}_{s \in S}$
- $A \cap B = \{A_s \cap B_s\}_{s \in S}$
- $A \times B = \{A_s \times B_s\}_{s \in S}$
- $A \setminus B = \{A_s \setminus B_s\}_{s \in S}$
- $A \uplus B = \{A_s \uplus B_s\}_{s \in S}$, where $A_s \uplus B_s = (A_s \times \{1\}) \cup (B_s \times \{2\})$
- $A \subseteq B$ iff $A_s \subseteq B_s$ for all $s \in S$
- $\mathtt{card}(A) = \sum_{s \in S} \mathtt{card}(A_s)$

Definition 2 (Many-sorted function)

Let $A = \{A_s\}_{s \in S}$ and $B = \{B_s\}_{s \in S}$. An *S-sorted function* $f : A \to B$ is an S-indexed family of functions $f = \{f_s : A_s \to B_s\}_{s \in S}$.

- If $f : A \to B$ and $g : B \to C$ be two S-sorted functions then the composition $f ; g = \{f_s ; g_s\}_{s \in S}$ is considered in diagrammatic order.
- An S-sorted function $f = \{f_s : A_s \to B_s\}_{s \in S}$ is an *identity (an inclusion, an injection or a surjection)* if $f_s : A_s \to B_s$ is an identity (an inclusion, an injection or a surjection) for all $s \in S$.

- if S is a set then we denote by S^* the set of strings over S
- we let ε to denote the empty string
- notice that $S \subseteq S^*$
- if $S = \{a, b, c\}$ then
  $a, b, c, aa, ab, ac, ba, bb, bc, ca, cb, cc, aaa, \dots \in S^*$

Signatures/Vocabularies

Definition 3 (First-order signature)

A many-sorted first-order signature is tuple $\Sigma = (S, F, P)$, where

- S is a set of sorts,
- $F = \{F_{w \to s}\}_{(w,s) \in S^* \times S}$ is an $(S^* \times S)$-sorted set of function symbols
- $P = \{P_w\}_{w \in S^*}$ is an S^*-sorted set of relation symbols

- If $\sigma \in F_{w \to s}$ then we call
  - σ the name of the function symbol,
  - w the arity of σ, and
  - s the sort of σ.
- We may write $(\sigma : w \to s) \in F$ instead of $\sigma \in F_{w \to s}$. This means that we can regard the $(S^* \times S)$-sorted set F as an ordinary set of elements of the form $\sigma : w \to s$.
- A similar remark holds for P, too.
- If $P = \emptyset$ then we write $\Sigma = (S, F)$ instead of $\Sigma = (S, F, \emptyset)$, and we say that Σ is an algebraic signature.

Notation

We let Σ, Σ' and Σ_i to range over signatures of the form (S, F, P), (S', F', P') and (S_i, F_i, P_i), respectively.

Examples of signatures

Example 4 (Natural numbers)

```
spec NAT is
 sort Nat .
 op 0 : -> Nat  .
 op s_ : Nat -> Nat .
end
```

- $\Sigma_{\mathtt{NAT}} = (S_{\mathtt{NAT}}, F_{\mathtt{NAT}})$
- $S_{\mathtt{NAT}} = \{\mathtt{Nat}\}$
- $F_{\mathtt{NAT}} = \{0 :\to \mathtt{Nat}, \mathtt{s\_} : \mathtt{Nat} \to \mathtt{Nat}\}$

Example 5 (Integers)

```
spec INT is
 sort Int .
 op 0 : -> Int .
 op s_ : Int -> Int .
 op p_ : Int -> Int .
end
```

- $\Sigma_{\mathtt{INT}} = (S_{\mathtt{INT}}, F_{\mathtt{INT}})$
- $S_{\mathtt{INT}} = \{\mathtt{Int}\}$
- $F_{\mathtt{INT}} = \{0 :\to \mathtt{Int}, \mathtt{s\_} : \mathtt{Int} \to \mathtt{Int}, \mathtt{p\_} : \mathtt{Int} \to \mathtt{Int}\}$

Example 6 (Lists)

```
spec LIST is
 sorts Elem List .
 op empty : -> List  .
 op con : Elem List -> List .
end
```

- $\Sigma_{\mathtt{LIST}} = (S_{\mathtt{LIST}}, F_{\mathtt{LIST}})$
- $S_{\mathtt{LIST}} = \{\mathtt{Elem}, \mathtt{List}\}$
- $F_{\mathtt{LIST}} = \{\mathtt{empty} :\to \mathtt{List},$
  $\mathtt{con} : \mathtt{Elem}\ \mathtt{List} \to \mathtt{List}\}$

Example 7 (Automata)

```
spec AUTOM is
 sorts Input Output State .
 op init : -> State .
 op f : Input State -> State .
 op g : State -> Output .
end
```

- $\Sigma_{\mathtt{AUTOM}} = (S_{\mathtt{AUTOM}}, F_{\mathtt{AUTOM}})$
- $S_{\mathtt{AUTOM}} = \{\mathtt{Input}, \mathtt{Output}, \mathtt{State}\}$
- $F_{\mathtt{AUTOM}} = \{\mathtt{s0} :\to \mathtt{State},$
  $\mathtt{f} : \mathtt{Input}\ \mathtt{State} \to \mathtt{State},$
  $\mathtt{g} : \mathtt{State} \to \mathtt{Output}\}$

Example 8 (Graphs)

```
spec GRAPH is
 sorts Edge Node .
 ops @0 @1 :  Edge -> Node .
end
```

Example 9 (Groups)

```
spec GROUP is
 sort Group .
 op 0 : -> Group .
 op _+_ : Group Group -> Group .
 op -_ : Group -> Group .
end
```

First-order structures/Models

Definition 10 (Σ-models)

A Σ-*model* $\mathfrak{A}$ interprets

- each $s \in S$ as a (non-empty) set $\mathfrak{A}_s$, called the *carrier set* of sort of s; $|\mathfrak{A}| = \{\mathfrak{A}_s\}_{s \in S}$ is called the *universe* of $\mathfrak{A}$.
- each $(\sigma\colon w \to s) \in F$ as a function $\sigma^{\mathfrak{A}}\colon \mathfrak{A}_w \to \mathfrak{A}_s$, where $\mathfrak{A}_w = \mathfrak{A}_{s_1} \times \cdots \times \mathfrak{A}_{s_n}$ if $w = s_1 \ldots s_n$;
- each $(\pi : w) \in P$ as a relation $\pi^{\mathfrak{A}} \subseteq \mathfrak{A}_w$.

Definition 11 (Σ-homomorphisms)

A Σ-*homomorphism* $h\colon \mathfrak{A} \to \mathfrak{B}$ is an S-sorted function $h\colon |\mathfrak{A}| \to |\mathfrak{B}|$ such that

- the following diag. is commutative $\begin{tikzcd} \mathfrak{A}_w \arrow[r, "\sigma^{\mathfrak{A}}"] \arrow[d, "h_w"'] & \mathfrak{A}_s \arrow[d, "h_s"] \\ \mathfrak{B}_w \arrow[r, "\sigma^{\mathfrak{B}}"'] & \mathfrak{B}_s \end{tikzcd}$ for all $(\sigma\colon w \to s) \in F$;
- $h_w(\pi^{\mathfrak{A}}) \subseteq \pi^{\mathfrak{B}}$ for all $(\pi : w) \in P$.

The Σ-homomorphisms form a category under the obvious composition (component-wise as many-sorted functions). The category of Σ-models is denoted by $\mathtt{Mod}(\Sigma)$.

Examples of models and homomorphisms

Example 12

$\Sigma_{\mathtt{NAT}}$-models (see Example 4):

- $\mathbb{N}$ - natural numbers
- $\mathbb{Z}$ - integers
- $\mathbb{Z}_{\mathtt{n}}$ - integers modulo $\mathtt{n}$
- $\mathbb{N}'$ - natural no. with two zeros
  - $\mathbb{N}'_{\mathtt{Nat}} = \omega \cup \{0'\}$
  - $\mathtt{s}\ \ 0' = 1$
  - $\mathtt{s}\ \ 0 = 1$

Example 13

Are the following mappings $\Sigma_{\mathtt{NAT}}$-homomorphisms?

- $h : \mathbb{N} \to \mathbb{Z}$, defined by $h(n) = n$ for all $n \in |\mathbb{N}|$
- $h : \mathbb{Z} \to \mathbb{N}$, defined by $h(z) = |z|$ for all $z \in |\mathbb{Z}|$
- $h : \mathbb{Z} \to \mathbb{Z}_{\mathtt{n}}$, defined by $h(z) = \widehat{z}$ for all $z \in |\mathbb{Z}|$

Example 14

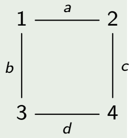
$\Sigma_{\mathtt{AUTOM}}$-model $\mathbb{A}$:

- $\mathbb{A}_{\mathtt{Input}} = \mathbb{A}_{\mathtt{Output}} = \mathbb{A}_{\mathtt{State}} = \omega$
- $\mathtt{init}^{\mathbb{A}} = 0$
- $\mathtt{f}^{\mathbb{A}}(m, n) = m + n$ for all $m, n \in \omega$
- $g^{\mathbb{A}}(n) = n + 1$ for all $n \in \omega$

Example 15

$\Sigma_{\mathtt{GRAPH}}$-model $\mathbb{G}$:

- $\mathbb{G}_{\mathtt{Node}} = \{1, 2, 3, 4\}$,
- $\mathbb{G}_{\mathtt{Edge}} = \{a, b, c, d\}$,
- $\mathtt{@}_0^{\mathbb{G}}(a) = 1$,
  $\mathtt{@}_1^{\mathbb{G}}(a) = 2$,
  ...



Sentences

Definition 16 (Σ-terms)

The S-sorted set of Σ-*terms* is constructed inductively:

- $(c :\to s) \in T_{\Sigma,s}$ for all constants $(c :\to s) \in F$;
- $\sigma(t_1, \ldots, t_n) \in T_{\Sigma,s}$ for all $(\sigma : s_1 \ldots s_n \to s) \in F$ and $t_i \in T_{\Sigma,s_i}$.

Definition 17 (Σ-sentences)

For all signatures Σ, the set of Σ-sentences, denoted $\mathtt{Sen}(\Sigma)$, is constructed inductively:

- *equations* $t_1 = t_2 \in \mathtt{Sen}(\Sigma)$, where $t_1, t_2 \in T_{\Sigma,s}$ and $s \in S$;
  relations $\pi(t_1, \ldots, t_n) \in \mathtt{Sen}(\Sigma)$, where $(\pi : s_1 \ldots s_n) \in P$ and $t_i \in T_{\Sigma,s_i}$;
- $\bigvee \Gamma \in \mathtt{Sen}(\Sigma)$, where $\Gamma \subseteq \mathtt{Sen}(\Sigma)$ is finite;
  $\neg\gamma \in \mathtt{Sen}(\Sigma)$, where $\gamma \in \mathtt{Sen}(\Sigma)$;
- $\exists X \cdot \gamma' \in \mathtt{Sen}(\Sigma)$, where (a) X is a finite S-sorted set of variables/special constants for Σ, (b) $\gamma' \in \mathtt{Sen}(\Sigma[X])$, (c) $\Sigma[X] = (S, F[X], P)$, and (d) $F[X]$ is obtained from F by adding the elements of X as constants to F.

The formal definition of a *variable* for Σ is that of a triple (x, s, Σ), where (a) x is the name of the variable, and (b) s is the sort of the variable.
Note that the qualification of the variables by their signature context guarantees automatically, by a simple set theoretic argument, that when added as new constants to the signature they indeed do not clash with the already existing constants.

More about sentences

Other sentence building operators are introduced using the classical definitions:

- $\bigwedge \Gamma := \neg(\bigvee_{\gamma \in \Gamma} \neg\gamma)$
- $\bot := \bigvee \emptyset$
- $\top := \bigwedge \emptyset$
- $\gamma_1 \Rightarrow \gamma_2 := \neg\gamma_1 \vee \gamma_2$
- $\gamma_1 \Leftrightarrow \gamma_2 := (\gamma_1 \Rightarrow \gamma_2) \wedge (\gamma_2 \Rightarrow \gamma_1)$
- $\forall X \cdot \gamma' := \neg\exists X \cdot \neg\gamma'$

Convention

Dealing with standard logical operators, we adopt the following convention about their binding strength:

- $\neg$ binds stronger than $\wedge$,
- which binds stronger than $\vee$,
- which binds stronger than $\Rightarrow$,
- which binds stronger than quantifiers;
- quantifiers $\exists$ and $\forall$ have the same binding strength.

Signature morphisms

Definition 18 (Signature morphisms)

A signature morphism $\chi = (\chi^{st}, \chi^{op}, \chi^{rl}) \colon \Sigma \to \Sigma'$ consists of

- a function between the set of sorts $\chi^{st} \colon S \to S'$,
- a family of function between the sets of function symbols $\chi^{op} = \{\chi^{op} \colon F_{w \to s} \to F'_{\chi^{st}(w) \to \chi^{st}(s)}\}_{(w,s) \in S^* \times S}$, and
- a family of function between the sets of relation symbols $\chi^{rl} = \{\chi^{rl} \colon P_w \to P'_{\chi^{st}(w)}\}_{w \in S^*}$;

When there is no danger of confusion we may drop the superscripts *st*, *op* or *rl* from the above notations.

Model reducts

Definition 19 (Model reducts)

Given a signature morphism $\chi\colon \Sigma \to \Sigma'$ as in Definition 18,

1. the reduct $\mathfrak{A}' \restriction_\chi$ of a Σ'-model $\mathfrak{A}'$ is a Σ-model defined as follows:
   - $(\mathfrak{A}' \restriction_\chi)_s = \mathfrak{A}'_{\chi^{st}(s)}$ for all sorts $s \in S$,
   - $(\mathfrak{A}' \restriction_\chi)_\sigma = \mathfrak{A}'_{\chi^{op}(\sigma)} : \mathfrak{A}'_{\chi^{st}(w)} \to \mathfrak{A}' \restriction_{\chi^{st}(s)}$ for all $(\sigma : w \to s) \in F$,
   - $(\mathfrak{A}' \restriction_\chi)_\pi = \mathfrak{A}'_{\chi^{rl}(\pi)}$ for all $(\pi : w) \in P$.
2. the reduct $h' \restriction_\chi$ of a Σ'-homomorphism h' is a Σ-homomorphism $h' \restriction_\chi = \{h'_{\chi(s}\}_{s \in S}$.

Fact 20

For each signature morphism $\chi\colon \Sigma \to \Sigma'$, the model reduct is a functor $\restriction_\chi\colon \mathtt{Mod}(\Sigma') \to \mathtt{Mod}(\Sigma)$ is a functor.

- If $\mathfrak{A}' \restriction_\chi = \mathfrak{A}$ then (a) $\mathfrak{A}'$ is called a χ-*expansion* of $\mathfrak{A}$, and (b) $\mathfrak{A}$ is called the χ-*reduct* of $\mathfrak{A}'$.
- If $\chi : \Sigma \hookrightarrow \Sigma'$ is an inclusion and $\mathfrak{A}'$ is a Σ'-model, we may write $\mathfrak{A}' \restriction_\Sigma$ instead of $\mathfrak{A}' \restriction_\chi$.

Examples of model reducts

Example 21

Let $\chi\colon \Sigma_{\mathtt{NAT}} \hookrightarrow \Sigma_{\mathtt{NAT+}}$, where

- $\Sigma_{\mathtt{NAT}}$ is defined in Example 4, and
- $\Sigma_{\mathtt{NAT+}} = (S_{\mathtt{NAT+}}, F_{\mathtt{NAT+}})$,
  $S_{\mathtt{NAT+}} = S_{\mathtt{NAT}} = \{\mathtt{Nat}\}$,
  $F_{\mathtt{NAT+}} = F_{\mathtt{NAT}} \cup \{\_+\_\colon \mathtt{Nat}\ \mathtt{Nat} \to \mathtt{Nat}\}$.

Let $\mathbb{N}$ be the $\Sigma_{\mathtt{NAT+}}$-model of natural numbers with addition.
Then $\mathbb{N}\restriction_\chi$ is the $\Sigma_{\mathtt{NAT}}$-model of natural numbers without addition:

- $(\mathbb{N}\restriction_\chi)_{\mathtt{Nat}} = \omega$,
- $\mathsf{s}^{(\mathbb{N}\restriction_\chi)} n = s^{\mathbb{N}} n = n + 1$ for all $n \in \omega$.

Example 22

Let $\chi\colon \Sigma_{\mathtt{NAT}} \hookrightarrow \Sigma_{\mathtt{LIST}}$, where

- $\Sigma_{\mathtt{NAT}}$ is defined in Example 4, and
- $\Sigma_{\mathtt{LIST}} = (S_{\mathtt{LIST}}, F_{\mathtt{LIST}})$,
  $S_{\mathtt{LIST}} = \{\mathtt{Nat}, \mathtt{List}\}$,
  $F_{\mathtt{LIST}} = F_{\mathtt{NAT}} \cup$
  $\{\mathtt{empty} :\to \mathtt{List}, \mathtt{con}\colon \mathtt{Nat}\ \mathtt{List} \to \mathtt{List}\}$.

Let $\mathbb{L}$ be the $\Sigma_{\mathtt{LIST}}$-model of the lists of natural numbers. Then $\mathbb{L}\restriction_\chi$ is the $\Sigma_{\mathtt{NAT}}$-model of natural numbers:

- $(\mathbb{L}\restriction_\chi)_{\mathtt{Nat}} = \omega$,
- $\mathsf{s}^{(\mathbb{L}\restriction_\chi)} n = n + 1$ for all $n \in \omega$.

Fact 23

For any signature inclusion $\chi\colon \Sigma \hookrightarrow \Sigma'$ such that $S \subsetneq S'$, the reduct functor $\restriction_\chi$ changes the universes of models: for any Σ'-model $\mathfrak{A}'$, the universe of the model reduct $|\mathfrak{A}'\restriction_\chi| = \{\mathfrak{A}'_s\}_{s \in S}$ is different from the universe of the initial model $|\mathfrak{A}'| = \{\mathfrak{A}'_{s'}\}_{s' \in S'}$.

Sentence translations

Definition 24 (Term translations)

Any signature morphism $\chi\colon \Sigma \to \Sigma'$ determines a function $\chi^{tm}\colon T_\Sigma \to T_{\Sigma'}$ inductively defined as follows:

- $\chi^{tm}(c) = \chi^{op}(c)$, for all constants $(c :\to s) \in F$;
- $\chi^{tm}(\sigma(t_1, \ldots, t_n)) = \chi^{op}(\sigma)(\chi^{tm}(t_1), \ldots, \chi^{tm}(t_n))$, for all terms $\sigma(t_1, \ldots, t_n) \in T_\Sigma$.

We may drop the superscript *tm* from the above notations when there is no danger of confusion.

Definition 25 (Sentence translations)

Any signature morphism $\chi\colon \Sigma \to \Sigma'$ determines a function $\mathtt{Sen}(\chi)\colon \mathtt{Sen}(\Sigma) \to \mathtt{Sen}(\Sigma)$ which replaces the symbols from Σ with the symbols from Σ' according to χ:

- $\mathtt{Sen}(\chi)(t_1 = t_2) = (\chi^{tm}(t_1) = \chi^{tm}(t_2))$
- $\mathtt{Sen}(\chi)(\pi(t_1, \ldots, t_n)) = \chi^{op}(\pi)(\chi^{tm}(t_1), \ldots, \chi^{tm}(t_n))$
- $\mathtt{Sen}(\chi)(\neg\gamma) = \neg\mathtt{Sen}(\chi)(\gamma)$
- $\mathtt{Sen}(\chi)(\bigvee \Gamma) = \bigvee \mathtt{Sen}(\chi)(\Gamma)$
- $\mathtt{Sen}(\chi)(\exists X \cdot \gamma') = \exists X' \cdot \mathtt{Sen}(\chi')(\gamma')$, where
  - $X' = \{(x, \chi(s), \Sigma') \mid (x, s, \Sigma) \in X\}$, and
  - $\chi'\colon \Sigma[X] \to \Sigma'[X']$ is the extension of $\chi\colon \Sigma \to \Sigma'$ mapping each variable $(x, s, \Sigma) \in X$ to $(x, \chi(s), \Sigma') \in X'$.

We denote $\mathtt{Sen}(\chi)$ simply by χ when there is no danger of confusion.

Satisfaction

Definition 26 (Interpretation of terms into models)

The interpretation of a Σ-term into a Σ-model $\mathfrak{A}$ is defined inductively as follows:

- $(\sigma(t_1, \ldots, t_n))^{\mathfrak{A}} = \sigma^{\mathfrak{A}}(t_1^{\mathfrak{A}}, \ldots, t_n^{\mathfrak{A}})$, where $(\sigma : s_1 \ldots s_n \to s) \in F$ and $t_i \in T_{\Sigma, s_i}$.

Definition 27 (Satisfaction relation)

The satisfaction between models and sentences is inductively defined:

- $\mathfrak{A} \models_\Sigma t_1 = t_2$ iff $t_1^{\mathfrak{A}} = t_2^{\mathfrak{A}}$
- $\mathfrak{A} \models_\Sigma \pi(t_1, \ldots, t_n)$ iff $(t_1^{\mathfrak{A}}, \ldots, t_n^{\mathfrak{A}}) \in \pi^{\mathfrak{A}}$
- $\mathfrak{A} \models_\Sigma \neg\gamma$ iff $\mathfrak{A} \not\models_\Sigma \gamma$
- $\mathfrak{A} \models_\Sigma \bigvee \Gamma$ iff $\mathfrak{A} \models_\Sigma \gamma$ for some $\gamma \in \Gamma$
- $\mathfrak{A} \models_\Sigma \exists X \cdot \gamma'$ iff $\mathfrak{A}' \models_{\Sigma[X]} \gamma'$ for some expansion $\mathfrak{A}'$ of $\mathfrak{A}$ to the signature $\Sigma[X]$

We drop the subscript Σ from the notation $\models_\Sigma$ when there is no danger of confusion.

Fact 28 (Interpretation of variables)

Let $\iota_X : \Sigma \hookrightarrow \Sigma[X]$.

- *A $\Sigma[X]$-model consists of a Σ-model $\mathfrak{A}$ and an S-sorted function $f : X \to |\mathfrak{A}|$.*
- *A ι_X-expansion of a Σ-model $\mathfrak{A}$ can be regarded as a pair $(\mathfrak{A}, f : X \to |\mathfrak{A}|)$; therefore, "for some ι_X-expansion" means "for some valuation $f : X \to |\mathfrak{A}|$".*

More about satisfaction

Example 29

Let $\mathbb{N}$ be the $\Sigma_{\texttt{NAT+}}$-model from Example 21.

- $\mathbb{N} \models \mathrm{s}\ 0 + \mathrm{s}\ 0 = \mathrm{s}\ \mathrm{s}\ 0$ iff $(\mathrm{s}\ 0 + \mathrm{s}\ 0)^{\mathbb{N}} = (\mathrm{s}\ \mathrm{s}\ 0)^{\mathbb{N}}$ iff $(1 + 1) = 2$
- $\mathbb{N} \models \forall x, y \cdot x + y = y + x$ iff
  $\mathbb{N}' \models x + y = y + x$ for all expansions $\mathbb{N}'$ of $\mathbb{N}$ to $\Sigma_{\texttt{NAT+}}[x, y]$ iff
  $x^{\mathbb{N}'} + y^{\mathbb{N}'} = x^{\mathbb{N}'} + y^{\mathbb{N}'}$ for all expansions $\mathbb{N}'$ of $\mathbb{N}$ to $\Sigma_{\texttt{NAT+}}[x, y]$

Problem 30

For any signature Σ and all Σ-models $\mathfrak{A}$ we have:

- $\mathfrak{A} \models_{\Sigma} \bigwedge \Gamma$ *iff* $\mathfrak{A} \models_{\Sigma} \gamma$ *for all* $\gamma \in \Gamma$
- $\mathfrak{A} \not\models_{\Sigma} \bot$ *and* $\mathfrak{A} \models_{\Sigma} \top$
- $\mathfrak{A} \models_{\Sigma} \gamma_1 \Rightarrow \gamma_2$ *iff* $\mathfrak{A} \models \gamma_1$ *implies* $\mathfrak{A} \models_{\Sigma} \gamma_2$
- $\mathfrak{A} \models_{\Sigma} \forall X \cdot \gamma'$ *iff* $\mathfrak{A}' \models_{\Sigma[X]} \gamma'$ *for all expansions of* $\mathfrak{A}'$ *of* $\mathfrak{A}$ *to* $\Sigma[X]$

Satisfaction condition

$$
\begin{tikzcd}
\Sigma' & \mathfrak{A}' \arrow[d, dotted] & \models_{\Sigma'} \arrow[d, Leftrightarrow] & \chi(\gamma) \\
\Sigma \arrow[u, "\chi"] & \mathfrak{A}'\restriction_{\chi} & \models_{\Sigma} & \gamma \arrow[u, dotted]
\end{tikzcd}
$$

Theorem 31 (Satisfaction is invariant w.r.t. change of notation)

For all signature morphisms $\chi\colon \Sigma \to \Sigma'$, all Σ'-models $\mathfrak{A}'$, and all Σ-sentences γ, we have

$$\mathfrak{A}' \models_{\Sigma'} \chi(\gamma) \textit{ iff } \mathfrak{A}'\restriction_{\chi} \models_{\Sigma} \gamma$$

In order to prove Theorem 31, we need two preliminary results, Lemma 32 and Lemma 33.

Lemma 32

For all signature morphisms $\chi\colon \Sigma \to \Sigma'$, *all* Σ'*-models* $\mathfrak{A}'$, *and all* Σ*-terms* t, *we have* $\chi(t)^{\mathfrak{A}'} = t^{\mathfrak{A}'\upharpoonright\chi}$.

Proof.

We proceed by induction on the structure of terms:

- $(c :\to s) \in F$: By definition, $\chi(c)^{\mathfrak{A}'} = c^{\mathfrak{A}'\upharpoonright\chi}$.
- $\sigma(t_1, \ldots, t_n) \in T_\Sigma$: We have $\chi(\sigma(t_1, \ldots, t_n))^{\mathfrak{A}'} = \chi(\sigma)(\chi(t_1), \ldots, \chi(t_n))^{\mathfrak{A}'} = \chi(\sigma)^{\mathfrak{A}'}(\chi(t_1)^{\mathfrak{A}'}, \ldots, \chi(t_n)^{\mathfrak{A}'}) \overset{IH}{=} \sigma^{\mathfrak{A}'\upharpoonright\chi}(t_1^{\mathfrak{A}'\upharpoonright\chi}, \ldots, t_n^{\mathfrak{A}'\upharpoonright\chi}) = \sigma(t_1, \ldots, t_n)^{\mathfrak{A}'\upharpoonright\chi}$.

$\square$

$$
\begin{tikzcd}
\mathfrak{B} \arrow[d, dotted] & \Sigma[X] \arrow[r, "\chi'"] & \Sigma'[X'] & {\color{blue}\mathfrak{B}'} \arrow[lll, dotted, bend right] \arrow[d, dotted] \\
\mathfrak{B}\restriction_{\Sigma} = \mathfrak{A}'\restriction_{\chi} & \Sigma \arrow[u, "\iota_X"] \arrow[r, "\chi"'] & \Sigma' \arrow[u, "\iota_{X'}"'] & \mathfrak{A}' \arrow[lll, dotted, bend left]
\end{tikzcd}
$$

Lemma 33

Assume a signature morphism $\chi \colon \Sigma \to \Sigma'$, a finite set of variables X for Σ, a Σ'-model $\mathfrak{A}'$, and a $\Sigma[X]$-model $\mathfrak{B}$ such that $\mathfrak{A}'\restriction_{\chi} = \mathfrak{B}\restriction_{\Sigma}$.
Then there exists a unique $\Sigma'[X']$-model $\mathfrak{B}'$ such that $\mathfrak{B}'\restriction_{\chi'} = \mathfrak{B}$ and $\mathfrak{B}'\restriction_{\Sigma'} = \mathfrak{A}'$, where (a) X' is the translation of X along χ, (b) $\chi' \colon \Sigma[X] \to \Sigma'[X']$ is the extension of χ mapping each variable $(x, s, \Sigma) \in X$ to $(x, \chi(s), \Sigma') \in X'$, and (c) $\iota_{X'} \colon \Sigma' \hookrightarrow \Sigma'[X']$ is an inclusion.

Proof.

We define $\mathfrak{B}'$ as follows: (a) $\mathfrak{B}'$ interprets each symbol in Σ' as $\mathfrak{A}'$, and
(b) $(x, \chi(s), \Sigma')^{\mathfrak{B}'} = (x, s, \Sigma)^{\mathfrak{B}}$ for all $(x, \chi(s), \Sigma') \in X'$.
It is straightforward to check that $\mathfrak{B}'\restriction_{\chi'} = \mathfrak{B}$ and $\mathfrak{B}'\restriction_{\Sigma'} = \mathfrak{A}'$.
For the uniqueness part, assume a $\Sigma'[X']$-model $\mathfrak{C}$ such that $\mathfrak{C}\restriction_{\Sigma'} = \mathfrak{A}'$ and $\mathfrak{C}\restriction_{\chi'} = \mathfrak{B}$.

1. Since $\mathfrak{C}\restriction_{\Sigma'} = \mathfrak{A}'$, $\mathfrak{C}$ interprets all symbols in Σ' as $\mathfrak{A}'$ and $\mathfrak{B}'$.
2. Since $\mathfrak{C}\restriction_{\chi'} = \mathfrak{B}$, we have $(x, \chi(s), \Sigma')^{\mathfrak{C}} = \chi(x, s, \Sigma)^{\mathfrak{C}} = (x, s, \Sigma)^{\mathfrak{C}\restriction_{\chi'}} = (x, s, \Sigma)^{\mathfrak{B}} = (x, s, \Sigma)^{\mathfrak{B}'\restriction_{\chi'}} = \chi(x, s, \Sigma)^{\mathfrak{B}'} = (x, \chi(s), \Sigma')^{\mathfrak{B}'}$, for all variables $(x, \chi(s), \Sigma') \in X'$.

By (1) and (2), $\mathfrak{B}' = \mathfrak{C}$. $\square$

$$
\begin{tikzcd}
\Sigma'[X'] & \Sigma' \arrow[l, hook', dotted, "\iota_{X'}"'] & \mathfrak{A}' \arrow[d, dotted] & \models_{\Sigma'} \arrow[d, Leftrightarrow] & \chi(\gamma) \\
\Sigma[X] \arrow[u, dotted, "\chi'"] & \Sigma \arrow[l, hook', dotted, "\iota_X"] \arrow[u, "\chi"'] & \mathfrak{A}'\restriction_\chi & \models_{\Sigma} & \gamma \arrow[u, dotted]
\end{tikzcd}
$$

Proof of Theorem 31.

We proceed by induction on the structure of sentences:

- $\mathfrak{A}' \models_{\Sigma'} \chi(t_1 = t_2)$ iff $\mathfrak{A}' \models_{\Sigma'} \chi(t_1) = \chi(t_2)$ iff $\chi(t_1)^{\mathfrak{A}'} = \chi(t_2)^{\mathfrak{A}'}$ iff $t_1^{\mathfrak{A}'\restriction_\chi} = t_2^{\mathfrak{A}'\restriction_\chi}$ iff $\mathfrak{A}'\restriction_\chi \models_\Sigma t_1 = t_2$.
  The case corresponding to $\pi(t_1, \ldots, t_n)$ is similar to the one above.
- $\mathfrak{A}' \models_{\Sigma'} \chi(\neg\gamma)$ iff $\mathfrak{A}' \models_{\Sigma'} \neg\chi(\gamma)$ iff $\mathfrak{A}' \not\models_{\Sigma'} \chi(\gamma)$ iff $\mathfrak{A}'\restriction_\chi \not\models_\Sigma \gamma$ iff $\mathfrak{A}'\restriction_\chi \models_\Sigma \neg\gamma$.
  The case corresponding to $\bigvee \Gamma$ is similar to the one above.
- $\mathfrak{A}' \models_{\Sigma'} \chi(\exists X \cdot \gamma)$ iff $\mathfrak{A}' \models_{\Sigma'} \exists X' \cdot \chi'(\gamma)$ iff $\mathfrak{B}' \models_{\Sigma'[X']} \chi'(\gamma)$ for some $\iota_{X'}$-expansion $\mathfrak{B}'$ of $\mathfrak{A}'$ iff $\mathfrak{B}'\restriction_{\chi'} \models_{\Sigma[X]} \gamma$ for some $\iota_{X'}$-expansion $\mathfrak{B}'$ of $\mathfrak{A}'$. By Lemma 33, this is equivalent to $\mathfrak{B} \models_{\Sigma[X]}$ for some ι_X-expansion $\mathfrak{B}$ of $\mathfrak{A}'\restriction_\chi$ iff $\mathfrak{A}'\restriction_\chi \models_\Sigma \exists X \cdot \gamma$.

$\square$