RELATIVISTIC KINEMATICS

LORENTZ TRANSFORMATIONS

- in special relativity, laws of physics equally hold for all inertial reference systems.

  - inertial system : Newton's 1st law

    objets in motion stay in motion, } unless a force exerted.
    objects at rest stay at rest }

  - any system moving with constant v with respect to an inertial system is also inertial.

  - for S, S' two inertial frames,

    $S' \rightarrow \bar{v}$ with respect to S
    $S \rightarrow -\bar{v}$ with respect to S'

  - Transformations:

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

  - Inverse transformations:

$$x = \gamma(x' + vt')$$

$$y = y'$$

$$z = z'$$

$$t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

  - consequences of the transformations:

1. relativity of simultaneity

2 events occuring at S, in the same time but at different places, don't happen at the same time in S'.

$$t_A = t_B$$

$$t_A' = t_B' + \frac{\gamma v}{c^2}(x_B - x_A)$$

events that are simultaneous in one inertial system are not simultaneous in others.

2. Lorentz Contraction

a moving object gets shortened by a factor of γ, compared to its length in the system at rest.

it only applies in the direction of the motion. perpendicular directions are not affected.

$$x' = L' \longrightarrow x' = L'/\gamma$$

3. Time Dilation

a difference of elapsed time between 2 events measured by observers either moving relative to each other, or differently situated from a gravitational mass.

clocks in S tick longer $(T = \gamma T')$

moving clocks run slow.

moving particles last longer than they would while at rest.

4. Velocity Addition

particle moving in x-dir with speed u', wrt S'. then its speed u wrt S would be:

$$\Delta x = \gamma(\Delta x' + v\Delta t') \qquad \Delta t = \gamma\,[\Delta t' + (v/c^2)\Delta x']$$

$$\frac{\Delta x}{\Delta t} = \frac{\Delta x' + v\Delta t'}{\Delta t' + (v/c^2)\Delta x'} = \frac{(\Delta x'/\Delta t') + v}{1 + (v/c^2)\Delta x'/\Delta t'} = u$$

$$\frac{\Delta x'}{\Delta t'} = u'$$

$$u = \frac{u' + v}{1 + (u'v/c^2)}$$

FOUR-VECTORS

- position-time 4-vector

$$x^{\mu}, \quad \mu = 0, 1, 2, 3$$

- lorentz transformations:

$$x^{0\prime} = \gamma(x^0 - \beta x^1)$$
$$x^{1\prime} = \gamma(x^1 - \beta x^0)$$
$$x^{2\prime} = x^2 \qquad\qquad \beta \equiv \frac{v}{c}$$
$$x^{3\prime} = x^3$$

or in compact form:

$$x^{\mu\prime} = \sum_{0}^{3} \Lambda^{\mu}_{\nu} x^{\nu} \qquad (\mu = 0, 1, 2, 3)$$

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{aligned} \Lambda^0_0 &= \Lambda^1_1 = \gamma \\ \Lambda^1_0 &= \Lambda^0_1 = -\gamma\beta \\ \Lambda^2_2 &= \Lambda^3_3 = 1 \end{aligned}$$

- ex: $x^{0\prime} = \Lambda^0_0 x^0 + \Lambda^0_1 x^1 + \Lambda^0_2 x^2 + \Lambda^0_3 x^3 = \gamma x^0 - \gamma\beta x^1$

$$= \gamma(x^0 - \beta x^1)$$

- invariance:

$$I \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x^{0\prime})^2 - (x^{1\prime})^2 - (x^{2\prime})^2 - (x^{3\prime})^2$$

can be thought as the rotational invariance of $r^2 = x^2 + y^2 + z^2$

- g metric $g_{\mu\nu}$

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

- now I can be written as:

$$I = \sum_{\mu=0} \sum_{\nu=0} g_{\mu\nu} x^{\mu} x^{\nu} = g_{\mu\nu} x^{\mu\nu} \longrightarrow I = x_{\mu} x^{\mu}$$

- covariant 4-vector: $x_{\mu} = g_{\mu\nu} x^{\nu}$

- contravariant 4-vector: $x^{\mu} = g^{\mu\nu} a_{\nu}$
- a^{μ}, b^{μ} any 4 vectors

$$a^{\mu} b_{\mu} = a_{\mu} b^{\mu} = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 \quad \rightarrow \text{invariant.}$$

- $a \cdot b \equiv a_{\mu} b^{\mu}$, but don't confuse this with $\underbrace{\bar{a} \cdot \bar{b}}_{\rightarrow \text{3 vector}}$, the scalar prod.

$$a \cdot b = a^0 b^0 - \bar{a} \cdot \bar{b}$$

$$a^2 \equiv a \cdot a = (a^0)^2 - \bar{a}^2$$

here, a^2 can be

$a^2 > 0$ $\quad a^{\mu} \longrightarrow$ timelike : 2 events same place, different time

$a^2 < 0$ $\quad a^{\mu} \longrightarrow$ spacelike : 2 events same time, different places

$a^2 = 0$ $\quad a^{\mu} \longrightarrow$ lightlike : BC. light can travel in this interval

Energy and Momentum

- moving clock runs slow, ground time (maybe observer time?) changes infinitesimally by $d\tau$

$$d\tau = \frac{dt}{\gamma}$$

- at "normal" speeds $\gamma \rightarrow 1$, $d\tau = dt$

- BUT in particle physics, there is a difference between lab time and particle time. we can transform and get both from one another but it's not convenient.

- Best thing to do is work with "proper time" where all observes can read the particle's watch at any given time and agree on the value. They own watches may be different.

- velocity of the particle means the distance it travels (lab frame) divided by the time it takes. (lab clock)

$$v = \frac{dx}{dt}$$

- proper velocity $\rightarrow \eta$ $\rightarrow$ distance travelled divided by proper time

$$\eta \equiv \frac{dx}{d\tau}$$

- two velocities are related by a factor of γ

$$\eta = \gamma v$$

- proper velocity as 4-vector:

$$\eta^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, v_x, v_y, v_z)$$

$$\eta^0 = \frac{dx^0}{d\tau} = \frac{d(ct)}{(1/\gamma)dt} = \gamma c$$

- invariant = $\eta_\mu \eta^\mu = \gamma^2(c^2 - v_x^2 - v_y^2 - v_z^2) = \gamma^2 c^2 (1 - v^2/c^2) = c^2$

- Now, we wanna define momentum relativisticly. It's velocity * mass, but should we use regular, or proper velocity?

regular $v = mv$ = conservation of momentum $\rightarrow$ if holds for one inertial system wouldn't hold in others.

proper $v = m\eta$ = conservation of momentum $\rightarrow$ holds for one system, holds for every inertial system.

- CAREFUL $\rightarrow$ this does not Guarantee that the momentum IS conserved. That depends on the experiments. It means that IF you're hoping to extend momentum cons. to the relativistic domain, you better use $m\eta$

- $p^\mu = m\eta^\mu$

- relativistic momentum three vector:

$$\bar{p} = \gamma m \bar{v} = \frac{m\bar{v}}{\sqrt{1 - v^2/c^2}}$$

- Relativistic energy:

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}}$$

- so p^0, 0^{th} component of $p^\mu = \frac{E}{c}$

$\longrightarrow p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right)$

$\longrightarrow \eta_\mu \eta^\mu = \gamma^2(c^2 - v_x^2 - v_y^2 - v_z^2) = \gamma^2 c^2(1 - v^2/c^2) = c^2$

$$\underbrace{p^\mu = m\eta^\mu \qquad p_\mu = m\eta_\mu}$$

$$p^\mu p_\mu = m^2 \eta^\mu \eta_\mu = m^2 c^2 = \frac{E^2}{c} - \underbrace{p_x^2 - p_y^2 - p_z^2}_{\bar{p}^2}$$

- Expand the energy using Taylor

$$E = mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots\right)$$

$$= \underbrace{mc^2}_{\text{cnst.}} + \underbrace{\frac{1}{2}mv^2}_{\text{classical K.E}} + \frac{3}{8}m\frac{v^4}{c^2} + \cdots$$

$E = mc^2$ (or R) $\longrightarrow$ rest energy