Problems.pdf

Question 1

(a) for $-2 \leq x \leq 0$:

$$E[X] = \int_{-2}^{0} x \left(-\frac{x}{4}\right) dx$$

$$= \left[\frac{-x^3}{12}\right]_{-2}^{0} = -\frac{0^3}{12} - \left(-\frac{(-2)^3}{12}\right) =$$

$$= 0 - \left(\frac{8}{12}\right) = -\frac{2}{3}$$

for $0 \leq x \leq 1$:

$$E[X] = \int_0^1 x \, x \, dx =$$

$$= \left[\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

$$E[X] = -\frac{2}{3} + \frac{1}{3} = -\frac{1}{3}$$

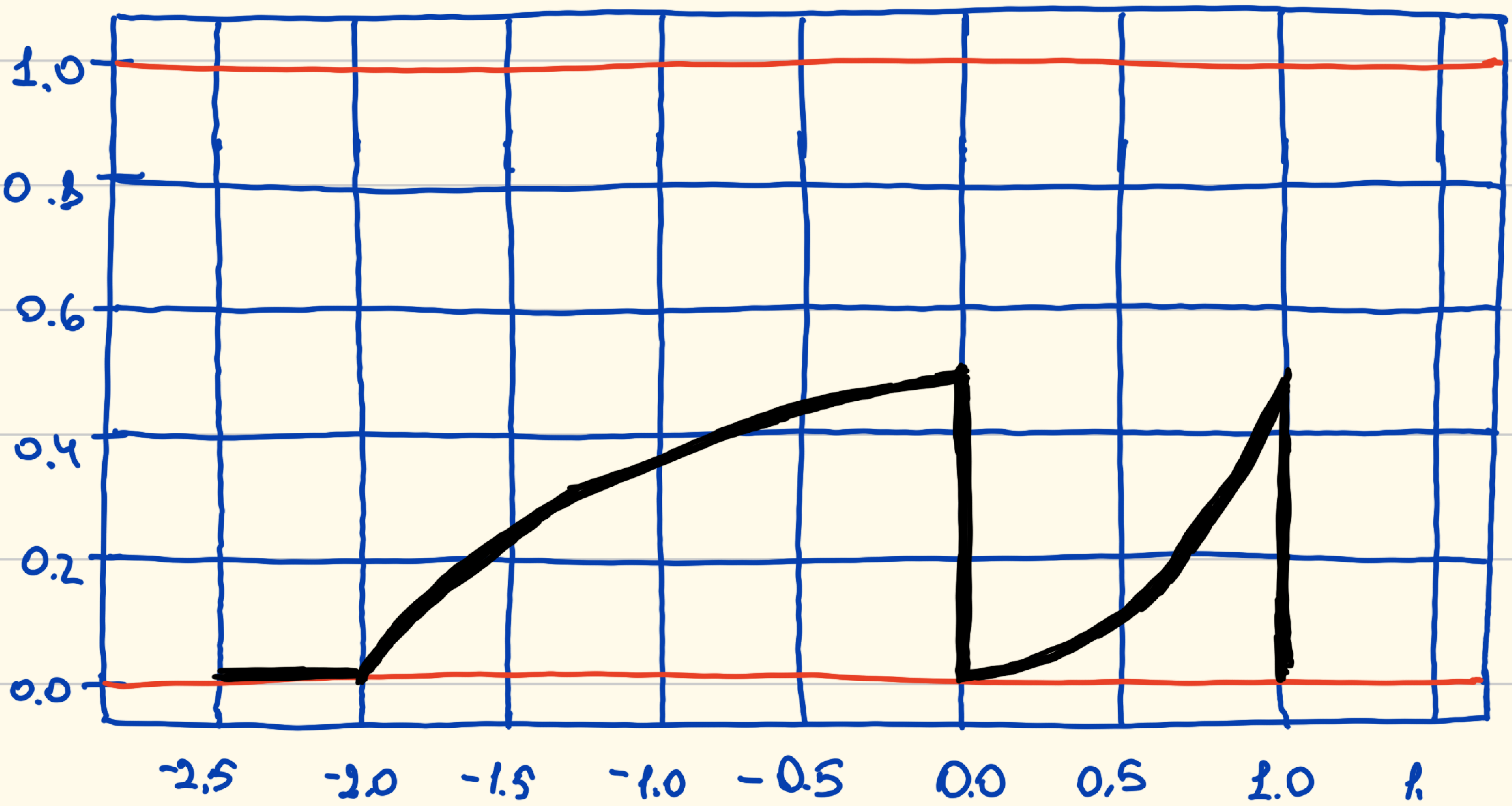
(b) for $-2 \leq x \leq 0$:

$$F_X(x) = \int_{-2}^{x} -\frac{t}{4} dt = \left.\frac{-t^2}{8}\right|_{-2}^{x} = \frac{-x^2}{8} - \left(\frac{-4}{8}\right) = \frac{-x^2}{8} + \frac{1}{2}$$

for $0 \leq x \leq 1$:

$$F_X(x) = \int_0^x t \, dt = \left.\frac{t^2}{2}\right|_0^x = \frac{x^2}{2} - 0 = \frac{x^2}{2}$$

$$F_X(x) = \begin{cases} -\frac{x^2}{8} + \frac{1}{2}, & \text{if } -2 \leq x \leq 0 \\ \frac{x^2}{2}, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{if } x < -2 \text{ or } x > 1 \end{cases}$$



— CDF $F_X(x)$

— Limit at $x=1$; $x<-2$ or $x>1$

ⓒ $P(X>0)$ → for finding $P(X>0)$, we can use CDF of the random variable X and then find the complement of the event $X \leq 0$

$$P(X>0) = 1 - P(X \leq 0)$$

1. For $x \leq 0$:

$$F_x(0) = \frac{-0^2}{8} + \frac{1}{2} = \frac{1}{2} \rightarrow \text{so, } P(x>0) = 1 - \frac{1}{2} = \frac{1}{2}$$

in this case, the probability that the random variable X is greater than 0 is $\frac{1}{2}$

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d) $$f_{X|A}(x|A) = \frac{f_X(x)}{P(X>0)}$$

We need to compute the conditional pdf $f_{X|A}(x|A)$ for $x>0$. For $0 \leq x \leq 1$, the pdf of X given event A $(X>0)$ will be:

$$f_{X|A}(x|A) = \frac{f_X(x)}{P(X>0)} = \frac{x}{P(A)} = \frac{x}{\frac{1}{2}} = 2x$$

In this case, the conditional pdf $f_{X|A}(x|A)$ for $X>0$ is $2x$ within the interval $0 \leq x \leq 1$.

Question 2

(a) $$f_p(p) = \begin{cases} \frac{1}{0.7-0.3} = \frac{1}{0.4}, & \text{for } 0.3 \le p \le 0.7 \\ 0, & \text{otherwise} \end{cases}$$

$$P(H) = \int_{0.3}^{0.7} p\, f_p(p)\, dp = \int_{0.3}^{0.7} \frac{1}{0.4} p\, dp = \frac{p^2}{0.8}\Big|_{0.3}^{0.7} =$$

$$= \frac{0.49}{0.8} - \frac{0.09}{0.8} = \frac{0.4}{0.8} = \frac{1}{2}$$

answer is $\frac{1}{2}$

(b) $$P(E) = \binom{10}{7} \times \left(\frac{1}{2}\right)^7 \times \left(\frac{1}{2}\right)^3 = \binom{10}{7} \cdot \left(\frac{1}{2}\right)^{10} =$$

$$= 120 \times \frac{1}{1024} = \frac{120}{1024} = \frac{30}{256} = \frac{15}{128} = P(E)$$

$$P(H|E) = \frac{P(E|H)\,P(H)}{P(E)} =$$

Question 3

(a) Each guest selects a box randomly out of 100 boxes. The probability of a guest not finding the prize in a box is 99/100, as there is only 1 box with the prize out of 100 boxes

The probability that none of the guests find the prize is (99/100) raised to the power of 50 (prob. of not finding the prize for each guest multiplied together for all 50 guests)

Prob. of the guests winning the game is the complement of the prob. that they lose. So:

Prob. of guests winning = 1 − prob of guests losing

Prob. of guests winning $= 1 - \left(\frac{99}{100}\right)^{50} \approx 0.395$

answer is 0.395.

(b) Each guest selects a unique number from 1 to 100. That's why the probability that all guests miss the prize in a single try is:

$$\underbrace{\frac{99}{100} \cdot \frac{98}{99} \cdot \frac{97}{98} \cdot \ldots \cdot \frac{50}{51}}_{\substack{\text{the prob. of} \\ \text{not finding the box} \\ \text{for each guest}}} = \frac{50}{100} = \frac{1}{2}$$

$1 - \frac{1}{2} = \frac{1}{2} \rightarrow$ the probability that guests win the game with the certain strategy

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(c) In the first strategy, if first player finds the prize, he/she takes it and the rest of the players gets nothing. If the last player finds the prize, he/she takes it and previous players don't get anything. If any player except the first and the last one finds the prize, both the first and the last players get nothing. So the expected value is 1 for the player who finds the prize and is 0 for the rest of the players who doesn't find the prize.

prob. of finding the prize for a single player $\rightarrow \frac{1}{100}$

value of finding the prize for a single player $\longrightarrow 100$

Expected value for a single player $\longrightarrow \frac{1}{100} \cdot 100 = 1$

In the second strategy, it doesn't matter who finds the price, all players have the same expectation value.

prob. of finding the prize for a single player $\rightarrow \frac{1}{100}$

value of finding the prize for a single player $\longrightarrow \frac{100}{50} = 2$

Expected value for a single player $\longrightarrow \frac{1}{100} \cdot 2 = 0.02$

Question 4

ⓐ $\lambda_1 = 0.3$

$\lambda_2 = 0.5$

$\lambda_3 = 0.7$

The expected number of accidents for a Poisson distribution is equal to its parameter λ.

for highway 1 : $\lambda_1 = 0.3$

for highway 2: $\lambda_2 = 0.5$

for highway 3: $\lambda_3 = 0.7$

Expected number of accidents that will happen on any of these highways can be found by adding the expected number of accidents for each highway:

Expected number of daily accidents on any highway $= \lambda_1 + \lambda_2 + \lambda_3 = 0.3 + 0.5 + 0.7 =$
$= 1.5$

So the expected number of daily accidents on any of these highways is 1.5

(b) The probability of no accidents occuring on a particular road that follows a Poisson distribution with parameter λ is given by the PMF of the Poisson distribution;

$$P(x=0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda}$$

for highway 1: $P(\text{no accidents}) = e^{-0.3}$

for highway 2: $P(\text{no accidents}) = e^{-0.5}$

for highway 3: $P(\text{no accidents}) = e^{-0.7}$

$$P(\text{no accident on any road}) = e^{-0.3} \cdot e^{-0.5} \cdot e^{-0.7} =$$
$$= e^{-1.5} \approx 0.223$$

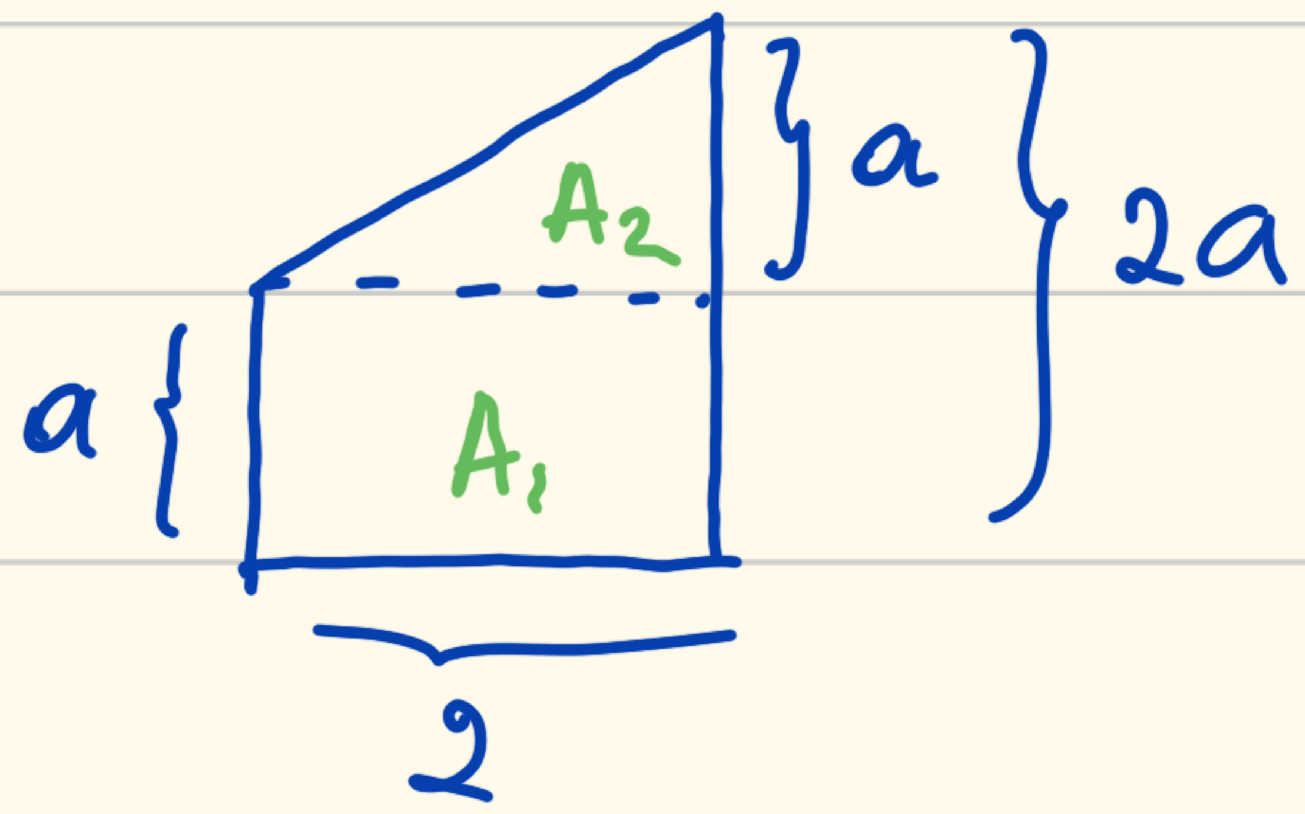
$$P\left(\begin{matrix}\text{at least one accident} \\ \text{on any road}\end{matrix}\right) = 1 - P(\text{no accident on any road})$$

$$P\left(\begin{matrix}\text{at least one accident} \\ \text{on any road}\end{matrix}\right) = 1 - 0.223 = 0.777$$

Answer is 0.777

Question 5

(a) The area under the curve is:



$A_1 = 2 \cdot a = 2a$

$A_2 = \frac{a \cdot 2}{2} = a$

The total area under the curve shoul be equal to 1. → in this case: $3a = 1$

$a = \frac{1}{3}$

answer is $a = \frac{1}{3}$

(b) $f_X(x) = \frac{ax}{2} + a$

$$E[X] = \int_0^2 x\, f(x)\, dx = \int_0^2 x \cdot \left(\frac{a}{2}x + a\right) dx =$$

$$= \int_0^2 \left(\frac{ax^2}{2} + ax\right) dx = \left[\frac{ax^3}{6} + \frac{ax^2}{2}\right]_0^2 =$$

$$= \left(\frac{8a}{6} + \frac{4a}{2}\right) - 0 = \frac{4a}{3} + \frac{4a}{2} = \frac{8a + 12a}{6} = \frac{20a}{6} =$$

$$= \frac{10}{3}a$$

$E[x] = \frac{10}{3} a \Rightarrow$ we found $a = \frac{1}{3}$ in a).

$\downarrow$

$\frac{10}{3} \cdot \frac{1}{3} = \frac{10}{9}$ $\qquad E[x] = \frac{10}{9}$

answer is $E[x] = \frac{10}{9}$

ⓒ $E[x^2] = \int_0^2 x^2 \cdot f(x)\, dx$ $\qquad f(x) = \frac{a}{2} x + a$

$$E[x^2] = \int_0^2 x^2 \cdot \left(\frac{a}{2} x + a\right) dx = \int_0^2 \left(\frac{a x^3}{2} + a x^2\right) dx =$$

$$= \left[\frac{a x^4}{8} + \frac{a x^3}{3}\right]_0^2 = \left(\frac{16a}{8} + \frac{8a}{3}\right) - 0 = 2a + \frac{8a}{3} =$$

$$= \frac{6a}{3} + \frac{8a}{3} = \frac{14a}{3} \rightarrow a = \frac{1}{3} \rightarrow \frac{14}{3} \cdot \frac{1}{3} = \frac{14}{9}$$

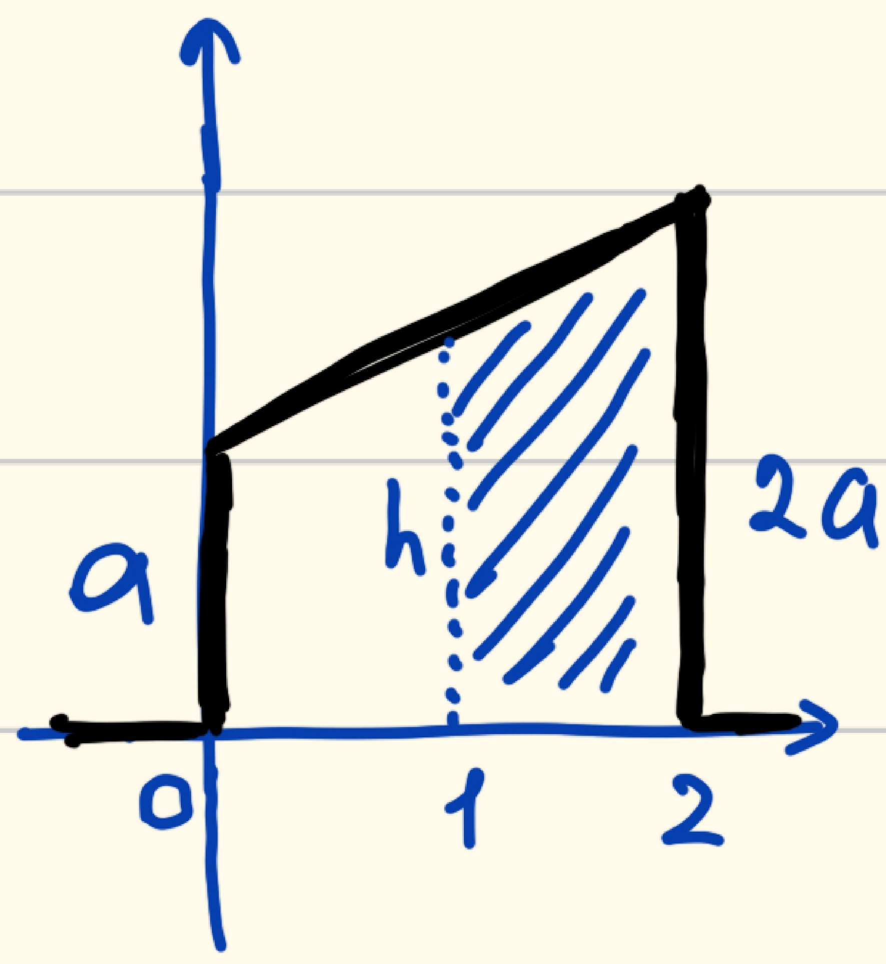
we found $E[x] = \frac{10}{9}$

$$\sigma_x^2 = E[x^2] - (E[x])^2 = \frac{14}{9} - \frac{100}{81} = \frac{126}{81} - \frac{100}{81} = \frac{26}{81}$$

$$\sigma_x^2 = \frac{26}{81}$$

ⓓ $f_{X|A}(x|A) = \dfrac{f_X(x)}{P(A)}$

The prob of the event A (i.e. $X>1$) is the area under the curve of the PDF from $X=1$ to $X=2$



$\rightarrow \quad h = \dfrac{a+2a}{2} = \dfrac{3a}{2}$

the area under the curve of the PDF from $X=1$ to $x=2$ $\Bigg\}$ $\dfrac{\frac{3a}{2}+2a}{2} \cdot 1 = \dfrac{\frac{7a}{2}}{2} \cdot 1 = \dfrac{7a}{4}$

we found $a = \dfrac{1}{3}$ from a).

so, $\dfrac{7a}{4} \Rightarrow \dfrac{7}{4} \cdot \dfrac{1}{3} = \dfrac{7}{12}$ $\rightarrow$ Therefore, $P(A) = \dfrac{7}{12}$

$$f_X(x) = \frac{ax}{2} + a \longleftarrow a = \frac{1}{3}$$

$$\downarrow$$

$$f_X(x) = \frac{x}{6} + \frac{1}{3}$$

$$\int_1^2 \left(\frac{x}{6} + \frac{1}{3}\right) dx = \left[\frac{x^2}{12} + \frac{1}{3}x\right]_1^2 = \left(\frac{4}{12} + \frac{2}{3}\right) - \left(\frac{1}{12} + \frac{1}{3}\right) =$$

$$= 1 - \frac{5}{12} = \frac{7}{12}$$

$$f_{X|A}(x|A) = \frac{f_X(x)}{P(A)} \longrightarrow \frac{\frac{7}{12}}{\frac{7}{12}} = 1$$

answer is 1.

Question 6

(a) The curve $y = \ln(x+1)$ intersects the x-axis when

$y=0$:

$0 = \ln(x+1) \rightarrow x+1 = e^0 \rightarrow x+1=1 \rightarrow x=0$

it intersects the line $x=1$ when:

$y = \ln(1+1) = \ln 2$

in this case, the limits of integration for x are from 0 to 1 and for y are from 0 to $\ln 2$

$$\int_0^1 \int_0^{\ln 2} c x e^y \, dy \, dx = 1$$

$$c \int_0^1 \left[x \int_0^{\ln 2} e^y \, dy \right] dx = 1 \longrightarrow c \int_0^1 \left[x \left(e^{\ln 2} - e^0 \right) \right] dx = 1$$

$$c \int_0^1 \left(x(2-1) \right) dx = 1 \longrightarrow c \int_0^1 x \, dx = 1$$

$$c \left[\frac{x^2}{2} \right]_0^1 = 1 \longrightarrow c \cdot \frac{1}{2} = 1 \qquad c=2 \qquad \text{answer: } c=2$$

(b) $f_X(x) = \int_0^{\ln 2} c x e^y \, dy$

$f_X(x) = c x \int_0^{\ln 2} e^y \, dy \longrightarrow f_X(x) = c x \left[e^y \right]_0^{\ln 2}$

$f_X(x) = c x \left(e^{\ln 2} - e^0 \right) \longrightarrow f_X(x) = \underset{2}{\underset{\downarrow}{c}}\, x$

$f_X(x) = 2x$

$f_Y(y) = c e^y \int_0^1 x \, dx \longrightarrow f_Y(y) = c e^y \left[\frac{x^2}{2} \right]_0^1$

$f_Y(y) = c e^y \cdot \frac{1}{2} \longrightarrow f_Y(y) = \frac{\overset{2}{\overset{\uparrow}{c}}}{2} e^y$

$f_Y(y) = e^y$

$f_X(x) = 2x$ for $0 \leq x \leq 1$

$f_Y(y) = e^y$ for $0 \leq y \leq \ln 2$

ⓒ $P(X>0.5 \text{ and } Y<0.5) = \int_0^{0.5}\int_{0.5}^{1} 2x e^y \, dx \, dy$

$\int_0^{0.5} e^y \left[x^2\right]_{0.5}^{1} dy = \int_0^{0.5} (e^y - 0.25 e^y) \, dy =$

$= \int_0^{0.5} (0.75 e^y) \, dy = 0.75 \int_0^{0.5} e^y \, dy = 0.75 \left[e^y\right]_0^{0.5} =$

$= 0.75 (e^{0.5} - e^0) = 0.75 (e^{0.5} - 1)$

$\approx 0.75 \cdot (1.6487 - 1) \approx 0.75 \cdot 0.6487 \approx 0.4865$

answer: $P(X>0.5 \text{ and } Y<0.5) \approx 0.4865$

$\frac{7}{10}$ $\quad 11 \cdot \frac{7}{10}$