Listing Maximal k-Plexes in Large Real-World Graphs

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April 1, 2022

1. Background
2. Algorithms
3. Implementation Techniques
4. Experiments

Finding Cohesive Groups

Finding ***cohesive groups*** (or ***communities***) has received attention from various areas.

- In the WWW, identify clients sharing similar interests and serve them with a common proxy to reduce network traffic.
- In social networks, discover closely related individuals.
- In biological networks, predict the structure and function of protein.

Clique Model

Naturally, cohesive groups can be modeled with ***Cliques***.
A clique is a subgraph where vertices are pairwise connected, i.e., a complete graph.

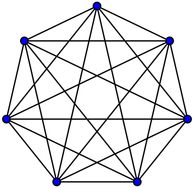


Figure 1: K_7

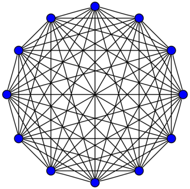


Figure 2: K_{12}

k-Plex Model

Due to various reasons like *data noise*, communities rarely appear as cliques.
k-Plexes allow every vertex missing at most $k-1$ links to other vertices.

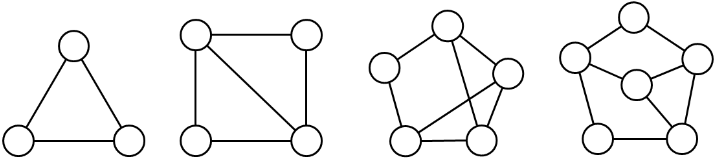


Figure 3: 1, 2, 3, 4-plex

Properties

Lemma (Hereditary Property)

- *Any induced subgraph of a k-plex is still a k-plex.*
- *A k-plex is maximal if it is not a subgraph of any larger k-plex.*

Lemma (Distance Property)

- *Any k-plex with at least $2k - 1$ vertices has its diameter at most 2.*
- *A k-plex with at most $2k - 2$ vertices may be unconnected.*

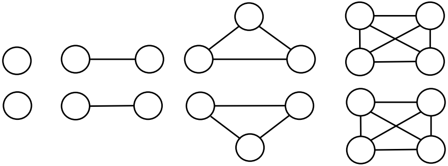


Figure 4: unconnected $2, 3, 4, 5$-plex

Translated Problems

We model cohesive groups with ***maximal k-plexes***.

Problem (Listing maximal k-plexes)

Given a graph G, a positive integer k, list all maximal k-plexes from G.

Problem (Listing large maximal k-plexes)

Given a graph G, two positive integers k and l where $l \geq 2k - 1$, list all maximal k-plexes with at least l vertices.

The Bron-Kerbosch Algorithm

Our algorithm stems from the Bron-Kerbosch algorithm, say *BKPlex*.
BKPlex accepts three sets P, C and X as parameters,

- P: ***plex vertices*** of the growing k-plex,
- C: ***candidate vertices*** for further branching,
- X: ***excluded vertices*** to avoid non-maximal solutions.

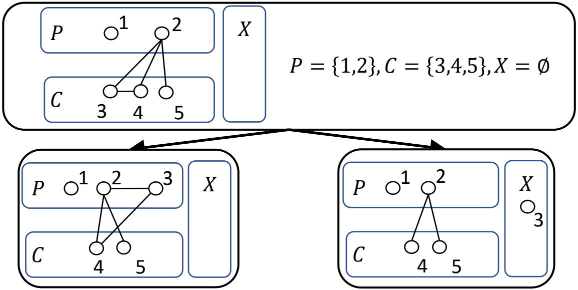
then lists maximal k-plexes $G[P']$ with three properties:

- $P \subseteq P'$
- $P' \subseteq P \cup C$
- $\forall v \in X$, the subgraph $G[\{v\} \cup P']$ is not a k-plex.

Hereditary Prop. Any induced subgraph of a k-plex is still a k-plex.

The Bron-Kerbosch Algorithm

BKPlex branches by doing bipartition recursively. [1]



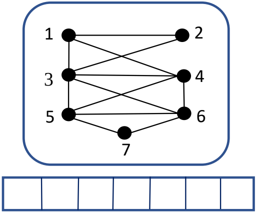
Vtx	1	2	3	4	5
Br1	P	P	**P**	**C**	**C**
Br2	P	P	**X**	**C**	**C**

Figure 5: An example of BKPlex.

[1] Simplified for clarity. In fact, a variant of bipartition.

Degeneracy Ordering

- $\eta = v_1, \ldots, v_n$ is called *degeneracy ordering* (*core ordering*) if each vertex v_i has the minimum degree in the induced subgraph $G[\{v_i, \ldots, v_n\}]$.
- Given a degeneracy ordering $\eta = v_1, \ldots, v_n$, the degree of v_i in $G[\{v_i, \ldots, v_n\}]$ is called the *core number* of v_i.
- For any degeneracy ordering of the same graph, the largest core number among all vertices is a constant D called *degeneracy*.
- Due to the sparsity of many real-world graphs, $D \ll \Delta \ll n$ where Δ is the maximum degree.



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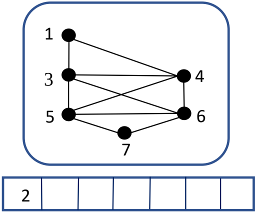

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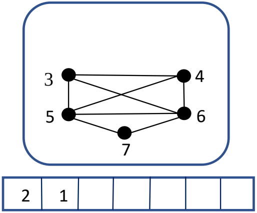

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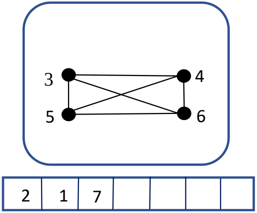

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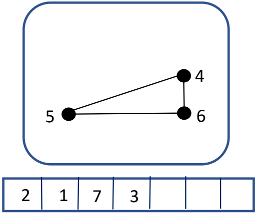

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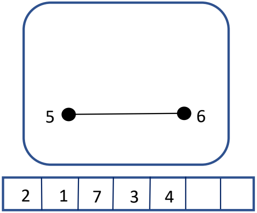

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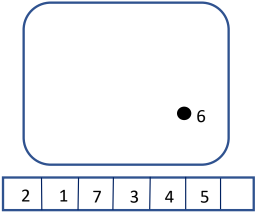

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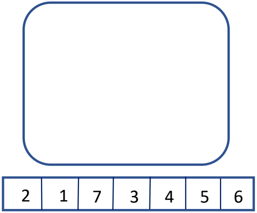

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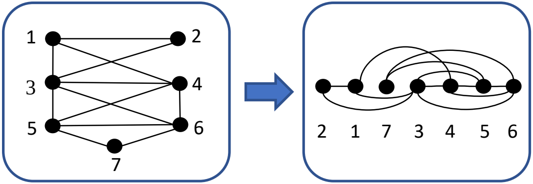

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Recent Algorithms [3]

Pivot Heuristic Zhou et al. (2020) proposed *BKPivot* with a *pivot* heuristic in the branch scheme of BKPlex, and improved its running time from $O^*(2^n)$ to $O^*(\gamma_k^n)$, where $\gamma_k < 2$. [2]

Graph Decomposition Conte et al. (2018) proposed *D2K*, a decomposition-based algorithm for listing k-plexes with the diameter at most 2.
For each vertex, D2K builds a local subgraph and then adopts BKPlex to list maximal k-plexes locally which runs in $O^*(2^{D\Delta})$.

Distance Prop. Any k-plex with at least $2k-1$ vertices has the diameter at most 2.

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We combine them and propose *ListPlex* running in $O^*(\gamma_k^D)$.

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[3] Simplified for clarity.

ListPlex
Listing All Maximal k-Plexes

Based on Distance Property, ListPlex divides its task into two parts.

- *Part I*: k-plexes with size at most $2k-2$. (Solved directly by BKPlex)
- ***Part II***: k-plexes with size at least $2k-1$.

Distance Prop. Any k-plex with at least $2k-1$ vertices has the diameter at most 2.

ListPlex
Part II

Procedure

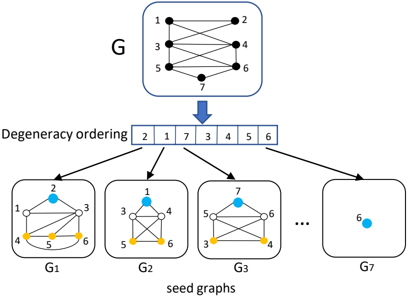
1) Sort V by a degeneracy ordering $\eta = v_1 \dots v_n$.

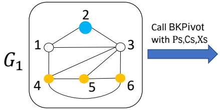
- $N^k(v)$ denotes k-hop neighbors of v.
- $N^k_{\succ}(v_i)$ denotes $N^k(v_i) \cap \{v_{i+1}, ..., v_n\}$, say ***forward*** k-hop neighbors of v_i

2) From v_1 to v_n, in the i-th iteration, list maximal k-plexes with i as the minimum index of vertices.

- Build ***seed graph*** $G_i = G[\{v_i\} \cup N_{\succ}(v_i) \cup N^2_{\succ}(v_i)]$.
- Call BKPivot with given combinations of $N^2_{\succ}(v_i)$, say ***seed set*** S.
- Validate maximality in G for k-plexes generated in G_i.

ListPlex
Part II



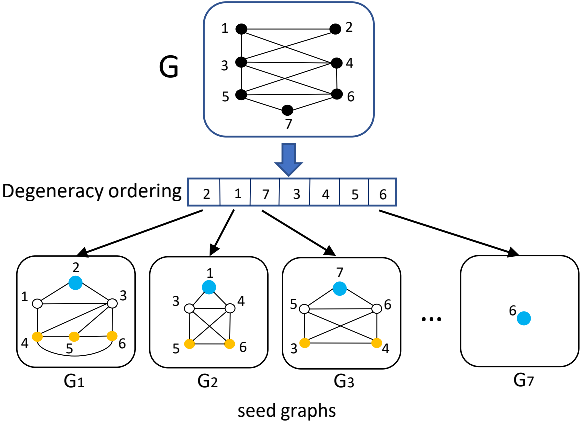


S	Ps	Cs	Xs
∅	{2}	{1,3}	{4,5,6}
{4}	{2,4}	{1,3}	{5,6}
{5}	{2,5}	{1,3}	{4,6}
{6}	{2,6}	{1,3}	{4,5}
{4,5}	{2,4,5}	{1,3}	{6}
{4,6}	{2,4,6}	{1,3}	{5}
{5,6}	{2,5,6}	{1,3}	{4}

Figure 7: An example of ListPlex's Part II.
(L) sort V in degeneracy ordering η and induce seed graphs G_i for each $v_i \in \eta$.
(R) enumerate $S \subseteq N^2_{\succ}(v_i)$ with bound $|S| \leq k-1$ ($k=3$) and call BKPivot with P_s, C_s, X_s.

ListPlex

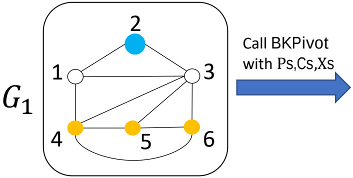
Graph Decompostion



- Cohesive groups appear locally in large real-world graphs,
- Parallelism,
- Smaller scale and better locality,
- $D \ll \Delta$.

ListPlex

Seed Set



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- At most $k-1$ vertices come from $N^2_{\succ}(v_i)$,
- Reducing candidates fast,
- $|N^2_{\succ}(v_i)|$ is potentially $D\Delta$,
- Pruning rules.

With lower bound l, a related lower bound l' can be derived.

Lemma

Assume $|P| \geq l$, for any vertex pair $u, v \in P$, $|N(u) \cap N(v) \cap P| \geq l'$

Removing unfruitful candidates from G_i reduces the scale of G_i.

- Consider vertex pair (v_i, u), $u \in V_i$.

Dropping unfruitful seed sets S saves the forthcoming exponential search.

- Consider vertex pair (u, v) from some S.

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Reducing cache misses

Validating maximality in G for maximal k-plex of G_i suffers a high amount of cache misses.

Alleviation:

- Build a bipartite graph B_i for each G_i, which serves as a cache of currently useful data of large G,
- Pruning bipartite graph just like seed graph.

Parallelization

ListPlex owns appealing parallel features.

How To:

- Parallelize searches of maximal k-plexes on each G_i, say T_i,
- Split T_i when some cores are idle for better load balance,
- Construct degeneracy ordering and perform generated T_i in parallel.

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Dataset

All graphs are taken from SNAP and LAW [4].

Table 1: Considered networks and their properties

Network	n	m	Δ	D
jazz	198	2742	100	29
ca-grqc	5241	14484	81	43
gnutella08	6301	41554	97	10
wiki-vote	7116	100763	1065	53
lastfm	7624	55612	216	20
as-caida	26475	53381	2628	22
soc-epinions	75888	405739	3044	67
soc-slashdot	82144	500480	2548	54
email-euall	265214	365569	7636	37
amazon0505	410236	2439436	2760	10
in-2004	1353703	13126172	21869	488
soc-pokec	1632803	22301964	14854	47
as-skitter	1696415	11095298	35455	111
soc-livejournal	4847571	68993773	14815	360
arabic-2005	22744080	639999458	575628	3247
uk-2005	39459925	936364282	1372171	584
it-2004	41291594	1150725436	1243927	3209
webbase-2001	118142155	1019903190	816127	1506

[4] http://law.di.unimi.it/

Listing All Maximal k-Plexes

Table 2: Listing all maximal k-plexes in small graphs

Network	k	#k-plexes	The running time (s)				Speedup
			BKPlex	BKPivot	ListPlex	ListPlex(16)	
jazz	2	35214	648.864	0.29	**0.086**	0.408	0.211
jazz	3	3602575	772.826	17.55	**6.477**	0.832	7.785
jazz	4	193056583	3226.746	829.40	**417.646**	26.187	15.949
ca-grqc	2	13718439	OOT	1858.02	**649.985**	40.880	15.899
gnutella08	2	19866959	1500.208	3627.57	**1117.858**	70.207	15.922
wiki-vote	2	66193264	10356.553	10671.92	**1526.884**	95.656	15.962
lastfm	2	29086855	2643.394	6676.89	**1989.701**	124.525	15.978

Listing Large Maximal k-Plexes

Table 3: The running time of listing large maximal k-plexes from small and medium graphs by CommuPlex[5], D2K[6] and ListPlex.

Graph	k	l	#k-plexes	The running time (s)		
$(\lvert V\rvert, \lvert E\rvert)$				CommuPlex	D2K	ListPlex
jazz (198, 2742)	4	12	2745953	25.218	33.054	**4.498**
lastfm (7624, 55612)	4	12	1827337	20.724	23.991	**4.586**
as-caida (26475, 53381)	3	12	281251	5.684	13.421	**0.867**
	4	12	15939891	300.388	785.506	**47.98**
amazon0505 (410236, 2439436)	2	12	376	1.825	0.641	**0.137**
	3	12	6347	11.359	0.77	**0.286**
	4	12	105649	47.049	5.338	**1.171**
email-euall (265214, 365569)	2	12	412779	8.793	11.199	**1.946**
	3	12	32639016	619.384	1043.266	**91.62**
		20	2637	10.754	53.691	**0.429**
	4	20	1707177	825.126	3800.889	**24.089**
soc-slashdot (82144, 500480)	2	12	27208777	376.071	213.141	**59.42**
		20	11411028	227.016	137.159	**32.988**
		30	453	10.77	16.481	**0.688**
	3	12	2807943240	OOT	26029.006	**7813.045**
		20	1303148522	28361.707	15308.777	**4538.022**
		30	1679468	699.876	2066.598	**51.364**
	4	30	502699966	OOT	OOT	**6680.261**
as-skitter (1696415, 11095298)	2	50	47969775	OOT	OOT	**520.884**
	2	100	0	1.793	2.951	**0.716**
	3	50	21070497438	OOT	OOT	OOT
	3	100	0	2.37	3.285	**0.718**
in-2004 (1353703, 13126172)	2	50	25855779	7663.843	576.06	**150.212**
	2	100	9978037	5899.638	256.225	**72.063**
	3	50	29045783792	OOT	OOT	OOT
	3	100	4257410159	OOT	OOT	**28384.76**

Graph	k	l	#k-plexes	The running time (s)		
$(\lvert V\rvert, \lvert E\rvert)$				CommuPlex	D2K	ListPlex
wiki-vote (7116, 100763)	2	12	2919931	75.871	115.757	**17.653**
		20	52	4.52	11.289	**0.591**
		30	0	1.033	**0.027**	0.091
	3	12	458153397	OOT	OOT	**2185.598**
		20	156727	595.636	1852.186	**9.384**
		30	0	1.072	**0.029**	0.1
	4	20	46729532	OOT	OOT	**1174.2**
		30	0	9.17	3.627	**0.112**
soc-pokec (1632803, 22301964)	2	12	7679906	1537.506	172.987	**47.475**
		20	94184	1064.371	20.03	**15.161**
		30	3	662.64	**8.637**	9.557
	3	12	520888893	OOT	OOT	**1607.285**
		20	5911456	1470.536	856.393	**46.262**
		30	5	717.425	**9.993**	10.127
	4	20	318035938	34048.155	OOT	**1825.216**
		30	4515	1140.117	111.987	**11.211**
soc-epinions (75888, 405739)	2	12	49823056	843.9	735.589	**193.307**
		20	3322167	137.427	180.061	**19.382**
		30	0	8.995	12.109	**0.492**
	3	20	548634119	27037.614	35525.693	**3072.267**
		30	16066	546.69	2591.439	**6.123**
	4	30	13172906	OOT	OOT	**661.103**
com-livejournal (4847571, 68993773)	2	340	650322	2284.435	OOT	**109.382**
	2	345	0	57.548	13589.487	**6.914**
	3	340	555718694	OOT	OOT	**22863.467**
	3	345	3963139	24861.871	OOT	**826.183**

[5] Zhou et al. (2020)
[6] Conte et al. (2018)

Listing Large Maximal k-Plexes

Table 4: The parallel running time of large networks by ListPlex and D2K with 16 threads.

Graph ($\lvert V\rvert$, $\lvert E\rvert$)	k	l	#k-plexes	The running time (s) D2K(16)	ListPlex(16)
arabic-2005 (22744080, 639999458)	2	800	224870903	2195.272	**714.159**
	2	1000	236897	151.328	**40.202**
	3	800	>25062182205	OOT	OOT
	3	1000	34155502	587.967	**128.737**
uk-2005 (39459925, 936364282)	2	250	106243475	OOT	**355.855**
	2	500	256406	318.118	**35.001**
	3	250	>18336111409	OOT	OOT
	3	500	28199814	9506.661	**121.726**
it-2004 (41291594, 1150725436)	2	2000	675111	340.904	**41.983**
	2	3000	675111	307.735	**38.468**
	3	2000	197679229	4254.456	**724.979**
	3	3000	197679229	4235.389	**715.002**
webbase-2001 (118142155, 1019903190)	2	800	1599005	374.134	**54.19**
	2	1000	1164383	346.393	**53.651**
	3	800	1785341050	36116.817	**5521.386**
	3	1000	1484341137	35005.343	**6960.816**

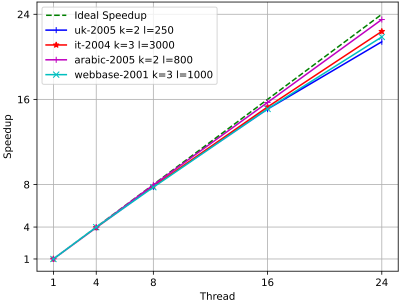

Figure 8: The speedup of ListPlex for the large graphs with different parameters.

Excluding unfruitful seed sets

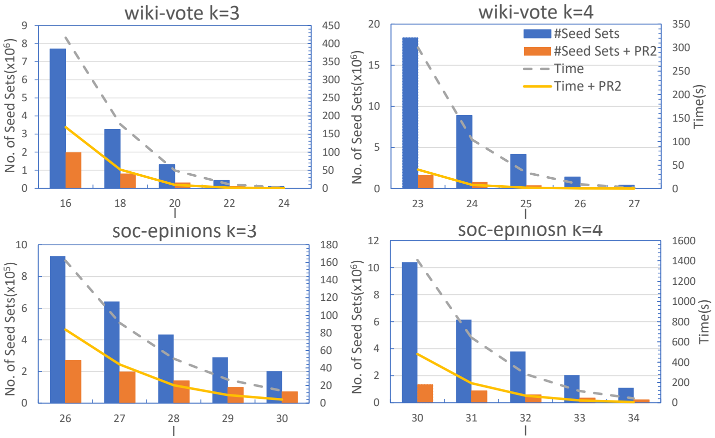

Figure 9: The number of seed sets and running time with and without Prune Rule 2.

Reducing cache misses

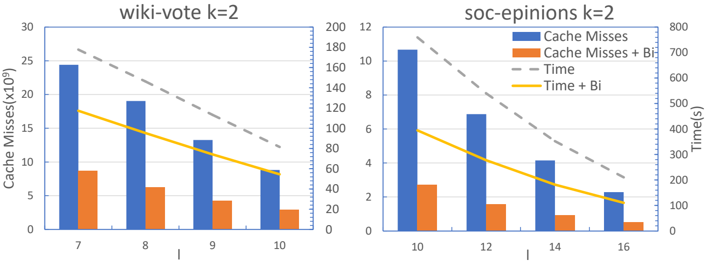

Figure 10: The total number of data cache misses and the running time with and without using bipartite graph B_i.

Q&A

Thanks!