# From KAM Tori to Quantum Localization: Classical Standard Map Diagnostics and Quantum Kicked Rotor via Floquet-FFT


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**Abstract**

We study the classical kicked rotor through the Chirikov standard map and track how the dynamics changes from mostly regular motion (KAM tori and resonance islands) to widespread chaos as the kick strength $K$ increases.[1] We combine qualitative phase-space diagnostics (Poincaré sections on the torus) with quantitative ones: finite-time Lyapunov exponent (FTLE) maps on a grid of initial conditions, estimates of the largest Lyapunov exponent $\lambda(K)$ (with special attention to the critical region around $K_c \simeq 0.971635$), and momentum transport on the cylinder via the mean-squared displacement (MSD) $\langle (p_n - p_0)^2 \rangle$.[1]

As a quantum counterpart, we implement the quantum kicked rotor using a one-kick Floquet operator and a symmetric split-operator FFT scheme on a finite basis ($N = 2048$) up to $n_{\max} = 4000$ kicks, with effective Planck constant $\hbar_{\text{eff}} = 1.0$.[2] In the classically chaotic regime we observe short-time diffusive-like growth followed by dynamical localization: $\langle p^2 \rangle$ saturates (estimated from a late-time window, here $n \geq 2000$) and the final momentum distribution develops approximately exponential tails.[1, 2] We fit the tail in the window $|m| \in [50, 400]$ to extract a localization length and report the fit quality ($R^2$).[1, 2]


## Contents

# 1 Motivation

Kicked systems provide a compact numerical testbed to explore how a deterministic Hamiltonian model moves from regular motion to chaos as a single control parameter is increased, and how the quantum dynamics can deviate from the classical picture because of interference.[1] The kicked rotor is especially convenient: stroboscopically (kick-to-kick) the classical dynamics reduces to a simple area-preserving map, while the quantum dynamics can be propagated efficiently by alternating diagonal operators in the momentum and angle representations, connected by FFT basis changes.[1, 2]

For small $K$, the standard map is largely regular and many trajectories lie on KAM tori (invariant curves) that act as barriers to global transport.[1] As $K$ increases, resonances generate island chains and chaotic layers develop around unstable structures, leading to a mixed phase space with coexisting regular and chaotic regions.[1]

A common reference point is the critical value $K_c \simeq 0.971635$, associated with the breakup of the last spanning (golden) invariant torus.[1] Beyond this threshold a large connected chaotic sea forms on the torus, and long-range transport pathways become available.[1]

In the quantum kicked rotor, the same parameter regime that is classically diffusive can show dynamical localization: the wavepacket first spreads in momentum, but after a characteristic time the spreading stops and the distribution approaches a localized stationary profile due to quantum interference.[1, 2] This contrast is the main point of the project: classical chaos promotes transport, while quantum interference can arrest it.[1, 2]

# 2 Problem statement and goals

The project goals are:

- Implement the Chirikov standard map and explore phase-space structure for a range of $K$ values (from near-integrable to strongly chaotic regimes).
- Produce phase-space plots $(\theta, p)$ and identify regular and chaotic regions.
- Quantify chaos via finite-time Lyapunov exponents (FTLE) and estimates of the largest Lyapunov exponent $\lambda(K)$, and relate these to the onset of global transport.
- Implement quantum kicked-rotor Floquet propagation and show signatures of dynamical localization in momentum space.

# 3 Theoretical background

## 3.1 Classical kicked rotor and the standard map (Santhanam Sec. 3)

We start from the classical kicked rotor, described by a free-rotor kinetic term interrupted by periodic, instantaneous kicks of a spatially periodic potential.[1] In dimensional variables, one common form is

$$H(\theta, p, t) = \frac{p^2}{2I} + k\cos(\theta) \sum_{n=-\infty}^{\infty} \delta(t - nT), \tag{1}$$

where $\theta$ is the angular coordinate, $p$ is the angular momentum, $I$ is the moment of inertia, $k$ is the kick strength, and $T$ is the time between consecutive kicks.[1]

To obtain a dimensionless map, one rescales time by the kick period, $t \to t/T$, and rescales momentum as $p \to pT$.[1] With this rescaling the Hamiltonian keeps the same structure but the dynamics depends on the single dimensionless control parameter

$$K = kT, \tag{2}$$

often called the stochasticity (or nonlinearity) parameter.[1]

Because the force is impulsive at integer times (in rescaled units), Hamilton's equations can be integrated exactly from one kick to the next, producing a discrete-time stroboscopic evolution known as the Chirikov standard map:[1]

$$p_{n+1} = p_n + K \sin(\theta_n), \tag{3}$$
$$\theta_{n+1} = \theta_n + p_{n+1} \pmod{2\pi}. \tag{4}$$

Here $n$ labels the kick number, $(\theta_n, p_n)$ is the phase-space point just after the $n$-th kick, and "mod $2\pi$" enforces angular periodicity.[1]

### 3.2 Order-to-chaos transition and the critical $K_c$ (Santhanam Sec. 3–4)

For small $K$, most irrational invariant tori survive (KAM scenario) and act as barriers to global transport.[1] As $K$ increases, resonance layers grow and overlap; eventually the last spanning invariant torus breaks, enabling global chaotic transport.[1] A widely used reference value for the breakup of the golden invariant torus is $K_c \approx 0.971635$.[1]

### 3.3 Chaos diagnostics: FTLE and Lyapunov exponent

To quantify sensitivity to initial conditions, we consider the linearized (tangent-map) dynamics and compute finite-time Lyapunov exponents (FTLE) over a grid of initial conditions.[1] Regions with high FTLE indicate strong local instability, whereas near-zero FTLE indicates regular motion on tori or within stable islands.[1]

As a global diagnostic, we estimate the largest Lyapunov exponent $\lambda(K)$ by averaging finite-time tangent-map growth rates over many initial conditions.[1]

### 3.4 Quantum kicked rotor and Floquet operator (Santhanam Sec. 5; Delande notes)

In the quantum model, one kick corresponds to a unitary Floquet step that factorizes into a kick operator (diagonal in angle representation) and a free-evolution operator (diagonal in momentum representation).[2] A central phenomenon is dynamical localization: in parameter regimes that are classically chaotic and diffusive, quantum dynamics can show initial diffusion in momentum followed by saturation of $\langle p^2 \rangle$ and exponentially localized momentum distributions.[1, 2]

### 3.5 Localization observables used in this project (Santhanam Sec. 5)

We focus on:

- $\langle p^2 \rangle(n)$: short-time growth followed by a plateau in the localized regime.[2]
- Final momentum distribution $P(m) = |\psi_m|^2$: approximately exponential tails, allowing a localization-length estimate from a linear fit of $\log P$ vs. $|m|$ over a tail window.[1]
- Time-resolved snapshots $P(m, n)$: visualizing the slowdown of spreading and the approach to a stationary localized profile.[2]

## 4 Code design and execution

### 4.1 Repository layout

The project is split into:

- **Classical (Fortran)**: standard-map iteration, chaos diagnostics (FTLE / Lyapunov), and transport on the cylinder (MSD).
- **Quantum (Python)**: Floquet split-operator FFT solver; CSV/NPZ outputs; plotting and poster-ready figures.

### 4.2 Classical pipeline (standard map)

The classical workflow is:

1. Choose $K$ values and simulation parameters (number of iterations, number of initial conditions, torus vs. cylinder).
2. Generate Poincaré data for multiple initial conditions by iterating the standard map and recording $(\theta_n, p_n)$ after each kick.
3. Compute FTLE on a $(\theta_0, p_0)$ grid to map regular versus chaotic regions.
4. Estimate $\lambda(K)$ by averaging finite-time tangent-map growth rates over the initial-condition grid (and report a spread as an uncertainty band).
5. On the cylinder, compute momentum transport via the mean-squared displacement (MSD) $\langle (p_n - p_0)^2 \rangle$ from ensembles.

**Poincaré sections and phase-space structure.** A Poincaré section here is the stroboscopic phase-space portrait obtained by plotting the sequence $\{(\theta_n, p_n)\}_{n=0}^{N}$ for many initial conditions.[1] Regular trajectories draw smooth invariant curves (KAM tori) or closed island chains, whereas chaotic trajectories fill extended two-dimensional regions.[1] In the torus view we apply modulo $2\pi$ to $\theta$ and, optionally, to $p$; in the Fortran code this behavior is controlled by the boolean flag `periodicp`.[1] For representative phase-space portraits we use

$$K \in \{0.2,\ 0.5,\ 0.8,\ 0.971,\ 1.8,\ 5.5\},$$

which spans near-integrable, mixed, near-critical, and strongly chaotic regimes.[1]

**Finite-time Lyapunov exponent (FTLE).** For each initial condition $(\theta_0, p_0)$ we evolve both the orbit and a small tangent perturbation using the linearized tangent-map dynamics.[1] Linearizing the standard map gives, for variations $(\delta\theta_n, \delta p_n)$,

$$\delta p_{n+1} = \delta p_n + K\cos(\theta_n)\,\delta\theta_n, \qquad \delta\theta_{n+1} = \delta\theta_n + \delta p_{n+1}.$$

[1] After $N$ kicks, the FTLE is estimated as

$$\mathrm{FTLE}_N(\theta_0, p_0) = \frac{1}{N}\ln\left(\frac{\|\delta\mathbf{x}_N\|}{\|\delta\mathbf{x}_0\|}\right), \quad \delta\mathbf{x}_n = (\delta\theta_n, \delta p_n),$$

with optional periodic renormalization of $\delta\mathbf{x}_n$ to avoid overflow while accumulating the log-growth.[1] For visualization we compute FTLE values on a $(\theta_0, p_0)$ grid and use a shared color normalization based on the 99th percentile across panels to keep different $K$ values comparable.[1] When overlaying Poincaré density on top of FTLE maps, low-density histogram bins are masked using a minimal count threshold (here `countclip`=3) so that only robust structures remain visible.[1]

**Estimating $\lambda(K)$.** For each $K$ we compute a set of finite-time Lyapunov exponents over the initial-condition grid and average them to obtain $\lambda(K)$.[1] We also report the standard deviation across initial conditions as a simple measure of the spread of local instability values.[1]

**Momentum transport on the cylinder (MSD).** On the cylinder we do not wrap $p$, so trajectories can drift along the momentum axis.[1] We quantify transport by the mean-squared displacement

$$\mathrm{MSD}(n) = \left\langle (p_n - p_0)^2 \right\rangle,$$

where $\langle\cdot\rangle$ denotes an ensemble average over many initial conditions.[1] In the strongly chaotic regime, $\mathrm{MSD}(n)$ is expected to grow approximately linearly in $n$, so we fit a late-time slope as a numerical diffusion estimate.[1]

### 4.3 Quantum pipeline (Floquet + split-operator FFT)

The quantum solver iterates a one-kick unitary step using:

- Free evolution in momentum space (diagonal phase factor).
- A kick in angle space (diagonal phase factor).
- FFT / inverse FFT to switch between representations.

**Hilbert space, grids, and initial state.** We work in a finite Hilbert space of dimension $N$ with a uniform angle grid $\theta_j \in [0, 2\pi)$ and an integer angular-momentum grid $m \in \{-N/2, \ldots, N/2-1\}$ centered at $m = 0$.[2] In this discretization the physical momentum is $p = m\,\hbar_{\text{eff}}$, hence $\langle p^2 \rangle = \hbar_{\text{eff}}^2 \langle m^2 \rangle$.[2] As initial condition we take a Gaussian wavepacket localized in momentum around $m_0 = 0$ with width $\sigma_{m_0} = 2$,

$$\psi_m(0) \propto \exp\left[-\frac{(m-m_0)^2}{2\sigma_{m_0}^2}\right],$$

and normalize to unit norm.[2]

**One-kick Floquet step and split-operator algorithm.** We use a symmetric split-operator for one Floquet period,

$$U = e^{-i\hat{p}^2/(4\hbar_{\text{eff}})}\, e^{-iK\cos(\hat{\theta})/\hbar_{\text{eff}}}\, e^{-i\hat{p}^2/(4\hbar_{\text{eff}})},$$

implemented via diagonal multiplications in momentum/angle space combined with FFT and inverse FFT.[2] In practice, one kick is applied as:

1. Transform $\psi(\theta)$ to $\psi(m)$ with a unitary FFT convention.
2. Multiply by the half free-evolution phase (diagonal in $m$):

   $$\psi_m \leftarrow e^{-i\,\hbar_{\text{eff}} m^2/4}\, \psi_m,$$

   using $p = m\hbar_{\text{eff}}$.[2]
3. Transform back to $\theta$ by inverse FFT.
4. Multiply by the kick phase (diagonal in $\theta$):

   $$\psi(\theta) \leftarrow e^{-iK\cos\theta/\hbar_{\text{eff}}}\, \psi(\theta).$$

5. Repeat steps 1–3 to apply the second half free-evolution factor.[2]

After each kick we renormalize $\psi$ to unit norm to remove small numerical drift, and we monitor the deviation of the norm from 1 as a unitarity diagnostic.[2]

**When does the evolution stop, and why?** We iterate up to a fixed maximum number of kicks $n_{\max}$ (default $n_{\max} = 4000$) because dynamical localization is a long-time effect: the evolution must extend well beyond the initial transient diffusion to observe saturation of $\langle p^2 \rangle$ and stable exponential tails.[2] For visualization of the approach to the stationary profile, we store momentum snapshots every fixed stride (default every 10 kicks) and we include the $n = 0$ snapshot.[2]

**Choice of parameters** $(K, \hbar_{\text{eff}}, N)$**.** The default run uses $N = 2048$, $n_{\max} = 4000$, $K = 5.5$, and $\hbar_{\text{eff}} = 1.0$.[2] We use $K = 5.5$ because the corresponding classical standard map is strongly chaotic and exhibits diffusive momentum transport on the cylinder, making it a useful benchmark for comparing classical diffusion with quantum localization.[1]

**Observables and fitting criteria.** For $\langle p^2 \rangle(n)$ we plot the raw curve and a centered moving average with a window of 101 kicks to reduce fluctuations.[2] The centered moving average $\mathrm{MA}_{101}(n)$ is defined as

$$\mathrm{MA}_{101}(n) = \frac{1}{101} \sum_{k=-50}^{50} \langle p^2 \rangle(n+k),$$

with edge handling near the beginning/end of the time series so that the smoothed curve has the same length as the original one.[2] We estimate the plateau by taking the mean and standard deviation over the late-time window $n \geq 2000$.[2]

To estimate a localization length from the final momentum distribution $P(m) = |\psi_m|^2$, we use the expectation that the dynamically localized regime produces approximately exponential momentum tails.[1, 2] We assume

$$P(m) \propto e^{-|m|/\xi},$$

so $\log P(m)$ is approximately linear in $|m|$ in the tail region.[1] By fitting $\log P$ vs. $|m|$ with $\log P \approx a + b|m|$ on $|m| \in [50, 400]$, we extract $\xi = -1/b$ as a localization length in momentum-index units.[1] We also report $R^2$ and the number of fitted points to quantify how consistently an exponential describes the tails over the chosen window.[1]

The solver writes:

- `krm2vsn.csv`: $n$, $\langle m^2 \rangle$, $\langle p^2 \rangle$, and unitarity/norm diagnostics.
- `krfinalpm.csv`: final momentum distribution $P(m)$.
- `krfinalptheta.csv`: final angle distribution.
- `krpmsnapshots.npz`: snapshots $(n_{\mathrm{snap}}, m, P(m, n))$ for 2D heatmaps / 3D stacks.

# 5 Figures and discussion

## 5.1 Classical results

**Poincaré sections (torus view).** The Poincaré sections in $(\theta, p)$ for $K = \{0.2, 0.5, 0.8, 0.971, 1.8, 5.5\}$ are displayed as 2D density histograms with logarithmic color normalization to reveal both dense regular curves and sparse chaotic regions.[1] For small $K$ the portrait is dominated by invariant curves (KAM tori), while increasing $K$ produces resonance islands and chaotic layers that progressively invade phase space.[1] Near $K \simeq 0.971$ the last spanning invariant structure is close to destruction, and for $K \gtrsim 1$ a large connected chaotic sea becomes visible.[1]

**FTLE maps and overlays.** We compute FTLE values over a grid of initial conditions and plot them as heatmaps, using a global color scale set by the 99th percentile across all panels to make comparisons between different $K$ meaningful.[1] To connect geometry and instability, we overlay the Poincaré density on top of FTLE maps and mask bins with very low counts (threshold `countclip`=3), so that the overlay highlights only robust structures.[1] High-FTLE regions align with the chaotic sea and separatrix layers, whereas near-zero FTLE traces regular islands and surviving tori.[1]

**$\lambda(K)$ and the critical region.** We estimate the largest Lyapunov exponent $\lambda(K)$ by averaging finite-time tangent-map growth rates over initial conditions and plot $\lambda(K)$ with a $1\sigma$ band.[1] We mark the reference critical value $K_c \simeq 0.971635$ (golden torus breakup) and highlight the closest simulated data point to this value in the plot inset.[1] The rise of $\lambda(K)$ around this region provides a quantitative signature consistent with the emergence of a connected chaotic sea.[1]

**Cylinder transport (MSD).** On the cylinder (unwrapped $p$), we measure transport through the mean-squared displacement $\langle (p_n - p_0)^2 \rangle$.[1] For a strongly chaotic case (here illustrated with $K = 5.5$), we fit a linear law $\mathrm{MSD}(n) \approx a + bn$ on a late-time window $800 \leq n \leq 5000$ and report the slope and $R^2$ as evidence of diffusive transport.[1]

### 5.2 Quantum results

**$\langle p^2 \rangle(n)$: diffusion then saturation.** We propagate a wavepacket for $n_{\max} = 4000$ kicks with $N = 2048$, $K = 5.5$, and $\hbar_{\text{eff}} = 1.0$ using the split-operator Floquet-FFT method.[2] The observable $\langle p^2 \rangle(n)$ shows an initial growth compatible with classical-like diffusion, followed by saturation.[2] To make the long-time trend easier to read, we show both $\langle p^2 \rangle(n)$ and the centered moving average $\mathrm{MA}_{101}(n)$, and we quantify saturation by a plateau mean computed over $n \geq 2000$.[2]

**Exponential momentum tails and localization length.** From the final momentum distribution $P(m) = |\psi_m|^2$, we test exponential localization by fitting $\log P$ vs $|m|$ in the tail window $|m| \in [50, 400]$.[1] Assuming $P(m) \propto e^{-|m|/\xi}$, the fitted slope gives $\xi = -1/b$, and we report the coefficient of determination $R^2$ to quantify how well an exponential describes the tails.[1] This provides a compact localization-length estimate consistent with dynamical localization.[1, 2]

**Snapshots $P(m, n)$.** To visualize the slowdown of spreading, we store snapshots of $P(m, n)$ and plot a heatmap of $\log_{10} P(m, n)$ using a viridis colormap over a momentum window $|m| \leq 300$ (with color range $[-16, 0]$).[2] The heatmap shows the early growth of support in momentum followed by convergence to a stationary localized profile.[2]

## 6 Conclusions and outlook

This project connects classical and quantum aspects of the kicked rotor using reproducible numerical diagnostics.[1] In the classical standard map, increasing $K$ progressively destroys invariant structures (KAM tori), enlarges chaotic regions visible in Poincaré plots and FTLE maps, increases the estimated largest Lyapunov exponent $\lambda(K)$, and enables diffusive transport on the cylinder once global pathways open around the critical region near $K_c \simeq 0.971635$.[1] In the quantum kicked rotor, for parameters corresponding to classically chaotic motion, momentum spreading is eventually suppressed by dynamical localization, visible as saturation of $\langle p^2 \rangle$ and exponential tails in $P(m)$.[1, 2]

Possible extensions include adding noise to study decoherence-induced delocalization, scanning $\hbar_{\text{eff}}$ to approach the classical limit, and analyzing Floquet eigenstates/eigenphases to connect localization to the spectral properties of the evolution operator.[1]

## References


[1] M. S. Santhanam, S. Paul, and J. B. Kannan. Quantum kicked rotor and its variants: Chaos, localization and beyond. *Physics Reports*, 956:1–187, 2022.

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