Lecture 19: Random Walks

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1 Introduction

Let $\{X_i\}$ be iid random variables with $E[|X_1|] < \infty$. Let $S_0 = 0$. Then the process

$$S_n = \sum_{k=1}^{n} X_i \quad n \geq 1$$

is called a **random walk (RW)** process. A random walk is called a Simple Random walk or (SRW) if

$$P[X_1 = 1] = p = 1 - P[X_1 = -1]$$

This has the interpretation of the winnings of a Gambler who plays a simple coin toss game and wins Re.1 if heads and loses Re. 1 if tails. Random walks are useful in analyzing GI/GI/1 Queues, Ruin systems and even stock prices !!

2 Duality in Random Walks

Essentially, if X_i are iid, then $X_1, X_2, \cdots, X_n$ has the same joint distribution as $X_n, X_n - 1, \cdots, X_1$. The first result we shall show that if $E[X_1] > 0$ then the random walk will become positive in finite steps.

Proposition 2.1. *Suppose $X_1, X_2, \cdots$ are iid random variables, $S_n = \sum_k = 1^n X_i$ with $E[X_i] > 0$. If*

$$N = \min\{n > 0 : S_n > 0\}$$

Then $E[N] < \infty$

Proof.

$$\begin{aligned}
E[N] &= \sum_{n=0}^{\infty} P[N > n] && (1)\\
&= \sum_{n=0}^{\infty} P[X_1 \leq 0, X_1 + X_2 \leq 0, \cdots, X_1 + X_2 + \cdots + X_n \leq 0] && (2)\\
&= \sum_{n=0}^{\infty} P[X_n \leq 0, X_n + X_{n-1} \leq 0, \cdots, X_n + X_{n-1} + \cdots + X_1 \leq 0] && (3)\\
&= \sum_{n=0}^{\infty} P[S_n \leq S_{n-1}, S_n \leq S_{n-2}, \cdots, S_n \leq 0] && (4)
\end{aligned}$$

Where we used the duality principle to get (3). Now let us count a renewal at time n when $S_n \leq S_{n-1}, S_n \leq S_{n-2}, \cdots, S_n \leq 0$. Then we get

$$E[N] = \sum_{n=0}^{\infty} P[\text{renewal happens at time n}] \tag{5}$$
$$= 1 + E[\text{No of renewals that occur}] \tag{6}$$

As $S_n \to \infty$ by Strong Law of Large numbers, it follows that the expected number of renewals that occur is finite. Thus $E[N] < \infty$. □

Define the range R_n as the number of distinct values of $(S_0, \cdots, S_n)$.

Proposition 2.2.

$$\lim_{n\to\infty} \frac{E[R_n]}{n} = P[\ \textit{Random walk never returns to zero}]$$

Proof. Define

$$I_k = \begin{cases} 1 & \text{if } S_k \neq S_{k-1}, S_k \neq S_{k-2}, \cdots, S_k \neq S_0 \\ 0 & \text{else} \end{cases}$$

Then

$$R_n = 1 + \sum_{k=1}^{n} I_k$$

Now by expanding and using the Duality principle, we get

$$E[R_n] = 1 + \sum_{k=1}^{n} P[X_1 \neq 0, X_1 + X_2 \neq 0, \cdots, X_1 + X_2 + \cdots + X_k \neq 0] \tag{7}$$
$$= 1 + \sum_{k=1}^{n} P[S_1 \neq 0, S_2 \neq 0, \cdots, S_k \neq 0] \tag{8}$$
$$= \sum_{k=0}^{n} P[T > k] \tag{9}$$

Where T is the first time the RW returns to 0. Since

$$\lim P[T > k] = P[\text{RW never returns to 0}]$$

, we get our result by dividing by n and taking limits. □

Example 2.3. Simple Random Walk. Consider the simple random walk with $P[X_i = 1] = p$. When $p = \frac{1}{2}$, the RW is recurrent and thus

$$P[\text{No Return to 0}] = 0$$

When $p > \frac{1}{2}$, let $\alpha = P[\text{return to } 0 | X_1 = 1]$. Since $P[\text{return to } 0 | X_1 = -1] = 1$ by Law of Large numbers, we have

$$P[\text{Return to 0}] = \alpha p + 1 - p$$

Conditioning on X_2 yields

$$\alpha = \alpha^2 p + 1 - p$$

Solving for α yields

$$\alpha = \frac{1-p}{p}$$

Thus for $p > 1/2$,

$$\frac{E[R_n]}{n} \to 2p - 1$$

And for $p \leq 1/2$

$$\frac{E[R_n]}{n} \to 2(1-p) - 1$$

Proposition 2.4. *In the symmetric RW ($p = 1/2$), the expected number of visits to state k before returning to origin is equal to 1 for all $k \neq 0$.*

Proof. For $k > 0$, let Y denote the number of visits to state k before the first return to origin. Then

$$Y = \sum_{n=1}^{\infty} I_n$$

where

$$I_n = \begin{cases} 1 & \text{if a visit is made to state k at time n and} \\ & \text{there is no return to origin before n} \\ 0 & \text{else} \end{cases}$$

Thus

$$\begin{aligned}
E[Y] &= \sum_{n=1}^{\infty} P[S_n > 0, S_{n-1} > 0, \cdots, S_1 > 0, S_n = k] && (10) \\
&= \sum_{n=1}^{\infty} P[X_1 + X_2 + \cdots X_n > 0, X_2 + X_3 + \cdots + X_n > 0, \cdots, X_n > 0, S_n = k] && (11) \\
&= \sum_{n=1}^{\infty} P[S_n > 0, S_n > S_1, \cdots, S_n > S_{n-1}, S_n = k] && (12) \\
&= \sum_{n=1}^{\infty} P[\text{Symmetric RW hits k for first time at time n}] && (13) \\
&= P[\text{Symmetric RW ever hits k}] = 1 \text{ due to recurrence} && (14)
\end{aligned}$$

$\square$