Lecture 7: Limiting Mean Excess Time, Branching Processes, Delayed Renewal Process

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1 Renewal thmry Contd. - Key Renewal thm-rem and Applications

1.1 Example:

Consider the number of commodities desired by customers at a store follows a distribution G. The ordering policy of the store is as follows: For some fixed $s,\ S$, if the inventory level after serving a customer is x, then the amount ordered is

$$\begin{cases} S-x & \text{if} x < s \\ 0 & \text{if} x \geq s \end{cases}$$

$X(t)$ denote the inventory level at time t. We are interested in finding $\lim_{t\to\infty} \mathbb{P}(X(t)/geqx)$. From alternating renewal process thmry, we have

$$\begin{aligned}\lim_{t\to\infty} \mathbb{P}(X(t) \geq x) &= \mathbb{E}[\text{ON time}] \\ &= \frac{\mathbb{E}[\sum_{i=1}^{N_x} X_i]}{\mathbb{E}[\sum_{i=1}^{N_s} X_i]} = \frac{\mathbb{E}[N_x]}{\mathbb{E}[N_s]}.\end{aligned}$$

where $N_x = \min\{n \in \mathbb{N} : \sum_{i=1}^{n} Y_i > s - x\}$ and $Y_1, Y_2 \dots$ denote the successive customer demands. Since Y_is are iid, we can interpret $N_x - 1$ as the number of renewals till time $S - x$. Y_is are the inter arrival times of the processes. Thus

$$\lim_{t\to\infty} \mathbb{P}(X(t) \geq x) = \frac{m_G(S-x)+1}{m_G(S-s)+1}, s \leq x \leq S.$$

1.2 Limiting Mean Excess Time

Consider a nonlattice renewal process and we are interested in computing the mean excess time of the process. We start by writing the renewal equation of mean excess life time, $\mathbb{E}[Y(t)]$.

$$\begin{aligned}\mathbb{E}[Y(t)] &= \mathbb{E}[Y(t)|S_{N(t)} = 0]F^c(t) + \int_0^t \mathbb{E}[Y(t)|S_{N(t)} = y]F^c(t-y)dm(y) \\ &= \mathbb{E}[X-t|X>t]F^c(t) + \int_0^t \mathbb{E}[X-(t-y)|X>t-y]F^c(t-y)dm(y).\end{aligned}$$

From Key Renewal thmrem, we have

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{E}[Y(t)] &= \frac{1}{\mu}\int_0^\infty \mathbb{E}[X-t|X-t>0]F^c(t)dt \\
&= \frac{1}{\mu}\int_{t=0}^\infty \int_{x=t}^\infty xdF(x)F^c(t)dt \\
&= \frac{1}{\mu}\int_{x=0}^\infty \int_{t=0}^x xdF(x)F^c(t)dt \\
&= \frac{\mathbb{E}[X^2]}{2\mu}.
\end{aligned}$$

Proposition 1.1. *If the inter arrival time is nonlattice and $\mathbb{E}[X^2] < \infty$, by corollary , we have $\mu(m(t)+1) = t + \mathbb{E}[Y(t)]$*

$$\lim_{t\to\infty}(m(t) - \frac{t}{\mu}) = \frac{\mathbb{E}[X^2]}{2\mu^2} - 1.$$

1.3 Age-dependent Branching Process

Suppose an organism lives upto a time period of $X \sim F$ and produces $N \sim P$ number of offspring. Let $X(t)$ denote the number of organisms alive at time t. The stochastic process $\{X(t),\ t \geq 0\}$ is called an age-dependent branching process. We are interested in computing $M(t) = \mathbb{E}[X(t)]$ when $m = \mathbb{E}[N] = \sum_{j\in\mathbb{N}} jP_j$.

Theorem 1.2. *If $X(0) = 1$, $m > 1$ and F is non lattice, then*

$$\lim_{t\to\infty} M(t) = \frac{m-1}{m^2\alpha\int_0^\infty xe^{-\alpha x dF(x)}},$$

where $\alpha > 0$ is unique such that $\int_0^\infty xe^{-\alpha x}dF(x) = \frac{1}{m}$.

Proof. Condition on T_1, the life time of first organism,

$$\begin{aligned}
M(t) &= \int_0^\infty \mathbb{E}[X(t)|T_1 = y]dF(y) \\
&\overset{(a)}{=} \int_{y=0}^t 1dF(y) + \int_{y=t}^\infty mM(t-y)dF(y).
\end{aligned}$$

Thus we get

$$M(t) = F^c(t) + m\int_0^t M(t-y)dF(y) \tag{1}$$

Let α denote the unique positive number such that $\int_0^\infty xe^{-\alpha x}dF(x) = \frac{1}{m}$ and $G(y) = m\int_0^y e^{-alphay}dF(y)$. Upon multiplying both sides of equation (1) by $e^{-\alpha t}$ and defining $f(t) = e^{-\alpha t}M(t)$, $h(t) = e^{-\alpha t}F^c(t)$,

$$
\begin{aligned}
f &= h + f * G \\
&= h + G * (h + f * G) \\
\vdots &= h + h * \sum_{i=1}^{\infty} G_i \\
&= h + h * m_G.
\end{aligned}
$$

Or, $f(t) = h(t) + \int_0^t h(t-s) dm_G(s)$. It can be shown that $h(t)$ is dRi and hence by Key Renewal thmrem,

$$
f(t) \to \frac{\int_o^\infty e^{-\alpha t} F^c(t) dt}{\int_0^\infty x dG(x)}.
$$

$$
\begin{aligned}
\int_o^\infty e^{-\alpha t} F^c(t) dt &= \int_0^\infty e^{-\alpha t} \int_t^\infty dF(x) dt \\
&= \int_0^\infty \int_0^x e^{-\alpha t} dt dF(x) \\
&= \int_0^\infty (1 - e^{-\alpha x}) dF(x) \\
&= \frac{1}{\alpha}(1 - \frac{1}{m}) \text{ (by the definition of } \alpha).
\end{aligned}
$$

Also $\int_0^\infty x dG(x) = m \int_0^\infty x e^{-\alpha x} dF(x)$. Hence the result follows.

□

1.4 Delayed Renewal Process

Let $\{X_n : n \in \mathbb{N}\}$ be independent but $X_1 \sim G$ and $X_i \sim F,\ i \geq 2$ then the counting process $\{N_D(t) : t \geq 0\}$ is called general renewal process or delayed renewal process. Let $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$. We have

$$
\begin{aligned}
N_D(t) &= \sup\{n \in \mathbb{N} : S_n \leq t\}, \\
P(N_D(t) = n) &= P(S_n \leq t) - P(S_{n+1} \leq t) \\
&= G * F^{n-1}(t) - G * F^n(t), \\
m_D(t) = \mathbb{E}[N_D(t)] &= \sum_{n \in \mathbb{N}} G * F^{n-1}(t).
\end{aligned}
$$

Taking the Laplace transform of $m_D(t)$, denoted as $\tilde{m}_D(s) = \frac{\tilde{G}(s)}{1 - \tilde{F}(s)}$.

Proposition 1.3. *The following holds:*

1. $\lim_{t \to \infty} \frac{N_D(t)}{t} = \frac{1}{\mu}$.
2. $\lim_{t \to \infty} \frac{m_D(t)}{t} = \frac{1}{\mu}$.
3. *If F is non-lattice, $lim_{t \to \infty} m_D(t+a) - m_D(t) = \frac{a}{\mu_F}$.*

4. *If F and G are lattice with period d, $\mathbb{E}[\ \#$of renewals at $nd\ ] = \frac{d}{\mu_F}$.*

5. *If F is nonlattice, $\mu < \infty$ and h dRi, then*

$$\lim_{t\to\infty} \int_0^t h(t-x)dm_D(x) = \frac{\int_0^\infty h(t)dt}{\mu}.$$

1.4.1 Example:

Let $\{X_n : n \in \mathbb{N}\}$ be iid discrete observed. A pattern $x_1, x_2 \ldots x_k$ is said to occur at time n if $X_n = x_k,\ X_{n-1} = x_{k-1}, \ldots X_{n-k+1} = x_1$.. If we have iid tosses and consider $N(n)$ as the number of times pattern $0,1,0,1$ appear in n tosses, with $P(H) = p = 1-q$, the process is a delayed renewal processes. To find the mean number of tosses for the first time the pattern $0,1,0,1$ appear,

$$\begin{aligned}\mathbb{E}[\text{first time pattern } 0,1,0,1 \text{ appears}] &= \mathbb{E}[\text{first time pattern } 0,1 \text{ appears}] \\ &+ \mathbb{E}[\text{time between patterns } 0,1,0,1] \\ &= p^{-1}q^{-1} + p^{-2}q^{-2}.\end{aligned}$$

Similarly we can show that $\mathbb{E}[\text{first time } k\text{heads}] = \sum_{i=1}^n p^{-i}$.