

STOPPING TIMES, RANDOM WALKS, BERNOULLI PROCESSES - SOME EXERCISES

E2 – 202 - Random Processes, Fall 2017, ECE, IISc.

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1. Prove:

(a) (*Adam's law*)

$$E[X] = E[E[X|Y]].$$

(b) (*Eve's law*)

$$\text{Var}(X) = E[\text{Var}(X|Y)] + \text{Var}(E[X|Y]),$$

$$\text{where } \text{Var}(X|Y) = E[(X - E[X|Y])^2|Y].$$

2. Suppose $\{\mathcal{F}_n : n \in \mathbb{N}\}$ is a filtration and Y_n is measurable with respect to \mathcal{F}_n . Let N be a stopping time with respect to this filtration. Show that Y_N is measurable with respect to \mathcal{F}_N .

Note: Here, \mathcal{F}_N denotes the stopping time σ -algebra. Use its definition to solve the problem.

3. Let $(X_n)_{n \geq 1}$ be a Markov process on the state space $\mathcal{S} = \{0, 1\}$. Associated with the process $(X_n)_{n \geq 1}$, consider the natural filtration $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$, $n \geq 1$. Define

$$N := \inf\{n \geq 1 : X_{n+1} = 1\}.$$

Assume that $P(N < \infty) = 1$.

(a) Is N a stopping time with respect to $(\mathcal{F}_n)_{n \geq 1}$? Justify your answer.

(b) Show that $(X_n)_{n \geq 1}$ fails to satisfy the strong Markov property with N .

4. Let $(X_i : i \in \mathbb{N})$ be iid with $E|X_1| < \infty$ and $S_n = \sum_{i=1}^n X_i$.

(a) Show that

$$ES_n = n \cdot EX_1.$$

(b) Let N be a random variable taking values in \mathbb{N} with $EN < \infty$ and N be independent of X_i s. Show that

$$ES_N = EN \cdot EX_1.$$

(c) Let $X_i \sim \text{Geo}(p)$ and $N \sim \text{Geo}(q)$ be a stopping time with respect to the process $(X_i : i \in \mathbb{N})$. Show that

$$ES_N = \frac{1}{pq}.$$

5. Consider an iid sequence $(X_n)_{n \geq 1}$ with each X_n being uniformly distributed over the set $\{1, 2, \dots, 10\}$. Think of X_1 as the bonus (in units of INR 10000) given at the end of first year, X_2 as the bonus given at the end of second year, and so on. Define N to be the first time a bonus of 7 is obtained, i.e.,

$$N := \inf\{n \geq 1 : X_n = 7\}.$$

Compute the expected total bonus received up to time N .

6. Let X_1, X_2, \dots be iid uniform on $(0, 1)$. For each $n \geq 1$, let $S_n = X_1 + \dots + X_n$. Define

$$N := \inf\{n \geq 1 : X_n > X_{n-1}\}.$$

(a) Show that $P(N > n) = \frac{1}{n!}$.

(b) Compute $E[S_N]$.

7. (*Gambler's ruin*) Two players A and B play a game with independent rounds where, in each round, one of the players wins \$1 from his opponent; A wins with probability p and B wins with probability $q = 1 - p$. A starts the game with \$ a and B with \$ b . The game ends when one of the players is ruined (i.e., the player's earnings becomes 0).

*(As a means of visualizing the above game, draw a straight line on a piece of paper, and mark $0, 1, 2, 3, \dots$ on it. Imagine yourself as player A . Then, according to the game, you start from the integer a and at each step either move one integer forward with probability p or move one integer backward with probability q . Such a movement is known as a **one-dimensional random walk**. The game ends when you have reached either 0 (which is when your opponent B has won) or $a + b$ (which is when you have won)).*

Assume that $p = q = 0.5$. Let

$$A[k] := P(A \text{ goes on to win the game, when his current earnings is } \$k).$$

(a) Evaluate $A[0]$ and $A[a + b]$.

(b) Express $A[k]$ in terms of $A[k - 1]$ and $A[k + 1]$.

(c) Solve the difference equation obtained in part (b) above using the initial conditions in part (a) to find a closed form expression for $A[k]$.

(d) What is the probability that A ruins B ?

8. Suppose there are n papers in a drawer. You draw a paper and sign it, and then, instead of filing it away, you place the paper back into the drawer. If any paper is equally likely to be drawn each time, independent of all other draws, what is the expected number of papers that you will draw before signing all n papers? You may leave your answer in the form of a summation.