# Lecture 16: Martingales

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## 1 Martingales

A martingale is a type of stochastic process whose definition formalizes the concept of a fair game.

**Definition 1.1.** A stochastic process $\{Z_n,\ n \geq 1\}$ is said to be a martingale process if

1. $E[|Z_n|] < \infty, \quad \text{for all} \quad \text{n}.$
2. $E[Z_{n+1}|Z_1, Z_2, \dots Z_n] = Z_n.$

Taking expectation on both sides of part 2 of the above definition, we get

$$E[Z_{n+1}] = E[Z_n],$$

and so

$$E[Z_{n+1}] = E[Z_1], \text{ for all } n.$$

**Example 1.2.** Let $\{X_i\}$ be a sequence of independent random variables with mean 0. Let $Z_n = \sum_{i=1}^n X_i$. Then, $\{Z_n,\ n \geq 1\}$ is a martingale. This is so because, $E[Z_n] = 0$ and

$$\begin{aligned}
E[Z_{n+1}|Z_1, Z_2 \dots Z_n] &= E[Z_n + X_{n+1}|Z_1, Z_2 \dots Z_n] \\
&= E[Z_n|Z_1, Z_2 \dots Z_n] + E[X_{n+1}|Z_1, Z_2 \dots Z_n] \\
&= Z_n.
\end{aligned}$$

**Example 1.3.** Let $\{X_i\}$ be a sequence of independent random variables with mean 1. Let $Z_n = \Pi_{i=1}^n X_i$. Then, $\{Z_n,\ n \geq 1\}$ is a martingale. This is so because, $E[Z_n] = 1$ and

$$\begin{aligned}
E[Z_{n+1}|Z_1, Z_2 \dots Z_n] &= E[Z_n X_{n+1}|Z_1, Z_2 \dots Z_n] \\
&= Z_n E[X_{n+1}|Z_1, Z_2 \dots Z_n] \\
&= Z_n E[X_{n+1}] \\
&= Z_n.
\end{aligned}$$

**Example 1.4.** Let $\{X_n\}$ be a branching process. Let $X_0 = 1$. Then,

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i,$$

where $Z_i$ represents the number of offspring of the $i^{\text{th}}$ individual of the $(n-1)^{\text{st}}$ generation. conditioning on $X_{n-1}$ yields, $E[X_n] = \mu^n$ where $\mu$ is the mean number of offspring per individual. Then $Y_n$ is a martingale when

$$Y_n = X_n/\mu^n.$$

This is true because $E[Y_n] = 1$ and

$$\begin{aligned} E[Y_{n+1}|Y_1, \dots Y_n] &= \frac{1}{\mu^{n+1}} E[X_{n+1}|Y_1, \dots Y_n] \\ &= \frac{1}{\mu^{n+1}} E[\sum_{i=1}^{X_n} Z_i|Y_1, \dots Y_n] \\ &= \frac{1}{\mu^{n+1}} X_n E[Z_i] = \frac{X_n}{\mu^n} = Y_n. \end{aligned}$$

**Example 1.5.** Let $X, Y_1, Y_2 \dots$ be arbitrary random variables such that $E[|X|] < \infty$. Then

$$Z_n = E[X|Y_1, Y_2, \dots Y_n]$$

is a martingale. The integrability condition is direct to be verified.

$$\begin{aligned} E[Z_{n+1}|Y_1, Y_2, \dots Y_n] &= E[E[X|Y_1, \dots Y_{n+1}]|Y_1, \dots Y_n] \\ &= E[X|Y_1, \dots Y_n]] = Z_n. \end{aligned}$$

Thus the result follows. The above martingale is called the Doob type martingale.

**Example 1.6.** For any sequence of random variables $X_1, X_2 \dots$, the random variables $X_i - E[X_i|X_1 \dots X_{i-1}]$ have zero mean. Define

$$Z_n = \sum_{i=1}^{n} X_i - E[X_i|X_1, X_2, \dots X_{i-1}]$$

is a martingale provided $E[|Z_n|] < \infty$. To verify the same,

$$\begin{aligned} E[Z_{n+1}|Z_1 \dots Z_n] &= E[Z_n + X_n - E[X_n|X_1 \dots X_{n-1}]] \\ &= Z_n + E[X_n - E[X_n|X_1 \dots X_{n-1}]] = Z_n. \end{aligned}$$

## 1.1 Stopping Times

**Definition 1.7.** The positive integer values, possibly infinite, random variable $N$ is said to be a random time for the process $\{Z_n\}$ if the event $\{N = n\}$ is determined by the random variables $Z_1 \dots Z_n$. If $Pr(N < \infty) = 1$, then the random time $N$ is said to be a stopping time.

**Definition 1.8.** Let $N$ be a random time for the process $\{Z_n\}$. Let

$$\bar{Z}_n = \begin{cases} Z_n & : n \le N \\ Z_N & : n > N. \end{cases}$$

$\{\bar{Z}_n\}$ is called the stopped process.

**Proposition 1.9.** *If $N$ is a random time for the martingale $\{Z_n\}$, then the stopped process $\{\bar{Z}_n\}$ is also a martingale.*

*Proof.* We claim that

$$\bar{Z}_n = \bar{Z}_{n-1} + 1_{N \geq n}(Z_n - Z_{n-1})$$

The above equation can be directly verified by considering the two cases separately viz.

1. $N \geq n$: $\bar{Z}_n = Z_n$.
2. $N < n : \bar{Z}_{n-1} = \bar{Z}_n = Z_N$

$$\begin{aligned} E[\bar{Z}_{n+1}|Z_1 \dots \bar{Z}_n] &= E[\bar{Z}_n + 1_{n \leq N}(Z_n - Z_{n-1})|Z_1 \dots \bar{Z}_n] \\ &\overset{(a)}{=} \bar{Z}_n + 1_{n \leq N} E[(Z_n - Z_{n-1})|Z_1 \dots \bar{Z}_n] \\ &= \bar{Z}_n, \end{aligned}$$

□

where in $(a)$ we have used the fact that $N$ is a random time. Also, we have $E[\bar{Z}_n] = E[Z_1]$, for all $n$. Now assume that $N$ is a stopping time. It is immediate

$$that \bar{Z}_n \to Z_N \text{ w.p 1.}$$

But is it true that

$$that E[\bar{Z}_n] \to E[Z_N] \text{ as n } \to \infty.$$

It so turns out that the above is true under some additional regularity constraints only. We state the following theorem without proof.

**Theorem 1.10.** *If either:*

1. *$\bar{Z}_n$ are uniformly bounded, or;*
2. *$N$ is bounded, or;*
3. *$E[N] < \infty$, and there is an $M < \infty$ such that*

$$E[|Z_{n+1} - Z_n||Z_1 \dots Z_n] < M,$$

*then $E[\bar{Z}_n] \to E[Z_N] = E[Z_1]$.*

**Corollary 1.11.** ***Wald's Equation**: If $X_i$, $i \geq 1$, are independent and identically distributed iid with $E[|X|] < \infty$ and if $N$ is a stopping time for $X_1, X_2 \dots$ with $E[N] < \infty$, then*

$$E[\sum_{i=1}^{N} X_i] = E[N]E[X].$$

*Proof.* Let $\mu = E[X]$. Since

$$Z_n = \sum_{i=1}^{n}(X_i - \mu)$$

is a martingale and hence from the previous theorem,

$$E[Z_N] = E[Z_1] = 0.$$

But

$$\begin{aligned} E[Z_N] &= E[\sum_{i=1}^{N}(X_i - \mu)] \\ &= E[\sum_{i=1}^{N}(X_i) - N\mu)] \\ &= E[\sum_{i=1}^{N}(X_i)] - E[N]\mu)]. \end{aligned}$$

Observe that condition 3 for Martingale stopping theorem to hold can be directly verified. Hence the result follows. □