Lecture 12 : Continuous Time Markov Chains

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1 Continuous Time Markov Chains

Consider a continuous time stochastic process $\{X(t),\ t \geq 0\}$ taking on values on the set of non-negative integers. The process $\{X(t),\ t \geq 0\}$ is continuous time Markov chain if for all $s,\ t \geq 0$ and $i,\ j \in \mathbb{N}_0$, $P(X(t+s)|X(u),\ u \in [0,s]) = P(X(t+s)|X(s))$. If the probability is independent of s, the continuous time Markov chain (CTMC) has homogeneous transitions and we denote the probability by $P_{ij}(t)$. Suppose $X(0) = i,\ X(u) = i,\ \forall u \in [0,s]$. We are interested in knowing probabilities of the form $P(X(v) = i,\ v \in [s, s+t]||X(u) = i,\ u \in [0,s])$. To that end, $\tau_i \triangleq \{t \geq 0 : X(t) \neq i | X(0) = i\}$. We could sample the process at these instants and construct a DTMC out of it and study the same. Observe that,

$$
\begin{aligned}
P(\tau_i \geq s+t | \tau_i > s) &= P(X(v) = i,\ v \in [s, s+t] | X(0) = i) \\
&= P(\tau_i \geq t | X(0) = i)
\end{aligned}
$$

τ_i is memoryless and exponentially distributed.

1.1 Alternative way of constructing CTMC

A CTMC is a stochastic process that each time it enters state i,

1. $\tau_i \sim \exp(\nu_i)$.

2. P(Entering state j— state i) $\equiv P_{ij}$ is such that $\sum_{i \neq j} P_{ij} = 1$.

Note that $P_{i,j}$ and τ_i are not dependent. ν_i is called rate of state i. $\nu_i < \infty$. Else we call the state to be instantaneous. A CTMC is a DTMC with exponential sojourn time in each state.

Definition: A CTMC is called "regular" if $P(\#transitions in[0,t] is \text{finite}) = 1,\ \forall t < \infty$. Consider the following example of a non-regular CTMC.
Example: $P_{i,i+1} = 1, \nu_i = i^2$. Show that it is regular.

Let Q be a matrix such that

1. $q_{ij} = \nu_i P_{ij}, \forall i \neq j$.

2. $q_{ii} = -\nu_i$.

Properties of Q :

1. $0 \leq -q_{ii} < \infty, \ \forall i.$

2. $q_{ij} \geq 0, \ \forall i \neq j.$

3. $\sum_j q_{ij} = 0, \ \forall i.$

From the Q matrix, we can construct the whole CTMC. In DTMC, we had the result $P^{(n)}(i,j) = (P^n)_{i,j}$. We can generalize this notion in the case of CTMC as follows: $P = e^Q \triangleq \sum_{k \in \mathbb{N}_0} \frac{Q^k}{k!}$. Observe that $e^{Q_1+Q_2} = e^{Q_1}e^{Q_2}, \ e^{nQ} = (e^Q)^n = P^n$.

Theorem 1.1. *Let Q be a finite sized matrix. Let $P(t) = e^{tQ}$. Then $\{P(t), \ t \geq 0\}$ has the following properties:*

1. *$P(s+t) = P(s)P(t), \ \forall s, \ t$ (semi group property).*
2. *$P(t), \ t \geq 0$ is the unique solution to the forward equation, $\frac{dP(t)}{dt} = P(t)Q, \ P(0) = I$.*
3. *And the backward equation $\frac{dP(t)}{dt} = QP(t), \ P(0) = I$.*
4. *For all $k \in \mathbb{N}$, $\frac{d^kP(t)}{d^k(t)}|_{t=0} = Q^k$.*

Proof. $\frac{dM(t)e^{-tQ}}{dt} = 0$, $M(t)e^{-tQ}$ is constant. $M(t)$ is any matrix satisfying the forward equation. $\square$

Theorem 1.2. *A finite matrix Q is Q matrix if and only if $P(t) = e^{tQ}$ is a stochastic matrix for all $t \geq 0$.*

Proof. $P(t) = I + tQ + O(t^2)$ $(f(t) = O(t) \Rightarrow \frac{f(t)}{t} \leq c$, for small $t, \ c < \infty$). $q_{ij} \geq 0$ if and only if $P_{ij}(t) \geq 0, \ \forall i \neq j$ and $t \geq 0$ sufficiently small. $P(t) = P(\frac{t}{n})^n$. Note that if Q has zero row sums, Q^n also has zero row sums.

$$\sum_j [Q^n]_{ij} = \sum_j \sum_k [Q^{n-1}]_{ik} Q_{kj} = \sum_j \sum_k Q_{kj} [Q^{n-1}]_{ik} = 0.$$
$$\sum_j P_{ij}(t) = 1 + \sum_{n \in \mathbb{N}} \frac{t^n}{n!} \sum_j [Q^n]_{ij} = 1 + 0 = 1.$$

Conversely $\sum_j P_{ij}(t) = 1, \ \forall t \geq 0$, then $\sum_j Q_{ij} = \frac{dP_{ij}(t)}{dt} = 0.$ $\square$

1.2 Kolmogorov Differential Equations

Lemma 1.3. *1.* $\lim_{t \to 0} \frac{1-P_{ii}(t)}{t} = \nu_i$.

2. $\lim_{t \to 0} \frac{P_{ij}(t)}{t} = q_{ij}$.

Lemma 1.4. *For all $s, \ t \geq 0$, $P_{ij}(t+s) = \sum_{k \in \mathbb{N}_0 P_{ik}(t)P_{kj}(s)}$*

1.3 Chapman Kolmogorov Equation for CTMC

Theorem 1.5. ***Kolmogorov Backward equation:*** *For all $i, j, t \geq 0$, $P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t)$, $\frac{dP(t)}{dt} = QP(t)$*

Proof. $P_{ij}(t+h) = \sum_k P_{ik}(h) P_{kj}(t)$.

$$P_{ij}(t+h) - P_{ij}(t) = \sum_{k \neq 1} P_{ik}(h) P_{kj}(t) - (1 - P_{ii}(h)) P_{ij}(t),$$

divide by h, $h \to 0$, we get $\frac{dP_{ij}(t)}{dt} = \lim_{h \to 0} P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t)$. Now the exchange of limit and summation has to be justified.

$$\begin{aligned}
&\liminf_{h \to 0} \sum_{k \neq 1} \frac{P_{ik}(h)}{h} P_{kj}(t) \geq \liminf_{h \to 0} \sum_{k \neq 1} \frac{P_{ik}(h)}{h} P_{kj}(t), k < N \\
&= \sum_{k \neq 1} q_{jk} P_{kj}(t), \;\; k < N.
\end{aligned}$$

This is true for any finite N. Take supremum over all N. We get $\liminf_{h \to 0} \sum_{k \neq 1} \frac{P_{ik}(h)}{h} P_{kj}(t) \geq \sum_{k \neq 1} q_{jk} P_{kj}(t)$. Suffices to show that $\limsup_{h \to 0} \sum_{k \neq 1} \frac{P_{ik}(h)}{h} P_{kj}(t) \leq \sum_{k \neq 1} q_{jk} P_{kj}(t)$. To that end,

$$\begin{aligned}
\text{LHS} &\leq \limsup_{h \to 0} [\sum_{k \neq 1, k < N} \frac{P_{ik}(h)}{h} P_{kj}(t) + \sum_{k \neq 1, k \geq N} \frac{P_{ik}(h)}{h}] \\
&= \limsup_{h \to 0} [\sum_{k \neq 1, k < N} \frac{P_{ik}(h)}{h} P_{kj}(t) + \frac{1 - P_{ii}(h)}{h} \sum_{k \neq 1, k \geq N} \frac{P_{ik}(h)}{h}] \\
&= [\sum_{k \neq 1, k < N} q_{ik} P_{kj}(t) + \nu_i - \sum_{k \neq 1, k \geq N} q_{ik}] \\
&= \sum_{k \neq 1} q_{ik} P_{kj}(t).
\end{aligned}$$

□

Theorem 1.6. ***Kolmogorov Forward Equation:*** *Under suitablle reqularity conditions, $P'_{ij}(t) = \sum_{k \neq i} P_{ik}(t) q_{kj} - P_{ij}(t) \nu_i$, i.e. $\frac{dP(t)}{dt} = P(t)Q$.*