Lecture 10: Examples of Discrete Time Markov Chain

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1 Discrete Time Markov Chains Contd.

1.1 Example 4.3(c) The Age of Renewal Process:

Consider a discret time slotted system. An item is put into use. When the item fails, it is replaced at the beginning of the next time period. P_i is the probability of the failure of the item in i^{th} time period. Assume that the life times are independent. P_i is aperiodic. $\sum iP_i < \infty$. The age of an item is the number of periods the item is in use. Let us denote the age of the item at time n as X_n. Let us denote

$$\lambda(i) \triangleq \frac{P_i}{\sum_{j=i}^{\infty} P_j}$$

$\lambda(i)$ is the probability that an i unit old item fails. $\{X_n\}$ is a Markov chain with transition probability,

$$P(X_n = i_n | X_{n-1} = i) = \begin{cases} \lambda(i) & : i_n = 1 \\ 1 - \lambda(i) & : i_n = i + 1, \end{cases}$$

for $i \geq 1$. To find the equilibrium distribution of this process we solve for $\pi = \pi P$.

$$\begin{aligned} \pi_1 &= \sum_i \pi_i \lambda(i) \\ \pi_{i+1} &= \pi_i (1 - \lambda(i)). \end{aligned}$$

We can solve the above set of equations iteratively to obtain $\pi_{i+1} = \pi_1(1 - \lambda(1))(1 - \lambda(2)) \ldots (1 - \lambda(i)) = \pi_1 \sum_{j=i+1}^{\infty} P_j = \pi_1 P(Y \geq i + 1)$, where Y is the life time of an item. Using $\sum_j \pi_j = 1$, we get $\pi_1 = \frac{1}{E[Y]}$ and $\pi_i = \frac{P(Y \geq i)}{E[Y]}$.

1.2 Example 4.3(D) Time slotted system:

Consider a time slotted system and in each slot every member of a population dies with probability p i.i.d. In each time slot, the number of new members that join the population is according to a Poisson (λ) process. Let X_n denote the number of members of the population at the beginning of time period n. Observe that $\{X_n\}$ is a Markov chain. We are interested in computing the stationary distribution of the Markov chain. To that end, let X_0 be a Poisson random variable with mean α. Then X_0 individuals will be alive at the beginning of slot 1 with probability $(1-p)$. Since the number of new members in the system will be a Poisson random variable with parameter λ, X_1 is distributed Poisson with parameter $\alpha(1-p)+\lambda$. If $\alpha = \alpha(1-p)+\lambda$, the chain would be stationary. Hence by the uniqueness of the stationary distribution is Poisson with mean $\frac{\lambda}{p}$.

1.3 Example 4.3(E)Gibbs sampler:

Gibbs sampler is used to generate values of a random vector $X_1, X_2 \dots X_n$ where it as such difficult to generate values from the mass function $P_{X_1,X_2 \dots X_n}(.)$ The idea is to form a Markov chain whose stationary distribution is given by $P_{X_1,X_2 \dots X_n}(.)$ The Gibbs sampler works as follows: Let $\mathbf{X}^0 = (x_1^0, x_2^0 \dots x_n^0)$ for which $p(x_!^0, x_2^0 \dots x_n^0) > 0$. Now generate x_k^1 for $k = 1, 2 \dots n$ according to $P_{X_k | X_j^0 = x_j^0,\ j \neq k}(.)$. Similarly generate $\mathbf{X}^k$ given $\mathbf{X}^{k-1} = (x_1^{k-1}, \dots x_n^{k-1})$. Observe the $\mathbf{X}^j,\ j \geq 0$ is a Markov chain. It is easy to see that $P_{X_1,X_2 \dots X_n}$ is the stationary probability distribution of the Markov chain.

Consider an irreducible positive recurrent Markov chain with stationary probability $\{\pi(i) : i \in \mathbb{N}_0\}$. Let N denote the number of transitions between successive visits to state 0. Visits to state 0 constitutes renewal instants. Then, the number of visits by time n to state 0, for large n is approximately normally distributed. i.e.

$$\frac{1}{n} \sum_{m=1}^{n} 1_{X_m=0} \rightarrow \mathcal{N}(\frac{n}{E[N]}, \frac{nVar(N)}{E[N]^3}).$$

Note that $n/E[N] = n\pi_0$ and $nVar(N)/(E[N])^3 = nVar(N)\pi_0^3$. Since $Var(N) = E[N^2] - \frac{1}{\pi + 0^2}$, we need to find $E[N^2]$. Let T_n be the number of transitions from n until next visit to state 0. Note that, $\frac{1}{n}\sum_{m=0}^{n} T_m$. If N is the number of transitions between successive visits to state 0, average reward of the process per unit time$=\frac{E[\frac{N(N+1)}{2}]}{E[N]} = \frac{1}{2} + \frac{1}{2}\frac{E[N^2]}{E[N]} = \sum_{i \in \mathbb{N}_0} \pi(i)\mu_{i,0}$. Here $\mu_{i,0} = E[T_n | X_n = i]$. Thus $\mu_{i,0} = 1 + \sum_{j \neq 0} P_{i,j}\mu_{j,0}$.

1.4 Transition Among Classes and Mean Times in Transient States

Proposition 1. *Let R be a recurrent class of states. If $i \in R$, $j \neq R$, then $P_{ij} = 0$.*

Proof. Suppose $P_ij > 0$. Then, as i and j do not communicate with each other, $P_{ji}^n = 0,\ \forall n$. Hence, starting from state i, there is a positive probability of at least P_{ij} that the process will never return to state i. This contradicts the fact that i is recurrent. Hence, $P_{ij} = 0$. □

Let T denote the set of all transient states. Let j be recurrent, $i \in T$. f_{ij} denote the probability of eventually hitting j starting from $i = P_i(T_j < \infty)$.

Proposition 2. *If j is recurrent, then the set of probabilities $\{f_{ij} : i \in T\}$ satisfies*

$$f_{ij} = \sum_{k \in T} P_{ik} f_{kj} + \sum_{k \in K} P_{ik},$$

where R denotes set of states communicating with j.

Proof.

$$\begin{aligned} f_{ij} &= \sum_k P_i(T_i < \infty, X_1 = k) \\ &= P_i(T_j < \infty, X_1 \in T) + P_i(T_j < \infty, X_1 \in R) \\ &= \sum_{k \in T} P_k(T_j < \infty) p_{ik} + \sum_{k \in R} p_{ik} \\ &= \sum_{k \in T} f_{kj} p_{ik} + \sum_{k \in R} p_{ik}. \end{aligned}$$

□

1.5 Gambler's Ruin Problem

Consider a gambler who at each play of the game has $p = P(\text{win}) = 1 - P(\text{loss}) = 1 - q$. Assume successive plays of the game are independent. We are interested in knowing the probability that starting with i units the gambler hits fortune N before hitting 0. Let X_n be the player's fortune at time n. $\{X_n\}$ is a Markov chain. Observe that $P_{00} = 1$. $P_{NN} = 1$. Also observe that $P_{i,i+1} = p = 1 - P_{i,i-1}$, $i = 2, 3 \ldots N-1$. Let $P(X_{n+1} = j | X_n = i)$. The Markov chain has three classes $\{0\}, \{N\}, \{1, 2 \ldots N-1\}$, the first two are recurrent and the last one is transient. Let $f_i \equiv f_{iN}$. We have, from the previous proposition, $f_i = p f_{i+1} + (1-p) f_{i-1}$, $f_0 = 0$, $f_N = 1$. Since $p + q = 1$, we can observe that $f_{i+1} - f_i = \frac{q}{p}(f_i - f_{i-1})$. Since $f_0 = 0$, we can write a recursion and observe that

$$f_i = \begin{cases} \frac{(1-(\frac{q}{p})^i)}{1-(\frac{q}{p})^N} & : p \neq \frac{1}{2} \\ \frac{i}{N} & : p = \frac{1}{2}. \end{cases}$$

As $N \to \infty$

$$f_i = \begin{cases} (1 - (\frac{q}{p})^i) & : p > \frac{1}{2} \\ 0 & : p \leq \frac{1}{2}. \end{cases}$$

Consider a finite state Markov chain with transient states with $T = \{1, hdotst\}$. Let Q be the associated transition matrix. Let m_{ij} denote the expected total number of time periods spent if state j stating form state i.

$$\begin{aligned} m_{ij} &= E[\sum_{n \in \mathbb{N}} 1_{X_n = j} | X_0 = i] \\ &= 1 i = j + \sum_{k \in T} P_{ik} m_{kj}. \end{aligned}$$

$[M]_{ij} = (I)_{ij} + (QM)_{ij}$, $M = [I - Q]^{-i}$. The matrix inverse exists.