Lecture 3: Compound Poisson Process

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1 Compound Poisson Process

One of the characterizations of Poisson process was single arrival in an infinitesimal time. We can generalize that definition to have a random number of arrivals X_n at every arrival instant S_n.

Definition 1.1 (Compound Poisson process). Let $\{X_i\}$ be *iid* random variables. Let $N(t), t \geq 0$ be a Poisson Process with parameter λ independent of $X_i, i \geq 1$. Then the process $X(t)$ defined as

$$X(t) = \sum_{i=1}^{N(t)} X_i$$

is called a **compound Poisson process**.

We derive some properties of compound Poisson Processes in the following.

1.1 Mean

$$\begin{aligned} E[X(t)] = E[\sum_{i=1}^{N(t)} X_i] &= E[E[\sum_{i=1}^{N(t)} X_i | N(t)]] \\ &= \sum_{k=0}^{\infty} E\left[\sum_{i=1}^{k} X_i | N(t) = k\right] \Pr\{N(t) = k\} \\ &= \sum_{k=0}^{\infty} \sum_{i=1}^{k} E[X_i] \Pr\{N(t) = k\} \\ &= E[N(t)]E[X_1] = \lambda t E[X_1]. \end{aligned}$$

1.2 MGF

We leave it as an exercise to show that $M_{X(t)}(\theta) = E[e^{\theta X(t)}] = e^{(M_X(\theta)-1)\lambda t}$.

1.3 A nice counterexample

A Poisson process is not uniquely determined by it's distribution. Let $X_t = Y_t + f(Z + t)$, where Y_t is a Poisson Process and

$$f(t) = t 1_{\{t \in \mathbb{Q}\}}.$$

Let Z be a continuous random variable. Then we can show that $\Pr\{X_t \neq Y_t) = 0$. This is true since

$$\begin{aligned}\Pr\{X_t \neq Y_t\} &= \Pr\{\omega \in \Omega : \quad t + Z(\omega) \in \mathbb{Q}\} \\ &= \Pr\{\omega \in \Omega : Z(\omega) \in \mathbb{Q} - t\} = 0.\end{aligned}$$

The last part follows since $\mathbb{Q} - t$ is a countable set of individual events with probability zero. We can also show that $X(t)$ and $Y(t)$ have same fdds.

$$\Pr\{X_{t_1} = Y_{t_1}, X_{t_2} = Y_{t_2}\} = \sum_{n_1, n_2} \Pr\{X_{t_1} = n_1, X_{t_2} = n_2, Y_{t_1} = n_1, Y_{t_2} = n_2\} = 1.$$

$\{X_t(\omega)\}$ can take non-integer values and is not non-decreasing. Two process can have same distribution but sample path behavior can be quite different.