Lecture 21: Martingales as Random Walks

Parimal Parag

1 Martingales for Random Walks

Let

$$S_n = \sum_{k=1}^{n} X_k \quad n \geq 1$$

denote a random walk. Then we have the following results.

Proposition 1.1. *If X_i can take on the integer values between (inclusive) $-M$ and M, for some finite M, then $S_n, n \geq 0$ is a recurrent DTMC iff $E[X] = 0$.*

Proof. If $EX \neq 0$, the random walk is clearly transient since, it will diverge to $\pm\infty$ depending on the sign of $E[X]$. Now when $E[X] = 0$, S_n is a martingale. Let

$$A = \{-M, -M+1, \cdots, -2, -1\}$$

Assume that the process starts in state i. Now for every $j > i$, define

$$A_j = \{j, j+1, \cdots, j+M\}$$

Let N denote the first time the process is in either A or A_j. Since N is a stopping time, by Doob's Stopping theorem, we have

$$E[S_N] = E[S_0] = i$$

Thus we have

$$\begin{aligned} i = E[S_N] &= E[S_N | S_N \in A] P[S_N \in A] && (1)\\ &+ E[S_N | S_N \in A] P[S_N \in A_j] P[S_N \in A_j] && (2)\\ &\geq -M P[S_N \in A] + j(1 - P[S_N \in A_j]) && (3) \end{aligned}$$

Rearranging this gives us

$$P[S_N \in A] \geq \frac{j-i}{j+M}$$

Thus we have

$$P[\text{process ever enters } A] \geq P[S_N \in A] \geq \frac{j-i}{j+M}$$

As j tends to ∞, we see

$$P[\text{process ever enters } A | \text{starts at } i] = 1 \quad i \geq 0$$

Now let $B = \{1, 2, \cdots, M\}$. Repeat the arguments to show

$$P[\text{process ever enters } B|\text{starts at } i] = 1 \quad i \geq 0$$

Hence we have

$$P[\text{process ever enters } A \cup B|\text{starts at } i] = 1 \quad i \geq 0$$

Thus the process is recurrent. $\square$

Now consider a random walk with $E[X] \neq 0$. For $A, B > 0$, we wish to compute the probability P_A that the walk hits at least A before it hits a value $\leq -B$. Let $\theta \neq 0$ s.t

$$E[e^{\theta X}] = 1$$

Now let $Z_n = e^{\theta S_n}$. We can see that Z_n is a martingale with mean 1. Define N as

$$N = \min\{S_n \geq A \text{ or } S_n \leq -B\}$$

From Doob's Theorem, $E[e^{S_N}] = 1$. Thus we get

$$1 = E[e^{\theta S_N}|S_N \geq A]P_A + E[e^{\theta S_N}|S_N \leq -B](1 - P_A)$$

We can obtain an approximation for P_A byneglecting the overshoots past A or $-B$. Thus we get

$$E[e^{\theta S_N}|S_N \geq A] \approx e^{\theta A}$$
$$E[e^{\theta S_N}|S_N \leq -B] \approx e^{-\theta B}$$

Hence we get,

$$P_A \approx \frac{1 - e^{-\theta B}}{e^{\theta A} - e^{\theta B}}$$

As an assignment, show that

$$E[N] \approx \frac{AP_A - B(1 - P_A)}{E[X]}$$

Example 1.2. Gambler Ruin Consider a simple random walk with probability of increment $= p$. As an exercise, show that $E\left[(q/p)^X\right] = 1$ and thus $e^\theta = q/p$. If A and B are integers, then there is no overshoot and hence, our approximations are exact. Thus

$$P_A = \frac{(q/p)^B - 1}{(q/p)^{A+B} - 1}$$

Suppose $E[X] < 0$ and we wish to know if the random walk ever crosses A. Then

$$\begin{aligned}1 = &E[e^{\theta S_N}|S_N \geq A]P[\text{process crossed } A \text{ before } -B]\\ &+ E[e^{\theta S_N}|S_N \leq -B]P[\text{process crossed } -B \text{ before } A]\end{aligned}$$

Now $E[X] < 0$ implies $\theta > 0$ (Why?). Hence we have

$$1 \geq e^{\theta A}P[\text{process crossed } A \text{ before } -B]$$

Taking B to ∞ yields

$$P[\text{Random walk ever crosses A}] \leq e^{-\theta A}$$

2 Application to G/G/1 Queues and Ruin

2.1 The G/G/1 Queue

For the G/G/1 queue, the limiting distribution of delay is

$$P[D_\infty \geq A] = P[S_n \geq A \text{ for some } n]$$

where

$$S_n = \sum_{k=1}^{n} U_k, \quad U_k = Y_k - X_{k+1}$$

Here Y_i is the service time of the ith customer and X_i is the interarrival duration between customer $i-1$ and customer i. Thus when $E[U] = E[Y] - E[X] < 0$, letting $\theta > 0$ such that

$$E[e^{\theta U}] = E[e^{\theta(Y-X)}] = 1$$

We get

$$P[D_\infty \geq A] \leq e^{-\theta A}$$

Now the exact distribution of D_∞ can be calculated when services are exponential. Hence assume $Y_i \sim exp(\mu)$. Once again,

$$\begin{aligned} 1 &= E[e^{\theta S_N} | S_N \geq A] P[S_n \text{ crossed } A \text{ before } -B] \\ &+ E[e^{\theta S_N} | S_N \leq -B] P[S_n \text{ crossed } -B \text{ before } A] \end{aligned}$$

Let us compute $E[e^{\theta S_N} | S_N \geq A]$ first. Let us condition this on $N = n$ and $X_{n+1} - \sum_{i=1}^{n-1}(Y_i - X_{i+1}) = c$. By the memoryless property, the conditional distribution of Y_n given $Y_n > c + A$ is just $c + A$ plus an exponential with rate μ. Thus we get

$$\begin{aligned} E[e^{\theta S_N} | S_N \geq A] &= E[e^{\theta(A+Y)}] \\ &= \frac{\mu e^{\theta A}}{\mu - \theta} \end{aligned}$$

Now substituting back, we get

$$\begin{aligned} 1 &= \frac{\mu e^{\theta A}}{\mu - \theta} P[S_n \text{ crossed } A \text{ before } -B] \\ &+ E[e^{\theta S_N} | S_N \leq -B] P[S_n \text{ crossed } -B \text{ before } A] \end{aligned}$$

Now as $\theta > 0$, let $B \to \infty$ to get

$$1 = \frac{\mu e^{\theta A}}{\mu - \theta} P[S_n \text{ ever crosses } A]$$

And hence

$$P[D_\infty \geq A] = \frac{\mu - \theta}{\mu} e^{-\theta A}$$

2.2 A Ruin Problem

Suppose claims made to an insurance company follow a renewal process with iid interarrival times $\{X_i\}$. Let the values of the claims also be iid and independent of the renewal process $N(t)$ of their occurence. Let Y_i be the ith claim value. Thus the total value of claims till time t is $\sum_{k=1}^{N(t)} Y_i$. Now let us suppose the insurance company receives money at constant rate c per unit time, $c > 0$. We wish to compute the probability of the insurance company, starting with capital A, will eventually be wiped out or **ruined**. Thus we require

$$p = P\left\{ \sum_{k=1}^{N(t)} Y_i > ct + A \text{ for some } t \geq 0 \right\}$$

As an assignment, show that the company will be ruined if $E[Y] \geq cE[X]$. So let us assume that $E[Y] < cE[X]$. Also the ruin occurs when a claim is made. After the nth claim, the company's fortune is

$$A + c\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} Y_k$$

Letting $S_n = \sum_{k=1}^{n} Y_i - cX_i$ and $p(A) = P[S_n > A \text{ for some } n]$. As S_n is a random walk, we see that

$$p(A) = P[D_\infty > A]$$

Now the results from the G/G/1 queue apply.

3 Blackwell Theorem on the Line

Let S_n denote a random walk where $0 < \mu = E[X] < \infty$. Let

$$U(t) = \#\{n : S_n \leq t\} = \sum_{n=1}^{\infty} I_n$$

Where $I_n = 1$ if $S_n \leq t$ and zero else. Observe that if X_n are nonnegative, then $U(t) = N(t)$. Let $u(t) = E[U(t)]$. Now we prove an analog of Blackwell Renewal Theorem.

Theorem 3.1. ***(Blackwell renewal theorem)*** *If $\mu > 0$ and X_i are not lattice, then*

$$u(t+a) - u(t) \to a/\mu \quad t \to \infty \quad \textit{for } a > 0$$

Let us define a few concepts. We say an **ascending ladder variable of ladder height** S_n occurs at time n when

$$S_n > \max(S_0, S_1, \cdots, S_{n-1})$$

where $S_0 = 0$. We may deduce that since X_i are iid random variables, then the random variables $(N_i, S_{N_i} - S_{N_{i-1}})$ are iid; where N_i denotes the time between the $(i-1)$th and ith random variable. We may analogously define descending ladder variables. Now let $p(p_*)$ denote the probability of ever achieving an ascending/descending ladder variable.

$$p = P\{S_n > 0 \text{ for some } n\}, \quad p_* = P\{S_n < 0 \text{ for some } n\}$$

At each ascension/descension there is a probability p (resp p_*) of achieving another one. Hence the number of ascensions/descensions is geometrically distributed. The number of ascending ladder variables (ascensions) will have finite mean iff $p < 1$. Now as $E[X] > 0$, by SLLN, we deduce that $w.p.1$, there will be infinitely many ascending ladder variables but finitely many descending ones. That is $p = 1$ and $p_* < 1$.

Proof. The successive ascending ladder heights are a renewal process. Let $Y(t)$ be the excess time. Now given the value of $Y(t)$, the distribution of $U(t+a) - U(t)$ is independent of t. (Why?). Hence let us denote

$$E[U(t+a) - U(t)|Y(t)] = g(Y(t))$$

for some function g. Now taking expectations yields

$$u(t+a) - u(t) = E[g(Y(t))]$$

Now since $Y(t) \to^d Y_\infty$ where Y_∞ has the equilibrium distribution, we have $E[g(Y(t))] \to E[g(Y_\infty)]$. The result would be true if we show g is continuous and bounded. We leave that as an exercise. For now, we deduce that the limit exists. Let

$$h(a) = \lim_{t\to\infty} u(t+a) - u(t)$$

This also implies $h(a+b) = h(a) + h(b)$. Thus for some constant c,

$$h(a) = ca$$

Now to get c, let N_t denote the first n for which $S_n > t$. If X_i are upper bounded by M, then

$$t < \sum_{i=1}^{N_t} X_i \leq t + M$$

Taking expectations, and using Wald's Lemma, yields

$$t < E[N_t]\mu \leq t + M$$

Thus

$$\frac{E[N_t]}{t} \to \frac{1}{\mu}$$

If X_i are unbounded, use the truncation arguments done while proving Elementary renewal theorem. Now $U(t)$ can be expressed as

$$U(t) = N_t - 1 + N_t^*$$

where N_t^* is the number of times $S_n \leq t$ after having crossed t. Since N_t^* is not greater than the number of points occuring after N_t when the random walk is less than S_{N_t}, we get

$$E[N_t^*] \leq E[\text{number of } n \text{ such that } S_n < 0]$$

Hence if we argue that RHS of above is finite, then

$$\frac{u(t)}{t} \to \frac{1}{\mu}$$

From the first proposition in Random walks, we have $E[N] < \infty$ where N is the first value of n for which $S_n > 0$. At time N, with positive probability $1 - p^*$, no future value of random walk will fall below S_N. Thus,

$$E[\text{number of } n \text{ where } S_n < 0] \leq \frac{E[N|X_1 < 0]}{1 - p^*} < \infty$$

Now follow the steps illustrated in the Blackwell renewal theorem (original) proof to arrive at the desired result.

$\square$