# Lecture 20: Queues As Random Walks

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## 0.1 GI/GI/1 Queueing Model

Consider a $GI/GI/1$ queue. Customers arrive in accordance with a renewal process having an arbitrary interarrival distribution $F$, and the service distribution is $G$. Let the interarrival times be $X_1,\ X_2\dots$ and let the service times be $Y_1,\ Y_2\dots$ and let $D_n$ denote the delay in queue of the $n^{\text{th}}$ arrival. The following recursion for $D_n$ is easy to verify:

$$D_{n+1} = \begin{cases} D_n + Y_n - X_{n+1} & \text{if } D_n + Y_n \geq X_{n+1} \\ 0 & \text{if } D_n + Y_n < X_{n+1} \end{cases}$$

Let $U_n \equiv Y_n - X_{n+1},\ n \geq 1$,

$$D_{n+1} = \max\{0, D_n + U_n\},\ n \geq 0.$$

Iterating the above relation yields

$$\begin{aligned} D_{n+1} &= \max\{0, D_n + U_n\} \\ &= \max\{0, U_n + \max\{0, D_{n-1} + U_{n-1}\}\} \\ &= \max\{0, U_n, U_n + U_{n-1} + D_{n-1}\} \\ &\vdots \\ &= \max\{0, U_n, U_n + U_{n-1}, \dots U_n + U_{n-1} + \dots U_1\}, \end{aligned}$$

where in the last step we have used the fact that $D_1 = 0$. Hence, for $c > 0$,

$$\begin{aligned} Pr(D_{n+1} \geq c) &= Pr(\max\{0, U_n, U_n + U_{n-1}, \dots, U_n + \dots + U_1\} \geq c) \\ &= Pr(\max\{0, U_1, U_2 + U_1, \dots, U_1 + \dots + U_n\} \geq c), \end{aligned}$$

where the last equality follows from duality. Thus the following proposition holds.

**Proposition 0.1.** *If $D_n$ is the delay in the queue of the $n^{th}$ customer in a GI/GI/1 queue with interarrival times $X_i,\ i \geq 1$, and service times $_i,\ i \geq 1$ then*

$$Pr(D_{n+1} \geq c) = Pr(\textit{the random walk } S_j,\ j \geq 1,\ \textit{crosses } c \textit{ by time } n), \quad (1)$$

*where*

$$S_j = \sum_{i=1}^{j} (Y_i - X_{i+1}).$$

From Proposition 0.1 that $Pr(D_{n+1} \geq c)$ is nondecreasing in $n$. Let

$$Pr(D_\infty \geq c) = \lim_{n\to\infty} Pr(D_n \geq c),$$

we have from 1

$$Pr(D_\infty \geq c) = Pr(\text{the random walk } S_j,\ j \geq 1,\ \text{ever crosses } c). \tag{2}$$

If $E[U] = E[Y] - E[X]$ is positive, then by Strong Law of Large Numbers (SLLN) the random walk will converge to positive infinity with probability 1. Hence,

$$Pr(D_\infty \geq c) = 1,\ \forall c \text{ if } E[Y] > E[X].$$

The above will also be true when $E[Y] = E[X]$ and hence we get that $E[Y] < E[X]$ will imply the existence of a stationary distribution.
Let $M_n = \max\{0, S_1, S_2 \ldots S_n\},\ n \geq 1$. We have the following proposition.

**Proposition 0.2.** ***Spitzer's Identity***

$$E[M_n] = \sum_{k=1}^{n} \frac{1}{k} E[S_k^+].$$

*Proof.* We represent $M_n$ as

$$M_n = 1_{\{S_n > 0\}} M_n + 1_{\{S_n \leq 0\}} M_n.$$

Consider first $1_{S_n > 0} M_n$.

$$1_{\{S_n>0\}} M_n = 1_{\{S_n>0\}} \max_{1 \leq i \leq n} S_i = 1_{\{S_n>0\}} (X_1 + \max\{0, X_2, \ldots X_2 + \ldots + X_n\})$$

Taking expectation,

$$E[1_{\{S_n>0\}} M_n] = E[1_{\{S_n>0\}} X_1] + E[1_{\{S_n>0\}} \max\{0, X_2, \ldots X_2 + \ldots + X_n\}]. \tag{3}$$

The joint distribution of $X_1, \ldots X_n$ and $X_n, X_1, \ldots X_{n-1}$ are the same.

$$E[1_{\{S_n>0\}} \max\{0, X_2, \ldots X_2 + \ldots + X_n\}] = E[1_{\{S_n>0\}} M_{n-1}]. \tag{4}$$

Since $X_i, S_n$ has the same joint distribution for all $i$,

$$E[S_n 1_{\{S_n>0\}}] = E[\sum_{i=1}^{n} X_i 1_{\{S_n>0\}}] = nE[X_1 1_{\{S_n>0\}}].$$

Hence,

$$E[X_1 1_{\{S_n>0\}}] = \frac{1}{n} = E[S_n 1_{\{S_n>0\}}] = \frac{1}{n} E[S_n^+]. \tag{5}$$

From equations 3, 4, 5, we have that

$$E[1_{\{S_n>0\}} M_n] = E[1_{\{S_n>0\}} M_{n-1}] + \frac{1}{n} E[S_n^+].$$

Also, $S_n \leq 0$ implies that $M_n = M_{n-1}$, it follows that

$$1_{\{S_n \leq 0\}} M_n = 1_{\{S_n \leq 0\}} M_{n-1}.$$

Thus,

$$E[M_n] = E[M_{n-1}] + \frac{1}{n} E[S_n^+].$$

Upon recursion, we get

$$E[M_n] = \sum_{k=2}^{n} \frac{1}{k} E[S_k^+] + E[M_1].$$

□

Since, $M_1 = S_1^+$, the result follows. From Proposition 0.1, with $M_n = \max\{0, S_1, \dots S_n\}$

$$Pr(D_{n+1} \geq c) = Pr(M_n \geq c).$$

Hence,

$$E[D_{n+1}] = E[M_n].$$

From Spitzer's identity we see that

$$E[D_{n+1}] = \sum_{k=1}^{n} \frac{1}{k} E[S_k^+].$$

## 0.2 Some Remarks Concerning Exchangeable Random Variables

**Definition 0.3.** $X_1, \dots, X_n$ is exchangeable if $X_{i_1}, \dots X_{i_n}$ has the same joint distribution for all permutations $(i_1, i_2 \dots i_n)$ of $(1, \dots, n)$. The infinite sequence of random variables $X_1, X_2 \dots$ is said to be exchangeable if every finite subsequence $X_1, \dots, X_n$ is exchangeable.

**Example 0.4.** Suppose balls are selected randomly, without replacement, from an urn consisting of $n$ balls of which $k$ are white. If we let

$$X_1 = \begin{cases} 1 & \text{if } i^{\text{th}} \text{ selection is white} \\ 0 & \text{otherwise,} \end{cases}$$

then $X_1, \dots X_n$ will be exchangeable but not independent.

**Example 0.5.** Let $\Lambda$ denote a random variable having distribution $G$. Given that $\Lambda = \lambda$, $X_1, X_2 \dots$ are *iid* with distribution $F_\lambda$. The random variables are exchangeable since

$$Pr(X_1 \leq x_1 \dots, X_n \leq x_n) = \int \Pi_{i=1}^n F_\lambda(x_i) dG(\lambda),$$

which is symmetric in $(x_1, \dots x_n)$. The are not independent.

**Theorem 0.6.** *(De Finetti's Theorem) To every infinite sequence of random variables $X_1, X_2 \dots$ taking values either $0$ or $1$, there corresponds a probability distribution $G$ on $[0,1]$ such that, for all $0 \leq k \leq n$,*

$$Pr(X_1 = X_2 = \dots X_k = 1, X_{k+1} = \dots X_n = 0) = \int_0^1 \lambda^k (1-\lambda)^{n-k} dG(\lambda).$$

*Proof.* Let $m \geq n$.

$$\begin{aligned} &Pr(X_1 = X_2 \dots X_k = 1, X_{k+1} = \dots X_n 0) \\ &= \textstyle\sum_{j=0}^m Pr(X_1 = \dots X_k = 1, X_{k+1} = X_n = 0 | S_m = j) Pr(S_m = j) \\ &= \textstyle\sum_j \frac{j(j-1)\dots(j-k+1)(m-j)(m-j-1)\dots(m-j-(n-k)+1)}{m(m-1)\dots(m-n+1)} Pr(S_m = j). \end{aligned}$$

The last equation follows by exchangeability as given $S_m = j$ each subset of size $j$ of $X_1 \dots X_m$ is equally likely to be the one consisting of all $1'$s. Letting $S_m = mY_m$, the above equation for large $m$ is roughly equal to $E[Y_m^k (1-Y_m)^{n-k}]$, and the theorem follows letting $m \to \infty$. Indeed, from a result known as Helly's theorem it can be shown that for some subsequence $m'$ converging to $\infty$, the distribution of $Y'_m$ will converge to a distribution $G$ and we get

$$E[Y_\infty^k(1-Y_\infty)^{n-k}] = \int_0^1 \lambda^k(1-\lambda)^{n-k} dG(\lambda).$$

$\square$