Lecture 8: Equilibrium Renewal Processes and Renewal Reward Processes

Parimal Parag

1 Renewal thmry Contd.

1.1 Example:

Consider two coins and suppose that each time is coin flipped, it lands tail with some unknown probability $p_i,\ i = 1, 2$. We are interested in coming up with a strategy that ensures that long term proportion of tails is $\min\{p_1,\ p_2\}$. One strategy is as follows: In n^{th} round of coin flipping, flip the first coin till n consecutive tails are obtained. Then flip the second coin till n consecutive tails are obtained. The proof is as follows:

Let $p = \max\{p_1, p_2\}$ and $\alpha p = \min\{p_1, p_2\}$. Call the coin with $P(T) = p$, the bad coin and the other, the good coin. Let B_m denote the number of flips in the n^textth round with bad coin and G_m denote the number of flips in the n^textth round with good coin.

Lemma 1.1. *$P(B_m \geq \epsilon G_m$ for infinitely many $m) = 0$.*

Proof.

$$\begin{aligned} P(G_m \leq \frac{B_m}{\epsilon}) &= \mathbb{E}[P(G_m \leq \frac{B_m}{\epsilon}|B_m)] \\ &= \mathbb{E}[\sum_{i=1}^{\frac{B_m}{\epsilon}} P(G_m = i|B_m)] \\ &\leq \mathbb{E}[\frac{B_m}{\epsilon}] \sum_{i=1}^{m} (\alpha p)^m \end{aligned}$$

$$= (\sum_{i=1}^{m} (\frac{1}{p^i}))(\alpha p)^m,$$

where the inequality follows from the fact that $\{G_m = i\}$ implies that $i \geq m$ and that cycle m coin flips numbered $i - m + 1$ to i are all tails. Hence by Borel-Cantelli lemma, it follows that $P(B_m \geq \epsilon G_m \text{for infinitely many m}) \to 0$ as $m \to \infty$. Hence $\frac{B}{B+G} < \frac{\epsilon}{1+\epsilon} < \epsilon$. ☐

1.2 Distribution of Last Renewal Time for Delayed Renewal Processes

$$P(S_{N(t)} \leq s) = G^c(t)P(S_{N(t)} \leq s|S_{N(t)=0}) + \int_0^t P(S_{N(t)} \leq s|S_{N(t)=s})F^c(t-u)dm(u)$$
$$= G^c(t) + \int_0^s F^c(t-u)dm(u).$$

Let $F_e(x) = frac\int_0^x F^c(y)dy\mu,\ x \geq 0$ equilibrium distribution of F. Observe that the moment generating function of $F_e(x)$ is $\tilde{F}_e(s) = \frac{1-\tilde{F}(s)}{s\mu}$. If $G = F_e$, then the delayed renewal process is called equilibrium renewal process. Suppose start observing a renewal process at some arbitrary time t, the observed renewal process is called equilibrium renewal process. Let $Y_D(t)$ denote the excess time for delayed renewal process.

Theorem 1.2. *For the equilibrium renewal process,*

1. $m_D(t) = \frac{t}{\mu}$.

2. $P(Y_D(t) \leq x) = F_e(x)$.

3. $\{N_D(t) : t \geq 0\}$ *has stationary increments.*

Proof. To prove i), observe that $\tilde{m_D}(s) = \frac{\tilde{G}(s)}{1-\tilde{F}(s)} = \frac{1}{s\mu}$. Hence, $m_D(t) = \frac{t}{\mu}$.
ii)

$$P(Y_D(t) > x) = P(Y_D(t) > x|S_{N(t)=0})P(S_{N(t)=0}) + P(Y_D(t) > x|S_{N(t)=s})F^c(t-s)\frac{ds}{\mu}$$
$$= P(X > t+x, X > t) + P(X_2 > t+x-s|X_2 > t-s)F^c(t-s)\frac{ds}{\mu}$$
$$= F^c(t+x) + \int_0^t F^c(t+x-s)\frac{ds}{\mu} = F_e^c(x).$$

iii) $N_D(t+s) - N_D(s) =$ Number of renewals in time interval of length t. When we start observing at s, the observed renewal process is delayed renewal process with initial distribution being the original distribution. □

1.3 Renewal Reward Process

Definition: A renewal process $\{N(t), t \geq 0\}$ with inter arrival times $\{X_n : n \in \mathbb{N}\}$ having distribution F and rewards $\{R_n : n \in \mathbb{N}\}$ where R_n is the reward at the end of X_n. Let (X_n, R_n) be iid. Then $R(t) = \sum_{i=1}^{N(t)} R_i$ is reward process.

Theorem 1.3. *Let* $\mathbb{E}[|R|]$ *and* $\mathbb{E}[|X|]$ *be finite.*

1. $\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]}$ *a.s.*

2. $\lim_{t\to\infty} \frac{\mathbb{E}[R(t)]}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]}$.

Proof.

$$\begin{aligned}R(t) &= \sum_{i=1}^{N(t)} R_i \\ &= (\frac{t}{N(t)}\sum_{i=1}^{N(t)} R_i)\frac{N(t)}{t}.\end{aligned}$$

Hence by Strong Law of Large Numbers, $\lim_{t\to\infty}\frac{R(t)}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]}$ *a.s.*
To prove the second part,

$\mathbb{E}[R(t)] = \mathbb{E}[\sum_{i=1}^{N(t)} R_i)] = (m(t)+1)\mathbb{E}[R] - \mathbb{E}[R_{N(t)+1}]$. Let $g(t) = \mathbb{E}[R_{N(t)+1}]$.

$$\begin{aligned}g(t) &= \mathbb{E}[R_{N(t)+1}1\{S_{N(t)}=0\}] + \mathbb{E}[R_{N(t)+1}1\{S_{N(t)}>0\}] \\ &= \mathbb{E}[R_1|X_1>t]P(X_1>t) + \int_0^t \mathbb{E}[R_1|X>t-u]F^c(t-u)dm(u). \quad = h(t) + \int_0^t h(t-u)dm(u).\end{aligned}$$

where $h(t) = \mathbb{E}[R_1|X>t]P(X_1>t)$. Since $\mathbb{E}[|R_1|]<\infty$, as $t\to\infty$, $h(t)\to 0$ as $t\to\infty$. Hence choose T such that $|h(T)|<\epsilon$, $t\geq T$.

$$\begin{aligned}\frac{g(t)}{t} &\leq \frac{|h(t)|}{t} + \int_0^{t-T}\frac{h(t-s)}{t}dm(s) + \int_{t-T}^{T}\frac{h(t-T)}{t}dm(s) \\ &\frac{\epsilon}{T} + \frac{\epsilon m(t-T)}{T} + \frac{\mathbb{E}[|R_1|]}{t}(m(t)-m(t-T)).\end{aligned}$$

Hence $\lim_{t\to\infty}\frac{g(t)}{t} = 0$ and the result follows. □

Remarks:

1. $R_{N(t)+1}$ has different distribution than R_1.
2. $R(t)$ is the gradual reward during a cycle,

$$\frac{\sum_{n=1}^{N(t)} R_n}{t} \leq \frac{R(t)}{t} \leq \frac{\sum_{n=1}^{N(t)+1} R_n}{t}.$$

1.3.1 Example:

Suppose for an alternating renewal process, we earn at a rate of one per unit time when the system is on and the reward for a cycle is the the time system is ON during that cycle.
Amount of on time in $\frac{[0,t]}{t} = \lim_{t\to\infty}\frac{R(t)}{t} = \frac{\mathbb{E}[X]}{\mathbb{E}[X]+\mathbb{E}[Y]} = \lim_{t\to\infty} P(\text{on at time t})$.