Lecture 4: Renewal Theory

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1 Renewal Theory

One of the characterization for the Poisson process is of it being a counting process with *iid* exponential inter-arrival times. Now we shall relax the "exponential" part. A counting process with *iid* general inter-arrival times is called a renewal process. As a result, we no longer have the nice properties such as Independent and stationary increments that Poisson processes had. However, we can still get some great results which also apply to Poisson Processes.

Definition 1.1 (Inter-arrival Times). Let $\{T_i : i \in \mathbb{N}\}$ be a sequence of *iid* random variables with a common distribution F. We interpret T_n as the time between $(n-1)^{\text{st}}$ and the n^{th} event. Assume

1. Positive inter-arrival time, i.e. $T_n \geq 0$,
2. Finite mean i.e. $(0 \leq \mu = E[T_1] < \infty)$, and
3. $F_n(0) = \Pr\{T_n \leq 0\} = \Pr\{T_n = 0\} < 1$.

Definition 1.2 (Event Instants). If we let S_n denote the time of n^{th} event, and assume $S_0 = 0$. Then, we have

$$S_n = \sum_{i=1}^{n} T_i, \quad n \in \mathbb{N}.$$

Definition 1.3 (Renewal process). Let $N(t)$ be the counting process that counts number of events by time t. Then,

$$N(t) = \sup\{n \in \mathbb{N}_0 : S_n \leq t\}.$$

This counting process $\{N(t), t \geq 0\}$ is called a renewal process.

Note 1 (Inverse Relationship). We note the inverse relationship between time of n^{th} event S_n, and the counting process $N(t)$, we have

$$\{S_n \leq t\} \iff \{N(t) \geq n\}, \tag{1}$$

since $N(t) = \sum_{n \in \mathbb{N}} 1_{\{S_n \leq t\}}$.

1.1 Time average of renewals

We are interested in knowing how many renewals occur per unit time. From SLLN, we have

$$\frac{S_n}{n} \to \mu \quad \text{a.s.}$$

Since $\mu > 0$, we must have S_n growing arbitrarily large as n increases. Thus, S_n can be finite for at most finitely many n. Therefore, $N(t)$ must be finite, and

$$N(t) = \max\{n \in \mathbb{N}_0 : S_n \leq t\}.$$

1.2 Distribution of N(t)

We need to know the distribution of $N(t)$. Denote $F_n = F^{*(n)}$ where $*$ denotes convolution. Essentially, $F^{*(n)}$ is the distribution of S_n. We are interested in the following two quantities:

$$m(t) = E[N(t)],$$
$$M_{N(t)}(\theta) = E[e^{\theta N(t)}].$$

From (1), we have

$$\Pr\{N(t) = n\} = \Pr\{S_n \leq t\} - \Pr\{S_{n+1} \leq t\} = F_n(t) - F_{n+1}(t).$$

Proposition 1.4.

$$m(t) = \sum_{n \in \mathbb{N}} F_n(t)$$

Proof.

$$E[N(t)] = \sum_{n \in \mathbb{N}} \Pr\{N(t) \geq n\} = \sum_{n \in \mathbb{N}} \Pr\{S_n \leq t\} = \sum_{n \in \mathbb{N}} F_n(t)$$

□

Alternatively one can prove the same result using indicator functions. Refer Ross for details.

Proposition 1.5.

$$m(t) < \infty \quad \forall 0 \leq t < \infty$$

Proof. Since we assumed that $\Pr\{T_n = 0\} < 1$, for some $\alpha > 0$, we have $\Pr\{T_n \geq \alpha\} > 0$. Define

$$\overline{T}_n = \alpha 1_{\{T_n \geq \alpha\}}.$$

Let $\overline{N}(t)$ denote the renewal process with inter-arrival times $\overline{T}_n$. Note that since T_i's are *iid*, so are $\overline{T}_i$ (Why?). In fact, the arrivals now happen at multiples of α. And yes, they stack. Moreover, $T_n \geq \overline{T}_n$

The number of arrivals till time t, therefore is Geometric with mean $\frac{1}{P[T_n \geq \alpha]}$. Thus

$$E[\overline{N}(t)] = \frac{\lceil \frac{t}{\alpha} \rceil + 1}{P[T_n \geq \alpha]} < \infty$$

Since $E[N(t)] \leq E[\overline{N}(t)]$ which follows from $N(t) \leq \overline{N}(t)$, we are done. □