Lecture 5: Limit theorems in Renewal Theory

Parimal Parag

1 Limit Theorems

Let $N(\infty) := \lim_{t\to\infty} N(t)$. Then, it is easy to see that $\Pr\{N(\infty) = \infty\} = 1$.

Proof. It suffices to show $\Pr\{N(\infty) < \infty\} = 0$. We have

$$\begin{aligned}
\Pr\{N(\infty) < \infty\} &= \Pr\{\bigcup_{n\in\mathbb{N}} \{N(\infty) < n\}\} \\
&= \Pr\{\bigcup_{n\in\mathbb{N}} \{S_n = \infty\}\} = \Pr\{\bigcup_{n\in\mathbb{N}} \{X_n = \infty\}\} \\
&\leq \sum_{n\in\mathbb{N}} \Pr\{X_n = \infty\} = 0.
\end{aligned}$$

The last step follows from the fact that $E[X_n] < \infty$. $\square$

1.1 Basic Renewal Theorem

We see that $N(t)$ increases to infinity with time. We are interested in rate of increase of $N(t)$ with t. Note that $S_{N(t)}$ represents the time of last renewal before t, and $S_{N(t)+1}$ represents the time of first renewal after time t.

Proposition 1.1.

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} \quad \textit{almost surely.}$$

Proof. Consider $S_{N(t)}$. By definition, we have

$$S_{N(t)} \leq t < S_{N(t)+1}$$

Dividing by $N(t)$, we get

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$$

By Strong Law of Large Numbers (SLLN) and the previous result, we have

$$\lim_{t\to\infty} \frac{S_{N(t)}}{N(t)} = \mu \quad \text{a.s.}$$

Also

$$\lim_{t\to\infty} \frac{S_{N(t)+1}}{N(t)} = \lim_{t\to\infty} \frac{S_{N(t)+1}}{N(t)+1}.\frac{N(t)+1}{N(t)}$$

Hence by Squeeze Theorem, the result follows. $\square$

1.1.1 Example

Suppose, you are in a casino with infinitely many games. Every game has a probability of win X, *iid* uniformly distributed between $(0, 1)$. One can continue to play a game or switch to another one. We are interested in a strategy that maximizes the long-run proportion of wins.

Let $N(n)$ denote the number of losses in n plays. Then fraction of wins $P_W(n)$ is given by

$$P_W(n) = \frac{n - N(n)}{n}.$$

We pick a strategy where any game is selected to play, and continue to be played till the first loss. Note that, time till first loss is geometrically distributed with mean $\frac{1}{1-X}$. We shall show that this fraction approaches unity as $n \to \infty$. By the previous proposition, we have:

$$\begin{aligned}\lim_{n\to\infty} \frac{N(n)}{n} &= \frac{1}{E[\text{Time till first loss}]} \\ &= \frac{1}{E\left[\frac{1}{1-X}\right]} = \frac{1}{\infty} = 0\end{aligned}$$

Hence Renewal theorems can be used to compute these long term averages. We'll have many such theorems in the following sections.

1.2 Wald's Lemma

Before we get into Wald's Lemma, let us first define what a stopping time is.

Definition 1.2 (Stopping Time). Let $\{X_n : n \in \mathbb{N}\}$ be independent random variables. Then T, an integer random variable, is called a stopping time wrt this sequence if $\{N = n\}$ depends only on $\{X_1, \cdots, X_n\}$ and is independent of $X_{n+1}, X_{n+2}, \cdots$.

Intuitively, if we observe the X_n's in sequential order and N denotes the number observed before stopping then. Then, we have stopped after observing, $X_1, \ldots, X_N$, and before observing $X_{N+1}, X_{N+2}, \ldots$. The intuition behind a stopping time is that it's value is determined by past and present events but NOT by future events.

Example 1.3. For instance, while traveling on the bus, the random variable measuring "Time until bus crosses Majestic and after that one stop" is a stopping time as it's value is determined by events before it happens. On the other hand "Time until bus stops before Majestic is reached" would not be a stopping time in the same context. This is because we have to cross this time, reach Majestic and then realise we have crossed that point.

Example 1.4. Consider X_n *iid* Bernoulli(1/2). Then $N = min\{n \in \mathbb{N} : \sum_{i=1}^{n} X_i = 10\}$ is a stopping time.

Example 1.5 (Random Walk Stopping Time). Consider X_n *iid* bivariate random variables with

$$\Pr\{X_n = 1\} = \Pr\{X_n = -1\} = \frac{1}{2}.$$

Then $N = min\{n \in \mathbb{N} : \quad \sum_{i=1}^{n} X_i = 1\}$ is a stopping time.

Exercise: Try to list out properties of stopping times. For instance, sum of two stopping times is a stopping time. Minimum of two stopping times is a stopping time. See how many can you find and prove/disprove.

Lemma 1.6 (Wald's Lemma). *Let* $\{X_i : \quad i \in \mathbb{N}\}$ *be* iid *random variables with finite mean* $E[X_1]$ *and let* N *be a stopping time with respect to this set of variables, such that* $E[N] < \infty$. *Then*

$$E\left[\sum_{n=1}^{N} X_n\right] = E[X_1]E[N]$$

Proof.

$$E\left[\sum_{n=1}^{N} X_n\right] = E\left[\sum_{n\in\mathbb{N}} X_n 1_{\{N\geq n\}}\right] \tag{1}$$

$$= \sum_{n\in\mathbb{N}} E\left[X_n 1_{\{N\geq n\}}\right] \tag{2}$$

I'd like to point out here that in step (2), you cannot always exchange infinite sums and expectations. But here you can do so. Refer Ross/Wolff if you are interested. Regardless, to proceed, we need to show that $N \geq n$ is independent of $X_k, \quad k \geq n$. To this end, observe that

$$\{N \geq k\} = \{N < k\}^c = \{N \leq k-1\}^c = \left(\bigcup_{i=1}^{k-1} \{N = i\}\right)^c.$$

Since, N is a stopping time and by definition $\{N = i\}$ depends only on $\{X_1, \ldots, X_i\}$. Therefore, $\{N \geq k\}$ depends only on $\{X_1, \ldots, X_{k-1}\}$, and is independent of the future and present samples. Therefore, we can write

$$\sum_{n\in\mathbb{N}} E\left[X_n 1_{\{N\geq n\}}\right] = \sum_{n\in\mathbb{N}} E\left[X_n\right] E\left[1_{\{N\geq n\}}\right]$$
$$= E\left[X_1\right] \sum_{n\in\mathbb{N}} \Pr\{N \geq n\} = E[X_1]E[N].$$

□

Proposition 1.7 (Wald's Lemma for Renewal Process). *Let* $\{X_n, n \in \mathbb{N}\}$ *be* iid *inter-arrival times of a renewal process* $N(t)$ *with* $E[X_1] < \infty$, *and let* $m(t) = E[N(t)]$. *Then,* $N(t)+1$ *is a stopping time and*

$$E\left[\sum_{i=1}^{N(t)+1} X_i\right] = E[X_1][1 + m(t)]$$

Proof. It is easy to see that $\{N(t) + 1 = n\}$ depends solely on $\{X_1, \ldots, X_n\}$ form the equation below.

$$\{N(t)+1 = n\} \iff \{S_{n-1} \leq t < S_n\} \iff \left\{\sum_{i=1}^{n-1} X_i \leq t < \sum_{i=1}^{n-1} X_i + X_n\right\}$$

Thus $N(t)+1$ is a stopping time, and the result follows from Wald's Lemma. □

1.3 Elementary Renewal Theorem

Basic renewal theorem implies $N(t)/t$ converges to $1/\mu$ almost surely. Now, we are interested in convergence of $E[N(t)]/t$. Note that this is not obvious, since almost sure convergence doesn't imply convergence in mean. Consider the following example.

Example 1.8.

$$Y_n = \begin{cases} n, & \text{w.p. } 1/n, \\ 0, & \text{w.p. } 1 - 1/n. \end{cases}$$

Then, $\Pr\{Y_n = 0\} = 1 - 1/n$. That is $Y_n \to 0$ a.s. However, $E[Y_n] = 1$ for all $n \in \mathbb{N}$. So $E[Y_n] \to 1$.

Even though, basic renewal theorem does **NOT** imply it, we still have $E[N(t)]/t$ converging to $1/\mu$.

Theorem 1.9 (Elementary Renewal Theorem). *Let $m(t)$ denote mean $E[N(t)]$ of renewal process $N(t)$, then under the hypotheses of basic renewal theorem, we have*

$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}.$$

Proof. Take $\mu < \infty$. We know that $S_{N(t)+1} > t$. Therefore, taking expectations on both sides and using Proposition 1.7, we have

$$\mu(m(t) + 1) > t.$$

Dividing both sides by μt and taking lim inf on both sides, we get

$$\liminf_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}. \tag{3}$$

We employ a truncated random variable argument to show the reverse inequality. We define truncated inter-arrival times $\{\bar{X}_n\}$ as

$$\bar{X}_n = X_n 1_{\{X_n \leq M\}} + M 1_{\{X_n > M\}}.$$

We will call $E[\bar{X}_n] = \mu_M$. Further, we can define arrival instants $\{\bar{S}_n\}$ and renewal process $\bar{N}(t)$ for this set of truncated inter-arrival times $\{\bar{X}_n\}$ as

$$\bar{S}_n = \sum_{k=1}^{n} \bar{X}_k, \qquad \bar{N}(t) = \sup\{n \in \mathbb{N}_0 : \bar{S}_n \leq t\}.$$

Note that since $S_n \geq \bar{S}_n$, number of arrivals would be higher for renewal process with truncated random variables, i.e.

$$N(t) \leq \bar{N}(t). \tag{4}$$

Further, due to truncation of inter-arrival time, next renewal happens with-in M units of time, i.e.

$$\bar{S}_{N(t)+1} \leq t + M.$$

Taking expectations on both sides in the above equation, using Proposition 1.7, dividing both sides by $t\mu_M$ and taking $\limsup$ on both sides, we obtain

$$\limsup_{t\to\infty} \frac{\bar{m(t)}}{t} \leq \frac{1}{\mu_M}.$$

Taking expectations on both sides of (4) and letting M go arbitrary large on RHS, we get

$$\limsup_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu}. \tag{5}$$

Result for finite μ follows from (3) and (3). When μ grows arbitrary large, results follow from (3), where RHS is zero. $\square$

1.4 Central Limit for Renewal Processes

Theorem 1.10. *Let X_n be* iid *random variables with $\mu = E[X_n] < \infty$ and $\sigma^2 = Var(X_n) < \infty$. Then*

$$\frac{N(t) - \frac{t}{\mu}}{\sigma\sqrt{\frac{t}{\mu^3}}} \to^d N(0,1)$$

Proof. Take $u = \frac{t}{\mu} + y\sigma\sqrt{\frac{t}{\mu^3}}$. We shall treat u as an integer and proceed, the proof for general u is an exercise. Recall that $\{N(t) < u\} \iff \{S_u > t\}$. By equating probability measures on both sides, we get

$$\Pr\{N(t) < u\} = \Pr\left\{\frac{S_u - u\mu}{\sigma\sqrt{u}} > \frac{t - u\mu}{\sigma\sqrt{u}}\right\} = \Pr\left\{\frac{S_u - u\mu}{\sigma\sqrt{u}} > -y\left(1 + \frac{y\sigma}{\sqrt{tu}}\right)^2\right\}.$$

By central limit theorem, $\frac{S_u - u\mu}{\sigma\sqrt{u}}$ converges to a normal random variable with zero mean and unit variance as t grows. Also, note that

$$\lim_{t\to\infty} -y\left(1 + \frac{y\sigma}{\sqrt{tu}}\right)^2 = -y.$$

These results combine with the symmetry of normal random variable to give us the result. $\square$