Lecture 9: Discrete Time Markov Chains

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1 Introduction

Definition 1.1 (DTMC). A stochastic process $\{X_n, n \in \mathbb{N}_0\}$, where $X_n \in \mathcal{S}$, where $\mathcal{S}$ is assumed at most countable, is called a DTMC (Discrete Time Markov Chain) if

$$P[X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0] = P[X_{n+1} = j | X_n = i] = p_{ij}^{(n)}$$
$$\forall i_0, i_1, \cdots i_{n-1}, i, j \in \mathcal{S}, \quad \forall n \in \mathbb{N}_0$$

If $p_{ij}^{(n)} = p_{ij}$ for every n then we say the DTMC is homogeneous. Henceforth we shall only work with homogeneous DTMCs.

Definition 1.2 (Transition Probability Matrix). $\{p_{ij}\}$ are called transition probabilities as they defnne the probability of transiting from state i to state j in one step. We can create a matrix P out of the transition probabilities where the i^{th} row and j^{th} column equal p_{ij}. Such a matrix is called a **Transition Probability Matrix** or **TPM**.

If a non-negative matrix P has the property that for all $i \in \mathcal{S}$, $\sum_{j \in \mathcal{S}} p_{ij} \leq 1$; then it is called a **sub-stochastic** matrix. If true with equality, then it is called a **stochastic** matrix. If in addition, P^T is stochastic, then P is called a **doubly stochastic**.

1.1 Example: M/G/1 Queue

In this queue, arrivals occurs as per a Poisson Process but service times are distributed *iid* G. And there is only one server. Let $X(t)$ count the number of customers in the system at time t. Consider $\{X(t), t \geq 0\}$. Let $X_n = X(D_n^+)$ be the number of customers left behind by nth departure.

Proposition 1.3. *X_n is a DTMC.*

To see this, let Y_n denote the number of customers arriving during service time of $(n+1)$th customer. Assume service time $\sim G$. Then

$$X_{n+1} = X(D_{n+1}^+) = (X_n - 1)^+ + Y_n$$

Hence writing the TPM (here take $i \geq 1$)

$$\begin{aligned}
&P[X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0] \\
&= P[i + Y_n - 1 = j | X_n = i, X_{n-1} = i_{n-1}, \cdots, X_0 = i_0] \\
&= P[Y_n = j + 1 - i] := p_{ij}
\end{aligned}$$

For $i = 0$, $p_{0j} = P[Y_n = j]$. As the arrival process is Poisson, we can compute the TPM hence.

1.2 Example: G/M/1 Queue

In this queue, arrivals occur as per a renewal process with interarrival times $\sim G$. The service times are exponentially distributed. Our strategy, therefore is the analogous one, where we sample the queue after the nth arrival. With this in mind, let $X_n = X(A_n^-)$ which is the number of customers seen by the nth arrival. We get

$$X_{n+1} = (X_n - Y_n + 1)^+$$

From this, the TPM may be calculated.

2 Chapman Kolmogorov Equations

defnne

$$p_{ij}^{(n)} = P[X_{n+m} = j | X_m = i] \quad \forall m \in \mathbb{N}, \quad i, j \in \mathcal{S}$$

Then we have by conditioning and the Markov property

$$p_{ij}^{(m+n)} = \sum_{k \in \mathcal{S}} p_{ik}^{(m)} p_{kj}^{(n)}$$

If P is the TPM, this result may be compactly written as $P^{(n)} = P^n$. As a result, $p_{ij}^{(n)} = P^n(i, j)$.

2.1 Strong Markov Property (SMP)

Let T be an integer valued stopping time. Then a process X_n that satisfies

$$P[X_{T+1} = j | X_T = i, \cdots, X_0 = i_0] = P[X_{T+1} = j | X_T = i]$$

is said to possess the strong Markov property (SMP). Markov chains, for instance, satisfy them (hence the name). Let X_n be a DTMC. Then

$$\begin{aligned}
&P[X_{T+1} = j | X_T = i, \cdots, X_0 = i_0] \\
&= \sum_{n \in \mathbb{N}_0} P[X_{T+1} = j, T = n | X_T = i, \cdots, X_0 = i_0] \\
&= \sum_{n \in \mathbb{N}_0} P[X_{n+1} = j | X_n = i, \cdots, X_0 = i_0, T = n] P[T = n | X_T = i, \cdots, X_0 = i_0] \\
&= \sum_{n \in \mathbb{N}_0} p_{ij} P[T = n | X_T = i, \cdots, X_0 = i_0] \\
&= p_{ij}
\end{aligned}$$

As an exercise, try to find a random variable T for which the SMP does not hold. A useful application of the SMP is as follows. Let i_0 be a fixed state. Let τ_n denote the time at which the DTMC visits i_0 for the nth time. Then $\{X_{\tau_n + m} : \quad m \in \mathbb{N}_0\}$ is a stochastic replica of $\{X_m : \quad m \in \mathbb{N}_0\}$ with $X_0 = i_0$ and can be studied as such.

3 Communicating classes

Definition 3.1. State j is said to be **accessible** from state i if $p_{ij}^{(n)} > 0$ for some $n \geq 1$. If two states are accessible to each other, they are said to **communicate** with each other. If state i and j communicate, we denote it by $i \iff j$. A set of states that communicate are called a **communicating class**.

Proposition 3.2. *Communication is an equivalence relation.*

Proof. Reflexivity and Symmetry are obvious. For transitivity, suppose $i \iff j$ and $j \iff k$. Suppose $p_{ij}^{(m)} > 0$ and $p_{jk}^{(n)} > 0$. Then by Chapman Kolmogorov, we have

$$p_{ik}^{(m+n)} = \sum_{l \in \mathbb{N}_0} p_{il}^{(m)} p_{lk}^{(n)} \geq p_{ij}^{(m)} p_{jk}^{(n)} > 0$$

Hence transitivity is assured. $\square$

3.1 Irreducibility and Periodicity

A consequence of the previous result is that communicating classes are disjoint or identical. A DTMC with only one class is called an **irreducible** markov chain.

Definition 3.3. The period of state i is defnned as

$$d(i) = \gcd\{n \in \mathbb{N}_0 : p_{ii}^{(n)} > 0\}$$

If the period is 1, we say the state is **aperiodic**.

Proposition 3.4. *If $i \iff j$, then $d(i) = d(j)$. Basically periodicity is a class property.*

Proof. Let m and n be such that $p_{ij}^{(m)} p_{ji}^{(n)} > 0$. Suppose $p_{ii}^{(s)} > 0$. Then

$$p_{jj}^{(n+m)} \geq p_{ji}^{(n)} p_{ij}^{(m)} > 0$$

$$p_{jj}^{(n+s+m)} \geq p_{ji}^{(n)} p_{ii}^{(s)} p_{ij}^{(m)} > 0$$

Hence $d(j)|n+m$ and $d(j)|n+s+m$ which implies $d(j)|s$. Hence $d(j)|d(i)$. By symmetrical arguments, we get $d(i)|d(j)$. Hence $d(i) = d(j)$. Source of proof is from *Sheldon Ross: Stochastic Processes.* $\square$

4 Transient and Recurrent States

Definition 4.1. Let $f_{ij}^{(n)}$ denote the probability that starting from state i, the first transition into state j happens at time n. Then let

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$$

Here f_{ij} would therefore denote the probability of ever entering state j given that we start at state i. State j is said to be **transient** if $f_{jj} < 1$ and **recurrent** if $f_{jj} = 1$.

A few remarks:

1. A transient state is visited a finite amount of times almost surely.
2. A recurrent state is visited infinitely often almost surely.
3. In a finite state DTMC, not all states may be transient.

Proposition 4.2. *A state j is recurrent iff*

$$\sum_{k=1}^{\infty} p_{jj}^{(k)} = \infty$$

Try to prove the above.

Proposition 4.3. *Transience and recurrence are class properties.*

Proof. Let us start with proving recurrence is a class property. Let i be a recurrent state and let $i \iff j$. Hence for some $m, n > 0$, $p_{ij}^{(m)} > 0$ and $p_{ji}^{(n)} > 0$. As a consequence of the recurrence, $\sum_{s \in \mathbb{N}_0} p_{ii}^{(s)} = \infty$. Now

$$\begin{aligned} \sum_{s \in \mathbb{N}_0} p_{jj}^{(m+n+s)} &\geq \sum_{s \in \mathbb{N}_0} p_{ji}^{(n)} p_{ii}^{(s)} p_{ij}^{(m)} \\ &\geq \infty \end{aligned}$$

Hence j is recurrent. Now, if i were transient instead, we have

$$\begin{aligned} \sum_{s \in \mathbb{N}_0} p_{jj}^{(s)} &\leq \frac{\sum_{s \in \mathbb{N}_0} p_{ii}^{(m+n+s)}}{p_{ji}^{(n)} p_{ij}^{(m)}} \\ &< \infty \end{aligned}$$

Hence j is transient. □

Corollary 4.4. *If j is recurrent, then for any state i, $f_{ij} = 1$.*

5 Limit Theorems

Let $N_j(t)$ denote the number of transitions into state j by time t. If j recurrent and $X_0 = j$ then $N_j(t)$ is a renewal process with interarrival distribution $\{f_{jj}^{(n)}\}_{n \geq 1}$.

If $X_0 = i$, $i \iff j$ and j recurrent, then $N_j(t)$ is a delayed renewal process with interarrival distribution $\{f_{ij}^{(n)}\}_{n \geq 1}$.

Hence from Renewal Theory, we have the following results. If $i \iff j$ then

1. $$P\left[\lim_{t \to \infty} \frac{N_j(t)}{t} = \frac{1}{\mu_{jj}} \middle| X_0 = i\right] = 1$$

   where

   $$\mu_{jj} = \begin{cases} \infty & j \text{ transient} \\ \sum_n n f_{jj}^{(n)} & j \text{ recurrent} \end{cases}$$

2. $$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} p_{ij}^{(k)}}{n} = \frac{1}{\mu_{jj}}$$

   Here μ_{jj} is the expected number of transitions needed to return to state j.

3. If j is aperiodic, then
$$\lim_{n\to\infty} p_{ij}^{(n)} = \frac{1}{\mu_{jj}}$$

4. If j is periodic with period d, then
$$\lim_{n\to\infty} p_{jj}^{(nd)} = \frac{d}{\mu_{jj}}$$

6 Positive and Null recurrence

A recurrent state j is said to be **positive recurrent** if $\mu_{jj} < \infty$ and **null recurrent** if $\mu_{jj} = \infty$. Let
$$\pi_j = \lim_{n\to\infty} p_{jj}^{(nd)}$$
where d is the period of state j. Then $\pi_j > 0$ if and only if j is positive recurrent and $\pi_j = 0$ if j is null-recurrent.

Proposition 6.1. *Prove that Positive recurrence and Null recurrence are class properties.*

Definition 6.2. An aperiodic positive recurrent state is also called **ergodic**.

Definition 6.3. A probability distribution $\{P_j\}_{j\in\mathbb{N}_0}$ is said to be **stationary** for the DTMC if
$$P_j = \sum_{k\in\mathbb{N}_0} P_k p_{kj} \quad \forall j \in \mathbb{N}_0$$
More compactly, π is stationary if $\pi = \pi P$ where P is the TPM.

A thing to note is that if we start the DTMC with the stationary distribution, the distribution remains the same throughout the chain. That is, if we started the chain with $X_0 \sim \pi$ where π is the stationary distribution, we would have for every $n \geq 1$, $X_n \sim \pi$. Moreover since X_n is a DTMC, we have $X_n, X_{n+1}, ...X_{n+m}$ have the same joint distribution. Hence it is a stationary process.

Theorem 6.4. *An irreducible aperiodic DTMC is of one of the following two types:*

1. *All states are transient or null recurrent, in which case*
$$\lim_{n\to\infty} p_{ij}^{(n)} = 0 \quad \forall i, j \in \mathbb{N}_0$$
   and there exists no stationary distribution.

2. All states are positive recurrent wherein

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j > 0 \quad \forall i,j \in \mathbb{N}_0$$

Here π_j is the unique stationary distribution that can achieve this.

Proof. Consider the first statement. Suppose a stationary distribution existed (say π), then since $\lim_{n\to\infty} p_{ij}^{(n)} = 0 \quad \forall i,j \in \mathbb{N}_0$, $\pi_j = 0$ for every state. This is clearly not a distribution, hence the contradiction and hence no stationary distribution exists. For the second statement, observe that

$$\sum_{j=0}^{M} p_{ij}^{(n)} \le \sum_{j\in\mathbb{N}_0} p_{ij}^{(n)} = 1$$

Taking limits as $n \to \infty$ on both sides yields

$$\sum_{j=0}^{M} \pi_j \le 1$$

Now we have

$$\begin{aligned}
p_{ij}^{(n+1)} &= \sum_{k\in\mathbb{N}_0} p_{ik}^{(n)} p_{kj} \\
\lim_{n\to\infty} p_{ij}^{(n+1)} &= \lim_{n\to\infty} \sum_{k\in\mathbb{N}_0} p_{ik}^{(n)} p_{kj} \\
&\ge \lim_{n\to\infty} \sum_{k=0}^{M} p_{ik}^{(n)} p_{kj} \\
\Rightarrow \pi_j &\ge \sum_{k=0}^{M} \pi_k p_{kj} \\
\Rightarrow \pi_j &\ge \sum_{k\in\mathbb{N}_0} \pi_k p_{kj}
\end{aligned}$$

To show equality, suppose for some j, it is strict. Then observe that

$$\sum_{j\in\mathbb{N}_0} \pi_j > \sum_{j\in\mathbb{N}_0} \sum_{k\in\mathbb{N}_0} \pi_k p_{kj} = \sum_{k\in\mathbb{N}_0} \pi_k$$

which is a contradiction. Hence it is an equality.

To prove Uniqueness, suppose λ is another stationary distribution that works here. Since λ is stationary, suppose we start the DTMC with this distribution. Then we get

$$\begin{aligned}
\lambda_j &= P[X_n = j] \\
&= \sum_{i=0}^{\infty} P[X_n = j | X_0 = i] P[X_0 = i] \\
&= \sum_{i=0}^{\infty} p_{ij}^{(n)} \lambda_i \\
&\ge \sum_{i=0}^{M} p_{ij}^{(n)} \lambda_i
\end{aligned}$$

Let n and then M approach ∞. This yields

$$\lambda_j \geq \sum_{i=0}^{\infty} \pi_j \lambda_i = \pi_j$$

Now consider

$$\begin{aligned}
\lambda_j &= \sum_{i=0}^{\infty} p_{ij}^{(n)} \lambda_i \\
&= \sum_{i=0}^{M} p_{ij}^{(n)} \lambda_i + \sum_{i=M+1}^{\infty} p_{ij}^{(n)} \lambda_i \\
&\leq \sum_{i=0}^{M} p_{ij}^{(n)} \lambda_i + \sum_{i=M+1}^{\infty} \lambda_i
\end{aligned}$$

Once again, letting n and then M approach ∞ yields the result. $\square$