Lecture-29: GI/GI/1 Queues

1 GI/GI/1 Queueing Model

Consider a $GI/GI/1$ queue. Customers arrive in accordance with a renewal process having an arbitrary inter-arrival distribution F, and the service time for each customer is *i.i.d.* with a common distribution G. We assume that the service discipline is FCFS. We denote the random *i.i.d.* inter-arrival sequence by $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, and the random *i.i.d.* service time sequence by $Y : \Omega \to \mathbb{R}_+^{\mathbb{N}}$. Then, the inter-arrival time between nth and $(n+1)$th customer is X_{n+1}, and the service time of customer n is Y_n.

Proposition 1.1 (Lindley's equation). *If we denote the waiting time (before service) for customer n in the queue by W_n, then we have*

$$W_n = (W_{n-1} + Y_{n-1} - X_n) \vee 0, \quad n \in \mathbb{N}.$$

We denote $W_0 = Y_0 = 0$, and the customer 1 arrives at time X_1.

Definition 1.2. For a GI/GI/1 queue with *i.i.d.* inter-arrival sequence $X : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ and independent *i.i.d.* service time sequence $Y : \Omega \to \mathbb{R}_+^{\mathbb{N}}$, we associate a random walk sequence $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ with *i.i.d.* step-size sequence $U : \Omega \to \mathbb{R}^{\mathbb{N}}$ such that

$$U_n \triangleq Y_{n-1} - X_n, \quad n \in \mathbb{N}.$$

Proposition 1.3. *Let $W : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ be the random waiting time sequence for customers in a GI/GI/1 queue with associated random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$. Then, we have for some $c \geqslant 0$*

$$P\{W_n \geqslant c\} = P\left(\cup_{j \in [n]} \{S_j \geqslant c\}\right). \tag{1}$$

Proof. From the Lindley's recursion for waiting times and the definition of the associated random walk, we get

$$W_n = \max\{0, W_{n-1} + U_n\}.$$

Iterating the above relation with $W_1 = 0$, and using the definition of random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ yields

$$W_n = \max\{0, U_n + \max\{0, W_{n-2} + U_{n-1}\}\} = \max\{0, U_n, U_n + U_{n-1} + W_{n-2}\} = \max\{0, S_n - S_{n-1}, \dots, S_n\}.$$

Using the duality principle for exchangeable random sequence U, we can rewrite the following equality in distribution $W_n = \max\{0, S_1, \dots, S_n\}$. □

Corollary 1.4. *If $\mathbb{E}U_n \geqslant 0$, then we have $P\{W_\infty \geqslant c\} \triangleq \lim_{n \in \mathbb{N}} P\{W_n \geqslant c\} = 1$ for all $c \in \mathbb{R}$.*

Proof. It follows from Proposition 1.3 that $P\{W_n \geqslant c\}$ is non-decreasing in n. Hence, by monotone convergence theorem, the limit exists and is denoted by $P\{W_\infty \geqslant c\} \triangleq \lim_{n \in \mathbb{N}} P\{W_n \geqslant c\}$. Therefore, by continuity of probability and Eq. (1), we have

$$P\{W_\infty \geqslant c\} = P\{S_n \geqslant c \text{ for some } n\}. \tag{2}$$

If $\mathbb{E}U_n = \mathbb{E}Y_n - \mathbb{E}X_{n+1}$ is positive, then by strong law of large numbers the random walk S will converge almost surely to positive infinity. The above will also be true when $E[U_n] = 0$, then the random walk is recurrent. □

Remark 1. It follows from this corollary, that the stability condition $\mathbb{E}Y_n < \mathbb{E}X_{n+1}$ is necessary for the existence of a stationary distribution.

Proposition 1.5 (Spitzer's Identity). *Let $M_n \triangleq \max\{0, S_1, S_2, \dots, S_n\}$ for all $n \in \mathbb{N}$, then*

$$\mathbb{E}M_n = \sum_{k=1}^{n} \frac{1}{k} \mathbb{E}S_k^+.$$

Proof. We can write $M_n = \mathbb{1}_{\{S_n>0\}} M_n + \mathbb{1}_{\{S_n \leqslant 0\}} M_n$. If $S_n \leqslant 0$, then $M_n = M_{n-1}$. That is,

$$\mathbb{1}_{\{S_n \leqslant 0\}} M_n = \mathbb{1}_{\{S_n \leqslant 0\}} M_{n-1}.$$

If $S_n > 0$, then $M_n = \max\{S_1, \dots, S_n\}$. Therefore, we can rewrite the first term in decomposition, as

$$\mathbb{1}_{\{S_n>0\}} M_n = \mathbb{1}_{\{S_n>0\}} \max_{i \in [n]} S_i = \mathbb{1}_{\{S_n>0\}} (U_1 + \max\{0, S_2 - S_1, \dots, S_n - S_1\}).$$

Hence, taking expectation and using exchangeability of the *i.i.d.* sequence U, we get

$$\mathbb{E}[M_n \mathbb{1}_{\{S_n>0\}}] = \mathbb{E}[U_1 \mathbb{1}_{\{S_n>0\}}] + \mathbb{E}[M_{n-1} \mathbb{1}_{\{S_n>0\}}].$$

Since U is an *i.i.d.* sequence and $S_n = \sum_{i=1}^n U_i$, the tuple (U_i, S_n) has an identical joint distribution for all $i \in [n]$. It follows that

$$\frac{1}{n} \mathbb{E} S_n^+ = \frac{1}{n} \mathbb{E}[S_n \mathbb{1}_{\{S_n>0\}}] = \frac{1}{n} \mathbb{E} \sum_{i=1}^n U_i \mathbb{1}_{\{S_n>0\}}] = \mathbb{E}[U_1 \mathbb{1}_{\{S_n>0\}}].$$

Combining the above results, we obtain the following recursion

$$\mathbb{E}[M_n] = \mathbb{E}[M_{n-1}] + \frac{1}{n} \mathbb{E}[S_n^+].$$

Result follow from the fact that $M_1 = S_1^+$. $\square$

Remark 2. Since $W_n = M_n$ in distribution, we have $\mathbb{E}[W_n] = \mathbb{E}[M_n] = \sum_{k=1}^n \frac{1}{k} \mathbb{E}[S_k^+]$.

2 Martingales for Random Walks

Proposition 2.1. *Consider an* i.i.d. *step-size sequence $X : \Omega \to \mathbb{Z}^{\mathbb{N}}$ such that $|X_n| \leqslant M \in \mathbb{Z}_+$. A random walk $S : \Omega \to \mathbb{Z}^{\mathbb{N}}$ with the step size sequence X is a recurrent Markov chain iff $\mathbb{E} X_n = 0$.*

Proof. If $\mathbb{E} X_n \neq 0$, the random walk is clearly transient since, it will diverge to $\pm\infty$ depending on the sign of $\mathbb{E} X_n$.

Conversely, if $\mathbb{E} X_n = 0$, then the random walk S is a martingale. Assume that the random walk starts at state $S_0 = i \in \mathbb{Z}_+$. We define sets

$$A \triangleq \{-M, -M+1, \cdots, -2, -1\}, \qquad A_j \triangleq \{j+1, \dots, j+M\}, \quad j > i.$$

Let τ denote the hitting time to either the set A or the set A_j by the random walk S, i.e.

$$\tau \triangleq \inf\{n \in \mathbb{N} : S_n \in A \cup A_j\}.$$

It follows that τ is a stopping time with respect to the natural filtration of the step-size sequence X. Further, $S_{\tau \wedge n} \leqslant M + j$. This is not a uniform bound in j. Can we still apply OST? From the optional stopping theorem, we have $\mathbb{E}_i[S_\tau] = \mathbb{E}_i[S_0] = i$. Thus, we have

$$i = \mathbb{E}_i[S_\tau] = \mathbb{E}_i[S_\tau \mathbb{1}_{\{S_\tau \in A\}} + S_\tau \mathbb{1}_{\{S_\tau \in A_j\}}] \geqslant -M P_i\{S_\tau \in A\} + j(1 - P_i\{S_\tau \in A\}).$$

Rearranging the above equation, we get a bound on probability of random walk S hitting A over A_j as

$$P_i\{S_n \in A \text{ for some } n\} \geqslant P_i\{S_\tau \in A\} \geqslant \frac{j-i}{j+M}.$$

Since the choice of $j \in \mathbb{Z}_+$ was arbitrary, taking limit $j \to \infty$, we see that for any $i \in \mathbb{Z}_+$, we have $\mathbb{P}_i\{S_n \in A \text{ for some } n\} = 1$. Similarly taking $B \triangleq \{1, 2, \cdots, M\}$, we can show that $P_i\{S_n \in B \text{ for some } n\} = 1$ for any $i \leqslant 0$. Result follows from combining the above two arguments to see that for any $i \in \mathbb{Z}$

$$P_i\{S_n \in A \cup B \text{ for some } n\} = 1.$$

$\square$

Proposition 2.2. *Consider a random walk $S : \Omega \to \mathbb{R}^{\mathbb{N}}$ with* i.i.d. *step-size sequence $X : \Omega \to \mathbb{R}^{\mathbb{N}}$ with common mean $\mathbb{E}[X_1] \neq 0$. For $a, b > 0$, we define the hitting time of the walk S exceeding a positive threshold a or going below a negative threshold $-b$ as*

$$\tau \triangleq \{n \in \mathbb{N} : S_n \geqslant a \text{ or } S_n \leqslant -b\}.$$

Let P_a denote the probability that the walk hits a value greater than a before it hits a value less than $-b$. That is,

$$P_a \triangleq P\{S_\tau \geqslant a\}.$$

Then, for $\theta \neq 0$ such that $\mathbb{E}e^{\theta X_1} = 1$, we have

$$P_a \approx \frac{1 - e^{-\theta b}}{e^{\theta a} - e^{-\theta b}}.$$

The above approximation is an equality when step size is unity and a and b are integer valued.

Proof. For any $a, b > 0$, we can define stopping times

$$\tau_a = \inf\{n \in \mathbb{N} : S_n \geqslant a\}, \qquad \tau_{-b} = \inf\{n \in \mathbb{N} : S_n \leqslant -b\}.$$

Then, $\tau = \tau_a \wedge \tau_{-b}$, and we are interested in computing the probability $P_a = P\{\tau_a < \tau_{-b}\}$. We define a random sequence $Z : \Omega \to \mathbb{R}_+^{\mathbb{N}}$ such that $Z_n \triangleq e^{\theta S_n}$ for all $n \in \mathbb{N}$, where $\mathbb{E}e^{\theta X_1} = 1$. Hence, it follows that Z is a martingale with unit mean. From the optional stopping theorem, we get $\mathbb{E}e^{\theta S_\tau} = 1$. Thus, we get

$$1 = \mathbb{E}[e^{\theta S_\tau} \mathbb{1}_{\{\tau_a < \tau_{-b}\}}] + \mathbb{E}[e^{\theta S_\tau} \mathbb{1}_{\{\tau_a > \tau_{-b}\}}].$$

We can approximate $e^{\theta S_\tau} \mathbb{1}_{\{\tau_a < \tau_{-b}\}}$ by $e^{\theta a} \mathbb{1}_{\{\tau_a < \tau_{-b}\}}$ and $e^{\theta S_\tau} \mathbb{1}_{\{\tau_a > \tau_{-b}\}}$ by $e^{-\theta b} \mathbb{1}_{\{\tau_a > \tau_{-b}\}}$, by neglecting the overshoots past the thresholds a and $-b$. Therefore, we have

$$1 \approx e^{\theta a} P_a + e^{-\theta b}(1 - P_a).$$

□