Lecture 11 : Time Reversibility of Discrete Time Markov Chains

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1 Discrete Time Markov Chains Contd.

1.1 Time Reversibility of Discrete Time Markov Chains

Consider a discrete time Markov chain with transition probability matrix P and stationary probability vector π.
Claim: The reversed process is a Markov chain.

Proof.

$$P(X_{m-1}=i|X_m=j,X_{m+1}=i_{m+1}\dots)=\frac{P(X_{m-1}=i,X_m=j,\dots)}{P(X_m=j,X_{m+1}=i_{m+1}\dots)}$$
$$=\frac{P(X_{m-1}=i,X_m=j)P(X_{m+1}=i_{m+1}\dots|X_{m-1}=i,X_m=j\dots)}{P(X_m=j)P(X_{m+1}=i_{m+1}\dots|X_m=j)}$$
$$\overset{(a)}{=}\frac{P(X_{m-1}=i,X_m=j)P(X_{m+1}=i_{m+1}\dots|X_m=j\dots)}{P(X_m=j)P(X_{m+1}=i_{m+1}\dots|X_m=j)}$$
$$=P(X_{m-1}=i|X_m=j).$$

where (a) follows from the Markov property. □

The sequence $\{X_n,X_{n-1}\dots\}$ is called reverse process. Let P^* denote the transition probability matrix.

$$P^*_{ij}=P(X_{n-1}=i|X_n=j)=\frac{P(X_{n-1}=i,X_n=j)}{P(X_n=j)}$$
$$=\frac{P(X_{n-1}=i)P(X_n=j|X_{n-1}=i)}{P(X_{n-1}=i)}$$
$$=\frac{P(X_{n-1}=i)}{P(X_n=j)}P_{ji}$$

suppose we are considering a stationary Markov chain, $P(X_n=l)=P(X_{n-1}=l)=\pi(l),\ \forall l,\ \pi(i){P^*}_{ij}=\pi(j)(P)_{ji}$. If $P^*=P$ then the Markov chain is called time reversible. Thus the condition for time reversibility is given by $\pi P_{ij}=\pi_j P_{ji}$. Any non-negative vector X satisfying $X_iP_{ij}=X_jP_{ji},\ \forall i,j$ and $\sum_{j\in\mathcal{N}_0}X_j=1$ is stationary distribution of time-reversible Markov chain. This is true because,

$$\sum_i X_iP_{ij}=\sum_i X_jP_{ji}=X_j\sum X_i=1.$$

Since stationary probabilities are the unique solution of the above, the claim follows.

Example 4.7(A) An Ergodic Random Walk: Any ergodic, positive recurrent random walk is time reversible. The transition probability matrix is $P_{i,i+1} + P_{i-1,i} = 1$. For every two transitions from $i+1$ to i, there must be one transition from i to $i+1$. The rate of transitions from $i+1$ to i must hence be same as the number of transitions from i to $i+1$. So the process is time reversible.

If we try to prove the equations necessary for time reversibility, $X_i P_{ij} = X_j P_{jk}$ for all i, j, for any arbitrary Markov chain, one may not end up getting any solution. This is so because, if $P_{ij} P_{jk} > 0$, then $\frac{X_i}{X_k} = \frac{P_{ji} P_{kj}}{P_{jk} P_{ij}} \neq \frac{P_{kj}}{P_{jk}}$.

Thus we see that a necessary condition for time reversibility is $P_{ij} P_{jk} P_{ki} = P_{ik} P_{kj} P_{ji}, \ \forall i, j, k$. In fact we can show the following.

Theorem 1.1. *A stationary Markov chain is time reversible if and only if starting in state i , any path back to state i has the same probability as the reversed path, for all i. That is, if*

$$P_{ii_1} P_{i_1 i_2} \dots P_{i_k i} = P_{i,i_k} P_{i_k i_{k-1}} \dots P_{i_1,i}.$$

Proof. The proof of necessity is as indicated above. To see the sufficiency part, fix states i, j

$$\sum_{i_1,i_2,\dots i_k} P_{ii_1} \dots P_{i_k,j} P_{j,i} = \sum_{i_1,i_2,\dots i_k} P_{i,j} P_{j,i_k} \dots P_{i_1 i}$$

$$(P^k)_{ij} P_{ji} = P_{ij} (P^k)_{ji}$$

$$\frac{\sum_{k=1}^n (P^k)_{ij} P_{ji}}{n} = \frac{\sum_{k=1}^n P_{ij} (P^k)_{ji}}{n}$$

As limit $n \to \infty$, we get the desire result. □

Theorem 1.2. *Consider irreducible Markov chain with transition matrix P. If one can find non-negative vector π and other transition matrix P^* such that $\sum_j \pi_j = 1$ and $\pi_i P_{ij} = \pi_j P^*_{ji}$ then π is the stationary probability vector and P^* is the transition matrix for the reversed chain.*

Proof. Summing $\pi_i P_{ij} = \pi_j P^*_{ji}$ over i gives, $\sum_i \pi_i P_{ij} = \pi_j$. Hence π_is are the stationary probabilities of the forward and reverse process. Since $P^*_{ji} = \frac{\pi_i P_{ij}}{\pi_j}$, P^*_{ij} are the transition probabilities of the reverse chain. □

1.2 Example 4.7(E): Example 4.3(C) revisited

Example 4.3(C) was with regard to the age of a renewal process. X_n the forward process there was such that it increases in steps of 1 until it hits a value chosen by the inter arrival distribution. Hence the reverse process should be such that it decreases in steps of 1 until it hits 1 and then jumps to a state as chosen by the inter arrival distribution. Thus letting P_i as the probability of inter arrival, it seems likely that $P_{1i}* = P_i, \ P_{i,i-1} = 1, \ i > 1$. We have that

$P_{i,1} = \frac{P_i}{\sum_{j\geq 1} P_j} = 1 - P_{i,i+1},\ i \geq 1$. For the reversed chain to be given as above, we would need

$$\begin{aligned}
&\pi_i P_{ij} = \pi_j P^*_{ji} \\
&\pi_i \frac{P_i}{\sum_j P_j} = \pi_1 P_i \\
&\pi_i = \pi_1 P(X \geq i) \\
&1 = \sum_i \pi_i = \pi_1 E[X]; \pi_i = \frac{P(X \geq i)}{E[X]},
\end{aligned}$$

where X is the inter arrival time. We need to verify that $\pi_i P_{i,i+1} = \pi_{i+1} P^*_{i+1,i}$ or equivalently $P(X \geq i)(1 - \frac{P_i}{P(X \geq i)}) = P(X \geq i)$ to complete the proof that the reversed process is the excess process and the limiting distributions are as given above. But that is immediate.

1.3 Semi Markov Processes

Consider a stochastic process with states $0, 1, 2 \ldots$ such that whenever it enters state i,

1. The next state it enters is state j with probability P_{ij}.

2. Given the next state is going to be j, the time until the next transition from state i to state j has distribution F_{ij}. If we denote the state at time t to be $Z(t)$, $\{Z(t), t \geq 0\}$ is called a Semi Markov process.

Markov chain is a semi Markov process with

$$F_{ij}(t) = \begin{cases} 0 & : t \leq 1 \\ 1 & : t > 1. \end{cases}$$

Let H_i the distribution of time the semi Markov process stays in state i before transition. We have $H_j(t) = \sum_i P_{ij} F_{ij}(t)$, $\mu_i = \int_0^\infty X dH_i(x)$. Let X_n denote the n^{th} state visited. Then $\{X_n\}$ is a Markov chain with transition probability P called the embedded Markov chain of the semi Markov process.
Definition: If the embedded Markov chain is irreducible, then Semi Markov process is said to be irreducible. Let T_{ii} denote the time between successive transitions to state i. Let $\mu_{ii} = E[T_{ii}]$.

Theorem 1.3. *If semi Markov process ius irreducible and if T_{ii} has non-lattice distribution with $\mu_{ii} < \infty$ then,*

$$P_i = \lim_{t\to\infty} P(Z(t) = i | Z(0) = j)$$

exists and is independent of the initial state. Furthermore, $P_i = \frac{\mu_i}{\mu_{ii}}$.

Corollary 4.8.2 If the semi-Markov process is irreducible and $\mu_{ii} < \infty$, then with probability 1, $\frac{\mu_i}{\mu_{ii}} = \frac{\lim_{t\to\infty} \text{Amount of time in [0,t]}}{t}$.

Theorem 1.4. *Suppose conditions of the previous thmrem hold and the embedded Markov chain is positive recurrent. Then* $P_i = \frac{\pi_i \mu_i}{\sum_i \pi_j \mu_j}$.

Proof. Define the notation as follows:

$Y_i(j)$ = amount of time spent in state i during j^{th} visit to that state. $i, j \geq 0$.
$N_i(m)$ = number of visits to state i in the first m transitions of the semi-Markov process.

The proportion of time in i during the first m transitions:

$$\begin{aligned} P_{i=m} &= \frac{\sum_{j=1}^{N_i(m)} Y_i(j)}{\sum_l \sum_{j=1}^{N_i(m)} Y_i(j)} \\ &= \frac{\frac{N_i(m)}{m} \sum_{j=1}^{N_i(m)} Y_i(j)}{\sum_l \frac{N_i(m)}{m} \sum_{j=1}^{N_i(m)} Y_i(j)} \end{aligned}$$

Since $N_i(m) \to \infty$ as $m \to \infty$, it follows from the strong law of large numbers that $\frac{\sum_{i=2}^{N_i(m)} Y_i(j)}{N_i(m)} \to \mu_i$ and $\frac{N_i(m)}{m} \to (E[\text{number of transitions between visits to state } i])^{-1} = \pi_i$. Letting $m \to \infty$, result follows. □

Theorem 1.5. *If Semi Markov process is irreducible and non lattice, then* $\lim_{t\to\infty} P(Z(t) = i, Y(t) > x, S(t) = j | Z(0) = k) = \frac{P_{ij} \int_x^\infty F_{ij}^c(y) d(y)}{\mu_{ii}}$. *Let* $Y(t)$ *denote the time from* t *until the next transition.* $S(t)$ *state entered at the first transition after* t.

Proof. The trick lies in defining the "ON" time.

$$E[\text{ON time in a cycle}] = E[(X_{ij} - x)^+].$$

□

Corollary:

$$\lim_{t\to\infty} P(Z(t) = i, Y(t) > x | Z(0) = k) = \frac{\int_x^\infty H_i^c(y) d(y)}{\mu_{ii}}.$$