Lecture 1: The Poisson Process

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1 The Poisson Process

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space.

Definition 1.1 (Point Process). A stochastic process $\{N(t), t \geqslant 0\}$ is a **point process** if

1. $N(0) = 0$.
2. $t \mapsto N(t)(\omega)$ is non-decreasing, integer valued, right continuous and at points of discontinuity (wherever it has jumps) $(N(t) - N(t^-)) \leqslant 1, \ \forall \quad \omega \in \Omega$.

Definition 1.2 (Simple Point Process). A **simple point process** is a point process of jump size 1.

Definition 1.3 (Stationary Increment Point Process). A point process $\{N(t), t \geqslant 0\}$ is called **stationary increment point process** , if for any collection of $0 \leqslant t_1 < t_2 ..., < t_n$, we have $(N(t_n) - N(t_{n-1}), N(t_{n-1}) - N(t_{n-2}), ..., N(t_1))$ having the same joint distribution as $(N(t_n + t) - N(t_{n-1} + t), ..., N(t_1 + t)), \ \forall t \geqslant 0$.

Definition 1.4 (Stationary Independent Increment Point Process). A point process $\{N(t), t \geqslant 0\}$ is called **stationary independent increment process**, if it has stationary increments and the increments are independent random variables.

The points of discontinuity corresponds to the arrival instants of the point process. Let X_n denote the inter arrival time between $(n-1)^{th}$ and n^{th} arrival. Further, let, $S_0 = 0, \ S_n = \sum_{k=1}^{n} X_k$. S_n is the arrival instant of of n^{th} point. The arrival at time zero is not counted.

Definition 1.5 (Poisson Process). A simple point process $\{N(t), \ t \geqslant 0\}$ is called a **Poisson process** with rate $0 < \lambda < \infty$, if inter-arrival times $\{X_n, \ n \geqslant 1\}$ are *iid* $\exp(\lambda)$ random variables, i.e.

$$\Pr\{X_1 \leqslant x\} = \begin{cases} 1 - e^{-\lambda x}, & x \geqslant 0 \\ 0, & \text{else.} \end{cases}$$

Remarks: Observe that

$$\begin{aligned} \{S_n \leqslant t\} &= \{N(t) \geqslant n\}, \\ \{S_n \leqslant t, S_{n+1} > t\} &= \{N(t) = n\}, \quad \text{and} \\ \Pr\{X_n = 0\} &= \Pr\{X_n \leqslant 0\} = 0. \end{aligned}$$

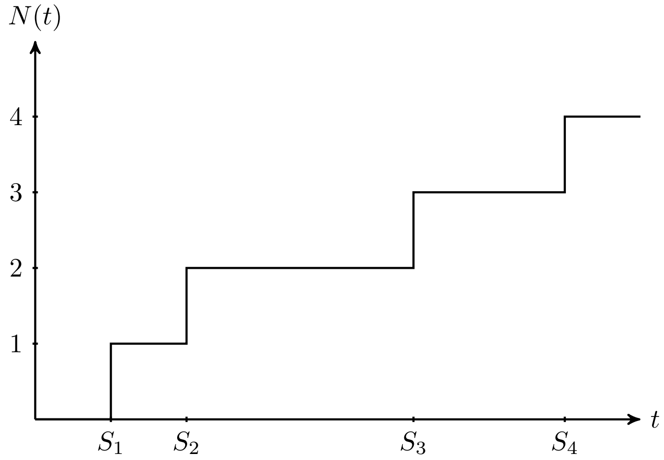

Figure 1: Sample path of a Poisson process.

Also, by Strong Law of Large Numbers (SLLN),

$$\lim_{n\to\infty} \frac{S_n}{n} = E[X_1] = \frac{1}{\lambda} \quad \text{a.s.}$$

Therefore, we have $S_n \to \infty$, a.s. This implies $\Pr\{\omega : N(t)(\omega) < \infty\} = 1$. To see this, let's pick an $\omega \in \Omega$ such that $N(t)(\omega) = \infty$, then $S_n(\omega) \leqslant t, \quad \forall n$. This implies $S_\infty(\omega) \leqslant t$ and $\omega \notin \{\omega : S_n(\omega) \to \infty\}$. Hence, probability measure for such *omega*'s is zero and the claim follows.

1.1 Moment Generating Function and Density Function of S_n

We know that time of n^{th} event S_n is sum of n consecutive *iid* inter-arrival times X_k, i.e. $S_n = \sum_{k=1}^n X_k$. Therefore, moment generating function $\mathbb{E}[e^{\alpha S_n}]$ of S_n is given by

$$\mathbb{E}[e^{\alpha S_n}] = \left(\mathbb{E}[e^{\alpha X_1}]\right)^n.$$

Since each X_k is *iid* exponential with rate λ, it is easy to see that moment generating function of intr-arrival time X_1 is

$$\mathbb{E}[e^{\alpha X_1}] = \begin{cases} \frac{\lambda}{\lambda-\alpha}, & \alpha < \lambda \\ \infty, & \alpha \geqslant \lambda. \end{cases}$$

Substituting the moment generating function of inter-arrival time X_1 in moment generating function of n^{th} event time S_n, we obtain

$$\mathbb{E}[e^{\alpha S_n}] = \begin{cases} \left(\frac{\lambda}{\lambda-\alpha}\right)^n, & \alpha < \lambda, \\ \infty, & \text{else.} \end{cases}$$

Theorem 1.6 (Arrival Time). *Density function of S_n is Gamma distributed with parameters n and λ. That is,*

$$f_{S_n}(s) = \frac{\lambda(\lambda s)^{n-1}}{(n-1)!} e^{-\lambda s}.$$

Proof. Notice that X_i's are *iid* and $S_1 = X_1$. In addition, we know that $S_n = X_n + S_{n-1}$. Since, X_n is independent of S_{n-1}, we know that distribution of S_n would be convolution of distribution of S_{n-1} and X_1. Since X_n and S_1 have identical distribution, we have $f_{S_n} = f_{S_{n-1}} * f_{S_1}$. The result follows from straightforward induction. □

Process $N(t)$ is of real interest, and we can compute the distribution of $N(t)$ for each t from the distribution of S_n in the following.

Theorem 1.7 (Poisson process). *Process $N(t)$ is Poisson distributed with parameter λ for each t. That is,*

$$\Pr\{N(t) = n)\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}.$$

Proof.

$$\begin{aligned}
\Pr\{N(t) = n\} &= \Pr\{S_n \leqslant t, S_{n+1} > t) \\
&= \int_0^t \Pr\{S_{n+1} > t | S_n = s\} f_{S_n}(s) ds \\
&\stackrel{(a)}{=} \int_0^t \Pr\{X_{n+1} > t - s\} f_{S_n}(s) ds \\
&= \int_0^t e^{-\lambda(t-s)} \frac{\lambda^n s^{n-1}}{(n-1)!} e^{-\lambda s} ds \\
&= \frac{e^{-\lambda t}(\lambda t)^n}{n!}.
\end{aligned}$$

where (a) follows from the memoryless property of exponential distribution. □

Remark: The Poisson process is not a stationary process. That is, the finite dimensional distributions (fdd) are not shift invariant. In the following section, we show that the Poisson process is a *stationary, independent increment* process. To this end, we will use an important property of exponential distribution-namely memoryless property. Memoryless property of exponential distribution will facilitate the computation of fdd of the Poisson process via one dimensional marginal distribution.

Proposition 1.8 (Memoryless Distribution). *Exponential distribution with continuous support is the only distribution satisfying memoryless property.*

Proof. Let X be a random variable with a distribution function F with memoryless property defined on $\mathbb{R}^+$. Let $g(t) \triangleq \Pr\{X > t\} = 1 - F(t)$. Due to memoryless property of F, we notice that

$$\begin{aligned}
\Pr\{X > s\} &= \Pr\{X > t + s | X > t\} \\
\Pr\{X > s\} &= \frac{\Pr\{X > t + s, X > t\}}{\Pr\{X > t\}}.
\end{aligned}$$

Since $\{X > t + s\} = \{X > t + s, X > t\}$, we have $g(t + s) = g(t)g(s)$ and hence $g(0) = g^2(0)$. Therefore, $g(0)$ is either unity or zero. Note, that g is a right continuous (RC) function and is non-negative.

We will show that g is an exponential function. That is, $g(t) = e^{\alpha t}$ for some $\alpha \geqslant 0$. We will prove this in stages. First, we show this is true for $t \in \mathbb{Z}^+$. Notice that we can obtain via induction

$$\begin{aligned} g(2) &= g(1)g(1) = g^2(1), \text{and} \\ g(m) &= [g(1)]^m. \end{aligned}$$

Since $g(1)$ is non negative, there exists a β such that $g(1) = e^{\beta}$ and $g(m) = e^{m\beta}, m \in \mathbb{Z}_+$. Next we show that for any $n \in \mathbb{Z}_+$,

$$g(1) = g\left(\frac{1}{n} + ..., + \frac{1}{n}\right) = \left[g\left(\frac{1}{n}\right)\right]^n.$$

Therefore, for same β we used for $g(1)$, we have $g\left(\frac{1}{n}\right) = e^{\frac{\beta}{n}}$. Now, we show that g is exponential for any $t \in \mathbb{Q}^+$. To this end, we see that for any $m, n \in \mathbb{Z}_+$, we have

$$g\left(\frac{m}{n}\right) = \left[g\left(\frac{1}{n}\right)\right]^m = e^{\frac{m\beta}{n}}.$$

Now, we can show that g is exponential for any real positive t by taking a sequence of rational numbers $\{t_n\}$ decreasing to t. From right continuity of g, we obtain

$$g(t) \overset{(a)}{=} \lim_{n\to\infty} g(t_n) = \lim_{n\to\infty} e^{\beta t_n} = e^{\beta t}.$$

Since $\Pr\{X > x\}$ is decreasing with x, β is negative. $\square$

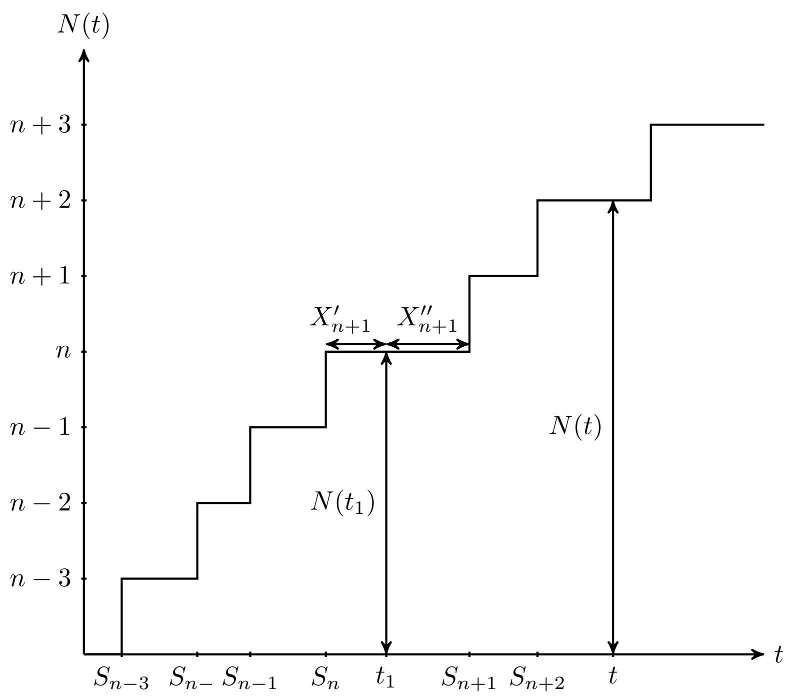

Figure 2: Stationary independent increment property of Poisson process.

Proposition 1.9 (Stationary Independent Increment Property). *Poisson process $N(t), t \geqslant 0$ has stationary independent increment property.*

Proof. To show that $N(t)$ has stationary independent increment property, it suffices to show that $N_t - N(t_1) \perp N(t_1)$ and $N(t) - N(t_1) \sim N(t - t_1)$. Since, we can use induction to show this stationary independence increment property for for any finite disjoint time-intervals. The memoryless property of exponential distribution is crucially used. And, we see that independent increment holds only if inter-arrival time is exponential. We can see in Figure 2 that t_1 divides X_{n+1} in two parts such that, $X_{n+1} = X^{'}_{n+1} + X^{''}_{n+1}$. Here, $X^{''}_{n+1}$ is independent of $X^{'}_{n+1}$ and has same distribution as X_{n+1}. Therefore,

$$\{N(t_1) = n\} \iff \{S_n = t_1 + X'_{n+1}\},$$
$$\{N(t) - N(t_1) \geqslant m\} \iff \{X''_{n+1} + \sum_{i=n+2}^{n+m} X_i \leqslant t - t_1\}.$$

Since, $\{X_i : i \geqslant n+2\} \cup \{X^{''}_{n+1}\}$ are independent of $\{X_i : i \leqslant n\} \cup X^{'}_{n+1}$, we have $N(t) - N(t_1) \perp N(t_1)$. Further, since $X^{''}_{n+1}$ has same distribution as X_{n+1}, we get $N(t) - N(t_1) \sim N(t - t_1)$. By induction we can extend this result to $(N(t_n) - N(t_{n-1}), ..., N(t_1))$. $\square$