Fourier Analysis

Half Range Fourier Series

Fourier Cosine and Sine Series

Conducted By

Partho Sutra Dhor

Faculty, Mathematics and Natural Sciences

BRAC University, Dhaka, Bangladesh

Half Range Series

What is Half Range Series and How to Identify

$(-L, L)$

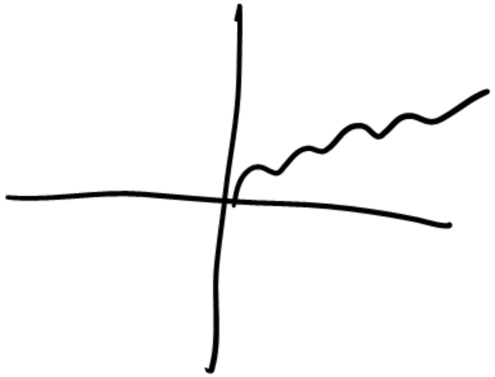
Even $\longrightarrow$ cosine

$f(-x) = f(x)$

odd $\longrightarrow$ sine

$f(-x) = -f(x)$

$(0, L)$



Half Range Cases of General Fourier Series

General Fourier Series

Let $f(x)$ be defined on the interval $(-L, L)$ with period $2L$. The Fourier series expansion of $f(x)$ is defined to be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where the Fourier coefficients a_n and b_n are

$$\begin{cases} a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx\,, & n = 0,1,2,3,\ldots \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\,, & n = 1,2,3,\ldots\ldots \end{cases}$$

Half Range Fourier Cosine Series

Let $f(x)$ be a function defined on the half interval $(0, L)$. The Fourier Cosine series expansion of $f(x)$ is defined to be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} \right)$$

where the Fourier coefficients a_n are

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx \,, \qquad\qquad n = 0,1,2,3, \ldots$$

Half Range Fourier Sine Series

Let $f(x)$ be a function defined on the half interval $(0, L)$. The Fourier Sine series expansion of $f(x)$ is defined to be

$$f(x) = \sum_{n=1}^{\infty} \left(b_n \sin \frac{n\pi x}{L} \right)$$

where the Fourier coefficients b_n are

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \,, \qquad\qquad n = 1,2,3, \ldots$$

How to calculate L

$2L$ = interval length

half Range / cosine series / sine series

L = given half Range

Problems

PROBLEM Expand $f(x) = \cos x, \quad 0 < x < \pi$ in a Fourier sine series.

$\therefore \ L = \pi.$

$\therefore$ Fourian sine series $f(x) = \sum_{n=1}^{\infty} \left(b_n \sin(nx) \right)$

$$\therefore \ b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)$$

$$\therefore \quad b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \cdot \sin(nx)\,dx \qquad \text{for } n \neq 1 .$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\sin(nx)\cos(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\sin(nx+x) + \sin(nx-x)\right]dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(nx+x)}{n+1} + \frac{-\cos(nx-x)}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos(nx+x)}{n+1} + \frac{\cos(nx+x)}{n-1}\right]_{\pi}^{0}$$

$$= \frac{1}{\pi}\left[\left(\frac{1}{n+1} + \frac{1}{n-1}\right) - \left(\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n+1} + \frac{1}{n-1} - \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{2n}{n^2-1} - \frac{(-1)\cdot(-1)^n}{n+1} - \frac{(-1)^n\cdot(-1)^{-1}}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{n}{n^2-1} + \frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{n}{n^2-1} + \frac{n\cdot(-1)^n}{n^2-1}\right]$$

$$= \frac{n\left(1+(-1)^n\right)}{\pi\left(n^2-1\right)}.$$

$$\therefore\ b_1 = \frac{2}{\pi}\int_0^{\pi} \cos x \cdot \sin x\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} (\sin 2x)\, dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos 2x}{2}\right]_{\pi}^{0}$$

$$= \frac{1}{\pi}\left[\frac{1}{2} - \frac{1}{2}\right]$$

$$= 0.$$

$$\therefore \quad b_1 = 0$$

$$\therefore \quad b_n = \frac{n\left(1+(-1)^n\right)}{\pi\left(n^2-1\right)}$$

PROBLEM Expand $f(x) = x, \quad 0 < x < 2$ in a half range series of cosine.

$\therefore L = 2.$

$\therefore$ fourier cosine series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos\left(\frac{n\pi x}{2}\right)\right)$

$\therefore a_n = \frac{2}{2} \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx$

$$\therefore \quad a_0 = \int_0^2 f(x)\,dx$$

$$= \int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2 = 2.$$

$$\therefore \quad a_n = \int_0^2 x \cdot \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \left[x \cdot \frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} - 1 \cdot \frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}} \right]_0^2$$

$$= \left[\frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right) \right]_0^2$$

x	$\cos\left(\frac{n\pi x}{2}\right)$
1	$\dfrac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}$
0	$\dfrac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}}$

$$= \left[\left(\frac{4}{n\pi} \cancel{\sin(n\pi)} + \frac{4}{n^2\pi^2} \cos(n\pi) \right) - \left(0 + \frac{4}{n^2\pi^2} 1 \right) \right]$$

$$= \frac{4}{n^2\pi^2} (-1)^n - \frac{4}{n^2\pi^2}$$

$$= \frac{4(-1)^n - 4}{n^2\pi^2}.$$

PROBLEM Find the Fourier cosine series expansion of the function

$$f(x) = 3\sin x \qquad 0 < x < \pi$$

$\therefore L = \pi.$

$\therefore$ Fourier cosine series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos(nx)$

$$\therefore a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx$$

$$\therefore \quad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} 3\sin x\,dx$$

$$= \frac{2}{\pi}\Big[-3\cos x\Big]_0^{\pi}$$

$$= \frac{2}{\pi}\Big[3\cos x\Big]_{\pi}^{0}$$

$$= \frac{2}{\pi}\left(3\cdot 1 - 3\cdot(-1)\right)$$

$$= \frac{2}{\pi}6 \qquad = \frac{12}{\pi}.$$

$$\therefore \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx \qquad \text{for } n \neq 1$$

$$= \frac{2}{\pi}\int_0^{\pi} 3\sin x \cos(nx)\,dx$$

$$= \frac{3}{\pi}\int_0^{\pi} 2\sin x \cos(nx)\,dx$$

$$= \frac{3}{\pi}\int_0^{\pi} \left[\sin(x+nx) + \sin(x-nx)\right]dx$$

$$= \frac{3}{\pi}\left[\frac{-\cos(x+nx)}{1+n} + \frac{-\cos(x-nx)}{1-n}\right]_0^\pi$$

$$= \frac{3}{\pi}\left[\frac{\cos(x+nx)}{1+n} + \frac{\cos(x-nx)}{1-n}\right]_\pi^0$$

$$= \frac{3}{\pi}\left[\left(\frac{1}{1+n} + \frac{1}{1-n}\right) - \left(\frac{\cos(1+n)\pi}{1+n} + \frac{\cos(1-n)\pi}{1-n}\right)\right]$$

$$= \frac{3}{\pi}\left[\frac{2}{1-n^2} - \frac{(-1)^{1+n}}{1+n} - \frac{(-1)^{1-n}}{1-n}\right]$$

$$= \frac{3}{\pi}\left[\frac{2}{1-n^2} - \frac{(-1)\cdot(-1)^n}{1+n} - \frac{(-1)\cdot(-1)^{-n}}{1-n}\right]$$

$$(-1)^{-n}$$

$$= \left((-1)^{-1}\right)^n$$

$$= (-1)^n$$

$$= \frac{3}{\pi}\left[\frac{2}{1-n^2} + \frac{(-1)^n}{1+n} + \frac{(-1)^n}{1-n}\right]$$

$$= \frac{3}{\pi}\left[\frac{2}{1-n^2} + \frac{2\cdot(-1)^n}{1-n^2}\right).$$

$$\therefore \quad a_1 = \frac{2}{\pi}\int_0^{\pi} 3\sin x \cdot \cos x \, dx$$

$$= \frac{3}{\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{3}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi}$$

$$= \frac{3}{\pi}\left[\frac{\cos 2x}{2}\right]_{\pi}^{0}$$

$$= \frac{3}{\pi}\left[\frac{1}{2} - \frac{1}{2}\right]$$

$$= 0$$

$$\therefore \quad a_0 = \frac{12}{\pi}$$

$$\therefore \quad a_1 = 0$$

$$\therefore \quad a_n = \frac{3}{\pi}\left[\frac{2}{1-n^2} + \frac{2 \cdot (-1)^n}{1-n^2} \right) .$$

PROBLEM Expand $f(x) = A - \frac{Ax}{P},\ 0 < x < P$ in a half range series of Sine.

odd

$\therefore L = P$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{P}\right)$$

$$\therefore b_n = \frac{2}{P}\int_0^P f(x)\sin\left(\frac{n\pi x}{P}\right)dx$$

$$\therefore \quad b_n = \frac{2}{p}\int_0^p \left(A - \frac{Ax}{p}\right)\sin\left(\frac{n\pi x}{p}\right)dx$$

$$\begin{array}{c|c} A - \dfrac{Ax}{p} & \sin\left(\dfrac{n\pi x}{p}\right) \\ \hline -\dfrac{A}{p} & \dfrac{-\cos\left(\frac{n\pi x}{p}\right)}{\frac{n\pi}{p}} \\ \hline 0 & \dfrac{-\sin\left(\frac{n\pi x}{p}\right)}{\frac{n^2\pi^2}{p^2}} \end{array}$$

$$= \frac{2}{p}\left[\left(A - \frac{Ax}{p}\right)\frac{-\cos\left(\frac{n\pi x}{p}\right)}{\frac{n\pi}{p}} - \left(\frac{-A}{p}\right)\frac{-\sin\left(\frac{n\pi x}{p}\right)}{\frac{n^2\pi^2}{p^2}}\right]_0^p$$

$$= \left[\frac{2}{\cancel{p}}\left(A - \frac{Ax}{p}\right)\frac{-\cancel{p}}{n\pi}\cos\left(\frac{n\pi x}{p}\right) - \frac{2}{\cancel{p}}\cdot\frac{A}{\cancel{p}}\cdot\frac{\cancel{p^2}}{n^2\pi^2}\sin\left(\frac{n\pi x}{p}\right)\right]_0^p$$

$$= \left[-\frac{2}{n\pi}\left(A - \frac{Ax}{p}\right)\cos\left(\frac{n\pi x}{p}\right) - \frac{2A}{n^2\pi^2}\sin\left(\frac{n\pi x}{p}\right)\right]_0^p$$

$$= \left[\frac{2}{n\pi}\left(A - \frac{Ax}{p}\right)\cos\left(\frac{n\pi x}{p}\right) + \frac{2A}{n^2\pi^2}\sin\left(\frac{n\pi x}{p}\right)\right]_p^0$$

$$= \left(\frac{2}{n\pi}(A-0)\cdot 1 + \frac{2A}{n^2\pi^2}\cdot 0\right) - \left(\frac{2}{n\pi}\cdot(0) + \frac{2A}{n^2\pi^2}\cdot \cancel{\sin(n\pi)}\right)$$

$$= \frac{2A}{n\pi}.$$

$$\therefore \quad b_n = \frac{2A}{n\pi}.$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2A}{n\pi} \sin\left(\frac{n\pi x}{p}\right).$$


Thank you!