Fourier Analysis

Complex Fourier Series

Proof of Complex Fourier Series

Conducted By

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Basic Complex Arithmetic

$i = \sqrt{-1}$

$a + bi$

$i^2 = -1$

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$e^{i\pi} = \cos\pi + i\sin\pi$$

$$\Rightarrow e^{i\pi} = -1$$

$$\Rightarrow \boxed{e^{i\pi} + 1 = 0}$$

Complex form of Sine and Cosine

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \quad \text{(Adding)}$$

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta \quad \text{(subtracting)}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Complex Fourier Series

General Fourier Series

Let $f(x)$ be defined in an interval with period $2L$. The Fourier series expansion of $f(x)$ is defined to be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where the Fourier coefficients a_n and b_n are

$$\begin{cases} a_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \cos \dfrac{n\pi x}{L} \, dx \,, & n = 0,1,2,3, \dots \\ b_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \sin \dfrac{n\pi x}{L} \, dx \,, & n = 1,2,3, \dots \dots \end{cases}$$

Complex Fourier Series

Let $f(x)$ be defined in an interval with period $2L$. The complex Fourier series expansion of $f(x)$ is defined to be

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$

where the Fourier coefficients c_n are

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\ e^{-i\frac{n\pi x}{L}} dx\,, \qquad n = \cdots - 3, -2, -1, 0, 1, 2, 3, \ldots$$

$f(x) \longrightarrow$ Period $2L$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{+i\frac{n\pi x}{L}}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) \cdot e^{-i\frac{n\pi x}{L}} \, dx$$

Proof of Complex Fourier Series

PROBLEM Use Euler's Identity to prove that, the complex form of Fourier series can be expressed as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$

In General Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \quad \text{--- (1)}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad \text{--- (2)}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \text{--- (3)}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\therefore \cos\left(\frac{n\pi x}{L}\right) = \frac{e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}}{2}$$

$$\sin\theta = \frac{e^{i\theta} + e^{-i\theta}}{2i}$$

$$\therefore \sin\left(\frac{n\pi x}{L}\right) = \frac{e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}}{2i}$$

$\therefore$ from (1),

$$\therefore\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cdot \frac{e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}}{2} + b_n \frac{e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}}{2i} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{a_n}{2} + \frac{b_n}{2i} \right) e^{i\frac{n\pi x}{L}} + \left(\frac{a_n}{2} - \frac{b_n}{2i} \right) e^{-\frac{n\pi x}{L}} \right]$$

$$= \frac{a_0}{2} \cdot e^{i\frac{0\cdot\pi x}{L}} + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} + \frac{b_n}{2i} \right) e^{i\frac{n\pi x}{L}} + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} - \frac{b_n}{2i} \right) e^{-\frac{n\pi x}{L}}$$

$$= \sum_{n=1}^{\infty} \left(\frac{a_n}{2} - \frac{b_n}{2i} \right) e^{-i\frac{n\pi x}{L}} + \frac{a_0}{2} \cdot e^{i\frac{0 \cdot \pi x}{L}} + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} + \frac{b_n}{2i} \right) e^{i\frac{n\pi x}{L}}$$

$$= \sum_{n=-\infty}^{-1} \left(\frac{a_{-n}}{2} - \frac{b_{-n}}{2i} \right) e^{+i\frac{n\pi x}{L}} + \frac{a_0}{2} \cdot e^{i\frac{0 \cdot \pi x}{L}} + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} + \frac{b_n}{2i} \right) e^{i\frac{n\pi x}{L}}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$

where

$$c_n = \begin{cases} \dfrac{a_{-n}}{2} - \dfrac{b_{-n}}{2i} & n < 0 \\ \dfrac{a_0}{2} & n = 0 \\ \dfrac{a_n}{2} + \dfrac{b_n}{2} & n > 0 \end{cases}$$

PROBLEM Use Euler's Identity to prove that, the complex form of Fourier series can be expressed as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$

where the Fourier coefficients c_n are

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\ e^{-i\frac{n\pi x}{L}} dx\,, \qquad n = \cdots -3, -2, -1, 0, 1, 2, 3, \ldots$$

In General Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \qquad \text{--- } (1)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad \text{--- } (2)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad \text{--- } (3)$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\therefore \cos\left(\frac{n\pi x}{L}\right) = \frac{e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}}{2}$$

$$\sin\theta = \frac{e^{i\theta} + e^{-i\theta}}{2i}$$

$$\therefore \sin\left(\frac{n\pi x}{L}\right) = \frac{e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}}{2i}$$

$\therefore$ from (1),

$$\therefore\ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cdot \frac{e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}}{2} + b_n \frac{e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}}{2i} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{a_n}{2} + \frac{b_n}{2i} \right) e^{i\frac{n\pi x}{L}} + \left(\frac{a_n}{2} - \frac{b_n}{2i} \right) e^{-\frac{n\pi x}{L}} \right]$$

$$= \frac{a_0}{2} \cdot e^{i\frac{0\cdot\pi x}{L}} + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} + \frac{b_n}{2i} \right) e^{i\frac{n\pi x}{L}} + \sum_{n=1}^{\infty}\left(\frac{a_n}{2} - \frac{b_n}{2i} \right) e^{-\frac{n\pi x}{L}}$$

$$= \sum_{n=1}^{\infty} \left(\frac{a_n}{2} - \frac{b_n}{2i} \right) e^{-i\frac{n\pi x}{L}} + \frac{a_0}{2} \cdot e^{i\frac{0\cdot\pi x}{L}} + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} + \frac{b_n}{2i} \right) e^{i\frac{n\pi x}{L}}$$

$$= \sum_{n=-\infty}^{1} \left(\frac{a_{-n}}{2} - \frac{b_{-n}}{2i} \right) e^{+i\frac{n\pi x}{L}} + \frac{a_0}{2} \cdot e^{i\frac{0\cdot\pi x}{L}} + \sum_{n=1}^{\infty} \left(\frac{a_n}{2} + \frac{b_n}{2i} \right) e^{i\frac{n\pi x}{L}}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$

$$\therefore \quad f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{i\frac{n\pi x}{L}}.$$

$$\int_{-L}^{L} f(x) \cdot e^{-i\frac{m\pi x}{L}}\, dx = \int_{-L}^{L} \left(\sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}} \cdot e^{-i\frac{m\pi x}{L}} \right) dx.$$

$$\Rightarrow \int_{-L}^{L} f(x)\, e^{-i\frac{m\pi x}{L}}\, dx = \sum_{n=-\infty}^{\infty} \left(c_n \cdot \underbrace{\int_{-L}^{L} e^{i\frac{n\pi x}{L}} \cdot e^{-i\frac{m\pi x}{L}}\, dx}_{} \right) \quad \text{——— } (1)$$

For $\underline{n = m}$

$$\int_{-L}^{L} e^{i\frac{n\pi x}{L}} \cdot e^{-i\frac{m\pi x}{L}}\, dx$$

$$= \int_{-L}^{L} e^{i\frac{n\pi x}{L}} \cdot e^{-i\frac{n\pi x}{L}}\, dx \qquad [\because\ n = m]$$

$$= \int_{-L}^{L} 1\, dx = 2L.$$

when $n \neq m$:

$$\int_{-L}^{L} e^{i\frac{n\pi x}{L}} \cdot e^{-i\frac{m\pi x}{L}} \, dx$$

$$= \int_{-L}^{L} e^{i\frac{(n-m)\pi x}{L}} \, dx$$

$$= \left[\frac{e^{i\frac{(n-m)\pi x}{L}}}{\frac{(n-m)\pi}{L}} \right]_{-L}^{L}$$

$$= 0.$$

Hw

$$\Rightarrow \int_{-L}^{L} f(x)\, e^{-i\frac{m\pi x}{L}}\, dx = \sum_{n=-\infty}^{\infty} \left(c_n \cdot \int_{-L}^{L} e^{i\frac{n\pi x}{L}} \cdot e^{-i\frac{m\pi x}{L}}\, dx \right) \quad \text{——— } 1$$

$$\Rightarrow \int_{-L}^{L} f(x)\, e^{-i\frac{m\pi x}{L}}\, dx = 0 + 0 + 0 + \cdots + c_m \cdot (2L) + 0 + 0 + \cdots$$

$$\Rightarrow c_m = \frac{1}{2L} \int_{-L}^{L} f(x)\, e^{-i\frac{m\pi x}{L}}\, dx$$

$$\Rightarrow c_n = \frac{1}{2L} \int_{-L}^{L} f(x)\, e^{-i\frac{n\pi x}{L}}\, dx$$

$$\therefore \quad f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{i \frac{n\pi x}{L}}$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) \cdot e^{-i \frac{n\pi x}{L}} \, dx$$

Problems

PROBLEM Using complex form, find the Fourier series of the function

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$

$2L = 2\pi$

$\Rightarrow L = \pi$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{i\frac{n\pi x}{L}}$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{i(nx)}$$

$$\therefore \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\cdot e^{-i\frac{n\pi x}{L}}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0}(-1)\,e^{-inx}\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 1\cdot e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[+\frac{e^{-inx}}{+in}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_{0}^{\pi}$$

$$= \frac{1}{2\pi}\left[\left(\frac{e^{0}}{in}\right) - \left(\frac{e^{-in\pi}}{in}\right)\right] + \frac{1}{2\pi}\left[\left(\frac{e^{-in\pi}}{-in}\right) - \left(\frac{e^{0}}{-in}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{in} - \frac{\cos(n\pi) - i\cancel{\sin(n\pi)}}{in}\right] + \frac{1}{2\pi}\left[\frac{\cos(n\pi) - i\cancel{\sin(n\pi)}}{-in} + \frac{1}{in}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{in} - \frac{(-1)^n}{in}\right] + \frac{1}{2\pi}\left[\frac{(-1)^n}{-in} + \frac{1}{in}\right]$$

$$= \frac{1}{2\pi} \cdot \frac{2}{in} - \frac{1}{2\pi} \frac{2(-1)^n}{in}$$

$$= \frac{1}{\pi i n} - \frac{(-1)^n}{\pi i n} = \frac{1 - (-1)^n}{\pi i n}.$$


Thank you!