Undergraduate Course in Mathematics

Linear Algebra

Topic: Eigenvalue

Conducted By

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Consider $\boldsymbol{T}(\boldsymbol{x}, \boldsymbol{y}) = (\mathbf{3}\boldsymbol{x} - \mathbf{2}\boldsymbol{y}, \mathbf{2}\boldsymbol{x} - \mathbf{2}\boldsymbol{y})$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x-2y \\ 2x-2y \end{pmatrix}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$T\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$T\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

Idea of Eigenvalue and Eigenvector

Consider the Linear Transformation $\boldsymbol{T(x,y) = (3x - 2y, 2x - 2y)}$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\cdot\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Eigenvalue.

Eigenvector

$$T\begin{pmatrix}6\\3\end{pmatrix}=\begin{pmatrix}12\\6\end{pmatrix}$$

$$T\begin{pmatrix}6\\3\end{pmatrix}=2\cdot\begin{pmatrix}6\\3\end{pmatrix}$$

Eigenvector

Eigenvalue.

$$T\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}-1\\-2\end{pmatrix}$$

$$T\begin{pmatrix}1\\2\end{pmatrix}=-1\cdot\begin{pmatrix}1\\2\end{pmatrix}$$

Eigenvector

Eigenvalue

Eigenvalue and Eigenvector of a Linear Transformation

Let $\boldsymbol{T}$ be linear transformation from $\boldsymbol{V}$ to $\boldsymbol{V}$. A scalar number $\boldsymbol{\lambda}$ is called an **eigenvalue** of T if there exists a vector $\vec{v}$ such that $\vec{v} \neq 0$ with

$$T(\vec{v}) = \lambda\vec{v}$$

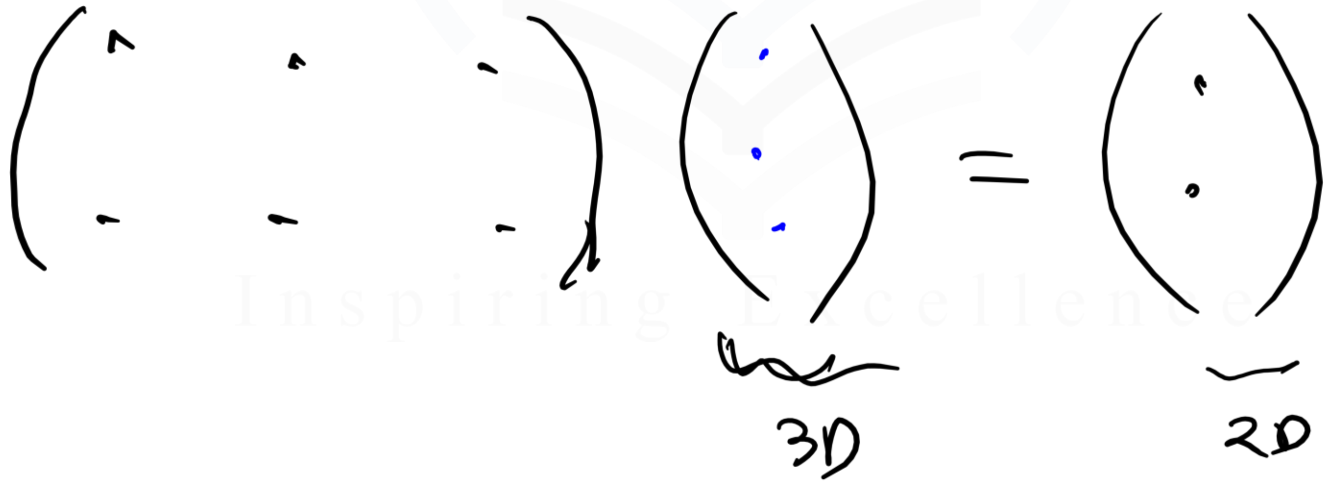
Here the vector $\vec{v}$ is called an eigenvector of the eigenvalue λ.

Eigenvalue and Eigenvector of a Matrix

Let $\boldsymbol{A}$ be a square matrix of size $n \times n$. A scalar number $\boldsymbol{\lambda}$ is called an **eigenvalue** of $\boldsymbol{A}$ if there exists a vector $\vec{v}$ such that $\vec{v} \neq 0$ with

$$A\vec{v} = \lambda\vec{v}$$

Here the vector $\vec{v}$ is called an eigenvector of the eigenvalue λ.



$$A \cdot \vec{v} = \lambda \cdot \vec{v}$$

$$\vec{v} \neq 0$$

$$\Rightarrow A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$\Rightarrow \underbrace{(A - \lambda I)}\vec{v} = \vec{0}$$

$$\boxed{\det(A - \lambda I) = 0}$$

Characteristic Equation.

Finding Eigenvalue *(Characteristic Polynomial Method)*

Let $\boldsymbol{A}$ be a square matrix of size $n \times n$. A scalar number $\boldsymbol{\lambda}$ is called an **eigenvalue** of $\boldsymbol{A}$ if and only if

$$\boxed{det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0} \quad \cdots\cdots\cdots\cdots\cdots \quad (1)$$

Here $\boldsymbol{I}$ is the $n \times n$ identity matrix. The equation (1) is called the characteristic equation of A.

Find the characteristic equation and eigenvalues of the matrix A.

$$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$$

Let λ be an eigenvalue of A.

$\therefore \quad A - \lambda I$

$$= \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} - \lambda \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix}$$

$$= \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} \ldots$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\cdot(-2-\lambda) - (2)\cdot(-2) = 0$$

$$-6 - 3\lambda + 2\lambda + \lambda^2 + 4 = 0$$

$$\lambda^2 - \lambda + 2 = 0$$

$$(\lambda + 1)(\lambda - 2) = 0$$

$\Rightarrow \lambda = -1, 2.$

Find the characteristic equation and eigenvalues of the matrix A.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

let λ be an eigenvalue of A.

$$A - \lambda I = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\cdot\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} - (1)\cdot\begin{vmatrix} 0 & 2 \\ -1 & 3-\lambda \end{vmatrix} + (2)\cdot\begin{vmatrix} 0 & 2-\lambda \\ -1 & 1 \end{vmatrix} = 0$$

$$(1-\lambda)\cdot[(2-\lambda)(3-\lambda)-2] - 1\cdot[0-(-1)(2)] + 2\cdot[0-(-1)(2-\lambda)] = 0$$

$$(1-\lambda)\cdot[4-5\lambda+\lambda^2] - 1\cdot[2] + 2\cdot[(2-\lambda)] = 0$$

$$4-5\lambda+\lambda^2-4\lambda+5\lambda^2-\lambda^3-2+4-2\lambda = 0$$

$$-\lambda^3+6\lambda^2-11\lambda+6 = 0$$

$$\lambda^3-6\lambda^2+11\lambda-6 = 0 \longrightarrow$$ Characteristic equation.

$$\boxed{\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0}$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

$\therefore \lambda = 1, 2, 3$

$a\lambda^3 + b\lambda^2 + c\lambda + d = 0$

Find the characteristic equation and eigenvalues of the matrix A.

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\cdot\begin{vmatrix} 5-\lambda & -2 \\ 1 & 2-\lambda \end{vmatrix} - (1)\cdot\begin{vmatrix} 2 & -2 \\ 1 & 2-\lambda \end{vmatrix} + (-1)\cdot\begin{vmatrix} 2 & 5-\lambda \\ 1 & 1 \end{vmatrix} = 0$$

$$(4-\lambda)\cdot[(5-\lambda)(2-\lambda) - (-2)] - 1\cdot[2(2-\lambda) - (-2)] + (-1)\cdot[2 - (5-\lambda)] = 0$$

$$(4-\lambda)\cdot[12 - 7\lambda + \lambda^2] - 1\cdot[6 - 2\lambda] + (-1)\cdot[-3 + \lambda] = 0$$

$$48 - 28\lambda + 4\lambda^2 - 12\lambda + 7\lambda^2 - \lambda^3 - 6 + 2\lambda + 3 - \lambda = 0$$

$$-\lambda^3 + 11\lambda^2 - 39\lambda + 45 = 0$$

$$\lambda^3 - 11\lambda^2 + 39\lambda - 45 = 0$$

$$\lambda^3 - 11\lambda^2 + 39\lambda - 45 = 0$$

$$(\lambda - 5)(\lambda - 3)(\lambda - 3) = 0$$

$\lambda = 5, 3, 3$

Find the characteristic equation and eigenvalues of the matrix A.

$$A = \begin{pmatrix} 0 & -6 & -4 \\ 5 & -11 & -6 \\ -6 & 9 & 4 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -\lambda & -6 & -4 \\ 5 & -11-\lambda & -6 \\ -6 & 9 & 4-\lambda \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 0-\lambda & -6 & -4 \\ 5 & -11-\lambda & -6 \\ -6 & 9 & 4-\lambda \end{vmatrix} = 0$$

$$(0-\lambda)\cdot\begin{vmatrix} -11-\lambda & -6 \\ 9 & 4-\lambda \end{vmatrix} - (-6)\cdot\begin{vmatrix} 5 & -6 \\ -6 & 4-\lambda \end{vmatrix} + (-4)\cdot\begin{vmatrix} 5 & -11-\lambda \\ -6 & 9 \end{vmatrix} = 0$$

$$(-\lambda)\cdot[(-11-\lambda)(4-\lambda)+54] + 6\cdot[5(4-\lambda)-36] - 4\cdot[54-(-6)(-11-\lambda)] = 0$$

$$(-\lambda)\cdot[10+7\lambda+\lambda^2] + 6\cdot[-16-5\lambda] - 4\cdot[-21-6\lambda] = 0$$

$$-10\lambda - 7\lambda^2 - \lambda^3 - 96 - 30\lambda + 84 + 24\lambda = 0$$

$$-\lambda^3 - 7\lambda^2 - 16\lambda - 12 = 0$$

$$\lambda^3 + 7\lambda^2 + 16\lambda + 12 = 0$$

$$\lambda^3 + 7\lambda^2 + 16\lambda + 12 = 0$$

$$(\lambda + 2)(\lambda + 2)(\lambda + 3) = 0$$

$$\lambda = \underbrace{-2, -2, -3}$$

Find the characteristic equation and eigenvalues of the matrix A.

$$A = \begin{pmatrix} 3 & 0 & -5 \\ \frac{1}{5} & -1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 3-\lambda & 0 & -5 \\ \frac{1}{5} & -1-\lambda & 0 \\ 1 & 1 & -2-\lambda \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 0 & -5 \\ \frac{1}{5} & -1-\lambda & 0 \\ 1 & 1 & -2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\cdot\begin{vmatrix} -1-\lambda & 0 \\ 1 & -2-\lambda \end{vmatrix} - (0)\cdot\begin{vmatrix} \frac{1}{5} & 0 \\ 1 & -2-\lambda \end{vmatrix} + (-5)\cdot\begin{vmatrix} \frac{1}{5} & -1-\lambda \\ 1 & 1 \end{vmatrix} = 0$$

$$(3-\lambda)\cdot[(-1-\lambda)(-2-\lambda) - 0] - 0 - 5\cdot\left[\frac{1}{5} - (1)(-1-\lambda)\right] = 0$$

$$(3-\lambda)\cdot[2 + 3\lambda + \lambda^2] - 1 - 5 - 5\lambda = 0$$

$$6 + 9\lambda + 3\lambda^2 - 2\lambda - 3\lambda^2 - \lambda^3 - 6 - 5\lambda = 0$$

$$-\lambda^3 + 2\lambda = 0$$

$$\lambda^3 - 2\lambda = 0$$

$$\lambda^3 - 2\lambda = 0$$

$$\lambda(\lambda^2 - 2) = 0$$

$\lambda = 0$,

$\lambda^2 - 2 = 0$

$\Rightarrow \lambda^2 = 2$

$\Rightarrow \lambda = \sqrt{2}, -\sqrt{2}$

Properties of Eigenvalues

Let A be a $n \times n$ square matrix and $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ be the eigenvalues of A then

- $Trace(A) = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n$.
- $Determinant(A) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \cdots \cdot \lambda_n$.

Verify the previous statement for $A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$

with $\lambda_1 = 1$ and $\lambda_2 = 4$

$\text{Trace}(A) = 3+2 = 5$ ✓

$\lambda_1 + \lambda_2 = 5$ ✓

$\det(A) = 4$

$\lambda_1 \cdot \lambda_2 = 4$

Verify the previous statement for $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$

with $\underline{\lambda_1 = 1}$, $\underline{\lambda_2 = 2}$ and $\underline{\lambda_3 = 3}$

$\lambda_1 + \lambda_2 + \lambda_3 = 6$

$Tr(A) = 6$

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 6$

$\det = ??$

$\underline{\underline{6}}$

Eigenvalues of a Triangular Matrix

Let A be an Upper or Lower Triangular Matrix.

$$A = \begin{pmatrix} \lambda_1 & * & * \\ 0 & \lambda_2 & * \\ 0 & 0 & \lambda_3 \end{pmatrix} \text{ or } A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ * & \lambda_2 & 0 \\ * & * & \lambda_3 \end{pmatrix}$$

Then the diagonal elements are the Eigenvalues.

Find the Eigenvalues of $A = \begin{pmatrix} 6 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix} \longrightarrow$ upper triangle

$$\lambda_1 = 6, \quad \lambda_2 = 2, \quad \lambda_3 = 4$$

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Find the Eigenvalues of $A = \begin{pmatrix} 5 & 0 & 0 \\ 6 & 3 & 0 \\ 9 & 7 & 2 \end{pmatrix}$ $\longrightarrow$ Lower Triangular

$\lambda = 5, 3, 2$


Thank you!