

# Fourier Analysis

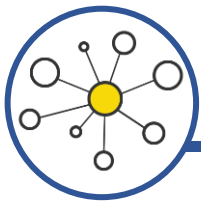
## Fourier Cosine & Sine Transform

### Evaluating Integrals using Inverse Transform

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# General Fourier Transform

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$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

F.T

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$$

I.F.T

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$= \int_{-\infty}^{\infty} f(x) (e^{i\omega x} - i \sin \omega x) dx$$

$$= \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx - i \int_{-\infty}^{\infty} f(x) \sin \omega x dx$$



## Fourier Cosine Transform

( $f(x)$  Even)

Let  $f(x)$  be a continuous signal. The Fourier Cosine transform of  $f(x)$  is denoted by  $F_c(\omega)$ , defined by

$$F_c(\omega) = \int_0^{\infty} f(x) \cos(\omega x) dx$$

And the Inverse Cosine transform of  $F_c(\omega)$ , defined by

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(\omega) \cos(\omega x) d\omega$$



## Fourier Sine Transform $\left( f(x) \text{ odd} \right)$

Let  $f(x)$  be a continuous signal. The Fourier Sine transform of  $f(x)$  is denoted by  $F_s(\omega)$ , defined by

$$F_s(\omega) = \int_0^{\infty} f(x) \sin(\omega x) dx$$

And the Inverse Cosine transform of  $F_s(\omega)$ , defined by

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(\omega) \sin(\omega x) d\omega$$

**PROBLEM** Find the Fourier Cosine transform of

$$f(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & x \geq 1 \end{cases}$$

Fourier cosine Transform

$$F_c(\omega) = \int_0^{\infty} f(x) \cos \omega x \, dx$$

$$= \int_0^1 1 \cdot \cos \omega x \, dx + 0$$

$$= \left[ \frac{\sin \omega x}{\omega} \right]_0^1$$

$$= \frac{\sin \omega}{\omega} - 0$$

$$= \frac{\sin \omega}{\omega}.$$



**PROBLEM** Find the Fourier Sine transform of

$$f(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & x \geq 1 \end{cases}$$

Hence, evaluate

$$\int_0^{\infty} \frac{\sin^3 x}{x} dx$$

$$f\left(\frac{1}{2}\right) = 1.$$



Fourier sine Transform

$$F_s(\omega) = \int_0^{\infty} f(x) \sin \omega x \, dx$$

$$= \int_0^1 1 \sin \omega x \, dx$$

$$= \left[ \frac{-\cos \omega x}{\omega} \right]_0^1$$

$$= \left[ \frac{\cos \omega x}{\omega} \right]_1^0$$

$$= \frac{1}{\omega} - \frac{\cos \omega}{\omega}$$

$$= \frac{1 - \cos \omega}{\omega}$$

Applying I.F.S.T

$$\int_0^{\infty} \frac{\sin^3 x}{x} dx$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(\omega) \sin \omega x d\omega$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{1 - \cos \omega}{\omega} \cdot \sin \omega x d\omega$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{2 \sin^2\left(\frac{\omega}{2}\right)}{\omega} \cdot \sin(\omega x) d\omega$$

Putting  $n = \frac{1}{2}$

$$\Rightarrow f\left(\frac{1}{2}\right) = \frac{2}{\pi} \int_0^{\infty} \frac{2 \sin^2 \frac{\omega}{2}}{\omega} \cdot \sin \frac{\omega}{2} d\omega$$

$$\Rightarrow 1 = \frac{4}{\pi} \int_0^{\infty} \frac{\sin^3 \frac{\omega}{2}}{\omega} d\omega$$

$$\Rightarrow \int_0^{\infty} \frac{\sin^3\left(\frac{\omega}{2}\right)}{\omega} d\omega = \frac{\pi}{4}$$

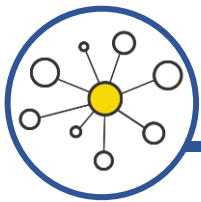
$$\int_0^{\infty} \frac{\sin^3(x)}{x} dx$$

$$\frac{\omega}{2} = u$$

$$\omega = 2u$$

$$d\omega = 2 du$$

$$\Rightarrow \int_0^{\infty} \frac{\sin^3 u}{2u} \cdot 2 \, du = \frac{\pi}{4} \quad \Rightarrow \quad \int_0^{\infty} \frac{\sin^3 u}{u} \, du = \frac{\pi}{4} .$$

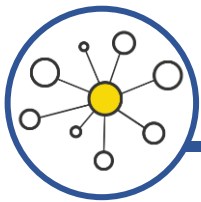


# UV Integration Shortcut

$$\int (u^3 - u) \cos 2u \, du$$

$$= (u^3 - u) \cdot \frac{\sin 2u}{2} - (3u^2 - 1) \cdot \frac{\cos 2u}{4} + 6u \cdot \frac{\sin 2u}{8} - 6 \cdot \frac{\cos 2u}{16}$$

$u^3 - u$	$\cos 2u$
$3u^2 - 1$	$\frac{\sin 2u}{2}$
$6u$	$-\frac{\cos 2u}{4}$
$6$	$-\frac{\sin 2u}{8}$
$0$	$\frac{\cos 2u}{16}$

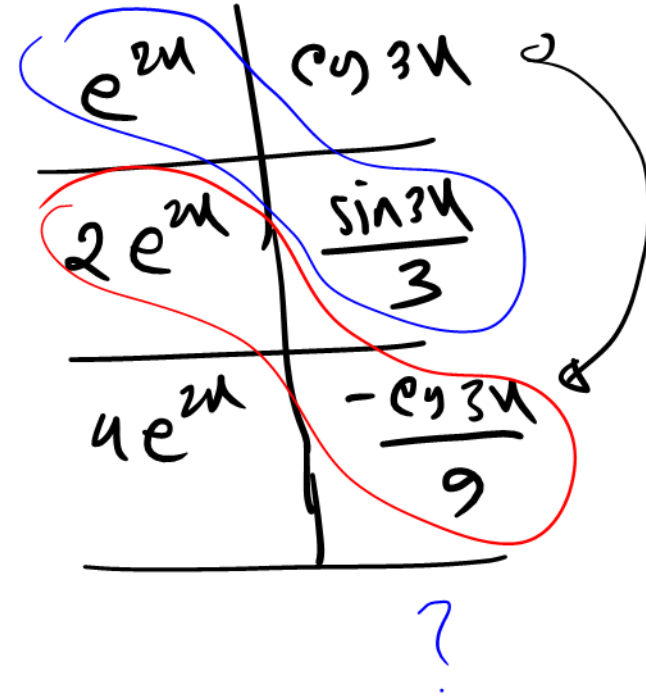


## UV Integration Shortcut (recurring type)

$$I = \int e^{2u} \cos 3u \, du$$

$$= e^{2u} \cdot \frac{\sin 3u}{3} - 2e^{2u} \frac{-\cos 3u}{9}$$

$$+ \int 4e^{2u} \frac{-\cos 3u}{9} \, du$$



$$I = \frac{1}{3} e^{2u} \sin 3u + \frac{2}{9} e^{2u} \cos 3u - \frac{4}{9} I$$

**PROBLEM** Find the Fourier Cosine transform of  $e^{-x}$ ,  $x \geq 0$ .

Hence, show that

$$\int_0^{\infty} \frac{\cos(mx)}{x^2 + 1} dx = \frac{\pi}{2} e^{-m}, m > 0$$



Fourier cosine Transform.

$$F_c(\omega) = \int_0^{\infty} f(x) \cos \omega x \, dx$$

$$I = \int_0^{\infty} e^{-x} \cos \omega x \, dx$$

$$= \left[ e^{-x} \frac{\sin \omega x}{\omega} \right]_0^{\infty} - \int_0^{\infty} (-e^{-x}) \cdot \frac{\sin \omega x}{\omega} \, dx$$

$$= 0 - 0 + \int_0^{\infty} e^{-x} \frac{\sin \omega x}{\omega} \, dx$$

$$\begin{aligned}
&= \frac{1}{\omega} \int_0^{\infty} e^{-x} \sin \omega x \, dx \\
&= \frac{1}{\omega} \left[ \left( e^{-x} - \frac{e^{-x} \cos \omega x}{\omega} \right) \right]_0^{\infty} - \int_0^{\infty} \left( -e^{-x} \right) \frac{\cos \omega x}{\omega} \, dx \\
&= \frac{1}{\omega} \left[ 0 - 1 \cdot \frac{-1}{\omega} - \frac{1}{\omega} \int_0^{\infty} e^{-x} \cos \omega x \, dx \right]
\end{aligned}$$

$$I = \frac{1}{\omega^2} - \frac{1}{\omega^2} I$$

$$\omega^2 I = 1 - 1$$

$$\Rightarrow I = \frac{1}{1 + \omega^2}.$$

$$\therefore f_c(\omega) = \frac{1}{1 + \omega^2}.$$

Applying I.f.e.T

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(\omega) \cos \omega x \, d\omega$$

$$\Rightarrow e^{-x} = \frac{2}{\pi} \int_0^{\infty} \frac{1}{1+\omega^2} \cos \omega x \, d\omega$$

$$\Rightarrow \int_0^{\infty} \frac{\cos \omega x}{1+\omega^2} \, d\omega = \frac{\pi}{2} e^{-x}$$

$$\int_0^{\infty} \frac{\cos(mx)}{x^2+1} \, dx = \frac{\pi}{2} e^{-m}, m > 0$$

substitution  $u$  by  $m$ ,

$$\int_0^{\infty} \frac{\cos(mx)}{x^2 + 1} dx = \frac{\pi}{2} e^{-m}, m > 0$$

$$\int_0^{\infty} \frac{\cos(\omega \cdot m)}{1 + \omega^2} d\omega = \frac{\pi}{2} e^{-m}$$

$$\Rightarrow \int_0^{\infty} \frac{\cos(mu)}{1 + u^2} du = \frac{\pi}{2} e^{-m}$$

**PROBLEM** Find the Fourier Sine transform of  $e^{-x}$ ,  $x \geq 0$ .

Hence, show that

$$\int_0^{\infty} \frac{x \sin(mx)}{x^2 + 1} dx = \frac{\pi}{2} e^{-m}, m > 0$$

fourier sin Transform,

$$f_s(\omega) = \int_0^{\infty} f(x) \sin \omega x dx$$

$e^{-x}$	$\sin \omega x$
$-e^{-x}$	$-\frac{e^{-x} \sin \omega x}{\omega}$
$e^{-x}$	$-\frac{\sin \omega x}{\omega^2}$

$$I = \int_0^{\infty} e^{-x} \sin \omega x dx$$

$$= \left[ e^{-x} \frac{-e^{-x} \sin \omega x}{\omega} - (-e^{-x}) \frac{-\sin \omega x}{\omega^2} \right]_0^{\infty} + \int_0^{\infty} e^{-x} \cdot \frac{-\sin \omega x}{\omega^2} dx$$

$$= \int \left[ -e^{-u} \frac{\cos \omega u}{\omega} - e^{-u} \frac{\sin \omega u}{\omega^2} \right]_0^{\infty} - \frac{1}{\omega^2} \int_0^{\infty} e^{-u} \sin \omega u \, du$$

$$I = \left[ (-0-0) - \left( \frac{-1}{\omega} - 0 \right) \right] - \frac{1}{\omega^2} \cdot I$$

$$I = \frac{1}{\omega} - \frac{1}{\omega^2} I$$

$$\Rightarrow \omega^2 I = \omega - I$$

$$\Rightarrow (\omega^2 + 1) I = \omega$$

$$\Rightarrow I = \frac{\omega}{1 + \omega^2}$$

$$\therefore f_s(\omega) = \frac{\omega}{1 + \omega^2}$$



Applying I. F. S. T,

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(\omega) \sin \omega x d\omega$$

$$e^{-x} = \frac{2}{\pi} \int_0^{\infty} \frac{\omega}{\omega^2 + 1} \sin \omega x d\omega$$

$$\Rightarrow \int_0^{\infty} \frac{\omega \cdot \sin \omega x}{\omega^2 + 1} d\omega = \frac{\pi}{2} e^{-x}$$

$$\int_0^{\infty} \frac{x \sin(mx)}{x^2 + 1} dx = \frac{\pi}{2} e^{-m}, m > 0$$

substituting  $x=m$ ,

$$\Rightarrow \int_0^{\infty} \frac{\omega \cdot \sin(m \cdot \omega)}{\omega^2 + 1} d\omega = \frac{\pi}{2} e^{-m}$$

$$\Rightarrow \int_0^{\infty} \frac{x \sin(mx)}{x^2 + 1} dx = \frac{\pi}{2} e^{-m}.$$

$$\int_0^{\infty} \frac{x \sin(mx)}{x^2 + 1} dx = \frac{\pi}{2} e^{-m}, m > 0$$

**PROBLEM** Find the Fourier Cosine transform of  $e^{-mx}$ ,  $x \geq 0$ .

Hence, show that

$$\int_0^{\infty} \frac{\beta \cos(\rho v)}{v^2 + \beta^2} dv = \frac{\pi}{2} e^{-\rho\beta}, \rho > 0, \beta > 0$$

Fourier cosine transform,

$$F_c(\omega) = \int_0^{\infty} e^{-m\lambda} \cos \omega \lambda \, d\lambda$$

$e^{-m\lambda}$	$\cos \omega \lambda$
$-m e^{-m\lambda}$	$\frac{\sin \omega \lambda}{\omega}$
$m^2 e^{-m\lambda}$	$-\frac{\cos \omega \lambda}{\omega^2}$

$$I = \left[ e^{-m\lambda} \frac{\sin \omega \lambda}{\omega} - (-m e^{-m\lambda}) \frac{-\cos \omega \lambda}{\omega^2} \right]_0^{\infty} + \int m^2 e^{-m\lambda} \frac{-\cos \omega \lambda}{\omega^2} d\lambda$$

$$I = \left[ (0) - \left( 0 - m \cdot 1 \cdot \frac{1}{\omega^2} \right) \right] - \frac{m^2}{\omega^2} \cdot I$$

$$\Rightarrow I = \frac{m}{\omega^2} - \frac{m^2}{\omega^2} I$$

$$\Rightarrow \omega^2 I = m - m^2 I$$

$$\Rightarrow I = \frac{m}{\omega^2 + m^2}.$$

$$\therefore F_c(\omega) = \frac{m}{\omega^2 + m^2}.$$

Applyn 1. F. C-7

$$\int_0^{\infty} \frac{\beta \cos(\rho v)}{v^2 + \beta^2} dv = \frac{\pi}{2} e^{-\rho\beta}$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{m}{\omega^2 + m^2} \cos(\omega x) d\omega$$

$$\Rightarrow \int_0^{\infty} \frac{m \cdot \cos(\omega x)}{\omega^2 + m^2} d\omega = \frac{\pi}{2} e^{-mx}$$

replacn  $x$  by  $\rho$

$$\Rightarrow \int_0^{\infty} \frac{\underline{m} \cos(\omega \rho)}{\omega^2 + m^2} d\omega = \frac{\pi}{2} e^{-\underline{m}\rho}$$

$$\int_0^{\infty} \frac{\underline{\beta} \cos(\rho v)}{v^2 + \beta^2} dv = \frac{\pi}{2} e^{-\rho \underline{\beta}}$$

$$\Rightarrow \int_0^{\infty} \frac{\beta \cos(\rho \omega)}{\omega^2 + \beta^2} d\omega = \frac{\pi}{2} e^{-\rho \beta} \quad [m \text{ by } \beta]$$

$$\Rightarrow \int_0^{\infty} \frac{\beta \cos(\rho v)}{v^2 + \beta^2} dv = \frac{\pi}{2} e^{-\rho \beta} \quad \checkmark$$

**PROBLEM** Find the Fourier Sine transform of  $e^{-mx}$ ,  $x \geq 0$ .

Hence, show that

$$\int_0^{\infty} \frac{x \sin(\rho x)}{x^2 + m^2} dx = \frac{\pi}{2} e^{-\rho m}, \rho > 0, m > 0$$



Fourier sine transform

$$F_s(\omega) = \int_0^{\infty} e^{-m\eta} \sin \omega \eta \, d\eta$$

$$I = \dots$$

$$\therefore I = \frac{\omega}{\omega^2 + m^2}.$$

$$\int_0^{\infty} \frac{x \sin(\rho x)}{x^2 + m^2} dx = \frac{\pi}{2} e^{-\rho m}$$

Applying I.F.T.,

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\omega}{\omega^2 + m^2} \sin \omega x d\omega$$

$$\Rightarrow \int_0^{\infty} \frac{\omega \cdot \sin \omega x}{\omega^2 + m^2} d\omega = \frac{\pi}{2} \cdot e^{-mx}$$

$$\Rightarrow \int_0^{\infty} \frac{\omega \cdot \sin(\rho \omega)}{\omega^2 + m^2} d\omega = \frac{\pi}{2} e^{-\rho m}$$

[x by rho]

$$\Rightarrow \int_0^{\infty} \frac{x \sin(\rho x)}{x^2 + m^2} dx = \frac{\pi}{2} e^{-\rho m}.$$

✓

$$\int_0^{\infty} \frac{x \sin(\rho x)}{x^2 + m^2} dx = \frac{\pi}{2} e^{-\rho m}$$

