

# Fourier Analysis

Series Sum using Fourier Series

with Parseval's Identity of Series

Conducted By

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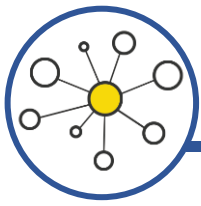
# General Fourier Series

Let  $f(x)$  be defined in an interval  $(-L, L)$  with period  $2L$ . The Fourier series expansion of  $f(x)$  is defined to be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \quad \checkmark$$

where the Fourier coefficients  $a_n$  and  $b_n$  are

$$\begin{cases} a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, & n = 0, 1, 2, 3, \dots \\ b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, & n = 1, 2, 3, \dots \end{cases}$$



# Convergence of Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

Two arrows point to the equation: one from the bottom left pointing to  $f(x)$ , and another from the bottom right pointing to the summation term.

**PROBLEM** Consider

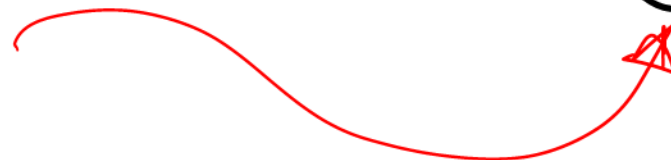
$$f(x) = \begin{cases} 3x + 1 & -\pi < x < 0 \\ 5 - x & 0 < x < \pi \end{cases}$$

$f(0)$  = undefined

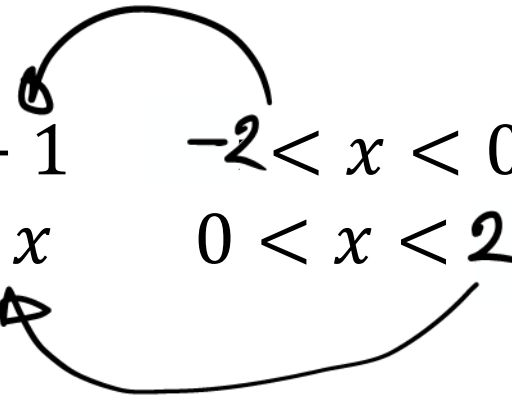
$$f(0) = \frac{L \cdot H \cdot L + R \cdot H \cdot L}{2}$$

$$= \frac{1 + 5}{2}$$

$$= 3$$



**PROBLEM** Consider

$$f(x) = \begin{cases} 3x + 1 & -2 < x < 0 \\ 5 - x & 0 < x < 2 \end{cases}$$


$$f(-\pi) = \text{undefined}$$

$$f(\pi) = \text{undefined}$$

$$\frac{L \cdot H \cdot L + R \cdot H \cdot L}{2}$$

$$= \frac{(-5) + (3)}{2} = -1$$

Problems

**PROBLEM** Find the Fourier series of  $f(x) = x^2$  for  $-\pi < x < \pi$ . Then using the Fourier Series, show that

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots = \frac{\pi^2}{12}$$

$$f(-x) = (-x)^2 = x^2 = f(x)$$

Even

$$\therefore 2L = 2\pi$$

$$\Rightarrow L = \pi$$

$$\therefore f(x) = \frac{a_0}{2} + \sum \left( a_n \cos(nx) \right)$$

$$\therefore a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx$$

$$\therefore a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left[ \frac{x^3}{3} \right]_0^{\pi} = \frac{2\pi^2}{3}.$$



$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx) dx$$

$$= \frac{2}{\pi} \left[ x^2 \frac{\sin(nx)}{n} - 2x \cdot \frac{\sin(nx)}{n^2} + 2 \cdot \frac{-\cos(nx)}{n^3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[ \left( \pi^2 \frac{\sin(n\pi)}{n} - 2 \cdot \pi \frac{\cos(n\pi)}{n^2} - 2 \cdot \frac{\sin(n\pi)}{n^3} \right) - (0 + 0 - 0) \right]$$

$x^2$	$\cos(nx)$
$2x$	$\frac{\sin(nx)}{n}$
$2$	$-\frac{\cos(nx)}{n^2}$
$0$	$-\frac{\sin(nx)}{n^3}$

$$= \frac{2}{\pi} \cdot 2\pi \cdot \frac{(-1)^n}{n^2} = \frac{4 \cdot (-1)^n}{n^2}.$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx) \quad \checkmark$$

at  $x=0$

$$f(0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cdot 1.$$

$$\Rightarrow 0 = \frac{\pi^2}{3} + 4 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\Rightarrow -\frac{\pi^2}{3} = 4 \cdot \left( \underbrace{\frac{-1}{1} + \frac{1}{2^2} + \frac{-1}{3^2} + \frac{1}{4^2} + \frac{-1}{5^2} + \dots}_{\text{alternating series}} \right)$$

$$\Rightarrow \frac{\pi^2}{3} = 4 \cdot \left( 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots \right)$$

$$\Rightarrow 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots = \frac{\pi^2}{12} \quad \checkmark$$

**PROBLEM** Find the Fourier series of  $f(x) = x^2$  for  $-\pi < x < \pi$ . Then using the Fourier Series, show that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{6}$$

$$\begin{aligned}\therefore f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)\end{aligned}$$

$$\frac{L.H.L + R.H.L.}{2} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(n\pi)$$

$$\frac{(-\pi)^2 + (\pi)^2}{2} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4 \cdot (-1)^n}{n^2} \cdot (-1)^n$$

$$\Rightarrow \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\Rightarrow \pi^2 - \frac{\pi^2}{3} = 4 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\begin{aligned} (-1)^n \cdot (-1)^n \\ &= (-1)^{2n} \\ &= 1. \end{aligned}$$

$$\Rightarrow \frac{2\pi^2}{3} = 4 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{2\pi^2}{3} = 4 \cdot \left( 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right)$$

$$\therefore 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6} \quad \checkmark$$

**PROBLEM** Find the Fourier Sine series of  $f(x) = \underline{x(\pi - x)}$  for  $0 < x < \pi$ .

Then using the Fourier Series, show that

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots = \frac{\pi^3}{32}$$

$$\therefore L = \pi$$

$$f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin(nx)$$

$$\therefore b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

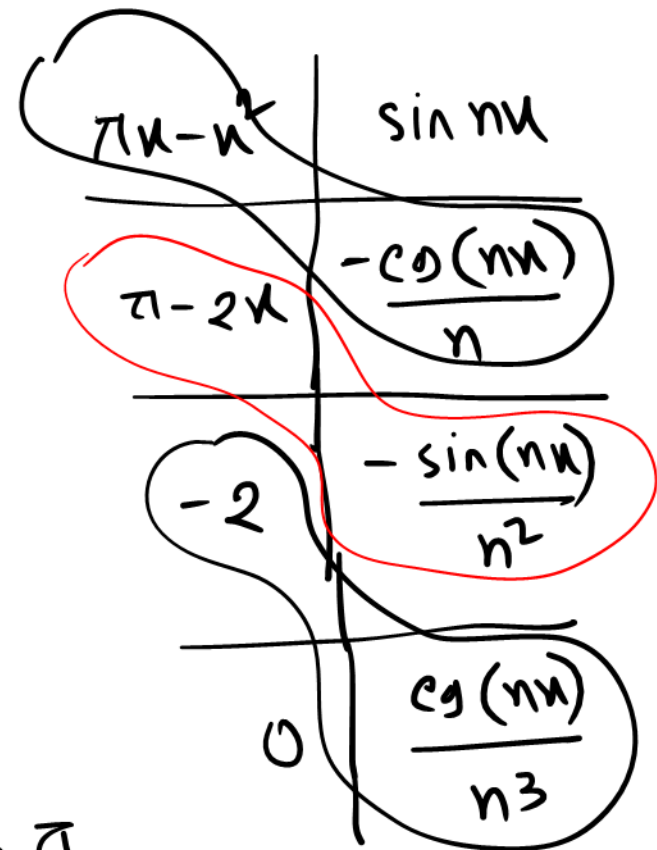


$$b_n = \frac{2}{\pi} \int_0^{\pi} f(u) \sin(nu) du$$

$$= \frac{2}{\pi} \int_0^{\pi} u(\pi-u) \sin(nu) du$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi u - u^2) \sin nu du$$

$$= \frac{2}{\pi} \left[ \underline{(\pi u - u^2)} \frac{-\cos(nu)}{n} + (\pi - 2u) \frac{\sin(nu)}{n^2} + (-2) \frac{\cos(nu)}{n^3} \right]_0^{\pi}$$



$$= \frac{2}{\pi} \left[ \left( 0 \cdot \frac{-e_y(n\pi)}{n} + (\pi - 2\pi) \frac{\cancel{\sin(n\pi)}}{n^2} - 2 \frac{e_y(n\pi)}{n^3} \right) - \left( 0 + \pi \cdot 0 - 2 \cdot \frac{1}{n^3} \right) \right]$$

$$= \frac{-4}{\pi n^3} e_y(n\pi) + \frac{4}{\pi n^3}$$

$$\therefore b_n = 4 \frac{1 - (-1)^n}{\pi n^3}$$

$$\therefore b_1 = 4 \cdot \frac{1 - (-1)}{\pi \cdot 1^3} = \frac{8}{\pi}$$

$$\therefore b_2 = 4 \cdot \frac{1 - 1}{\pi \cdot 2^3} = 0$$

$$\therefore b_3 = 4 \cdot \frac{1 - (-1)}{\pi \cdot 3^3} = \frac{8}{27\pi}$$

$$\therefore b_4 = 0$$

$$f(x) = \sum_{n=1}^{\infty} 4 \cdot \frac{1 - (-1)^n}{\pi n^3} \cdot \sin(nx)$$

$$= \frac{8}{\pi} \sin(x) + 0 + \frac{8}{\pi \cdot 3^3} \sin(3x) + 0 + \frac{8}{\pi \cdot 5^3} \sin(5x) + \dots$$

at  $x = \frac{\pi}{2}$ ,

$$f\left(\frac{\pi}{2}\right) = \frac{8}{\pi} \sin\left(\frac{\pi}{2}\right) + \frac{8}{\pi \cdot 3^3} \sin\left(3 \cdot \frac{\pi}{2}\right) + \frac{8}{\pi \cdot 5^3} \sin\left(5 \cdot \frac{\pi}{2}\right) + \frac{8}{\pi \cdot 7^3} \sin\left(7 \cdot \frac{\pi}{2}\right) + \dots$$

$$\Rightarrow \frac{\pi}{2} \cdot \left( \pi - \frac{\pi}{2} \right) = \frac{8}{\pi} + \frac{8}{\pi \cdot 3^3} (-1) + \frac{8}{\pi \cdot 5^3} 1 + \frac{8}{\pi \cdot 7^3} (-1) + \dots$$

$$\Rightarrow \frac{\pi^2}{4} = \frac{8}{\pi} \left( 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots \right)$$

$$\Rightarrow 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots = \frac{\pi^3}{32} \quad \text{Q.E.D.}$$

# Parseval's Identity



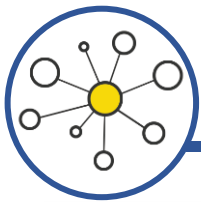
# General Fourier Series

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where the Fourier coefficients  $a_n$  and  $b_n$  are

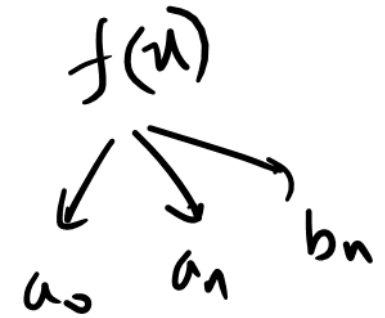
$$\begin{cases} a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, & n = 0, 1, 2, 3, \dots \\ b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, & n = 1, 2, 3, \dots \end{cases}$$



## Parseval's Identity for Fourier Series

Let  $f(x)$  be defined in an interval  $(-L, L)$  with period  $2L$ . then

$$\frac{1}{L} \int_{-L}^L [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$



**PROBLEM** Expand  $f(x) = x$ ,  $\underline{0 < x < 2}$  in a half range series of cosine.

Then using Parseval's Identity of Fourier Series, show that

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}$$

$$\therefore L = 2.$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{2}\right) \right)$$

$$\therefore a_n = \frac{2}{2} \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx$$

Even



$$\therefore a_0 = \int_0^2 f(x) dx = \int_0^2 x dx = \left[ \frac{x^2}{2} \right]_0^2 = 2.$$

$$a_n = \frac{2}{2} \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \int_0^2 x \cdot \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \left[ x \cdot \frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} - 1 \cdot \frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}} \right]_0^2$$

$x$	$\cos\left(\frac{n\pi x}{2}\right)$
1	$\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}$
0	$\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}}$

$$= \left[ \frac{2u}{n\pi} \sin\left(\frac{n\pi u}{2}\right) + \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi u}{2}\right) \right]_0^2$$

$$= \left[ \left( \frac{4}{n\pi} \sin(n\pi) + \frac{4}{n^2\pi^2} \cos(n\pi) \right) - \left( 0 + \frac{4}{n^2\pi^2} 1 \right) \right]$$

$$= \frac{4}{n^2\pi^2} (-1)^n - \frac{4}{n^2\pi^2}$$

$$= \frac{4}{n^2\pi^2} \left( (-1)^n - 1 \right)$$

$$\therefore f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2} \left( (-1)^n - 1 \right) \cos(nx)$$

Using Parseval's Identity

$$\frac{1}{L} \int_{-L}^L (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\Rightarrow \frac{2}{L} \int_0^L (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + 0^2)$$

$$\Rightarrow \int_0^2 (x)^2 dx = \frac{2^2}{2} + \sum_{n=1}^{\infty} \left[ \frac{4}{n^2 \pi^2} ((-1)^n - 1) \right]^2$$

$$\Rightarrow \left[ \frac{x^3}{3} \right]_0^2 = 2 + \sum_{n=1}^{\infty} \frac{16}{\pi^4 \cdot n^4} ((-1)^n - 1)^2$$

$$\Rightarrow \frac{8}{3} - 2 = \frac{16}{\pi^4 \cdot 1^4} \cdot (-2)^2 + 0 + \frac{16}{\pi^4 \cdot 3^4} (-2)^2 + 0 + \frac{16}{\pi^4 \cdot 5^4} (-2)^2 + 0 + \dots$$

$$\Rightarrow \frac{2}{3} = \frac{16(-2)^2}{\pi^4} \left( 1^4 + 0 + \frac{1}{3^4} + 0 + \frac{1}{5^4} + 0 + \frac{1}{7^4} + 0 + \dots \right)$$

$$\Rightarrow 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots = \frac{2}{3} \frac{\pi^4}{16 \cdot 4}$$

$$= \frac{\pi^4}{96} \quad \checkmark$$

**PROBLEM** Expand  $f(x) = x$ ,  $0 < x < 2$  in a half range series of cosine.

Then using Parseval's Identity of Fourier Series, show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\Rightarrow 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{90}$$

$$\therefore f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2} \left( (-1)^n - 1 \right) \cos(nx)$$

$$a_0 = 2$$

$$a_n = \frac{4}{n^2 \pi^2} \left( (-1)^n - 1 \right)$$

$$b_n = 0$$

Using Parseval's Identity

$$\frac{1}{L} \int_{-L}^L (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\Rightarrow \frac{2}{L} \int_0^L (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + 0^2)$$



$$\Rightarrow \int_0^2 (x)^2 dx = \frac{2^2}{2} + \sum_{n=1}^{\infty} \left[ \frac{4}{n^2 \pi^2} ((-1)^n - 1) \right]^2$$

$$\Rightarrow \left[ \frac{x^3}{3} \right]_0^2 = 2 + \sum_{n=1}^{\infty} \frac{16}{\pi^4 \cdot n^4} ((-1)^n - 1)^2$$

$$\Rightarrow \frac{8}{3} - 2 = \frac{16}{\pi^4 \cdot 1^4} \cdot (-2)^2 + 0 + \frac{16}{\pi^4 \cdot 3^4} (-2)^2 + 0 + \frac{16}{\pi^4 \cdot 5^4} (-2)^2 + 0 + \dots$$

$$\Rightarrow \frac{2}{3} = \frac{16(-2)^2}{\pi^4} \left( 1^4 + 0 + \frac{1}{3^4} + 0 + \frac{1}{5^4} + 0 + \frac{1}{7^4} + 0 + \dots \right)$$

$$\Rightarrow 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots = \frac{2}{3} \frac{\pi^4}{16 \cdot 4}$$

$$= \frac{\pi^4}{96} \quad \checkmark$$

$$\text{let } S = \underbrace{1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \dots}$$

$$= \left( 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots \right) + \left( \frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \dots \right)$$

$$= \frac{\pi^4}{96} + \left( \frac{1}{(2 \cdot 1)^4} + \frac{1}{(2 \cdot 2)^4} + \frac{1}{(2 \cdot 3)^4} + \frac{1}{(2 \cdot 4)^4} + \dots \right)$$

$$S = \frac{\pi^4}{96} + \frac{1}{2^4} \left( \underbrace{1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots} \right)$$

$$\therefore S = \frac{\pi^2}{96} + \frac{1}{16} \cdot S \quad \Rightarrow \quad S = \frac{\pi^2}{96} \times \frac{16}{15}$$

$$\Rightarrow S - \frac{1}{16} S = \frac{\pi^2}{96} \quad = \frac{\pi^2}{90}$$

$$\Rightarrow \frac{15}{16} S = \frac{\pi^2}{96}$$

$$S = \frac{\pi^2}{90}$$

$$\therefore 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots = \frac{\pi^4}{90}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \quad \checkmark$$

