# Fourier Analysis

## Series Sum using Fourier Series

## with Parseval's Identity of Series

**Conducted By**

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# General Fourier Series

Let $f(x)$ be defined in an interval $(-L, L)$ with period $2L$. The Fourier series expansion of $f(x)$ is defined to be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where the Fourier coefficients $a_n$ and $b_n$ are

$$\begin{cases} a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx\,, & n = 0,1,2,3,\ldots \\[2ex] b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\,, & n = 1,2,3,\ldots\ldots \end{cases}$$

# Convergence of Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

**PROBLEM** Consider

$$\boxed{f(x)} = \begin{cases} 3x + 1 & -\pi < x < 0 \\ 5 - x & 0 < x < \pi \end{cases}$$

$f(0) =$ undefined

$$f(0) = \frac{L.H.L + R.H.L}{2}$$

$$= \frac{1 + 5}{2}$$

$$= 3$$

**PROBLEM** Consider

$$f(x) = \begin{cases} 3x+1 & -2 < x < 0 \\ 5-x & 0 < x < 2 \end{cases}$$

$f(-\pi) = \text{undefined}$

$f(\pi) = \text{undefined}$

$$\frac{L.H.L + R.H.L}{2}$$

$$= \frac{(-5)+(3)}{2} = -1$$

# Problems

**PROBLEM** Find the Fourier series of $f(x) = \boldsymbol{x^2}$ for $\boldsymbol{-\pi < x < \pi}$. Then using the Fourier Series, show that

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots = \frac{\pi^2}{12}$$

$f(-x) = (-x)^2 = x^2 = f(x)$

Even

$\therefore \; 2L = 2\pi$

$\Rightarrow L = \pi$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum \left( a_n \cos(nx) \right)$$

$$\therefore \quad a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx$$

$$\therefore \quad a_0 = \frac{2}{\pi} \int_0^{\pi} x^2\, dx = \frac{2}{\pi} \left[ \frac{x^3}{3} \right]_0^{\pi} = \frac{2\pi^2}{3} .$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

| $x^2$ | $\cos(nx)$ |
| --- | --- |
| $2x$ | $\frac{\sin(nx)}{n}$ |
| $2$ | $-\frac{\cos(nx)}{n^2}$ |
| $0$ | $-\frac{\sin(nx)}{n^3}$ |

$$= \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left[x^2\,\frac{\sin(nx)}{n} + 2x\cdot\frac{+\cos(nx)}{n^2} + 2\cdot\frac{-\sin(nx)}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\pi^2\,\frac{\cancel{\sin n\pi}}{n} + 2\cdot\pi\,\frac{\cos(n\pi)}{n^2} - 2\cdot\frac{\cancel{\sin(n\pi)}}{n^3}\right) - \left(0 + 0 - 0\right)\right]$$

$$= \frac{2}{\pi} \cdot 2\pi \cdot \frac{(-1)^n}{n^2} = \frac{4 \cdot (-1)^n}{n^2}.$$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx) \quad //$$

at $x = 0$

$$f(0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \, 1.$$

$$\Rightarrow \quad 0 = \frac{\pi^2}{3} + 4 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\Rightarrow \quad -\frac{\pi^2}{3} = 4 \cdot \left( \frac{-1}{1} + \frac{1}{2^2} + \frac{-1}{3^2} + \frac{1}{4^2} + \frac{-1}{5^2} + \cdots \right)$$

$$\Rightarrow \quad \frac{\pi^2}{3} = 4 \cdot \left( 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots \right)$$

$$\Rightarrow \quad 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots = \frac{\pi^2}{12} \checkmark$$

PROBLEM Find the Fourier series of $f(x) = x^2$ for $-\pi < x < \pi$. Then using the Fourier Series, show that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6}$$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)$$

$$\frac{L.H.L + R.H.L.}{2} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(n\pi)$$

$$(-1)^n \cdot (-1)^n$$
$$= (-1)^{2n}$$
$$= 1.$$

$$\frac{(-\pi)^2 + (\pi)^2}{2} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4 \cdot (-1)^n}{n^2} \cdot (-1)^n$$

$$\Rightarrow \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\Rightarrow \pi^2 - \frac{\pi^2}{3} = 4 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{2\pi^2}{3} = 4 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{2\pi^2}{3} = 4 \cdot \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right)$$

$$\therefore 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6} .$$

**PROBLEM** Find the Fourier Sine series of $f(x) = \boldsymbol{x(\pi - x)}$ for $\mathbf{0} < \boldsymbol{x} < \boldsymbol{\pi}$.

Then using the Fourier Series, show that

$$\mathbf{1} - \frac{\mathbf{1}}{\mathbf{3^3}} + \frac{\mathbf{1}}{\mathbf{5^3}} - \frac{\mathbf{1}}{\mathbf{7^3}} + \cdots = \frac{\boldsymbol{\pi}^3}{\mathbf{32}}$$

$\therefore \quad L = \pi$

$$f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin(nx)$$

$$\therefore \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[ (\pi x - x^2)\frac{-\cos(nx)}{n} + (\pi - 2x)\frac{+\sin(nx)}{n^2} + (-2)\frac{\cos(nx)}{n^3} \right]_0^{\pi}$$

| | |
|---|---|
| $\pi x - x^2$ | $\sin nx$ |
| $\pi - 2x$ | $\frac{-\cos(nx)}{n}$ |
| $-2$ | $\frac{-\sin(nx)}{n^2}$ |
| $0$ | $\frac{\cos(nx)}{n^3}$ |

$$= \frac{2}{\pi}\left[\left(0\cdot\frac{-\cos(n\pi)}{n} + (\pi-2\pi)\cancel{\frac{\sin(n\pi)}{n^2}} - 2\,\frac{\cos(n\pi)}{n^3}\right) - \left(0+\pi\cdot 0 - 2\cdot\frac{1}{n^3}\right)\right]$$

$$= \frac{-4}{\pi n^3}\cos(n\pi) + \frac{4}{\pi n^3}$$

$$\therefore\ b_n = 4\,\frac{1-(-1)^n}{\pi n^3}$$

$$\therefore\ b_1 = 4\cdot\frac{1-(-1)}{\pi\cdot 1^3} = \frac{8}{\pi}$$

$$\therefore\ b_2 = 4\cdot\frac{1-1}{\pi\cdot 2^3} = 0.$$

$$\therefore\ b_3 = 4\cdot\frac{1-(-1)}{\pi\cdot 3^3} = \frac{8}{27\pi}$$

$$\therefore\ b_4 = 0$$

$$f(x) = \sum_{n=1}^{\infty} 4 \cdot \frac{1-(-1)^n}{\pi n^3} \cdot \sin(nx)$$

$$= \frac{8}{\pi} \sin(x) + 0 + \frac{8}{\pi \cdot 3^3} \sin(3x) + 0 + \frac{8}{\pi \cdot 5^3} \sin 5x + \cdots$$

at $x = \frac{\pi}{2}$,

$$f\left(\frac{\pi}{2}\right) = \frac{8}{\pi} \sin\left(\frac{\pi}{2}\right) + \frac{8}{\pi \cdot 3^3} \sin\left(3 \cdot \frac{\pi}{2}\right) + \frac{8}{\pi \cdot 5^3} \sin\left(5 \frac{\pi}{2}\right) + \frac{8}{\pi 7^3} \sin\left(7 \frac{\pi}{2}\right) + \cdots$$

$$\Rightarrow \frac{\pi}{2}\cdot\left(\pi-\frac{\pi}{2}\right) = \frac{8}{\pi} + \frac{8}{\pi\cdot 3^3}(-1) + \frac{8}{\pi\cdot 5^3}1 \cdot + \frac{8}{\pi 7^3}\cdot(-1)) + \cdots$$

$$\Rightarrow \frac{\pi^2}{4} = \frac{8}{\pi}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots\right)$$

$$\Rightarrow 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}\cdot \text{u}$$

# Parseval’s Identity

# General Fourier Series

Let $f(x)$ be defined in an interval $(-L, L)$ with period $2L$. The Fourier series expansion of $f(x)$ is defined to be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where the Fourier coefficients $a_n$ and $b_n$ are

$$\begin{cases} a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx\,, & n = 0,1,2,3,\ldots \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\,, & n = 1,2,3,\ldots\ldots \end{cases}$$

# Parseval's Identity for Fourier Series

Let $f(x)$ be defined in an interval $(-L, L)$ with period $2L$. then

$$\frac{1}{L}\int_{-L}^{L}[f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$f(x)$

$a_0$ $a_n$ $b_n$

**PROBLEM** Expand $f(x) = x, \quad \underline{0 < x < 2}$ in a half range series of $\underline{\text{cosine}}$.

Even

Then using Parseval's Identity of Fourier Series, show that

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$$

$\therefore L = 2.$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{2}\right)\right)$$

$$\therefore a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$\therefore \quad a_0 = \int_0^2 f(x)\,dx = \int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2 = 2.$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

| $x$ | $\cos\left(\frac{n\pi x}{2}\right)$ |
|---|---|
| $1$ | $\dfrac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}$ |
| $0$ | $\dfrac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}}$ |

$$= \int_0^2 x\cdot\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[x\cdot\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} - 1\cdot\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}}\right]_0^2$$

$$= \left[ \frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right) \right]_0^2$$

$$= \left[ \left( \frac{4}{n\pi} \sin\cancel{(n\pi)} + \frac{4}{n^2\pi^2} \cos(n\pi) \right) - \left( 0 + \frac{4}{n^2\pi^2} 1 \right) \right]$$

$$= \frac{4}{n^2\pi^2} (-1)^n - \frac{4}{n^2\pi^2}$$

$$= \frac{4}{n^2\pi^2} \left( (-1)^n - 1 \right)$$

$$\therefore \quad f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left((-1)^n - 1\right)\cos(nx)$$

Using Parseval's Identity

$$\frac{1}{L}\int_{-L}^{L} (f(x))^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$\Rightarrow \quad \frac{2}{L}\int_{0}^{L} (f(x))^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + 0^2\right)$$

$$\Rightarrow \int_0^2 (x)^2 \, dx = \frac{2^2}{2} + \sum_{n=1}^{\infty} \left[ \frac{4}{n^2 \pi^2} \left( (-1)^n - 1 \right) \right]^2$$

$$\Rightarrow \left[ \frac{x^3}{3} \right]_0^2 = 2 + \sum_{n=1}^{\infty} \frac{16}{\pi^4 \cdot n^4} \left( (-1)^n - 1 \right)^2$$

$$\Rightarrow \frac{8}{3} - 2 = \frac{16}{\pi^4 1^4} \cdot (-2)^2 + 0 + \frac{16}{\pi^4 3^4} (-2)^2 + 0 + \frac{16}{\pi^4 \cdot 5^4} (-2)^2 + 0 + \cdots$$

$$\Rightarrow \quad \frac{2}{3} = \frac{16\,(-2)^2}{\pi^4}\left(1^4 + 0 + \frac{1}{3^4} + 0 + \frac{1}{5^4} + 0 + \frac{1}{7^4} + 0 + \cdots\right)$$

$$\Rightarrow \quad 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{2}{3}\,\frac{\pi^4}{16 \cdot 4}$$

$$= \frac{\pi^4}{96}$$

**PROBLEM** Expand $f(x) = x, \quad 0 < x < 2$ in a half range series of cosine.

Then using Parseval's Identity of Fourier Series, show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\Rightarrow \quad 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{90}$$

$$\therefore\ f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left((-1)^n - 1\right)\cos(nx)$$

$$a_0 = 2 \qquad a_n = \frac{4}{n^2\pi^2}\left((-1)^n - 1\right) \qquad b_n = 0$$

Using Parseval's Identity

$$\frac{1}{L}\int_{-L}^{L}\left(f(x)\right)^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$\Rightarrow \frac{2}{L}\int_{0}^{L}\left(f(x)\right)^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + 0^2\right)$$

$$\Rightarrow \int_0^2 (x)^2 dx = \frac{2^2}{2} + \sum_{n=1}^{\infty} \left[\frac{4}{n^2\pi^2}\left((-1)^n - 1\right)\right]^2$$

$$\Rightarrow \left[\frac{x^3}{3}\right]_0^2 = 2 + \sum_{n=1}^{\infty} \frac{16}{\pi^4 \cdot n^4}\left((-1)^n - 1\right)^2$$

$$\Rightarrow \frac{8}{3} - 2 = \frac{16}{\pi^4 1^4} \cdot (-2)^2 + 0 + \frac{16}{\pi^4 3^4}(-2)^2 + 0 + \frac{16}{\pi^4 \cdot 5^4}(-2)^2 + 0 + \cdots$$

$$\Rightarrow \quad \frac{2}{3} = \frac{16(-2)^2}{\pi^4}\left(1^4 + 0 + \frac{1}{3^4} + 0 + \frac{1}{5^4} + 0 + \frac{1}{7^4} + 0 + \cdots\right)$$

$$\Rightarrow \quad 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{2}{3}\,\frac{\pi^4}{16 \cdot 4}$$

$$= \frac{\pi^4}{96}$$

let $S = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots$

$$= \left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \cdots\right)$$

$$= \frac{\pi^4}{96} + \left(\frac{1}{(2\cdot 1)^4} + \frac{1}{(2\cdot 2)^4} + \frac{1}{(2\cdot 3)^4} + \frac{1}{(2\cdot 4)^4} + \cdots\right)$$

$$S = \frac{\pi^4}{96} + \frac{1}{2^4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots\right)$$

$$\therefore \quad S = \frac{\pi^2}{96} + \frac{1}{16} \cdot S$$

$$\Rightarrow \quad S - \frac{1}{16} S = \frac{\pi^4}{96}$$

$$\Rightarrow \quad \frac{15}{16} S = \frac{\pi^4}{96}$$

$$\Rightarrow \quad S = \frac{\pi^4}{96} \times \frac{16}{15}$$

$$= \frac{\pi^4}{90}$$

$$S = \frac{\pi^4}{90}$$

$$\therefore \quad 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$


Thank you!