Fourier Analysis

Fourier Cosine & Sine Transform

Evaluating Integrals using Inverse Transform

Conducted By

Partho Sutra Dhor

Faculty, Mathematics and Natural Sciences

BRAC University, Dhaka, Bangladesh

General Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx \qquad \text{F.T}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, d\omega \qquad \text{I.F.T}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \int_{-\infty}^{\infty} f(x) \left(\cos \omega x - i \sin \omega x \right) dx$$

$$= \int_{-\infty}^{\infty} f(x) \cos \omega x \, dx - i \int_{-\infty}^{\infty} f(x) \sin \omega x \, dx$$

Fourier Cosine Transform (f(x) Even)

Let $f(x)$ be a continuous signal. The Fourier Cosine transform of $f(x)$ is denoted by $F_c(\omega)$, defined by

$$F_c(\omega) = \int_0^\infty f(x)\cos(\omega x)\,dx$$

And the Inverse Cosine transform of $F_c(\omega)$, defined by

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos(\omega x)\,d\omega$$

Fourier Sine Transform (f(x) odd)

Let $f(x)$ be a continuous signal. The Fourier Sine transform of $f(x)$ is denoted by $F_s(\omega)$, defined by

$$F_s(\omega) = \int_0^\infty f(x)\sin(\omega x)\,dx$$

And the Inverse Cosine transform of $F_s(\omega)$, defined by

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\omega)\sin(\omega x)\,d\omega$$

PROBLEM Find the Fourier Cosine transform of

$$f(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & x \ge 1 \end{cases}$$

Fourier cosine Transform

$$F_c(\omega) = \int_0^\infty f(x) \cos \omega x \, dx$$

$$= \int_0^1 1 \cdot \cos \omega x \, dx + 0$$

$$= \left[\frac{\sin \omega x}{\omega} \right]_0^1$$

$$= \frac{\sin \omega}{\omega} - 0$$

$$= \frac{\sin \omega}{\omega}.$$

PROBLEM Find the Fourier Sine transform of

$$f(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & x \geq 1 \end{cases}$$

Hence, evaluate

$$\int_0^\infty \frac{\sin^3 x}{x} dx$$

$f\left(\frac{1}{2}\right) = 1.$

Fourier sine Transform

$$F_s(\omega) = \int_0^\infty f(x) \sin \omega x \, dx$$

$$= \int_0^1 1 \sin \omega x \, dx$$

$$= \left[\frac{-\cos \omega x}{\omega} \right]_0^1$$

$$= \left[\frac{\cos \omega x}{\omega} \right]_1^0$$

$$= \frac{1}{\omega} - \frac{\cos \omega}{\omega}$$

$$= \frac{1 - \cos \omega}{\omega}.$$

$$\int_0^\infty \frac{\sin^3 x}{x}\,dx$$

Applying I.F.S.T

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\omega)\sin\omega x\,d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1-\cos\omega}{\omega}\cdot\sin\omega x\,d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{2\sin^2\left(\frac{\omega}{2}\right)}{\omega}\cdot\sin(\omega x)\,d\omega$$

Putting $x = \frac{1}{2}$

$$\int_0^\infty \frac{\sin^3 x}{x}\,dx$$

$$\Rightarrow f\left(\frac{1}{2}\right) = \frac{2}{\pi}\int_0^\infty \frac{2\sin^2\frac{\omega}{2}}{\omega}\cdot\sin\frac{\omega}{2}\,d\omega$$

$$\frac{\omega}{2} = x$$

$$\omega = 2x$$

$$\Rightarrow 1 = \frac{4}{\pi}\int_0^\infty \frac{\sin^3\frac{\omega}{2}}{\omega}\,d\omega$$

$$d\omega = 2\,dx$$

$$\Rightarrow \int_0^\infty \frac{\sin^3\left(\frac{\omega}{2}\right)}{\omega}\,d\omega = \frac{\pi}{4}$$

$$\Rightarrow \int_0^\infty \frac{\sin^3 x}{2x} \cdot 2\,dx = \frac{\pi}{4} \qquad \Rightarrow \int_0^\infty \frac{\sin^3 x}{x}\,dx = \frac{\pi}{4}.$$

UV Integration Shortcut

$$\int (x^3 - x)\cos 2x \, dx$$

$$= (x^3 - x)\cdot\frac{\sin 2x}{2} - (3x^2-1)\,\frac{-\cos 2x}{4}$$

$$+ 6x\,\frac{-\sin 2x}{8} - 6\cdot\frac{\cos 2x}{16}$$

$x^3 - x$	$\cos 2x$
$3x^2 - 1$	$\frac{\sin 2x}{2}$
$6x$	$-\frac{\cos 2x}{4}$
6	$-\frac{\sin 2x}{8}$
0	$\frac{\cos 2x}{16}$

UV Integration Shortcut *(recurring type)*

$$I = \int e^{2x} \cos 3x \, dx$$

$$= e^{2x} \cdot \frac{\sin 3x}{3} - 2e^{2x} \frac{-\cos 3x}{9}$$

$$+ \int 4e^{2x} \frac{-\cos 3x}{9} \, dx$$

e^{2x}	$\cos 3x$
$2e^{2x}$	$\frac{\sin 3x}{3}$
$4e^{2x}$	$\frac{-\cos 3x}{9}$

?

$$I = \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x - \frac{4}{9} I$$

PROBLEM Find the Fourier Cosine transform of $e^{-x},\ x \geq 0.$

Hence, show that

$$\int_0^\infty \frac{\cos(mx)}{x^2+1}\,dx = \frac{\pi}{2}e^{-m}\,, m > 0$$

Fourier cosine Transform.

$$F_c(\omega) = \int_0^\infty f(x) \cos\omega x \, dx$$

$$I = \int_0^\infty e^{-x} \cos\omega x \, dx$$

$$= \left[e^{-x} \frac{\sin\omega x}{\omega} \right]_0^\infty - \int_0^\infty \left(-e^{-x}\right) \cdot \frac{\sin\omega x}{\omega} \, dx$$

$$= 0 - 0 + \int_0^\infty e^{-x} \frac{\sin\omega x}{\omega} \, dx$$

$$= \frac{1}{\omega}\int_0^\infty e^{-x} \sin\omega x \, dx$$

$$= \frac{1}{\omega}\left[\left[e^{-x} \; \frac{-\cos\omega x}{\omega}\right]_0^\infty - \int_0^\infty \left(-e^{-x}\right) \frac{-\cos\omega x}{\omega}\, dx \right]$$

$$= \frac{1}{\omega}\left[0 - 1 \cdot \frac{-1}{\omega} - \frac{1}{\omega}\int_0^\infty e^{-x} \cos\omega x \, dx \right.$$

$$I = \frac{1}{\omega^2} - \frac{1}{\omega^2} I$$

$$\omega^2 I = 1 - I$$

$$\Rightarrow I = \frac{1}{1+\omega^2}.$$

$$\therefore \quad F_c(\omega) = \frac{1}{1+\omega^2}.$$

$$\int_0^\infty \frac{\cos(mx)}{x^2+1}\,dx = \frac{\pi}{2}e^{-m}, m > 0$$

Applyin I.F.C.T

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos\omega x\,d\omega$$

$$\Rightarrow e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\omega^2}\cos\omega x\,d\omega$$

$$\Rightarrow \int_0^\infty \frac{\cos\omega x}{1+\omega^2}\,d\omega = \frac{\pi}{2}e^{-x}$$

$$\int_0^\infty \frac{\cos(mx)}{x^2+1}\,dx = \frac{\pi}{2}e^{-m}, m > 0$$

subititution x by m,

$$\int_0^\infty \frac{\cos(\omega\cdot m)}{1+\omega^2}\,d\omega = \frac{\pi}{2}e^{-m}$$

$$\Rightarrow \int_0^\infty \frac{\cos(mx)}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}$$

PROBLEM Find the Fourier Sine transform of $e^{-x},\ x \geq 0$.

Hence, show that

$$\int_0^\infty \frac{x\sin(mx)}{x^2+1}\,dx = \frac{\pi}{2}e^{-m}, m > 0$$

fourier sin Transform,

e^{-x}	$\sin\omega x$
$-e^{-x}$	$-\frac{\cos\omega x}{\omega}$
e^{-x}	$-\frac{\sin\omega x}{\omega^2}$

$$f_s(\omega) = \int_0^\infty f(x)\sin\omega x\,dx$$

$$I = \int_0^\infty e^{-x}\sin\omega x\,dx$$

$$= \left[e^{-x}\,\frac{-\cos\omega x}{\omega} - (-e^{-x})\,\frac{-\sin\omega}{\omega^2}\right]_0^\infty + \int_0^\infty e^{-x}\cdot\frac{-\sin\omega x}{\omega^2}\,dx$$

$$= \left[-e^{-x} \frac{\cos \omega x}{\omega} - e^{-x} \frac{\sin \omega x}{\omega^2} \right]_0^\infty - \frac{1}{\omega^2} \int_0^\infty e^{-x} \sin \omega x \, dx$$

$$I = \left[(-0-0) - \left(\frac{-1}{\omega} - 0 \right) \right] - \frac{1}{\omega^2} \cdot I$$

$$I = \frac{1}{\omega} - \frac{1}{\omega^2} I$$

$$\Rightarrow \omega^2 I = \omega - I$$

$$\Rightarrow (\omega^2 + 1) I = \omega$$

$$\Rightarrow I = \frac{\omega}{1+\omega^2}.$$

$$\therefore F_s(\omega) = \frac{\omega}{1+\omega^2}.$$

$$\int_0^{\infty} \frac{x\sin(mx)}{x^2+1}\,dx = \frac{\pi}{2}e^{-m}, m > 0$$

Applyin I.F.S.T,

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(\omega)\sin\omega x\, d\omega$$

$$e^{-x} = \frac{2}{\pi}\int_0^{\infty} \frac{\omega}{\omega^2+1}\sin\omega x\, d\omega$$

$$\Rightarrow \int_0^{\infty} \frac{\omega \cdot \sin\omega x}{\omega^2+1}\, d\omega = \frac{\pi}{2} e^{-x}$$

$$\int_0^{\infty} \frac{x \sin(mx)}{x^2+1}\, dx = \frac{\pi}{2} e^{-m}, m > 0$$

substituting $x = m$,

$$\Rightarrow \int_0^{\infty} \frac{\omega . \sin(m . \omega)}{\omega^2 + 1}\, d\omega = \frac{\pi}{2} e^{-m}$$

$$\Rightarrow \int_0^{\infty} \frac{x \sin(mx)}{x^2+1}\, dx = \frac{\pi}{2} e^{-m} .$$

PROBLEM Find the Fourier Cosine transform of e^{-mx}, $x \geq 0$.

Hence, show that

$$\int_0^\infty \frac{\beta \cos(\rho v)}{v^2 + \beta^2} dv = \frac{\pi}{2} e^{-\rho\beta}, \rho > 0, \beta > 0$$

Fourier cosine transform,

e^{-mx}	$\cos\omega x$
$-me^{-mx}$	$\frac{\sin\omega x}{\omega}$
m^2e^{-mx}	$\frac{-\cos\omega x}{\omega^2}$

$$F_c(\omega) = \int_0^\infty e^{-mx}\cos\omega x\,dx$$

$$I = \left[e^{-mx}\frac{\sin\omega x}{\omega} - (-me^{-mx})\frac{-\cos\omega x}{\omega^2}\right]_0^\infty + \int m^2e^{-mx}\,\frac{-\cos\omega x}{\omega^2}\,dx$$

$$I = \left[(0) - \left(0 - m\cdot 1\cdot\frac{1}{\omega^2}\right)\right] - \frac{m^2}{\omega^2}\cdot I$$

$$\Rightarrow I = \frac{m}{\omega^2} - \frac{m^2}{\omega^2} I$$

$$\Rightarrow \omega^2 I = m - m^2 I$$

$$\Rightarrow I = \frac{m}{\omega^2 + m^2}.$$

$$\therefore F_c(\omega) = \frac{m}{\omega^2 + m^2}.$$

$$\int_0^\infty \frac{\beta\cos(\rho v)}{v^2+\beta^2}\,dv=\frac{\pi}{2}e^{-\rho\beta}$$

Apply I.F.C.T

$$f(x)=\frac{2}{\pi}\int_0^\infty \frac{m}{\omega^2+m^2}\cos(\omega x)\,d\omega$$

$$\Rightarrow \int_0^\infty \frac{m.\cos(\omega \underline{\underline{x}})}{\omega^2+m^2}\,d\omega=\frac{\pi}{2}e^{-m\underline{\underline{x}}}$$

replace x by ρ

$$\int_0^\infty \frac{\beta\cos(\rho v)}{v^2+\beta^2}\,dv = \frac{\pi}{2}e^{-\rho\beta}$$

$$\Rightarrow \int_0^\infty \frac{m\cdot\cos(\omega\rho)}{\omega^2+m^2}\,d\omega = \frac{\pi}{2}e^{-m\rho}$$

$$\Rightarrow \int_0^\infty \frac{\beta\cos(\rho\omega)}{\omega^2+\beta^2}\,d\omega = \frac{\pi}{2}e^{-\rho\beta} \quad [m \text{ by } \beta]$$

$$\Rightarrow \int_0^\infty \frac{\beta\cos(\rho v)}{v^2+\beta^2}\,dv = \frac{\pi}{2}e^{-\rho\beta}.$$

PROBLEM Find the Fourier Sine transform of $e^{-mx},\ x \geq 0$.

Hence, show that

$$\int_0^{\infty} \frac{x \sin(\rho x)}{x^2 + m^2}\, dx = \frac{\pi}{2} e^{-\rho m}, \rho > 0, \beta > 0$$

Fourier sine Transform

$$F_s(\omega) = \int_0^\infty e^{-mx} \sin \omega x \, dx$$

$I =$ - - - - - - - -

- - - - .

$$\therefore \quad I = \frac{\omega}{\omega^2 + m^2}.$$

$$\int_0^\infty \frac{x \sin(\rho x)}{x^2+m^2} dx = \frac{\pi}{2} e^{-\rho m}$$

Applying I.F.S.T,

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\omega}{\omega^2+m^2} \sin \omega x \, d\omega$$

$$\Rightarrow \int_0^\infty \frac{\omega \cdot \sin \omega x}{\omega^2+m^2} d\omega = \frac{\pi}{2} \cdot e^{-mx}$$

$$\Rightarrow \int_0^\infty \frac{\omega \cdot \sin(\rho \omega)}{\omega^2+m^2} d\omega = \frac{\pi}{2} e^{-\rho m} \qquad [x \text{ by } \rho]$$

$$\int_0^{\infty} \frac{x \sin(\rho x)}{x^2 + m^2} dx = \frac{\pi}{2} e^{-\rho m}$$

$$\Rightarrow \int_0^{\infty} \frac{x \sin(\rho x)}{x^2 + m^2} dx = \frac{\pi}{2} e^{-\rho m}.$$


Thank you!