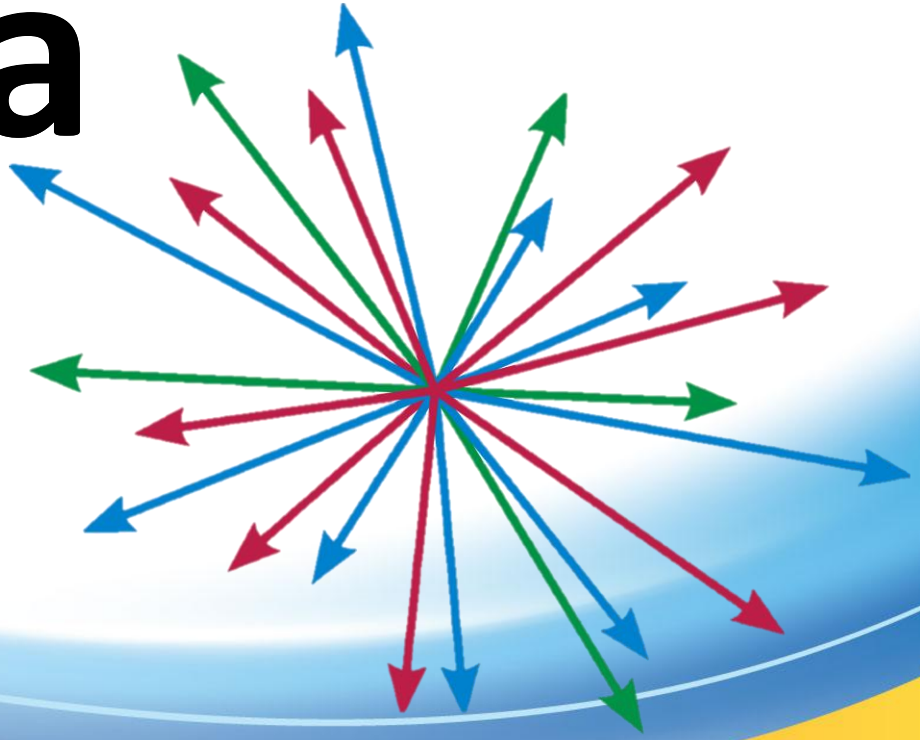
Undergraduate Course in Mathematics

Linear Algebra



Topic: Diagonalization

Conducted By

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Diagonalization

Similar Matrices

Let A and B be an $n \times n$ matrices. If there exits an Invertible $n \times n$ matrix P such that

$$A = PBP^{-1}$$

Then A and B are called similar matrices.

$$A = PBP^{-1}$$

A, B similar

What are Common in Similar Matrices

A, B similar

1. Eigenvalues same

2. Det same

3. Trace same

Idea of Diagonalization?

$$A = P D P^{-1}$$

diagonal

Diagonalizability

Let A be an $n \times n$ matrix. A is called diagonalizable If there exits an Invertible $n \times n$ matrix P such that

$$A = PDP^{-1}$$

Where D is an $n \times n$ diagonal matrix.

How to Diagonalize?

Let A be a square matrix of size $n \times n$. If A has n linearly independent eigenvectors then, A is diagonalizable by

$$\boxed{A = PDP^{-1}}$$

Where $\boldsymbol{D}$ is a diagonal matrix with **eigenvalues in the main diagonal**, and $\boldsymbol{P}$ is the matrix with **corresponding eigenvectors in the columns**.

Note: A $n \times n$ matrix is diagonalizable if and only if it has $\boldsymbol{n}$ **linearly independent eigenvectors**.

Check whether that A is diagonalizable or not. If diagonalizable then diagonalize A.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

Let λ be an eigenvalue of A,

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\cdot\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} - (1)\cdot\begin{vmatrix} 0 & 2 \\ -1 & 3-\lambda \end{vmatrix} + (2)\cdot\begin{vmatrix} 0 & 2-\lambda \\ -1 & 1 \end{vmatrix} = 0$$

$$(1-\lambda)\cdot[(2-\lambda)(3-\lambda)-2] - 1\cdot[0-(-1)(2)] + 2\cdot[0-(-1)(2-\lambda)] = 0$$

$$(1-\lambda)\cdot[4-5\lambda+\lambda^2] - 1\cdot[2] + 2\cdot[(2-\lambda)] = 0$$

$$4-5\lambda+\lambda^2-4\lambda+5\lambda^2-\lambda^3-2+4-2\lambda = 0$$

$$-\lambda^3+6\lambda^2-11\lambda+6 = 0$$

$$\boxed{\lambda^3-6\lambda^2+11\lambda-6 = 0}$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$(\lambda - 3)(\lambda - 2)(\lambda - 1) = 0$$

$$\lambda = \underline{\underline{3}}, \underline{\underline{2}}, \underline{\underline{1}}$$

For $\lambda = 3$, let $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigenvector

So $(A - \lambda I)\vec{v} = 0$

$$\begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The Augmented System

$$\left(\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 - 2x_3 = 0$

$x_2 - 2x_3 = 0$

x_3 is free

$\therefore$ let $x_3 = t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2t \\ 2t \\ t \end{pmatrix} = t \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$\therefore$ For $\lambda = 3$, $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector.

For $\lambda = 2$, let $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigenvector

So $(A - \lambda I)\vec{v} = 0$

$$\begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The Augmented System

$$\left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 - x_2 = 0$

$x_3 = 0$

free x_2.

let $x_2 = t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda = 2$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector.

For $\lambda = 1$, let $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigenvector

So $(A - \lambda I)\vec{v} = 0$

$$\begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The Augmented System

$$\left(\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 = 0$

$x_2 + 2x_3 = 0$

x_3 is free

let $x_3 = t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2t \\ t \end{pmatrix} = t \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

$\therefore$ For $\lambda = 1$, $\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector.

Eigenvalues	Eigenvectors
$\lambda_1 = 3$	$\vec{v_1} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$
$\lambda_2 = 2$	$\vec{v_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$
$\lambda_3 = 1$	$\vec{v_3} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$

$\Rightarrow$ A is diagonalizable

$n = 3$

Total eig vector $= 3$

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} \vdots & \vdots & \vdots \\ \vec{v_1} & \vec{v_2} & \vec{v_3} \\ \vdots & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

We can verify that

$$A = PDP^{-1}$$

$$P^{-1} = \begin{pmatrix} -1/2 & 1/2 & 1 \\ 2 & -1 & -2 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$

Check whether that A is diagonalizable or not. If diagonalizable then diagonalize A.

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}$$

Let λ be an eigenvalue of A,

$$A - \lambda I = \begin{pmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\cdot\begin{vmatrix} 5-\lambda & -2 \\ 1 & 2-\lambda \end{vmatrix} - (1)\cdot\begin{vmatrix} 2 & -2 \\ 1 & 2-\lambda \end{vmatrix} + (-1)\cdot\begin{vmatrix} 2 & 5-\lambda \\ 1 & 1 \end{vmatrix} = 0$$

$$(4-\lambda)\cdot[(5-\lambda)(2-\lambda) - (-2)] - 1\cdot[2(2-\lambda) - (-2)] + (-1)\cdot[2 - (5-\lambda)] = 0$$

$$(4-\lambda)\cdot[12 - 7\lambda + \lambda^2] - 1\cdot[6 - 2\lambda] + (-1)\cdot[-3 + \lambda] = 0$$

$$48 - 28\lambda + 4\lambda^2 - 12\lambda + 7\lambda^2 - \lambda^3 - 6 + 2\lambda + 3 - \lambda = 0$$

$$-\lambda^3 + 11\lambda^2 - 39\lambda + 45 = 0$$

$$\lambda^3 - 11\lambda^2 + 39\lambda - 45 = 0$$

$$\lambda^3 - 11\lambda^2 + 39\lambda - 45 = 0$$

$$(\lambda - 5)(\lambda - 3)(\lambda - 3) = 0$$

$\lambda = \underline{5}, \ \underline{3}, \ 3$

For $\lambda = 5$, let $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigenvector

So $(A - \lambda I)\vec{v} = 0$

$$\begin{pmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ 1 & 1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The Augmented System

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & 1 & -3 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 - x_3 = 0$$

$$x_2 - 2x_3 = 0$$

free x_3.

$\therefore$ Let $x_3 = t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ 2t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

For $\lambda = 5$, $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector.

For $\lambda = 3$, let $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigenvector

So $(A - \lambda I)\vec{v} = 0$

$$\begin{pmatrix} 4-\lambda & 1 & -1 \\ 2 & 5-\lambda & -2 \\ 1 & 1 & 2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The Augmented System

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 2 & -2 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} R_2' = R_2 - 2R_1 \\ R_3' = R_3 - R_1 \end{array}$$

$\Rightarrow x_1 + x_2 - x_3 = 0$

x_2 and x_3 free

let $x_2 = t_1$ and $x_3 = t_2$.

$x_1 = -x_2 + x_3 = -t_1 + t_2$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -t_1 + t_2 \\ t_1 \\ t_2 \end{pmatrix} = t_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ For $\lambda = 3$, $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent eigenvectors.

Eigenvalues	Eigenvectors
$\lambda_1 = 5$	$\vec{v_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$
$\lambda_2 = 3$	$\vec{v_2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$
$\lambda_3 = 3$	$\vec{v_3} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$n = 3$

$\#$ eig vectors $= 3$

$\therefore$ A is diagonalizable.

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$P = \begin{pmatrix} \vdots & \vdots & \vdots \\ \vec{v_1} & \vec{v_2} & \vec{v_3} \\ \vdots & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

We can verify that

$$A = PDP^{-1}$$

$$P^{-1} = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ -1 & 0 & 1 \\ -1/2 & -1/2 & 3/2 \end{pmatrix}$$

Check whether that A is diagonalizable or not. If diagonalizable then diagonalize A.

$$A = \begin{pmatrix} 0 & -6 & -4 \\ 5 & -11 & -6 \\ -6 & 9 & 4 \end{pmatrix}$$

Let λ be an eigenvalue of A,

$$A - \lambda I = \begin{pmatrix} 0-\lambda & -6 & -4 \\ 5 & -11-\lambda & -6 \\ -6 & 9 & 4-\lambda \end{pmatrix}$$

$$|A-\lambda I|=0$$

$$\begin{vmatrix} 0-\lambda & -6 & -4 \\ 5 & -11-\lambda & -6 \\ -6 & 9 & 4-\lambda \end{vmatrix}=0$$

$$(0-\lambda)\cdot\begin{vmatrix} -11-\lambda & -6 \\ 9 & 4-\lambda \end{vmatrix}-(-6)\cdot\begin{vmatrix} 5 & -6 \\ -6 & 4-\lambda \end{vmatrix}+(-4)\cdot\begin{vmatrix} 5 & -11-\lambda \\ -6 & 9 \end{vmatrix}=0$$

$$(-\lambda)\cdot[(-11-\lambda)(4-\lambda)+54]+6\cdot[5(4-\lambda)-36]-4\cdot[54-(-6)(-11-\lambda)]=0$$

$$(-\lambda)\cdot[10+7\lambda+\lambda^2]+6\cdot[-16-5\lambda]-4\cdot[-21-6\lambda]=0$$

$$-10\lambda-7\lambda^2-\lambda^3-96-30\lambda+84+24\lambda=0$$

$$-\lambda^3-7\lambda^2-16\lambda-12=0$$

$$\lambda^3+7\lambda^2+16\lambda+12=0$$

$$\lambda^3 + 7\lambda^2 + 16\lambda + 12 = 0$$

$$(\lambda + 3)(\lambda + 2)(\lambda + 2) = 0$$

$$\lambda = -3, -2, -2$$

For $\lambda = -3$, let $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigenvector

So $(A - \lambda I)\vec{v} = 0$

$$\begin{pmatrix} 0-\lambda & -6 & -4 \\ 5 & -11-\lambda & -6 \\ -6 & 9 & 4-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -6 & -4 \\ 5 & -8 & -6 \\ -6 & 9 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The Augmented System

$$\left(\begin{array}{ccc|c} 3 & -6 & -4 & 0 \\ 5 & -8 & -6 & 0 \\ -6 & 9 & 7 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 3 & -6 & -4 & 0 \\ 0 & 6 & 2 & 0 \\ 0 & -3 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 3 & -6 & -4 & 0 \\ 0 & 6 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 3 & 0 & -2 & 0 \\ 0 & 6 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -2/3 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 - \frac{2}{3}x_3 = 0$

$x_2 + \frac{1}{3}x_3 = 0$

x_3 is free

let $x_3 = t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}t \\ -\frac{1}{3}t \\ t \end{pmatrix} = t \cdot \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix} = \frac{1}{3}t \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$\therefore$ For $\lambda = -3$, $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ is an eigenvector.

For $\lambda = -2$, let $\vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigenvector

So $(A - \lambda I)\vec{v} = 0$

$$\begin{pmatrix} 0-\lambda & -6 & -4 \\ 5 & -11-\lambda & -6 \\ -6 & 9 & 4-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -6 & -4 \\ 5 & -9 & -6 \\ -6 & 9 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The Augmented System

$$\left(\begin{array}{ccc|c} 2 & -6 & -4 & 0 \\ 5 & -9 & -6 & 0 \\ -6 & 9 & 6 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & -6 & -4 & 0 \\ 0 & 12 & 8 & 0 \\ 0 & -9 & -6 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & -6 & -4 & 0 \\ 0 & 12 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 0 & 12 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 = 0$

$x_2 + \frac{2}{3}x_3 = 0$

x_3 free

let $x_3 = t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{2}{3}t \\ t \end{pmatrix} = \frac{1}{3}t \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$$

$\therefore$ For $\lambda = -2$, $\begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$ is an eigenvector.

Eigenvalues	Eigenvectors
$\lambda_1 = -3$	$\vec{v_1} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$
$\lambda_2 = -2$	$\vec{v_2} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$
$\lambda_3 = -2$	$\vec{v_3} = \times$

$n = 3$

vector = 2

$\Rightarrow$ A is not diagonalizable.

Large Integer Power on Matrix

Calculating Power on a Diagonal Matrix

$$D = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$$

$$D^2 = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 25 & 0 \\ 0 & 4 \end{pmatrix}$$

$$D^3 = \begin{pmatrix} 125 & 0 \\ 0 & 8 \end{pmatrix}$$

$$D^n = \begin{pmatrix} 5^n & 0 \\ 0 & 2^n \end{pmatrix}$$

$$D = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$D^{100} = \begin{pmatrix} 6^{100} & 0 & 0 \\ 0 & 5^{100} & 0 \\ 0 & 0 & 8^{100} \end{pmatrix}$$

Calculating Power on a Diagonalizable Matrix

$$A = PDP^{-1}$$

$$A^2 = PD\underbrace{P^{-1} \cdot P}DP^{-1} = PDIDP^{-1} = PDDP^{-1} = PD^2P^{-1}$$

$$A^3 = A^2 \cdot A = PD^2\underbrace{P^{-1} \cdot P}DP^{-1} = PD^2 \cdot DP^{-1} = PD^3P^{-1}$$

$$\therefore\ A^n = PD^nP^{-1}$$

Find the value of A^{100} where

$$A = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}$$

Let λ be an eigenvalue of A,

$$A - \lambda I = \begin{pmatrix} 6-\lambda & -1 \\ 2 & 3-\lambda \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)\cdot(3-\lambda) - (2)\cdot(-1) = 0$$

$$18 - 6\lambda - 3\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 9\lambda + 20 = 0$$

$$(\lambda - 5)(\lambda - 4) = 0$$

$$\lambda = \underline{\underline{5}}, \underline{\underline{4}}$$

For $\lambda = 5$, let $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigenvector

So $(A - \lambda I)\vec{v} = 0$

$$\begin{pmatrix} 6-\lambda & -1 \\ 2 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The Augmented System

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow x_1 - x_2 = 0$

due $x_2 = t$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

For $\lambda = 4$, let $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigenvector

So $(A - \lambda I)\vec{v} = 0$

$$\begin{pmatrix} 6-\lambda & -1 \\ 2 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The Augmented System

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow \quad x_1 - \frac{1}{2}x_2 = 0$

x_2 free, $x_2 = t$

$\therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}t \\ t \end{pmatrix} = \frac{1}{2}t\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

Eigenvalues	Eigenvectors
$\lambda_1 = 5$	$\vec{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\lambda_2 = 4$	$\vec{v_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\therefore$ A is diagonalizable.

$$A = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}$$

$$P = \begin{pmatrix} \vdots & \vdots \\ \vec{v_1} & \vec{v_2} \\ \vdots & \vdots \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

We can verify that

$$A = PDP^{-1}$$

$$P^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$A = PDP^{-1}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$A^{100} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}^{100} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 5^{100} & 0 \\ 0 & 4^{100} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \cdot 5^{100} & -5^{100} \\ -4^{100} & 4^{100} \end{pmatrix}$$

$$\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\cdot\begin{pmatrix}2\cdot 5^{100} & -5^{100}\\ -4^{100} & 4^{100}\end{pmatrix}$$

$$= \begin{pmatrix}2\cdot 5^{100}-4^{100} & -5^{100}+4^{100}\\ 2\cdot 5^{100}-2\cdot 4^{100} & -5^{100}+2\cdot 4^{100}\end{pmatrix}$$

Find the value of

$$\sqrt{\begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}}$$

Eigenvalues	Eigenvectors
$\lambda_1 = 5$	$\overrightarrow{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\lambda_2 = 4$	$\overrightarrow{v_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$A = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}$$

$$P = \begin{pmatrix} \vdots & \vdots \\ \overrightarrow{v_1} & \overrightarrow{v_2} \\ \vdots & \vdots \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

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$$A = PDP^{-1}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$A^{\frac{1}{2}} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{5} & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

Why Diagonalization is Important

$$e^{\begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}}$$

$$\sin\begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^A = \underbrace{I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots}_{\text{Matrix}}$$

Jordan canonical form

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