Fourier Analysis

General Fourier Transform

Evaluating Integrals using Inverse Transform

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Complex Fourier Series on $(-\boldsymbol{L}, \boldsymbol{L})$

Let $f(x)$ be defined in an interval with period $2L$. The complex Fourier series expansion of $f(x)$ is defined to be

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{i\frac{n\pi x}{L}}$$

where the Fourier coefficients c_n are

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) \; e^{-i\frac{n\pi x}{L}} dx \, , \qquad\qquad n = \cdots - 3, -2, -1, 0, 1, 2, 3, \ldots$$

Complex Fourier Series on $(-\pi, \pi)$

Let $f(x)$ be defined in an interval with period 2π. The complex Fourier series expansion of $f(x)$ is defined to be

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{i\,nx} \quad =$$

where the Fourier coefficients c_n are

$$c_n = \frac{1}{2\pi} \int_{-L}^{L} f(x)\; e^{-i\,nx} dx\,, \qquad\qquad n = \cdots - 3, -2, -1, 0, 1, 2, 3, \ldots$$

Idea of Fourier Transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$$

Inverse Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

Fourier Transform

Fourier Transform

Let $f(x)$ be a continuous signal. The Fourier transform of $f(x)$ is denoted by $F(\omega)$, defined by

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

Here, x is the time domain variable and ω is the frequency domain variable.

Inverse Fourier Transform

Let $F(\omega)$ be a frequency domain signal. The Inverse Fourier transform of $F(\omega)$ is denoted by $f(x)$, defined by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$$

Here, x is the time domain variable and ω is the frequency domain variable.

Physical Interpretation of Fourier transform

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

PROBLEM Find the Fourier transform of the function

$$f(x) = \begin{cases} \pi, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

$-1 < x < 1$

Hence, evaluate

$$\int_0^\infty \frac{\sin x}{x} dx$$

$f(0) = \pi$

Fourier Transform $F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$

$$= \int_{-1}^{1} \pi \cdot e^{-i\omega x}\, dx + 0$$

$$= \pi \cdot \left[\frac{e^{-i\omega x}}{-i\omega} \right]_{-1}^{1}$$

$$= \pi \left[\left(\frac{e^{-i\omega}}{-i\omega} \right) - \left(\frac{e^{i\omega}}{-i\omega} \right) \right]$$

$$= \pi \left(\frac{e^{i\omega}}{i\omega} - \frac{e^{-i\omega}}{i\omega} \right)$$

$$= \pi \cdot \frac{e^{i\omega} - e^{-i\omega}}{i\omega}.$$

$$= \pi \cdot \frac{2i \sin\omega}{i\omega} = \frac{2\pi \sin\omega}{\omega}.$$

Now, applying I.F.T,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, d\omega$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\pi \sin\omega}{\omega}\, e^{i\omega x}\, d\omega$$

$$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\pi \sin\omega}{\omega}\, e^{0}\, d\omega$$

substituting $x = 0$,

$$\Rightarrow \pi = \frac{1}{\cancel{2\pi}} \int_{-\infty}^{\infty} \frac{\cancel{2\pi} \sin\omega}{\omega} d\omega$$

$$\Rightarrow \pi = \int_{-\infty}^{\infty} \frac{\sin\omega}{\omega} d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin\omega}{\omega} d\omega = \pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

$$\Rightarrow 2 \cdot \int_{0}^{\infty} \frac{\sin x}{x} dx = \pi$$

$$\Rightarrow \therefore \int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

PROBLEM Find the Fourier transform of the function

$$f(x) = \begin{cases} \dfrac{1}{m}, & |x| < m \\ 0, & |x| > m \end{cases} \implies -m < x < m$$

Hence, evaluate

$$\int_0^\infty \frac{\sin x}{x}\, dx$$

Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \int_{-m}^{m} \frac{1}{m}\, e^{-i\omega x}\, dx$$

$$= \frac{1}{m}\left[\frac{e^{-i\omega x}}{-i\omega}\right]_{-m}^{m}$$

$$= \frac{1}{m}\left[\frac{e^{-i\omega m}}{-i\omega} - \frac{e^{i\omega m}}{-i\omega}\right]$$

$$= \frac{1}{m}\left[\frac{e^{i\omega m}}{i\omega} - \frac{e^{-i\omega m}}{i\omega}\right]$$

$$= \frac{1}{m}\, \frac{e^{i\omega m} - e^{-i\omega m}}{i\omega}$$

$$= \frac{1}{m}\, \frac{2i\sin(\omega m)}{i\omega} = \frac{2\sin(m\omega)}{m\omega}$$

Applying I.F.T

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin(m\omega)}{m\omega}\, e^{i\omega x}\, d\omega$$

at $x=0$,

$$f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\sin(m\omega)}{m\omega}\, 1 \cdot d\omega$$

$$\Rightarrow \frac{1}{\cancel{m}} = \frac{1}{\cancel{2}\pi} \int_{-\infty}^{\infty} \frac{\cancel{2}\sin(m\omega)}{\cancel{m}\cdot\omega}\, d\omega$$

$$\Rightarrow 1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(m\omega)}{\omega}\, d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin(m\omega)}{m\omega}\, d\omega = \pi$$

$$\Rightarrow 2 \cdot \int_0^\infty \frac{\sin(m\omega)}{\omega} d\omega = \pi$$

let $m\omega = x$

$\Rightarrow m\,d\omega = dx$

$$\Rightarrow \int_0^\infty \frac{\sin(m\omega)}{\omega} d\omega = \frac{\pi}{2}.$$

$$\Rightarrow \int_0^\infty \frac{\sin x}{\frac{x}{m}} \cdot \frac{dx}{m} = \frac{\pi}{2}$$

$$= \int_0^\infty \frac{\sin x}{x} \cdot m \cdot \frac{dx}{m} = \frac{\pi}{2}$$

$$\therefore \int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

PROBLEM Find the Fourier transform of the function

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases} \Rightarrow -a<x<a$$

Hence, evaluate

$$\int_0^\infty \frac{\sin(ax)\cos(ax)}{x}\,dx$$

Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \int_{-a}^{a} 1 \cdot e^{-i\omega x}\, dx$$

$$= \left[\frac{e^{-i\omega x}}{-i\omega} \right]_{-a}^{a}$$

$$= \frac{e^{-ia\omega}}{-i\omega} - \frac{e^{ia\omega}}{-i\omega}$$

$$= \frac{e^{ia\omega} - e^{-ia\omega}}{i\omega}$$

$$= \frac{2i \sin(a\omega)}{i\omega}$$

$$= \frac{2 \sin(a\omega)}{\omega}.$$

Applying I.F.T.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, d\omega$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin(a\omega)}{\omega}\, e^{i\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin(a\omega)}{\omega}\left(\cos\omega x + i\sin\omega x\right) d\omega$$

$$\Rightarrow f(x) + i \cdot 0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin(a\omega)}{\omega} \cos(\omega x)\, d\omega + i\, \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin(a\omega)}{\omega} \cdot \sin(\omega x)\, d\omega$$

$$\int_0^{\infty} \frac{\sin(ax)\cos(ax)}{x}\, dx$$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin(a\omega)}{\omega} \cos(\omega x)\, d\omega = f(x)$$

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin(a\omega)}{\omega} \cdot \cos(a\omega)\, d\omega = f(a)$$

$$\Rightarrow \frac{1}{\cancel{2}\pi} \int_{-\infty}^{\infty} \frac{\cancel{2} \sin(a\omega)}{\omega} \cos(a\omega) \, d\omega = \frac{1+0}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin(a\omega)}{\omega} \cdot \cos(a\omega) \, d\omega = \frac{\pi}{2}$$

$$\Rightarrow 2 \cdot \int_{0}^{\infty} \frac{\sin(a\omega)}{\omega} \cdot \cos(a\omega) \, d\omega = \frac{\pi}{2}$$

$$\therefore \int_{0}^{\infty} \frac{\sin(a\omega)}{\omega} \cos(a\omega) \, d\omega = \frac{\pi}{4}.$$

PROBLEM Find the Fourier transform of the function

$$f(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \Rightarrow \quad -1 < x < 1$$

Hence, evaluate

$$\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\left(\frac{x}{2}\right) dx$$

Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$1-x^2$	$e^{-i\omega x}$
$-2x$	$\dfrac{e^{-i\omega x}}{-i\omega}$
-2	$\dfrac{e^{-i\omega x}}{i^2\omega^2}$
0	$\dfrac{e^{-i\omega x}}{-i^3\omega^3}$

$$= \int_{-1}^{1} (1-x^2)\, e^{-i\omega x}\, dx$$

$$= \left[(1-x^2)\frac{e^{-i\omega x}}{-i\omega} - (-2x)\frac{e^{-i\omega x}}{i^2\omega^2} + (-2)\frac{e^{-i\omega x}}{-i^3\omega^3} \right]_{-1}^{1}$$

$$= \left[(1-x^2)\,\frac{e^{-i\omega x}}{-i\omega} - 2x \cdot \frac{e^{-i\omega x}}{\omega^2} - 2 \cdot \frac{e^{-i\omega x}}{i\omega^3} \right]_{-1}^{1}$$

$$= \left[(1-x^2)\,\frac{e^{-i\omega x}}{i\omega} + 2x\,\frac{e^{-i\omega x}}{\omega^2} + 2 \cdot \frac{e^{-i\omega x}}{i\omega^3} \right]_{1}^{-1}$$

$$= \left(0 \cdot \cancel{\frac{e^{i\omega}}{i\omega}} - 2 \cdot \frac{e^{i\omega}}{\omega^2} + 2 \cdot \frac{e^{i\omega}}{i\omega^3} \right) - \left(0 \cdot \cancel{\frac{e^{-i\omega}}{i\omega}} + 2 \cdot \frac{e^{-i\omega}}{\omega^2} + 2\,\frac{e^{-i\omega}}{i\omega^3} \right)$$

$$= -2 \cdot \frac{e^{i\omega}}{\omega^2} - 2 \cdot \frac{e^{-i\omega}}{\omega^2} + 2 \cdot \frac{e^{i\omega}}{i\omega^3} - 2 \cdot \frac{e^{-i\omega}}{i\omega^3}$$

$$= -2 \cdot \frac{e^{i\omega} + e^{-i\omega}}{\omega^2} + 2 \cdot \frac{e^{i\omega} - e^{-i\omega}}{i\omega^3}$$

$$= -2 \cdot \frac{2\cos\omega}{\omega^2} + 2 \cdot \frac{2i\sin\omega}{i\omega^3}$$

$$= \frac{-4\cos\omega}{\omega^2} + \frac{4\sin\omega}{\omega^3} = \frac{4\sin\omega - 4\omega\cos\omega}{\omega^3} .$$

Applying I.F.T.

$$\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\left(\frac{x}{2}\right) dx$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\cdot e^{i\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\sin\omega - 4\omega\cos\omega}{\omega^3}\, e^{i\omega x}\, d\omega$$

$$f(x) = \frac{4}{2\pi}\int_{-\infty}^{\infty} \frac{\sin\omega - \omega\cos\omega}{\omega^3}\left(\cos\omega x + i\sin\omega x\right) d\omega$$

$$\Rightarrow f(x) + i \cdot 0 = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin\omega - \omega\cos\omega}{\omega^3} \cos\omega x \, d\omega + i\,\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin\omega - \omega\cos\omega}{\omega^3} \sin\omega x \, d\omega$$

$$\therefore \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin\omega - \omega\cos\omega}{\omega^3} \cos\omega x \, d\omega = f(x)$$

substitution $x = \frac{1}{2}$

$$\Rightarrow \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin\omega - \omega\cos\omega}{\omega^3} \cos\left(\omega \cdot \frac{1}{2}\right) d\omega = f\left(\frac{1}{2}\right)$$

$$f(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

$$\int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\left(\frac{x}{2}\right)dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin\omega - \omega\cos\omega}{\omega^3}\cos\left(\frac{\omega}{2}\right)d\omega = \frac{\pi}{2}\cdot f\left(\frac{1}{2}\right).$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin x - x\cos x}{x^3}\cos\left(\frac{x}{2}\right)dx = \frac{\pi}{2}\cdot\left(1 - \left(\frac{1}{2}\right)^2\right)$$

$$\Rightarrow 2\cdot\int_0^{\infty} \frac{\sin x - x\cos x}{x^3}\cos\left(\frac{x}{2}\right)dx = \frac{\pi}{2}\cdot\left(1 - \frac{1}{4}\right) = \frac{\pi}{2}\cdot\frac{3}{4}$$

$$\Rightarrow \quad -2\int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = \frac{3\pi}{8}$$

$$\Rightarrow \quad \int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = \frac{-3\pi}{16}.$$

PROBLEM Find the Fourier transform of the function

$$f(x) = \begin{cases} 1 - |x|, & |x| < 1 \\ 0, & |x| > 1 \end{cases} \quad \Rightarrow \quad -1 < x < 1$$

Hence, evaluate

$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx$$

Fourier Transform

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$f(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \int_{-1}^{1} (1 - |x|)\, e^{-i\omega x}\, dx$$

$$= \int_{-1}^{0} (1 - (-x))\, e^{-i\omega x}\, dx + \int_{0}^{1} (1 - (x))\, e^{-i\omega x}\, dx$$

$1-x$	$e^{-i\omega x}$
-1	$\dfrac{e^{-i\omega x}}{-i\omega}$
0	$\dfrac{e^{-i\omega}}{i^2\omega^2}$

$$= \int_{-1}^{0} (1+x)\, e^{-i\omega x}\, dx + \int_{0}^{1} (1-x)\, e^{-i\omega x}\, dx$$

$$= \left[(1+x) \frac{e^{-i\omega x}}{-i\omega} - 1 \cdot \frac{e^{-i\omega x}}{i^2\omega^2} \right]_{-1}^{0}$$

$$+ \left[(1-x) \frac{e^{-i\omega x}}{-i\omega} - (-1) \frac{e^{-i\omega x}}{i^2\omega^2} \right]_{0}^{1}$$

$$= \left[(1+x)\frac{e^{-i\omega x}}{-i\omega} + \frac{e^{-i\omega x}}{\omega^2} \right]_{-1}^{0} + \left[(1-x)\frac{e^{-i\omega x}}{-i\omega} - \frac{e^{-i\omega x}}{\omega^2} \right]_{0}^{1}$$

$$= \left[(1+x)\frac{e^{-i\omega x}}{-i\omega} + \frac{e^{-i\omega x}}{\omega^2} \right]_{-1}^{0} + \left[(1-x)\frac{e^{-i\omega x}}{i\omega} + \frac{e^{-i\omega x}}{\omega^2} \right]_{1}^{0}$$

$$= \left(1 \cdot \cancel{\frac{1}{-i\omega}} + \frac{1}{\omega^2} \right) - \left(0 + \frac{e^{i\omega}}{\omega^2} \right) + \left(1 \cdot \cancel{\frac{1}{i\omega}} + \frac{1}{\omega^2} \right) - \left(0 + \frac{e^{-i\omega}}{\omega^2} \right)$$

$$= \frac{2}{\omega^2} - \frac{e^{i\omega} + e^{-i\omega}}{\omega^2}$$

$$= \frac{2}{\omega^2} - \frac{2\cos\omega}{\omega^2}$$

$$= \frac{2(1-\cos\omega)}{\omega^2}$$

$$= \frac{2}{\omega^2} 2 \cdot \sin^2\left(\frac{\omega}{2}\right)$$

$$= \frac{4}{\omega^2} \sin^2\left(\frac{\omega}{2}\right).$$

Applying I.F.T.

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\cdot e^{i\omega x}\cdot d\omega$$

$$f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{\omega^2}\cdot \sin^2\left(\frac{\omega}{2}\right) e^{i\omega x}\,d\omega$$

$$\Rightarrow f(0) = \frac{4}{2\pi}\int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{\omega}{2}\right)}{\omega^2}\cdot 1\,d\omega$$

$$\Rightarrow 1 - |0| = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2\left(\frac{\omega}{2}\right)}{\omega^2} \, d\omega$$

$$\frac{\omega}{2} = x$$

$$\frac{d\omega}{2} = dx$$

$$\Rightarrow 1 = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 x}{(2x)^2} \, 2\, dx$$

$$\Rightarrow 1 = \frac{2}{\pi} \cdot \int_{-\infty}^{\infty} \frac{\sin^2 x}{4x^2} \cdot 2\, dx$$

$$\Rightarrow 1 = \frac{2}{\pi} \cdot \frac{2}{4} \cdot \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \, dx$$

$$\Rightarrow \quad 1 = \frac{1}{\pi} \cdot \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx$$

$$\Rightarrow \quad 1 = \frac{2}{\pi} \cdot \int_{0}^{\infty} \frac{\sin^2 x}{x^2}\, dx$$

$$\therefore \int_{0}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}.$$


Thank you!