Naman’s Divisor Sum Theorem

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INTRODUCTION

Hello There, I am Naman Chaudhary. I am from a remote village of Nepal (i.e., Bindhi, Janakpurdham). I always wanted to make my country proud internationally. Today I am so close of it. I have made a theorem called 'Naman's Divisor Sum Theorem'. Obviously, named after me. This is one of my very first theorem. I don't know if this will get the recognition or not. I hope my theorem will contribute even a little bit in the field of mathematics. I am so excited to present my masterpiece to you.

I have developed this theorem with so much hard work and patience. This theorem, named after me is not just a theorem but start of a new phase of my life and start of an unpredictable journey. I want to leave a long-lasting impact in the field of mathematics.

I do not yet know that this theorem is even necessary or not and whether this will get widespread recognition or not. But I wholeheartedly hope that, even though small, but this theorem will contribute to the mathematics. Regardless of the outcomes, I will always continue working and contributing in the related fields. I am excited to share my work with the world, hopefully this will inspire others as well.

AUTHOR

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NAMAN'S DIVISOR SUM THEOREM

METHOD – I

Theorem: For any positive integer *n*, W(n) ≥ d(n), with equality if and only if *n* is square free.

Proof:

First,

$$W(n) = \sum_{d|n} V(d)$$

, where V(d) is the product of the exponents in the prime factorization of d.

Also,

$$d(n) = \sum_{d|n} 1$$

So,

$$W(n) - d(n) = \sum_{d|n} [V(d) - 1]$$

Now,

For each divisor *d* of *n*, if d = 1, then V(1) + 1, so V(1)-1 = 0.

For,

d>1, if d is square-free, meaning all the exponents in the d are 1, then V(d) = 1, since the products of exponents is $1^{W(d)}$, where W(d) is the number of distinct primes in d, but since exponents are 1, V(d) = 1.

More Precisely,

Since d is square-free, its prime factorization has exponents all equal to 1. So, V(d) = 11…×1 = 1.

Therefore, For square-free d, V(d) = 1, so V(d) – 1 = 0.

For, d that are not square-free, meaning there is at last one exponent greater than 1, then V(d) ≥ 2, since there is at least one exponent ≥ 2, and others ≥ 1, so the product is ≥ 2.

Thus, for d (i.e., not square-free), V(d) -1 ≥ 1.

Therefore, In the sum $\sum_{d|n}[V(d)-1]$, for d =1, its 0; for d>1 that are square-free, its 0; for that d that are not square-free, its ≥ 1.

So, If n is square-free, then all its divisors d are also square-free (since they are products of subsets of the primes dividing n, with exponents 0 or 1). So for all d|n, V(d)-1 = 0.

Thus,

$W(n) – d(n) = 0$

$W(n) = d(n)$

If n is not square-free, then there would be at least one divisor d of n that is not square-free.

Example:

$d = p^2$ for some p dividing n with exponent ≥ 2.

Then,

For such d, $V(d) – 1 ≥ 1$ since $d|n$, the $\sum_{d|n}[V(d) - 1]$ ≥1.

Moreover, Since other terms are ≥ 0. The total sum is ≥ 1. So,

$W(n) – d(n) ≥ 1$ or $W(n) > d(n)$.

Therefore, $W(n) ≥ d(n)$, with equality if and only if n is square-free.

Overall,

Simply this theorem helps connect the function V(n), which is defined in terms of exponent, to the divisor function and property of being square-free.

Mathematically,

$$W(n) = \sum_{d|n} V(d) \geq d(n)$$

NOTE: With equality if and only if n is square free.

NAMAN'S DIVISOR SUM THEOREM

METHOD – II

Theorem:

For any positive integer n, W(n) ≥ d(n), with equality if and only if n is square-free.

To Prove:

$W(n) = \sum_{d|n} V(d) \geq d(n)$ With equality if and only if n is square-free.

Proofs:

1. Exponent Product Function:

   V(n) = $a_1\ a_2 \ldots a_k$ (If n>1, otherwise V(1) = 1)
   Where, n = $p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$

2. Divisor Sum Function:

   $W(n) = \sum_{d|n} V(d)$

3. Inequality to compare with divisor function:

   W(n) ≥ d(n)
   [With equality if and only if n is square-free.

Let,

n = $p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ be the prime factorization of positive integer n.

Define,

V(n) = $\begin{cases} a_1\ a_2 \dots a_k, & if\ n > 1 \\ 1, & if\ n = 1 \end{cases}$

And,

$$W(n) = \sum_{d|n} V(d)$$

Where,

The sum runs over all positive divisors d of n. Let d(n) denote the number of positive divisors n.

Then,

W(n) ≥ d(n)

With equality if and only if n is square-free. (i.e., each a_i = 1)

Proofs:

1. Able to multiply and reduction to prime factors:

   Both functions V(n) and the divisor function d(n) are multiplicative. Consequently, the sum

   $$W(n) = \sum_{d|n} V(d)$$

   Is multiplicative as well. So,

   $$W(n) = \prod_{i=1}^{k} \left(\sum_{b=0}^{a_i} V(p_i^b) \right).$$

For each prime power p^a, the divisors are p^b with 0≤b≤a.

By Definition,

V(1) = 1 and V(p^b) = b for b ≥ 1.

Hence,

$$W(p^a) = 1 + \sum_{b=1}^{a} b = 1 + \frac{a(a+1)}{2}$$

2. Expressor for divisor function.

The number of divisor n is given by:

$$d(n) = \prod_{i=1}^{k} a_i + 1$$

3. Reduction of the inequality:

Reducing W(n) ≥ d(n),

$$\prod_{i=1}^{k} \left(1 + \frac{a_i(a_i+1)}{2}\right) \geq \prod_{i=1}^{k} (a_i + 1).$$

Since, The products on both side are over independent factors corresponding to each prime p_i, it proves that for each positive integer a ≥ 1.

1+ $\frac{a(a+1)}{2}$ ≥ a+1,
With equality if and only if a =1.

4. Verification for each prime factor:

Firstly,

$1+\frac{a(a+1)}{2} \geq a+1$

Multiplying both sides by 2,

$2+a(a+1) \geq 2(a+1)$
$a^2+a+2 \geq 2a+2$

Simplifying,

$a^2-a \geq 0$

Factorizing L.H.S.,

$a(a-1) \geq 0$

Thus, This inequality holds for all $a \geq 1$ and is equality if and only if $a = 1$.

Conclusion:

Since the inequality holds for each prime factor,

$$W(n) = \prod_{i=1}^{k} \left(1 + \frac{a_i(a_i+1)}{2}\right) \geq \prod_{i=1}^{k} (a_i + 1) = d(n),$$

With equality if and only if a_i = 1 for all I, i.e., when n is square-free.

Formulas Derived from This Theorem:

This theorem states that:

$$W(n) = \sum_{d|n} V(d) \geq d(n)$$

(Where equality hold if and only if n is square-free.)

DERIVATIONS:

1. Explicit Formula for W(n):

   For a prime power p^a:

   $W(p^a) = 1 + \sum_{b=1}^{a} b = 1 + \frac{a(a+1)}{2}$

   For general integer n = $p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$, Its multiplication property gives:

   $W(n) = \prod_{i=1}^{k} \left(1 + \frac{a_i(a_i+1)}{2}\right)$

   Therefore, This formula calculates W(n) for any n.

2. Inequality derived from Theorem:

   Since, W(n) ≥ d(n), it leads asymptotic bond.

$$\prod_{i=1}^{k}\left(1+\frac{a_i(a_i+1)}{2}\right) \geq \prod_{i=1}^{k}(a_i+1)$$

   NOTE: This can be used to study sum behavior and establish further inequalities in number theory.

3. Difference function D(n):

   D(n) = W(n) – d(n)

   From the theorem,

   (i) D(n) = 0, if and only if n is square free.
   (ii) If n is not square free then D(n)>0.

   NOTE: This function helps measure how non square-free n is.

4. Asymptotic growth of W(n):

   For large n, the divisor function:
   d(n)=O(n^{ϵ}),for any ϵ>0.
   Since, W(n) ≥ d(n), this states:

   $$W(n) = O(n^{\epsilon})$$

   Thus, W(n) grows at most polynomial with n, which is useful in number theory.

5. Corollary in square-free numbers:

   W(n) = d(n) [n is square-free]

   NOTE: This provides a new criterion to check if the number is square-free.

Possible Generalizations

1. Instead of summing over all divisors, we could sum over only square-free divisors or prime-power divisors.
2. Summing V(n) over multiplicative orders.

APPLICATIONS

(i) Improved divisor function bonds.
(ii) Square-free numbers detection.
(iii) Relation to highly composite numbers.
(iv) Prime exponent analysis in RSA Cryptography.
(v) Square-free modulus in cryptographic systems.
(vi) Keyspace reduction in lattice-based attacks.
(vii) Other summatory functions.
(viii) Extending this to other number systems.