MTH 9875 The Volatility Surface: Fall 2015

Lecture 2: Stochastic volatility and local volatility

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Outline of lecture 2

- Stochastic volatility
  - Motivation for stochastic volatility
  - Derivation of the valuation equation
  - The market price of volatility risk and market completeness
  - Solving the valuation equation
  - Examples of stochastic volatility models
- Local volatility
  - Historical development
  - The Dupire equation
  - Local volatility in terms of implied volatility
  - The BBF formula
  - Local variance as a conditional expectation of instantaneous variance

Stochastic volatility

Why stochastic volatility?

- Volatility obviously fluctuates
  - For example, recall the stock market crash of October 1987
- However, it may not be obvious what the benefits of going beyond Black-Scholes might be.
- Stochastic volatility (SV) models are useful because they explain in a self-consistent way why it is that options with different strikes and expirations have different Black-Scholes implied volatilities.
  - the “volatility smile”
- Unlike alternative models that can fit the smile (such as local volatility models for example), SV models assume realistic dynamics for the underlying.

Stochastic volatility as volatility in trading time

- SV price processes are sometimes accused of being *ad hoc*.
- On the contrary, they can be viewed as arising from Brownian motion subordinated to a random clock.
- This clock time may be identified with the volume of trades or the frequency of trading
  - The idea is that as trading activity fluctuates, so does volatility.
- As we saw in Lecture 1, this idea is not quite right empirically, but it is not altogether wrong either.

Engineering considerations

- Hedgers who use the Black-Scholes model must continuously change the volatility assumption in order to match market prices.
  - The BS hedge ratio changes uncontrollably every time the volatility changes.
- SV models bring some order into this chaos.
- The prices of exotic options given by models based on Black-Scholes assumptions can be wildly wrong.
- SV models take the volatility smile into account when pricing exotic options.
  - SV model prices are closer to market prices than BS or local volatility model prices.

Log returns of SPX

In [23]:
```
library(quantmod)

options("getSymbols.warning4.0"=FALSE)

getSymbols("^GSPC",from="1927-01-01") #Creates the time series obje
ct GSPC

ret.spx <- log(Cl(GSPC)/lag(Cl(GSPC)))
ret.spx <- ret.spx[!is.na(ret.spx)]  # Remove missing values
ret.spx <- ret.spx-mean(ret.spx)

plot(ret.spx,main=NA,type="l")
```

Out[23]: [1] "GSPC"

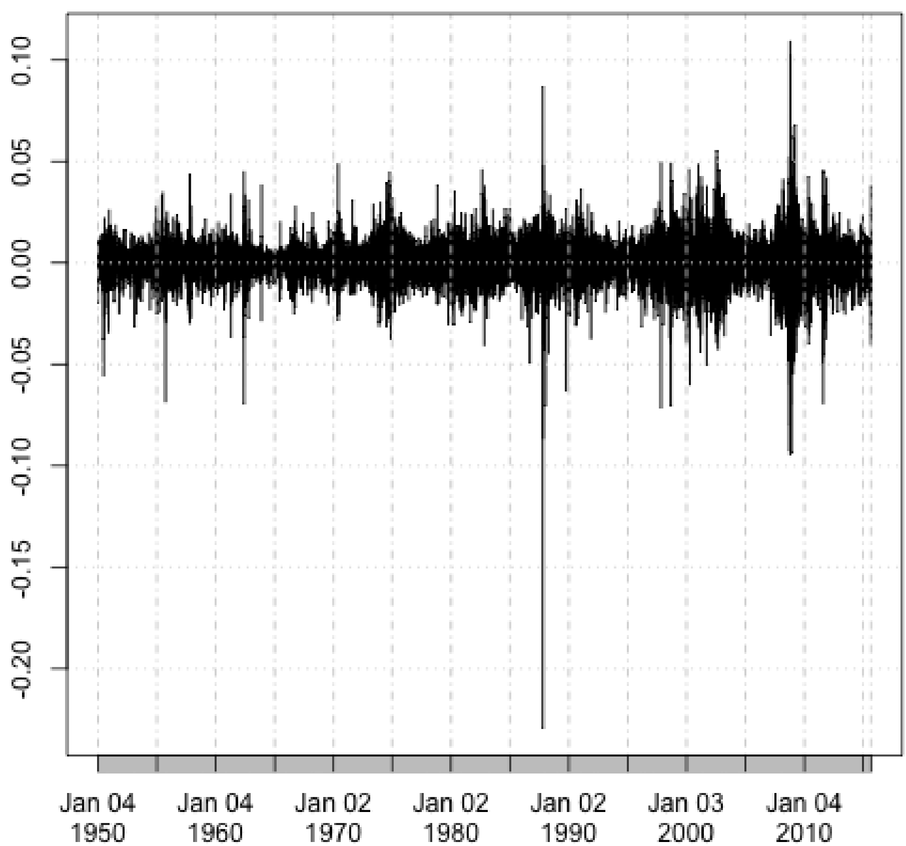

Figure1: Large moves follow large moves and small moves follow small moves: "volatility clustering".

Frequency distribution of log returns

In [24]:
```
breaks <- seq(-.235,.115,.002)
spx.hist <- hist(ret.spx,breaks=breaks,plot=F)
plot(spx.hist,xlim=c(-.05,.05),freq=F,main=NA,xlab="log return")
sig <- sd(as.numeric(ret.spx))
curve(dt(x*sqrt(3)/sig,df=3)*sqrt(3)/sig,from=-.05,to=.05,col="re
d", add=T)
curve(dnorm(x,mean=0,sd=sig),from=-.05,to=.05,col="blue", add=T)
```

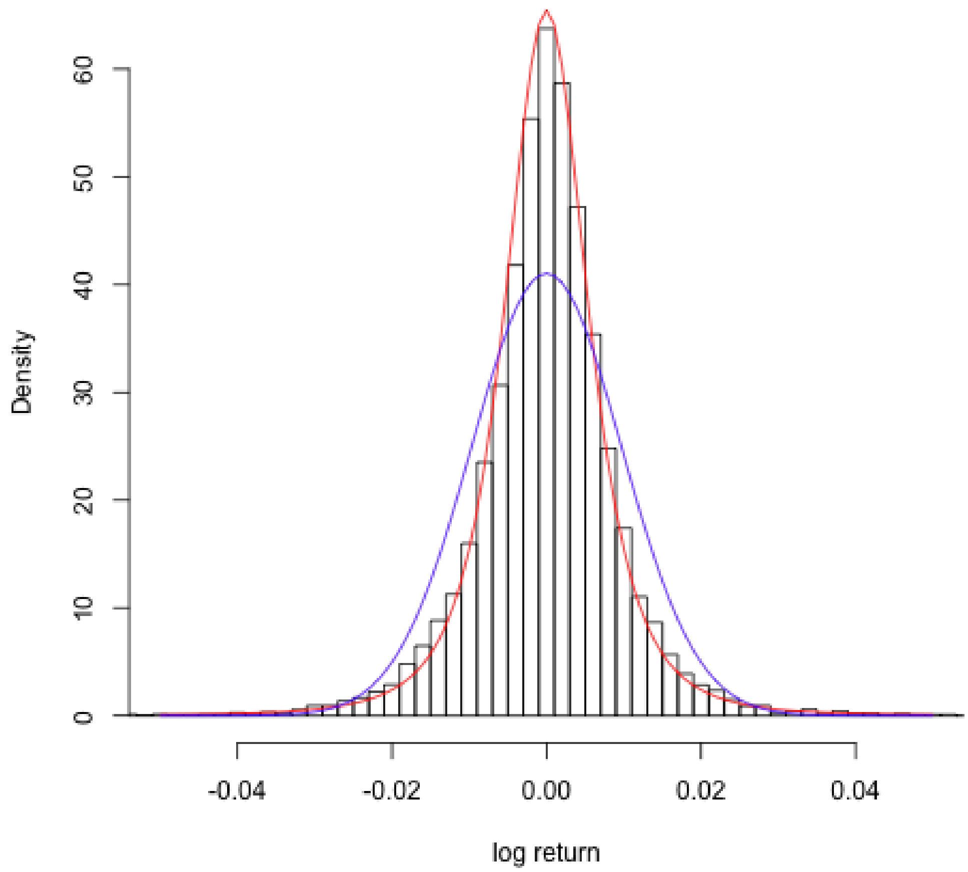

Figure 2: Frequency distribution of SPX daily log returns (since 1950) compared with the normal distribution (blue) and Student-t with 3 degrees of freedom (red).

- Note that the x-axis has been truncated; the -22.9% return on 10/19/1987 is not directly visible.

A toy regime-switching model

Suppose that realized volatility over a one-year period can be either 10% or 30%, each with probability $1/2$ (depending on an initial coin toss say).

In [13]:
```
d3010 <- function(x){(dnorm(x,sd=.1)+dnorm(x,sd=.3))/2}
curve(d3010(x),from=-.8,to=.8,col="blue",ylim=c(0,4))
curve(dnorm(x,sd=.1),from=-.8,to=.8,col="red",lty=2,add=T)
curve(dnorm(x,sd=.3),from=-.8,to=.8,col="green4",lty=2,add=T)
```

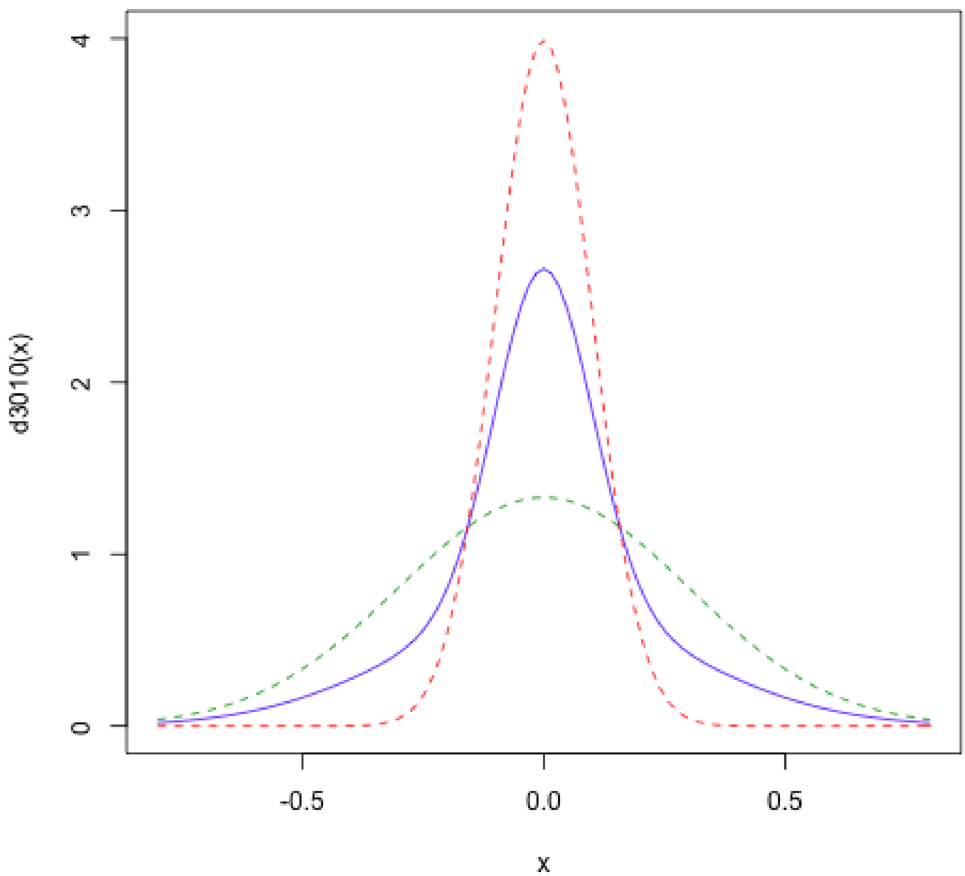

Figure 3: The two-regime density in blue; 10% and 30% volatility distributions in green and red respectively.

- In some sense, the peak of the distribution is driven by the low volatility regime and the tails by the high volatility regime.
- Fat tails and the high central peak are characteristics of mixtures of distributions with different variances.

Add more regimes

Consider the following 20-regime model:

In [19]:
```
d.20regimes <- function(x){
    res <- 0
    for (i in 1:20){ res <- res + dnorm(x,sd=i/20)}
    return(res/20)
}
```

The resulting density is:

In [21]:
```
curve(d.20regimes(x),from=-1.5,to=1.5,col="blue")
```

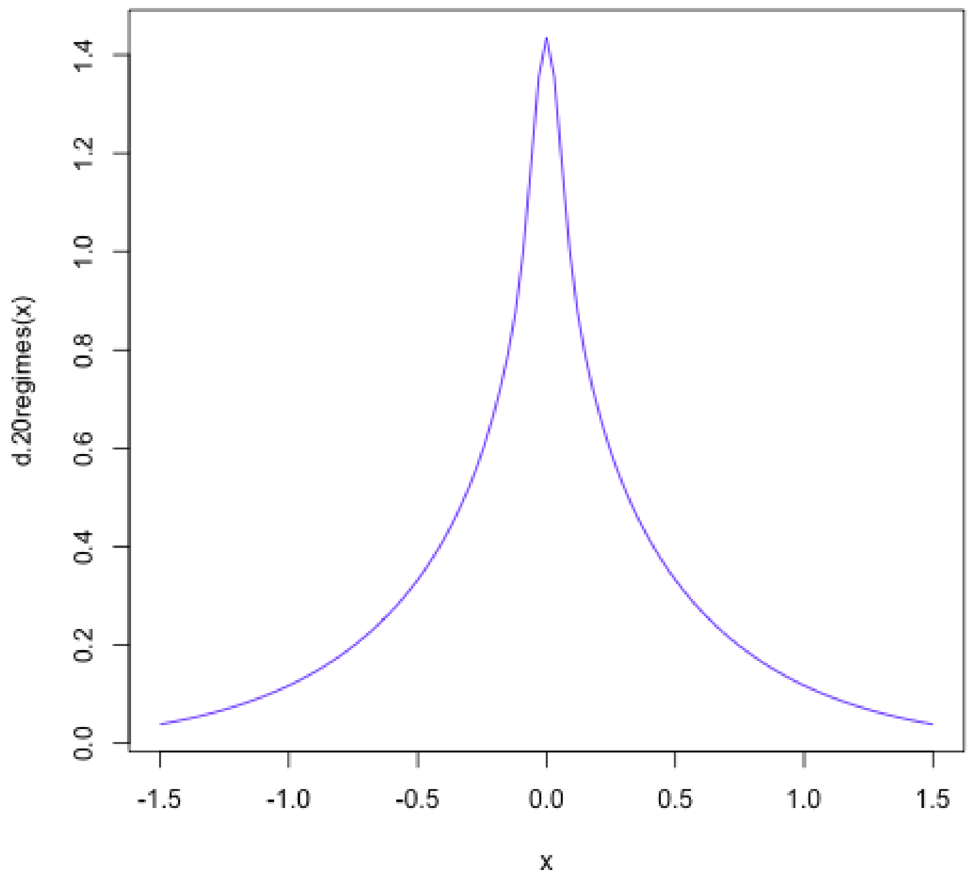

Figure 4: The 20-regime density in blue. Note the high central peak and fat tails.

Interpretation

- Fat tails and the high central peak are characteristics of mixtures of distributions with different variances.
  - This motivates us to model variance as a random variable.
- The volatility clustering feature implies that volatility (or variance) is auto-correlated.
  - In SV models, this is a consequence of the mean reversion of volatility.
- Note that simple jump-diffusion models do not have this volatility clustering property. After a jump, the stock price volatility does not change.

Economic argument for mean reversion

- Consider the distribution of the volatility of IBM in one hundred years time say.
- If volatility were not mean-reverting (* i.e.* if the distribution of volatility were not stable), the probability of the volatility of IBM being between 1% and 100% would be rather low.
- Since we believe that it is overwhelmingly likely that the volatility of IBM would in fact lie in that range, we deduce that volatility must be mean-reverting.

The SV process

We suppose that the stock price S and its variance v satisfy the following SDEs:

(1)

$$dS_t = \mu_t \, S_t \, dt + \sqrt{v_t} \, S_t \, dZ_t$$

(2)

$$dv_t = \alpha(S_t, v_t, t) \, dt + \eta \, \beta(S_t, v_t, t) \, \sqrt{v_t} \, dW_t$$

with

$$\langle dZ_t \, dW_t \rangle = \rho \, dt$$

where μ_t is the (deterministic) instantaneous drift of stock price returns, η is the volatility of volatility and ρ is the correlation between random stock price returns and changes in v_t. dZ and dW are Wiener processes.

The stock price process

- The stochastic process [(1)](#eq1:genstockprocess) followed by the stock price is equivalent to the Black-Scholes (BS) process.
    - This ensures that the standard time-dependent volatility version of the Black-Scholes formula may be retrieved in the limit $\eta \to 0$.
- In practical applications, this is desirable for a stochastic volatility option pricing model as practitioners' intuition for the behavior of option prices is invariably expressed within the framework of the Black-Scholes formula.

The variance process

- The stochastic process [(2)](#eq2:genvarprocess) followed by the variance is very general.
- We don't assume anything about the functional forms of $\alpha(\cdot)$ and $\beta(\cdot)$.
- In particular, we don't assume a square-root process for variance.

The hedge portfolio

Following closely the argument of [Wilmott][9]

- In the Black-Scholes case, there is only one source of randomness – the stock price, which can be hedged with stock.
- In the one-factor SV case, random changes in volatility also need to be hedged in order to form a riskless portfolio.
- We construct a portfolio Π containing the option being priced whose value we denote by $V(S, v, t)$, a quantity $-\Delta$ of the stock and a quantity $-\Delta_1$ of another asset whose value V_1 depends on volatility.
    - In an n-factor stochastic volatility model, we would have n different such assets in the hedge portfolio.
- The value of the hedge portfolio is given by

$$\Pi = V - \Delta \; S - \Delta_1 \; V_1$$

Evolution of the hedge portfolio

An application of Itô's Lemma gives the change in the value of this portfolio over some small time interval δt:

$$
\begin{aligned}
\delta\Pi \; = & \left\{ \frac{\partial V}{\partial t} + \frac{1}{2}\, v\, S^2 \; \frac{\partial^2 V}{\partial S^2} + \rho\, \eta\, v\, \beta\, S\, \frac{\partial^2 V}{\partial v\, \partial S} + \frac{1}{2} \eta^2 v \beta^2 \frac{\partial^2 V}{\partial v^2} \right\} \delta t \\
& - \Delta_1 \left\{ \frac{\partial V_1}{\partial t} + \frac{1}{2}\, v\, S^2\, \frac{\partial^2 V_1}{\partial S^2} \right. \\
& \qquad \left. + \rho\, \eta\, v\, \beta\, S\, \frac{\partial^2 V_1}{\partial v\, \partial S} + \frac{1}{2}\, \eta^2\, v\, \beta^2\, \frac{\partial^2 V_1}{\partial v^2} \right\} \delta t \\
& + \left\{ \frac{\partial V}{\partial S} - \Delta_1\, \frac{\partial V_1}{\partial S} - \Delta \right\} \delta S \\
& + \left\{ \frac{\partial V}{\partial v} - \Delta_1\, \frac{\partial V_1}{\partial v} \right\} \delta v
\end{aligned}
$$

Risk neutralization

To make the hedge portfolio instantaneously risk-free, we must choose

(3)

$$\frac{\partial V}{\partial S} - \Delta_1 \frac{\partial V_1}{\partial S} - \Delta = 0$$

to eliminate the δS term, and

(4)

$$\frac{\partial V}{\partial v} - \Delta_1 \frac{\partial V_1}{\partial v} = 0$$

to eliminate the δv term.

The rate of return on a risk neutral portfolio

This leaves us with

$$\begin{aligned}
\delta\Pi &= \left\{ \frac{\partial V}{\partial t} + \frac{1}{2}\, v\, S^2\, \frac{\partial^2 V}{\partial S^2} + \rho\, \eta\, v\, \beta\, S\, \frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2}\, \eta^2\, v\, \beta^2\, \frac{\partial^2 V}{\partial v^2} \right\} \delta t \\
&\quad - \Delta_1 \left\{ \frac{\partial V_1}{\partial t} + \frac{1}{2}\, v\, S^2\, \frac{\partial^2 V_1}{\partial S^2} \right. \\
&\qquad\qquad \left. + \rho\, \eta\, v\, \beta\, S\, \frac{\partial^2 V_1}{\partial v\, \partial S} + \frac{1}{2}\, \eta^2\, v\, \beta^2\, \frac{\partial^2 V_1}{\partial v^2} \right\} \delta t \\
&= r\, \Pi\, \delta t \\
&= r(V - \Delta S - \Delta_1 V_1)\, \delta t
\end{aligned}$$

where we have used the fact that the return on a risk-free portfolio must equal the risk-free rate r.

Collecting all V terms on the left-hand side and all V_1 terms on the right-hand side, and re-expressing Δ and Δ_1 using (3) and (4) we get

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2} v\, S^2\, \frac{\partial^2 V}{\partial S^2} + \rho \eta\, v\, \beta\, S \frac{\partial^2 V}{\partial v \partial S} + \frac{1}{2} \eta^2 v \beta^2\, \frac{\partial^2 V}{\partial v^2} + rS \frac{\partial V}{\partial S} - rV}{\frac{\partial V}{\partial v}}$$

$$= \frac{\frac{\partial V_1}{\partial t} + \frac{1}{2} v\, S^2\, \frac{\partial^2 V_1}{\partial S^2} + \rho \eta\, v \beta\, S \frac{\partial^2 V_1}{\partial v \partial S} + \frac{1}{2} \eta^2 v \beta^2\, \frac{\partial^2 V_1}{\partial v^2} + rS \frac{\partial V_1}{\partial S} - rV_1}{\frac{\partial V_1}{\partial v}}$$

The valuation equation

The left-hand side refers only to V and the right-hand side refers only to V_1 . The only way that this can be is for both sides to be equal to some function f of the *independent* variables S, v and t. We deduce that

(5) $$ \begin{eqnarray} &&\frac{\partial V}{\partial t}

- \frac{1}{2}\,v\,S^2\,\frac{\partial^2 V}{\partial S^2 }
- \rho\,\eta\,v\,\beta\,S\frac{\partial^2 V}{\partial v\,\partial S}
- \frac{1}{2}\,\eta^2\,v\,\beta^2\,\frac{\partial ^2 V}{\partial v^2} +r\,S\,\frac{\partial V}{\partial S}- r\,V\nonumber\ &=&- \left(\alpha - \phi\,\eta\,\beta\,\sqrt{v} \right)\,\frac{\partial V}{\partial v} \end{eqnarray} $$

where, without loss of generality, we have written the arbitrary function f of S, v and t as $\left(\alpha - \phi\,\eta\,\beta\,\sqrt{v}\right)$ where α and β are the drift and volatility functions from the SDE (2) for instantaneous variance.

The market price of volatility risk

$\phi(S, v, t)$ is called the market price of volatility risk . To see why, consider the portfolio Π_1 consisting of a delta-hedged (but not vega-hedged) option V. Then

$$\Pi_1 = V - \frac{\partial V}{\partial S}\,S$$

and again applying Itô’s Lemma,

$$\begin{aligned} \delta\Pi_1 &= \left\{ \frac{\partial V}{\partial t} + \frac{1}{2}\,v\,S^2\,\frac{\partial^2 V}{\partial S^2} + \rho\,\eta\,v\,\beta\,S\,\frac{\partial^2 V}{\partial v\,\partial S} + \frac{1}{2}\eta^2\,v\,\beta^2\,\frac{\partial^2 V}{\partial v^2} \right\}\,\delta t \\ &+ \left\{ \frac{\partial V}{\partial S} - \Delta \right\}\,\delta S + \left\{ \frac{\partial V}{\partial v} \right\}\,\delta v \end{aligned}$$

Because the option is delta-hedged, the coefficient of δS is zero and we are left with

$$\begin{aligned}
&\delta\Pi_1 - r\,\Pi_1\,\delta t\\
&= \left\{ \frac{\partial V}{\partial t} + \frac{1}{2}\,v\,S^2\;\frac{\partial^2 V}{\partial S^2} + \rho\,\eta\,v\,\beta\,S\,\frac{\partial^2 V}{\partial v\,\partial S}\right.\\
&\qquad\qquad \left. + \frac{1}{2}\eta^2\,v\,\beta^2\,\frac{\partial^2 V}{\partial v^2} - r\,S\,\frac{\partial V}{\partial S} - r\,V\right\}\,\delta t + \frac{\partial V}{\partial v}\,\delta v\\
&= \eta\,\beta\,\sqrt{v}\,\frac{\partial V}{\partial v}\,\{\phi(S,v,t)\,\delta t + \delta W\}
\end{aligned}$$

where we have used both the valuation equation (5) and the SDE (2) for v.

- The extra return per unit of volatility risk δZ_2 is given by $\phi(S,v,t)\,\delta t$
- In analogy with the Capital Asset Pricing Model, ϕ is known as the *market price of volatility risk*.

Transforming to the risk-neutral measure

Now, defining the *risk-neutral drift* as

$$\alpha' = \alpha - \eta\,\beta\,\sqrt{v}\,\phi$$

we see that as far as pricing of options is concerned, we could have started with the risk-neutral SDE for v

$$dv = \alpha'\,dt + \eta\,\beta\,\sqrt{v}\,dW$$

and got identical results with no explicit price of risk term (because we are in the risk-neutral world).

In slightly different language

Under the physical measure $\mathbb{P}$,

$$dv = \alpha\,dt + \eta\,\beta\,\sqrt{v}\,dW^{\mathbb{P}}.$$

Under the pricing measure $\mathbb{Q}$,

$$dv = \alpha\,dt + \eta\,\beta\,\sqrt{v}\,dW^{\mathbb{Q}}$$

where $dW^{\mathbb{Q}} = dW^{\mathbb{P}} - \phi(S,v,t)\,dt$.

- ϕ is the drift in the Girsanov transformation between the physical measure $\mathbb{P}$ and the pricing measure $\mathbb{Q}$.

Risk neutrality and market completeness

- In practical applications of SV models, we always assume that the SDEs for S and v are in risk-neutral terms because we are invariably interested in fitting models to option prices.
  - We effectively impute the risk-neutral measure by fitting the parameters of the process we are imposing.
  - We need n options in the hedge portfolio for an n-factor SV model.
- Only if we were interested in the connection between the pricing of options and the behavior of the time series of historical returns of the underlying, would we need to understand the connection between the statistical measure under which the drift of the variance process v is α and the risk-neutral process under which the drift of the variance process is α' .
- Academics often say that SV models are not complete. This is true only if there are no traded derivative securities.

Solving the valuation equation

Recall the valuation equation for a one-factor SV model (with $r = 0$ for simplicity):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\, v\, S^2\, \frac{\partial^2 V}{\partial S^2} + \rho\, \eta\, v\, \beta\, S \frac{\partial^2 V}{\partial v\, \partial S} + \frac{1}{2}\, \eta^2\, v\, \beta^2\, \frac{\partial^2 V}{\partial v^2} + \alpha\, \frac{\partial V}{\partial v} = 0$$

How easy this is to solve depends on $\alpha(\cdot)$, $\beta(\cdot)$ and the boundary conditions.

- Fast solutions for European options are needed for calibration to market option prices.
- The boundary condition for a European call option is

  $$V(S, K, t, T)|_{t=T} = (S - K)^+ .$$

- For general payoffs (*i.e.* boundary conditions), we solve the valuation equation using numerical PDE or Monte Carlo techniques.
- This is too slow for efficient calibration.
- Models for which fast exact or approximate European option solutions exist are consequently the most popular.

Examples of stochastic volatility models

The Heston model

In the Heston model,

$$\alpha = -\lambda\,(v - \bar{v});\ \beta = 1$$

So that (again with $r = 0$)

$$\begin{aligned} dS_t &= \sqrt{v_t}\, S_t\, dZ_1 \\ dv_t &= -\,\lambda\,(v - \bar{v})\,dt + \eta\,\sqrt{v_t}\, dZ_2 \end{aligned}$$

with

$$\langle dZ_1\ dZ_2 \rangle = \rho\, dt$$

The corresponding valuation equation with European boundary conditions may be solved using Fourier techniques leading to a quasi-closed form solution – the famous Heston formula.

The SABR model

The SABR model is usually written in the form

$$\begin{aligned} dS_t &= \sigma\, S_t^{\beta}\, dZ_1 \\ d\sigma_t &= \alpha\,\sigma\, dZ_2 \end{aligned}$$

with

$$\langle dZ_1\ dZ_2 \rangle = \rho\, dt$$

Hence the name ``stochastic alpha beta rho model".

- Note that this formulation is in general inconsistent with our original formulation [(1)](#eq1:genstockprocess) because the stock price is conditionally lognormal only if $\beta = 1$. We get the CEV model in the limit $\alpha \to 0$.
- There is an accurate asymptotic formula for BS implied volatility (the SABR formula) in terms of the parameters of the model permitting easy calibration to the volatility smile.

Extension to n factors

- In a one factor SV model, given the stock price S and the market price of *only one option*, in principle we know the prices of all options.
  - This is an extreme assumption.
- It may be desirable (essential if jointly calibrating to the prices of volatility derivatives) to include more factors. For example, in the Double Mean Reverting (DMR) model [Bayer, Gatheral and Karlsmark][2]:

  \begin{eqnarray} *dS&=&\sqrt{v_1}\,S\,dZ_1\ dv_1&=&-\kappa_1\,(v_1-v_2)\,dt+\eta_1\,{v_1}^{\alpha_1}\,dZ_2\ dv_2&=&-\kappa_2\,(v_2-\bar v)\,dt+\eta_2\,{v_2}^{\alpha_2}\,dZ_3 \end{eqnarray}*

  with

$$\langle dZ_i\,dZ_j\rangle = \rho_{ij}\,dt$$

```
-   In the DMR model, we know all option prices given the stock price a
nd the prices of two options.
```

Local Volatility

Local volatility: Historical development

- Stochastic volatility models are hard to handle
  - difficult to compute with and hard (impossible) to calibrate.
- Practitioners need a simple way of pricing exotic options consistently with the volatility skew.

Risk-neutral density from prices

The marginal risk-neutral density may be recovered by twice differentiating European call prices wrt strike: $p(K) = \frac{\partial^2 C}{\partial K^2}$.

Proof.

$$C(K) = \mathbb{E}[(S_T - K)^+]$$

so

$$-\frac{\partial C}{\partial K} = \mathbb{E}[\theta(S_T - K)] = \Pr(S_T > K)$$

and

$$\frac{\partial^2 C}{\partial K^2} = \mathbb{E}[\delta(S_T - K)] = p(K).$$

The trader version: The butterfly

- Consider the payoff of the following option combination:
  - Long a call struck at $K + \Delta K$,
  - short 2 calls struck at K,
  - long a call struck at $K - \Delta K$.

In [25]:
```
callPayoff <- function(x,K){pmax(x-K,0)}
putPayoff <- function(x,K){pmax(K-x,0)}

butterfly <- function(x){callPayoff(x,.99)-2*callPayoff(x,1)+callPa
yoff(x,1.01)}

curve(butterfly(x),from=.8,to=1.2,ylim=c(-0.005,0.015),col="red",xl
ab="Strike", ylab="Payoff")
```

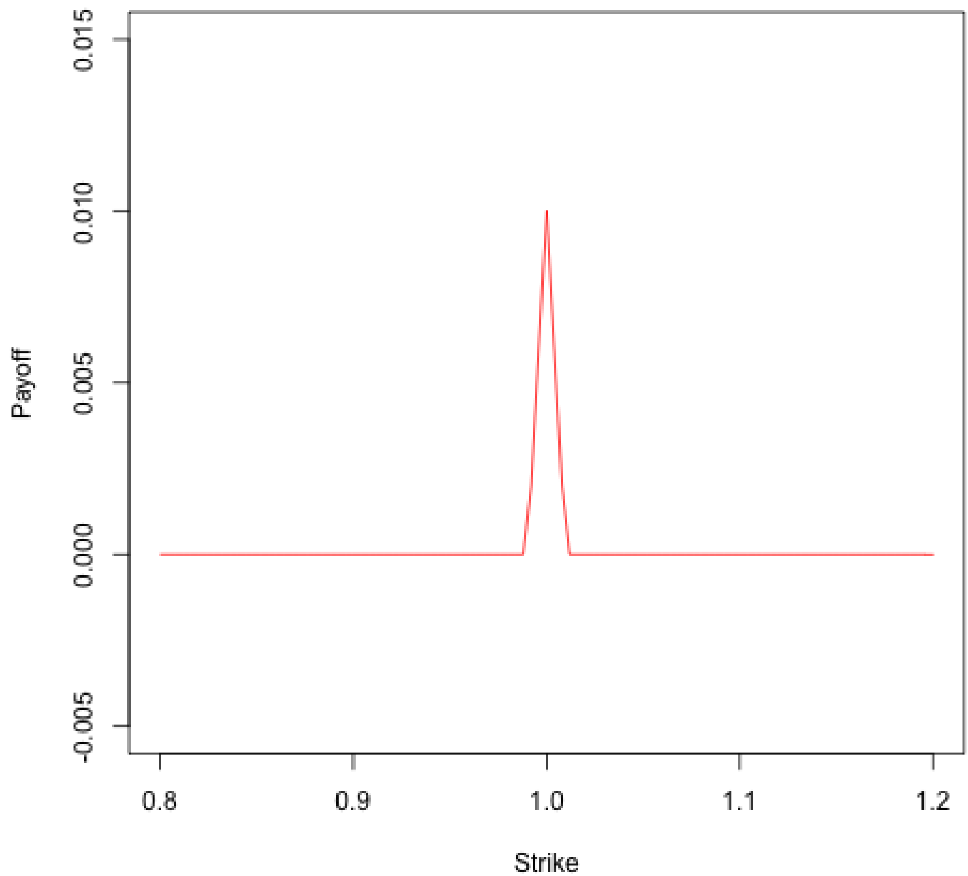

Figure 5: Payoff of a butterfly with strikes $\{0.99, 1.00, 1.01\}$.

The key idea

- For as long as options have been traded (probably), it has been understood that the risk-neutral density could be derived from the market prices of European options.
  - This is the famous Breeden-Litzenberger result.
- The breakthrough came when [Dupire][5]</sup> and [Derman and Kani][3] noted that under risk-neutrality, there was a unique diffusion process consistent with these distributions.
  - The corresponding unique state-dependent diffusion coefficient $\sigma_L(S,t)$ consistent with current European option prices is known as the *local volatility function*.

Local volatility: Philosophy

- It is unlikely that Dupire, Derman and Kani (DDK) ever thought of local volatility as representing a model of how volatilities actually evolve.
  - [Dumas, Fleming and Whaley][4]</sup> confirmed empirically that the dynamics of the implied volatility surface were not consistent with the assumption of constant local volatilities.
- They likely thought of local volatilities as representing some kind of average over all possible instantaneous volatilities in a stochastic volatility world (an “effective theory”).
  - We will prove a precise version of this idea later in this lecture, formalizing the DDK intuition.
- Local volatility is the simplest extension of Black-Scholes that allows practitioners to price exotic options consistently with the known prices of vanilla options.

A Brief Review of Dupire’s Work

- Given T-expiration European call prices $\{C(S_0,K,T)\}$, the risk neutral density function φ of the final spot S_T is given by

$$\varphi(K,T;S_0) = \frac{\partial^2 C}{\partial K^2}.$$

- Dupire showed that there is a unique risk-neutral diffusion process which generates a given marginal distribution of final spot prices S_T for each time T, conditional on some starting spot price S_0.
  - It has become the fashion in the literature to credit Gyöngy with this result.
- Equivalently, given the set of all European option prices, we may determine the functional form of the diffusion parameter (local volatility) of the unique risk neutral diffusion process which generates these prices.

The Fokker-Planck equation

Recall from Lecture 1 that if

$$dX_t = \mu(X_t,t)\,dt + \sigma(X_t,t)\,dW_t,$$

the density p evolves as

$$\frac{dp(x,t)}{dt} = -\frac{\partial}{\partial x}(\mu(x,t)\,p(x,t)) + \frac{1}{2}\,\frac{\partial^2}{\partial x^2}\left(\sigma^2(x,t)\,p(x,t)\right).$$

Thus, if the stock price evolves according to the local volatility SDE

$$\frac{dS_t}{S_t} = \mu_t\,dt + \sigma\left(S_t,t\right)\,dW_t,$$

the density evolves as

$$\frac{\partial p}{\partial T} = \frac{1}{2}\frac{\partial^2}{\partial S_T^2}\left(\sigma^2 S_T^2\,p\right) - \frac{\partial}{\partial S_T}(\mu_T\,S_T\,p)\,.$$

Derivation of the Dupire Equation

Suppose the stock price diffuses with risk-neutral drift $\mu_t\,(= r_t - D_t)$ and local volatility $\sigma\,(S,t)$ according to the equation:

$$\frac{dS_t}{S_t} = \mu_t\,dt + \sigma\left(S_t,t\right)\,dZ_t$$

The undiscounted risk-neutral value $C\left(S_t,K,T\right)$ of a European option with strike K and time to expiration T is given by

(6)

$$C\left(S_t,K,T\right) = \int_K^\infty dS_T\,p\left(S_T,T\right)\,\left(S_T-K\right)$$

Then, differentiating (6) with respect to T and applying the Fokker-Planck equation gives

$$\begin{aligned}\frac{\partial C}{\partial T} &= \int_K^\infty dS_T \, \left\{ \frac{\partial}{\partial T} p \, (S_T, T) \right\} (S_T - K) \\ &= \int_K^\infty dS_T \, \left\{ \frac{1}{2} \frac{\partial^2}{\partial S_T^2} \left(\sigma^2 S_T^2 \, p \right) - \frac{\partial}{\partial S_T} (\mu_T \, S_T \, p) \right\} (S_T - K)\end{aligned}$$

Integrating by parts gives:

(7)

$$\frac{\partial C}{\partial T} = \frac{\sigma^2 \, K^2}{2} \, p + \int_K^\infty dS_T \, \mu_T \, S_T \, p \; = \frac{\sigma^2 \, K^2}{2} \, \frac{\partial^2 C}{\partial K^2} + \mu_T \, \left(C - K \, \frac{\partial C}{\partial K} \right)$$

which is the Dupire equation when the underlying stock has risk-neutral drift μ_t.

Under the forward measure

As of time t, the time T forward price of the stock is given by

$$F_T = S_t \, \exp \left\{ \int_t^T \mu_s \, ds \right\} .$$

It is then straightforward to verify using Itô's Lemma that

$$dF_t = F_t \, \sigma(S_t, t) \, dt,$$

which with some abuse of notation, we may rewrite as

$$dF_t = F_t \, \sigma(F_t, t) \, dt.$$

In that case (or equivalenty if were to set rates and dividends to zero), we would get the get the same expression as (7) without the drift term. That is

$$\frac{\partial C}{\partial T} = \frac{1}{2} \, \sigma^2 \, K^2 \, \frac{\partial^2 C}{\partial K^2}$$

where C now represents $C \, (F_T, K, T)$.

Inverting this gives the Dupire equation in its simplest form:

The Dupire equation

(8)

$$\sigma^2(K,T,S_0) \;=\; \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}\;K^2\;\frac{\partial^2 C}{\partial K^2}}$$

- The right hand side of equation [(8)](#eq8:localvoldef) can in principle be computed from known European option prices.
  - Given a complete set of European option prices for all strikes and expirations, local volatilities are given uniquely by equation [(8)](#eq8:localvoldef).
  - In words, “local variance is given by the ratio of a calendar spread to a butterfly”.
- We can view equation [(8)](#eq8:localvoldef) as a *definition* of the local volatility function regardless of what kind of process (stochastic volatility for example) actually governs the evolution of volatility.

Local volatility in terms of implied volatility

Market prices of options are quoted in terms of Black-Scholes implied volatility $\sigma_{BS}\left(K,T;S_0\right)$. In other words, we may write

$$C\left(S_0,K,T\right)=C_{BS}\left(S_0,K,\sigma_{BS}\left(S_0,K,T\right),T\right)$$

It will be more convenient for us to work in terms of two dimensionless variables: the Black-Scholes *implied total variance* w defined by

$$w\left(S_0,K,T\right):=\sigma_{BS}^2\left(S_0,K,T\right)\,T$$

and the log-strike k defined by

$$k=\log\left(\frac{K}{F_T}\right)$$

where $F_T=S_0\,\exp\left\{\int_0^T\,\mu_t\,dt\right\}$ is the forward price.

Black-Scholes formula in terms of log-strike and total variance

In terms of these variables, the Black-Scholes formula for the undiscounted call price becomes

(9) \begin{eqnarray*} C_{BS} &=& F_T\, \left{N\left(d_1 \right) - e^k\, N\left(d_2 \right) \right}\ &=& F_T\, \left{ N \left(-\frac{k}{\sqrt{w}} +\frac{ \sqrt{w}}{2} \right)

- e^k \, N \left(-\frac{k}{ \sqrt{w} } -\frac{ \sqrt{w} }{2} \right) \right} \end{eqnarray*}

and the Dupire equation (7) becomes

(10) $$ \frac{\partial C}{\partial T} = \frac{v_L }{2}\,\left{ \frac{\partial ^2 C}{\partial k^2 } - \frac{\partial C}{\partial k} \right}

- \mu_T\, \,C $$

with $v_L=\sigma^2\left(S_0,K,T\right)$ representing the local variance.

Now, by taking derivatives of the Black-Scholes formula (9), we obtain

(11) \begin{eqnarray} \frac{\partial ^2 C{BS} }{\partial w^2 } &=& \left(- \frac{1}{8} - \frac{1}{2\,w} + \frac{k^2}{2\,w^2} \right)\, \frac{\partial C}{BS} }{\partial w} \nonumber\ \frac{\partial ^2 C{BS}}{\partial k \partial w} &=& \,\left(\frac{1}{2} - \frac{k}{w}\right)\, \frac{\partial C}{BS}}{\partial w}\nonumber\ \frac{\partial ^2 C{BS} }{\partial k^2 } -\frac{\partial C}{BS} }{\partial k} &=& 2\,\frac{\partial C_{BS} } {\partial w} \end{eqnarray}

Dupire equation in terms of implied variance

We may transform equation (10) into an equation in terms of implied variance by making the substitutions

$$\begin{aligned}
\frac{\partial C}{\partial k} &= \frac{\partial C_{BS}}{\partial k} + \frac{\partial C_{BS}}{\partial w}\,\frac{\partial w}{\partial k} \\
\frac{\partial^2 C}{\partial k^2} &= \frac{\partial^2 C_{BS}}{\partial k^2} + 2\,\frac{\partial^2 C_{BS}}{\partial k \partial w}\,\frac{\partial w}{\partial k} + \frac{\partial^2 C_{BS}}{\partial w^2}\left(\frac{\partial w}{\partial k}\right)^2 + \frac{\partial C_{BS}}{\partial w}\,\frac{\partial^2 w}{\partial k^2} \\
\frac{\partial C}{\partial T} &= \frac{\partial C_{BS}}{\partial T} + \frac{\partial C_{BS}}{\partial w}\,\frac{\partial w}{\partial T} = \frac{\partial C_{BS}}{\partial w}\,\frac{\partial w}{\partial T} + \mu_T\, C_{BS}
\end{aligned}$$

where the last equality follows from the fact that the only explicit dependence of the option price on T in equation (9) is through the forward price $F_T = S_0 \exp\left\{\int_0^T \mu_t\, dt\right\}$.

Canceling $\mu_T\, C$ terms on each side and using (11), equation (7) now becomes

$$\begin{aligned}
&\frac{\partial C_{BS}}{\partial w}\,\frac{\partial w}{\partial T} \\
&= \frac{v_L}{2}\left\{-\frac{\partial C_{BS}}{\partial k} + \frac{\partial^2 C_{BS}}{\partial k^2} - \frac{\partial C_{BS}}{\partial w}\,\frac{\partial w}{\partial k} + 2\,\frac{\partial^2 C_{BS}}{\partial k\,\partial w}\,\frac{\partial w}{\partial k} + \right. \\
&\quad \left.\frac{\partial^2 C_{BS}}{\partial w^2}\left(\frac{\partial w}{\partial k}\right)^2 + \frac{\partial C_{BS}}{\partial w}\,\frac{\partial^2 w}{\partial k^2}\right\} \\
&= \frac{v_L}{2}\,\frac{\partial C_{BS}}{\partial w}\left\{2 - \frac{\partial w}{\partial k} + 2\left(\frac{1}{2} - \frac{k}{w}\right)\frac{\partial w}{\partial k}\right. \\
&\quad \left. + \left(-\frac{1}{8} - \frac{1}{2\,w} + \frac{k^2}{2\,w^2}\right)\left(\frac{\partial w}{\partial k}\right)^2 + \frac{\partial^2 w}{\partial k^2}\right\}
\end{aligned}$$

Local variance in terms of implied variance

Then, taking out a factor of $\frac{\partial C_{BS}}{\partial w}$ and simplifying, we get

$$\frac{\partial w}{\partial T} = v_L \left\{ 1 - \frac{k}{w}\frac{\partial w}{\partial k} + \frac{1}{4}\left(-\frac{1}{4} - \frac{1}{w} + \frac{k^2}{w^2}\right)\left(\frac{\partial w}{\partial k}\right)^2 + \frac{1}{2}\frac{\partial^2 w}{\partial k^2} \right\}$$

Inverting this and rearranging slightly gives our final result:

Local volatility from implied volatility

(12)

$$v_L(k,T) = \frac{\frac{\partial w}{\partial T}}{\left(1 - \frac{1}{2}\frac{k}{w}\frac{\partial w}{\partial k}\right)^2 - \frac{1}{4}\left(\frac{1}{4} + \frac{1}{w}\right)\left(\frac{\partial w}{\partial k}\right)^2 + \frac{1}{2}\frac{\partial^2 w}{\partial k^2}}$$

Special Case: No Skew

If w is a function of T only and not of k (so no skew and no smile), then we must have

$$v_L = \frac{\partial w}{\partial T}$$

and the local variance reduces to the forward Black-Scholes implied variance. The solution to this is of course

$$w\left(T\right) = \int_0^T v_L\left(t\right)\, dt$$

The Berestycki-Busca-Florent (BBF) formula

Following [Berestycki, Busca and Florent][1], suppose $\sigma_{BS}(k,T) = \sigma_0(k) + \mathcal{O}(T)$. Then, to zeroth order in T, formula (12) reads:

(13)

$$\sigma(k,t)^2 = v_L(k,t) = \frac{\sigma_0^2(k) + \mathcal{O}(t)}{\left(1 - \frac{k}{2\,w}\,\partial_k w\right)^2 + \mathcal{O}(t)}.$$

Noting that

$$1 - \frac{k}{2\,w}\,\partial_k w = \sqrt{w}\,\frac{\partial}{\partial k}\left(\frac{k}{\sqrt{w}}\right) = \sigma_0(k)\,\frac{\partial}{\partial k}\left(\frac{k}{\sigma_0(k)}\right),$$

we may rearrange (13) to give (to zeroth order in t)

$$\frac{\partial}{\partial k}\left(\frac{k}{\sigma_0(k)}\right) = \frac{1}{\sigma(k,0).}$$

Integrating, we arrive at the solution:

The BBF formula

$$\frac{1}{\sigma_0(k)} = \frac{1}{k}\int_0^k \frac{dy}{\sigma(y,0)} = \int_0^1 \frac{d\alpha}{\sigma(\alpha\,k,0)}$$

The heat-kernel most-likely-path formula

In a more recent paper, [Gatheral and Wang][8] derived the following fancier approximation:

$$\frac{k^2}{\sigma_{BS}^2} \approx T \int_0^T \left[\frac{\dot{x}(t)}{\sigma(x(t),t)}\right]^2 dt$$

where the *variational most likely path (variational MLP)* $x(t)$ is the solution of the following variational problem:

(14)

$$\begin{cases} -\frac{d}{dt}\left[\frac{\dot{x}(t)}{\sigma(x,t)}\right] + \frac{\partial_t \sigma(x,t)}{\sigma^2(x,t)}\dot{x}(t) = 0, \\ x(0) = 0, \quad x(T) = k. \end{cases}$$

This formula reduces to the BBF formula when the local volatility function $\sigma(x,t)$ is time-homogeneous and is much more accurate otherwise.

A simplification

For simplicity, in what follows, we will assume zero interest rates and zero dividends.

Local variance as a conditional expectation of instantaneous variance

Following Derman and Kani, write

$$C\left(S_0, K, T\right) \; = \; \mathbb{E}\left[(S_T - K)^+\right]$$

As before, differentiating twice with respect to K gives

$$\frac{\partial^2 C}{\partial K^2} = \mathbb{E}\left[\delta\left(S_T - K\right)\right]$$

where $\delta(\cdot)$ is the Dirac δ function. Now, a formal application of Itô's Lemma to the terminal payoff of the option gives

(15)

$$d(S_T - K)^+ = \theta\left(S_T - K\right)\, dS_T + \frac{1}{2}\, v_T\, S_T^2\, \delta\left(S_T - K\right)\, dT$$

Taking conditional expectations of each side, and using the fact that S_T is a martingale, we get

$$dC = d\mathbb{E}\left[(S_T - K)^+\right] = \frac{1}{2}\,\mathbb{E}\left[v_T\, S_T^2\, \delta\,(S_T - K)\right]\, dT$$

Also, denoting the density of the final density by $p(S_T, T)$, we can write

$$\begin{aligned}\mathbb{E}\left[\, v_T S_T^2\, \delta\,(S_T - K)\right] &= \mathbb{E}\left[v_T\, |S_T = K\right]\, K^2\, \mathbb{E}\left[\, \delta\,(S_T - K)\right] \\ &= \mathbb{E}\left[v_T\, |S_T = K\right]\, K^2\, p(K, T) \\ &= \mathbb{E}\left[v_T\, |S_T = K\right]\, K^2\, \frac{\partial^2 C}{\partial K^2}\end{aligned}$$

Putting this together, we get

(16)

$$\frac{\partial C}{\partial T} = \mathbb{E}\left[v_T\, |S_T = K\right]\, \frac{1}{2} K^2\, \frac{\partial^2 C}{\partial K^2}$$

Comparing (16) with the definition (8) of local volatility, we get

Local variance as conditional expectation

$$\sigma^2(K, T, S_0) = \mathbb{E}\left[v_T\, |S_T = K\right]$$

That is, local variance is the risk-neutral expectation of the instantaneous variance conditional on the final stock price S_T being equal to the strike price K.

Tanaka’s formula

Sometimes in the literature, we see (15) written in integral form as

$$\begin{aligned}&(S_T - K)^+ \\ &= (S_0 - K)^+ + \int_0^T \mathbf{1}_{S_t > K}\, dS_t + \frac{1}{2} \int_0^T v_t\, S_t^2\, \delta\,(S_t - K)\, dt\end{aligned}$$

This is a form of Tanaka’s formula. It expresses the payoff of a European call as the proceeds of a stop-loss start-gain hedging strategy plus local time.

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In [7]:
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