REVISION : Week 21

Logic & Problem Solving

Relations , Functions, Permutation & Combination and Probability

Amazing Coin Tricks...

Schedule for Class test - 2

EXAM DATE: Sunday - Friday, 31 March - 5 April 2024

EXAM TIME : 3:15 pm to 4:30 pm

Revision : Relations

The Cartesian Product of Two Sets

Let **A** and **B** are two sets. The **Cartesian product** of sets **A** and **B** is denoted by **$A \times B$** and is the set of ordered pairs given by,

$$A \times B = \{ (a, b) : a \in A \text{ and } b \in B \}$$

Example:

If **A ={a, b, c}** and **B = {d}** then,

$A \times B = \{(a, d), (b, d), (c, d)\}$

$B \times A = \{(d, a), (d, b), (d, c)\}$

$A \times A = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$

Definition of Relation:

Let **A** be a non-empty set. Any **subset** **R** of the Cartesian product **A x A** is called a **relation** on the set **A**.

Notation:

If R is a relation on A (i.e., $R \subseteq A \times A$) and

if $(x, y) \in R$ then we say, 'x is related to y by R' and we write x R y.

if $(x, y) \notin R$ we write $x \not R\, y$

Examples...

Example 1:

Let A = {1, 2, 3, 4, 5}. Then

R = {(1, 2), (4, 2), (3, 5)}

S = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)} and

T = {(3,4)} are all relations on A.

We could write 1R2, 4R2 and 3R5

Similarly, 1S1,2S2 etc.

Example (Contd.):

Example 2 :

Let A = {1, 2, 3, 4}

We can define a subset R of A x A; and hence a relation on A, by

R = { (a , b): a $\in$ A, b $\in$ A, a$\leq$ b}

This is called the "is less than equal to" relation on A.

Another way of describing this relation is simply by saying ‘a R b if a $\leq$ b’.

In set listing notation

R = { (1,1) ,(1,2) (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}

Pictorial representation of relations.

Often the best way of illustrating the structure and properties of a relation is by means of a **diagrammatic** representation of the relation.

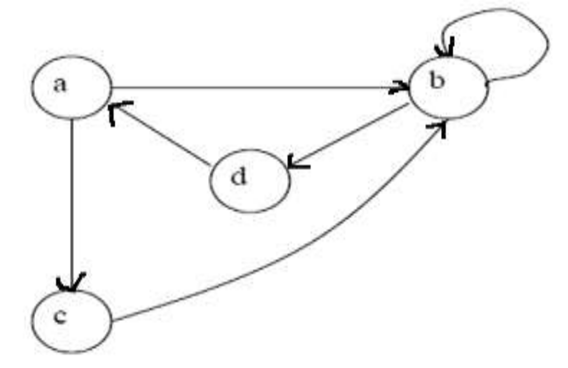
1. **Digraphs**

2. **Matrix Representation**

Digraphs:

In discrete mathematics a **Graph** is a set of points (called **Vertices**) some of which are connected by lines (or Arcs) called **Edges**.

For example:



Here there are the four vertices a, b, c and d with edges as shown.

Note that there is an edge joining b to itself. Such an edge is called a **Loop**.

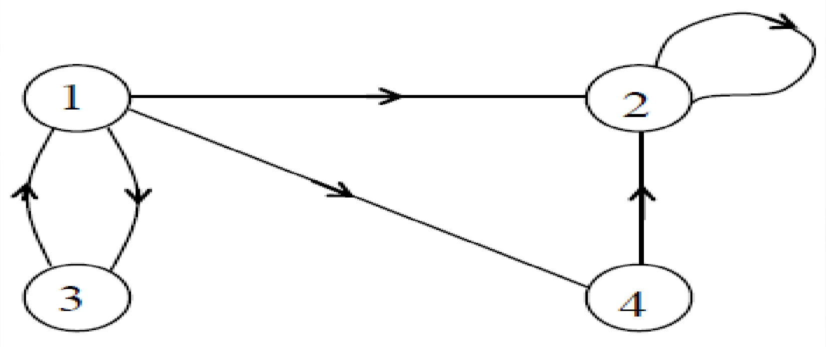
Digraphs (Contd.):

For example:

Let A = {1, 2, 3, 4} and

R = {(1, 2), (1, 3), (1, 4), (2, 2), (3, 1), (4, 2)}

The digraph of R is then:



Matrix Representation:

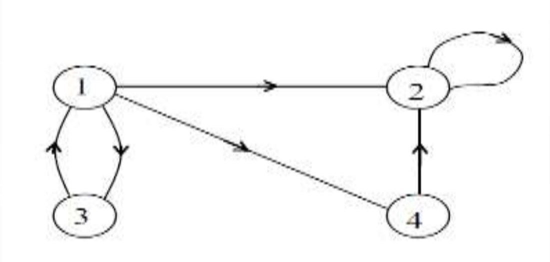
If R is a relation on a finite set A, with |A| = n, we construct an **n x n** matrix representation of R as follows:

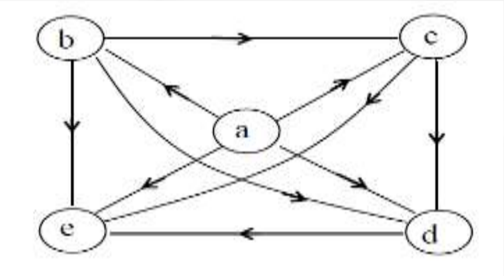
- Label the rows and columns of an n x n matrix M by the elements of A.
- For all ordered pairs (x, y) $\in$ A x A locate the unique position in M defined by row x and column y.
- Enter a **1** in this position if x R y and a **0** otherwise.

  We call **M** the matrix of the relation R or the relation matrix for R.

Matrix Representation (Contd.):

The two relations from above examples have the following relation matrices:





$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Some special relations:

1. Reflexive Relations
2. Symmetric Relations
3. Transitive Relations
4. Equivalence Relations
5. Anti-symmetric Relations
6. Asymmetric Relations
7. Irreflexive Relations

Revision Questions - Relations:

Question 1 :

For each of the following relations on set A = {1, 2, 3},

Write **digraph** and **matrix** representation .Also Check all the **seven** properties.

R = {(1,1) ,(1,2) , (1,3) , (3,3)}

T = { (1,1) ,(1,2) , (2,1) , (2,2),(3,3)}

P= {(1, 1), (2, 2), (3, 3)}

Question 2 :

Let A be the set {1, 2, 3, 4}

Describe the following relations as lists of ordered pairs of elements of relations and Check all the **seven properties**.

R_1= {(x,y) : X $\in$ A, y $\in$ A, x > y}

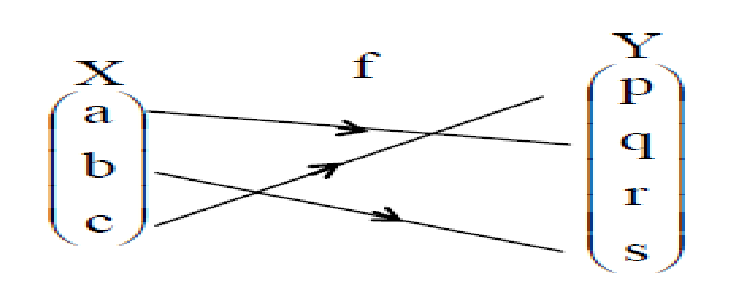
R_2 = {(x,y) :X $\in$ A, y $\in$ A , x + y is even}

R_3 = {(x,y) : x $\in$ A, y $\in$ A, x is divisible by y }

Revision : Functions

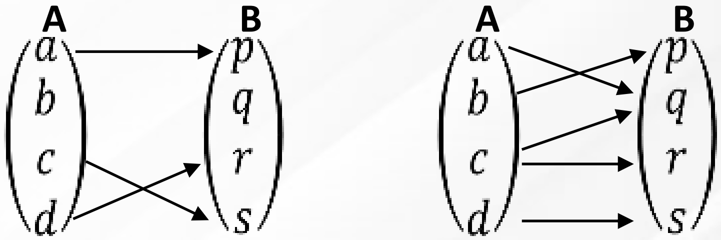
INTRODUCTION TO FUNCTIONS:

- A **function** is an association of exactly one object from one set (the **range**) with each object from another set (the **domain**).
- This means there must be **at least one arrow** leaving each point in the domain.
- Also, that there can be **no more than one arrow** leaving each point in the domain



Examples (Contd.):

Example **:** Neither of the diagrams



provide proper function definitions since

(i) f(b) is not defined

(ii) f(c) is not uniquely defined.

Composite Functions (1):

- Often one quantity is a function of a second quantity that depends, in turn, on a third quantity.

- **For example,** the cost of car trip is a function of the gasoline consumed. The amount of gasoline consumed, in turn, is a function of the number of miles driven.

- The chains of dependence are known as **Composition of functions.**

Composite Functions (2):

- Suppose that y = f(x) and y = g(x) define two functions.
- A value of x inserted in function g will produce an output g(x).
- This output g(x) is then inserted in function f to produce an output f(g(x))
- The final outcome f(g(x)) is written as,

$$(f \circ g)(x)$$

Composite function

Composite Functions (3):

To illustrate this, let f(x) = 5x + 1 and g(x) = 4x – 3

By definition,

$$(f \circ g)(x) = f(g(x))$$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(4x-3) \\ &= 5(4x-3)+1 \\ &= 20x-14\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(5x+1) \\ &= 4(5x+1)-3 \\ &= 20x+1\end{aligned}$$

$$(f \circ g)(x) \neq (g \circ f)(x)$$

Composite Functions (4):

Example: If $f(x) = 2x^2 + 1$ and $g(x) = x + 1$, then find $(f \circ g)(x)$ and $(f \circ g)(3)$

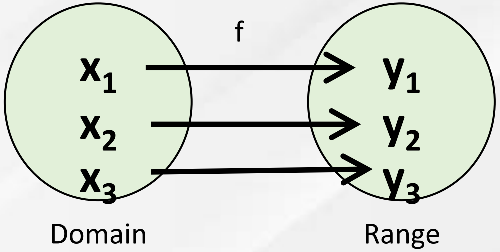
By definition,

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x+1) \\ &= 2(x+1)^2 + 1 \\ &= 2(x^2 + 2x + 1) + 1 \\ &= 2x^2 + 4x + 3\end{aligned}$$

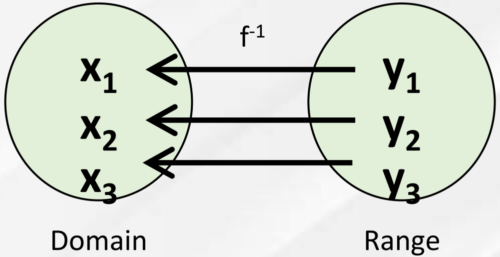
$$\begin{aligned}(f \circ g)(3) &= 2(3)^2 + 4(3) + 3 \\ &= 33\end{aligned}$$

Inverse Functions (1):

- Function **f** maps each number in the domain of **f** to the corresponding number in the range of **f**.



- **Inverse function**, symbol $\mathbf{f^{-1}}$, reverses the correspondence and maps each number in the range of **f** to the number in the domain.



Inverse Functions (2):

- To find f^{-1}, we follow the following steps:

1. Replace f(x) with y
2. Interchange variables x and y
3. Solve the resulting equation for y
4. Replace y with $f^{-1}(x)$

Inverse Functions (3):

Example: Find the inverse of $f(x) = 3x + 2$.

- Step 1: Replace $f(x)$ with y.

$$y = 3x + 2$$

- Step 2: Interchange variables y and x.

$$x = 3y + 2$$

- Step 3: Solve the resulting equation for y.

$$y = (x - 2)/3$$

- Step 4: Replace y with $f^{-1}(x)$.

$$f^{-1}(x) = (x - 2)/3$$

Inverse Functions (4):

- From the above example,

$$f(x) = 3x + 2 \quad \text{gives} \quad f^{-1}(x) = \frac{x-2}{3}$$

Consider,

$$\begin{aligned}(f \circ f^{-1})(x) &= f(f^{-1}(x)) \\ &= 3\left(\frac{x-2}{3}\right)+2 \\ &= x\end{aligned}$$

$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\ &= \frac{(3x+2)-2}{3} \\ &= x\end{aligned}$$

Hence,

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$$

If the inverse of any function f (x) is again a function, then f(x) is known as **Invertible Function**.

Revision questions - functions

Question 1:

Find the inverse of the following functions.

a. $f(x) = 2x - 1$

b. $f(x) = 4x^2 - 1$

c. $f(x) = \frac{x}{x-2}$

Question 2:

Functions $f: R \rightarrow R$ and $g: R \rightarrow R$ are defined by, $f(x) = 1+2x$, $g(x) = x^2-1$.

Find,

a. $(g \circ f)(x)$ b. $(f \circ g)(x)$ c. $(f \circ f)(x)$ and d. $(g \circ g)(x)$

Revision : Permutations & Combinations

Permutations

Permutation is an arrangement of items in a particular order.

ORDER MATTERS

To find the number of **Permutations** of n items, we can use the Fundamental Counting Principle or factorial notation.

Permutations

The **number of ways to arrange** the letters ABC:

_____ _____ _____

Number of choices for first blank? 3

Number of choices for second blank? 3 2

Number of choices for third blank? 3 2 1

$$3 \times 2 \times 1 = \mathbf{6\ ways}$$

ABC ACB BAC BCA CAB CBA

Permutations

To find the **number of Permutations** of **n** items chosen **r** at a time, you can use the formula

$$_n p_r = \frac{n!}{(n-r)!} \quad \text{where } 0 \le r \le n.$$

$$_5 p_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5.4.3 = 60$$

Combinations

Combination is an arrangement of items in which order does not matter.

ORDER DOES NOT MATTERS!!!

Since the order does not matter in combinations, there are **fewer combinations than permutations**.

The combinations are a **"subset "**of permutations.

Combinations

To find the **number of Combinations** of **n** items chosen **r** at a time, you can use the formula

$$_nC_r = \frac{n!}{r!(n-r)!} \quad \text{where } 0 \le r \le n.$$

$$_5C_3 = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} =$$

$$\frac{5.4.3.2.1}{3.2.1.2.1} = \frac{5.4}{2.1} = \frac{20}{2} = 10$$

Revision question – Permutation & Combination

Questions:

a) A club consisting of eight people must choose a president, vice-president, and a secretary. How many different arrangements are possible?

b) A student has to answer 10 questions, choosing at least 4 from each of Parts A and B. If there are 6 questions in Part A and 7 in Part B, in how many ways can the student choose 10 questions?

c) A group consists of 4 girls and 7 boys. In how many ways can a team of 5 members be selected if the team has

(i) no girls

(ii) at least three girls.

Revision question – Permutation & Combination

Questions:

d) In how many different ways can a committee of three people be selected from a total of eight people?

e) An ice cream parlour has 15 different flavours. George orders a sundae and has to select 3 flavours. How many different selections are possible?

f) In a company, Out of 20 candidates 14 men and 6 women apply for two vacancies. What is the probability that

(i) both men are selected? (ii) Both women are selected?

(iii) One man and one woman are selected?

Revision: Probability Theory

Classical definition of Probability

Definition :

The probability of an event **E** is denoted by **P(E)** and is given by

$$\text{Probability of event E, } P(E) = \frac{\textbf{Favorable Number}}{\textbf{Exhaustive Number}}$$

Basic Properties of Probabilities

<u>Property 1:</u>

The probability of an event is always **between 0 and 1, inclusive.**

<u>Property 2:</u>

The probability of an event that cannot occur is **0**. (An event that cannot occur is called an ***impossible event.***)

<u>Property 3:</u>

The probability of an event that must occur is **1**. (An event that must occur is called a ***certain event.***)

Basic Probability Examples...

Example :

If a single fair coin is tossed, what is the probability that it will land heads up?

Solution:

Exhaustive number = 2

Favorable number = 1

Probability of landing on its top $= \dfrac{Favourable\ number}{Exhaustive\ number} = \dfrac{1}{2}$

Basic Probability Examples…

Example :

Rolling a fair die, what is the probability of each event?

a) The number 3 is rolled.

b) A number not 3 is rolled.

c) The number 9 is rolled.

d) A number < 7 is rolled.

e) A prime number is rolled.

Basic Probability Examples...

Example :

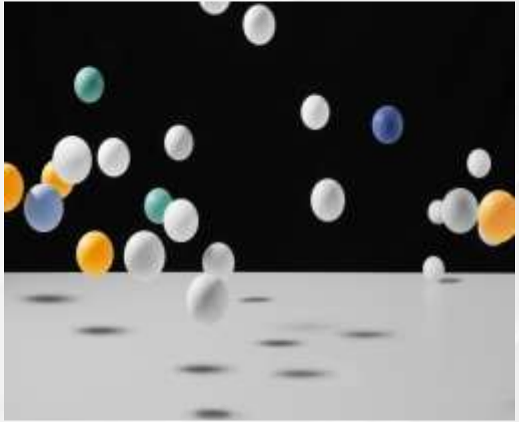
A bag contains 30 balls numbered 1 to 30 inclusive. A ball is drawn at random from the bag what is the probability of drawing

a) An even numbered ball.

b) A ball which is multiple of 7 .

c) A ball which is multiple of 3 or 7 .

d) A prime numbered ball.



Additional Theorem on Probability

Statement :

If A and B are any two events(**not mutually exclusive**), then the probability of occurrence of at least one of them is denoted by **P(A or B)** or **P(A U B)** and is given by

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

If A and B are **mutually exclusive**, then

$$P(A \text{ or } B) = P(A) + P(B)$$

Additional Theorem on Probability

Example:

What is the probability of drawing a king or a diamond from a standard deck of 52 cards?

P(king or diamond) =P(K) + P(D) – P(K and D)

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Additional Theorem on Probability

Example:

What is the probability of a 2 or an odd number being rolled on a fair die?

Mutually exclusive events

$$P(2 \text{ or odd}) = P(2) + P(\text{odd})$$

$$= \frac{1}{6} + \frac{3}{6}$$

$$= \frac{2}{3}$$

Multiplicative theorem on Probability

Statement :

If A and B are any two independent events , then the probability of occurrence of both of them is denoted by **P(A and B)** or **P(A $\cap$ B)** and is given by

$$\mathbf{P(A \text{ and } B) = P(A) \times P(B)}$$

Multiplicative theorem on Probability

Example :

What is the probability of all of these events occurring:

1. Flip a coin and get a head
2. Draw a card and get an ace
3. Throw a die and get a 1

Solution:

Here the events are independent so by multiplication theorem ,

P(A & B & C) = P(A) ·P(B) ·P(C)

$$= \frac{1}{2} \times \frac{4}{52} \times \frac{1}{6} = \frac{1}{156}$$

Probability…

Note :

The sum of the probability of happening of event A and probability of non happening of event A is always 1 .

$$P(A) + P(A') = 1$$

Therefore ,

$$P(A') = 1 - P(A)$$

$$P(A) = 1 - P(A')$$

Probability…

Example :

A and B are asked to solve a problem. The probability of A solving it is $\frac{2}{3}$ and B is $\frac{3}{4}$. Find the probability that,

i. both can solve the problem

ii. A can solve it, but B cannot

iii.None of them can solve

iv.At least one of them will solve it

v.Only one of them can solve it

Conditional Probability

Conditional Probability:

The probability of event A occurring given that event B has occurred is usually stated as "***the conditional probability of A given B has occurred*** *is denoted by* ***P(A/B)*** *and is given by*

$$\mathbf{P(A/B)} = \frac{P(A \text{ and } B)}{P(B)}$$

Conditional Probability

Question : Employee information of ABC company.

Department	Male	Female	Total
Manufacturing	280	220	500
Production	175	125	300
Quality Control	115	85	200
Total	570	430	1000

If an employee is chosen at random ,find the probability

a. Employee chosen **is male** given that he belongs to production department.

b. Department chosen **is production** given that an employee is male.

c. Employee chosen **is female** given that she belongs to manufacturing department.

Bayes's Theorem

Let I , II and III be the event of selecting 1st ,2nd and 3rd Urn, respectively.

Let R be the event of selecting red ball from the Urn

Now ,

a. Probability that the red ball is from the 1st urn is given by the Bayes's formula

$$P(I/R) = \frac{P(I) \text{ X } P(R/I)}{P(I) \text{ X } P(R/I) + P(II) \text{ X } P(R/II)+ P(III) \text{ X } P(R/III)}$$

Bayes's Theorem

b. Probability that the red ball is from the 2nd urn is given by the Bayes's formula

$$P(II/R) = \frac{P(II) \times P(R/II)}{P(I) \times P(R/I) + P(II) \times P(R/II) + P(III) \times P(R/III)}$$

c. Probability that the red ball is from the 3rd urn is given by the Bayes's formula

$$P(III/R) = \frac{P(III) \times P(R/III)}{P(I) \times P(R/I) + P(II) \times P(R/II) + P(III) \times P(R/III)}$$

(Bayes' Theorem)

Question:

A factory produces a certain types of output by three machines. The respectively daily production figures are:

Machine X: 1500 units, Machine Y: 3000 units, Machine Z: 4500 units

Experience shows that 1.5% of the output produced by machine X, 2% of the output produced by machine Y and 2.2% of the output produced by machine Z is defective. An item is drawn at random and found to be defective. **What is the probability that it comes from the output of machine Y?**

Revision questions - Probability

Questions :

- An urn contains 7 red and 4 blue balls. A ball is drawn and replaced. If another ball is drawn, find the probability of getting,

(i) 2 red balls (ii) 2 blue balls. (iii) One ball of each colour.

- The probability that a man will be alive 25 years hence is 0.3 and the probability that his wife will be alive 25 years hence is 0.4. Find the probability that 25 years hence,
  - Both will be alive.
  - Only the man will be alive.
  - Only the woman will be alive.
  - None of them will be alive.
  - At least one of them will be alive.

Revision questions - Probability

Questions :

- A bag contains 5 white and 3 black balls. Two balls are drawn at random one after the other without replacement. Find the probability that both balls drawn are

  (i) black and (ii) white.

- Let X is the random variable which represents the number of heads when 3 coins are tossed simultaneously .Construct the probability distribution for X and also calculates its Expected value and variance.

Binomial Probabilities

- Binomial probabilities are from **binomial experiment**.
- Binomial experiment or Bernoulli experiment is when there are exactly two possible outcomes (for each trial) of interest.
  - e.g., tossing a coin where outcomes are either head or tail

Computing probabilities for a binomial experiment.

If an experiment is performed **n** number of times with probability of success **p** and probability of failure **q** then the probability of getting **r** number of success is denoted by **P(X=r)** and is given by the formula,

$$\boxed{P(X=r) = {}^{n}C_{r} \times p^{r} \times q^{n-r}}$$

Where,

p = probability of success ,q = probability of failure

r = required number of success

n = no. of trials

n c r = combination

Its your turn,

Use Binomial formula to solve the following questions:

1. The probability that student entering a university will graduate is 0.4. If 'the student will graduate' is success, find the probability that out of 3 students at the university,

- **None of them will graduate.**
- **Exactly one will graduate.**
- **All of them will graduate.**

2. A die is thrown 6 times. If 'getting an odd number' is success. What is the probability of getting,

- **5 success.**
- **At least 5 success.**
- **At most 5 success.**
- **No success.**

Best of Luck for Class test – 2 !!!!