# Maximum - Likelihood Estimation in MLR

By,
P D Siva Rama Varma
I$^{st}$ MSc, 23907

The Maximum likelihood estimators for the model parameters in multiple linear regression when the model errors are normally and independently distributed are also least squares estimators.

The model is $\boxed{y = X\beta + \epsilon}$

where $y$: $n\times1$ vector of dependent var responses

$X$: $n\times p$ matrix of regressors

$\beta$: $p\times1$ vector of regression coefficients

$\epsilon$: $n\times1$ vector of error terms.

and the errors are normally and independently distributed with constant variance $\sigma^2$, or $\epsilon$ is distributed as $N(0, \sigma^2 I)$

i.e $\epsilon \sim N(0, \sigma^2 I)$

So, the normal density function is,

$$f(\epsilon_i) = \frac{1}{\sqrt{2\pi\sigma}} e^{\left(\frac{-1}{2\sigma^2}\epsilon_i^2\right)}$$

Now, let's consider the likelihood functions which is the joint density of $\epsilon_1, \epsilon_2, \dots, \epsilon_n$

or $\prod_{i=1}^{n} f(\epsilon_i)$

$$\Rightarrow L(\epsilon, \beta, \sigma^2) = \prod_{i=1}^{n} f(\epsilon_i)$$

$$= \frac{1}{(2\pi)^{n/2}\sigma^n} e^{\left(\frac{-1}{2\sigma^2}\epsilon'\epsilon\right)}$$

$$\because \epsilon = y - X\beta$$

$$L(y, X, \beta, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{\left(\frac{-1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right)}$$

Now, let's take log likelihood function.

ie $\ln(L(y, X, \beta, \sigma^2)) = \ln\left(\frac{1}{(2\pi\sigma^2)^{n/2}} e^{\left(\frac{-1}{2\sigma^2}(y - X\beta)'(y - X\beta)\right)}\right)$

w.k.t $\ln(ab) = \ln(a) + \ln(b)$

$$\Rightarrow \ln(L(y, X, \beta, \sigma^2)) = \ln\left(\frac{1}{(2\pi\sigma^2)^{n/2}}\right) + \ln\left(e^{\left(\frac{-1}{2\sigma^2}(y-X\beta)'(y-X\beta)\right)}\right)$$

$\because \ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$ & $\ln(1) = 0$ & $\ln e^x = x$

$$\Rightarrow \ln(L(y, X, \beta, \sigma^2)) = -\ln(2\pi\sigma^2)^{n/2} - \frac{1}{2\sigma^2}(y-X\beta)'(y-X\beta)$$

Doing differentiation w.r.t $\sigma^2$;

$$= -\frac{n}{2}(\ln(2\pi\sigma^2)) + \frac{1}{2(\sigma^2)^2}(y-X\beta)'(y-X\beta)$$

$$= -\frac{n}{2} \cdot \frac{(\cancel{2\pi})}{\cancel{2\pi}\sigma^2} + \frac{1}{2(\sigma^2)^2}(y-X\beta)'(y-X\beta)$$

$$= \frac{-n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}(y-X\beta)'(y-X\beta)$$

Now, equating it to zero

$$\Rightarrow \frac{n}{\cancel{2}\sigma^2} = \frac{1}{\cancel{2}(\sigma^2)^2}(y-X\beta)'(y-X\beta)$$

$$\boxed{\sigma^2 = \frac{(y-X\beta)'(y-X\beta)}{n}}$$

$\therefore$ The Maximum Likelihood estimator of $\sigma^2$ is $\boxed{\hat{\sigma}^2 = \frac{(y-X\hat{\beta})'(y-X\hat{\beta})}{n}}$