3 MATHEMATICAL LOGIC

: SYLLABUS :

Introduction to Logic, Truth table

1. Introduction :

We studied set theory in chapter - 1. There we came across some operations like $\cup$ (union), $\cap$ (intersection), ' (complementation), - (difference), $\times$ (Cartesian product), Δ (symmetric difference) etc. From it we can say that in study of Mathematics, we do require the particular type of logic. Hence we have always to proceed with the study in a logical manner. So now the question arises : What is logic ? It is to be noted that : **"Logic deals with methods of reasoning."** Logic provides rules and techniques of determining whether a given statement is valid or not.

In day-to-day life we find that logic is used at each and every stage. e.g. You might think, "Either I shall go to see my friend Bakul or shall go to see a movie. But I do not like to see Bakul. So I shall not go to see Bakul. Hence, I shall go to see a movie." This illustration is an example of the use of principles of logic.

2. Mathematical Logic :

We shall study mathematical logic by a sequential order in the following steps :

(I) Some definitions
(II) Laws of logical connectives
(III) Tautology and contradiction
(IV) Universal Quantifier and Existential Quantifier
(V) Implication and Double Implication
(VI) Valid Argument

3. Some Definitions :

In this article we shall take some definitions useful for the study of mathematical logic.

(1) Statement :** A declarative sentence is said to be a **statement** if it is possible to say definitely that it is true or that it is false but not both.

(2) Truth Value : The truth value of a statement is denoted by T if the statement is true, and the truth value of a statement is denoted by F if the statement is false.

e.g. (1) The sentence - "The sun is a star" is a statement and it is true. The truth value of this statement is T.

(2) The sentence - "2 + 2 = 5" is a statement and it is false. The truth value of this statement is F.

Usually statements are denoted by letters $p, q, r, \ldots$ or by letters $p_1, p_2, \ldots$

(3) Simple statement : A statement involving only one statement is called simple statement.

(4) Composite statement : A statement involving more than one statement is called a composite statement (or compound statement).

The simple statements which form a compound statement are called component statements (or elemental statements).

The composite statement S formed by the simple statements $p_1, p_2, \ldots p_n$ is denoted by S $(p_1, p_2, \ldots p_n)$ and the truth values of the component statements determine the truth value of the compound statement.

(5) Logical connectives :# Two or more simple statements can be joined by any of the five terms : "and", "or", "not", "if…then" and "if and only if."

• These five terms are called logical connectives or sentential connectives or simple connectives in logic.

4. Conjunction :*

The compound statement obtained by connecting two simple statements with the connective **"and"** is called the conjunction of these simple statements.

* Define with illustration : Logical connection AND. (April, 2010)
** Define with illustration : Statement (April, 2010)
# Define logical connections with example (Dec. 2015)

The conjunctions of p and q is denoted by $p \wedge q$ (and is read as "p and q").

The compound statement $p \wedge q$ is true only when p and q are both true and in all other cases it is false. The truth table for $p \wedge q$ is given below:

Truth Table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

5. Disjunction :*

The compound statement obtained by combining two simple stalements by the connective "**or**" is called the disjunction of these simple statements.

The disjunction of p and q is denoted by $p \vee q$ (and is read as "p or q").

The compound statement $p \vee q$ is false only when p and q are both false and in all other cases it is true. The truth table of $p \vee q$ is given below:

Truth Table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

6. Negation :

A statement whose truth value is opposite to that of a given statement and is also the opposite in its contents is called its **negation.**

The negation of p is denoted by $\sim p$.

The truth table for $\sim p$ is shown below :

Truth table for $\sim p$

p	$\sim p$
T	F
F	T

* Define with illustration : Logical connection OR. *(April, 2010)*

7. Logically equivalent statements :

If two statements $S_1(p, q, r, \ldots..)$ and $S_2(p, q, r, \ldots..)$ have the same truth value for all possible truth values of the componenent statements, they are said to be logically equivalent statements. We write this as

$S_1(p, q, r, \ldots..) = S_2(p, q, r, \ldots..)$.

8. Tautology and contradiction :

A statement which is always true is called a tautology. It is denoted by t. (t for tautology).

A statement which is always false is called a contradiction. It is denoted by c (c for contradiction).

Laws of Tautology and contradiction :

$p \vee t = t$	$p \vee c = p$
$p \wedge t = p$	$p \wedge c = c$
$p \vee (\sim p) = t$	$p \wedge (\sim p) = c$

9. Laws of logical connectives :

(1) **Commutative laws :**

$p \vee q = q \vee p$

$p \wedge q = q \wedge p$

(2) **Associative laws :**

$(p \vee q) \vee r = p \vee (q \vee r)$

$(p \wedge q) \wedge r = p \wedge (q \wedge r)$

(3) **De-Morgan's laws :**

$\sim (p \vee q) = (\sim p) \wedge (\sim q)$

$\sim (p \wedge q) = (\sim p) \vee (\sim q)$

(4) **Distributive laws :**

$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

(5) **Law for negation of a negation (or involution law) :**

$\sim (\sim p) = p$

(6) **Laws for absorption :**

$p \wedge (p \vee q) = p$

$p \vee (p \wedge q) = p$

It is to be noted that that all the laws can be proved by using truth table. We shall take now some examples.

Example 1 : Prove the following commutative laws :

(i) $p \vee q = q \vee p$

(ii) $p \wedge q = q \wedge p$ **by using truth table.**

Solution : Prepare the truth tables for $p \vee q$, $q \vee p$, $p \wedge q$ and $q \wedge p$.

		(1)	(2)	(3)	(4)
p	q	$p \vee q$	$q \vee p$	$p \wedge q$	$q \wedge p$
T	T	T	T	T	T
T	F	T	T	F	F
F	T	T	T	F	F
F	F	F	F	F	F

(i) From columns (1) and (2), we can say that for all possible truth values of p and q, $p \vee q$ and $q \vee p$ have the same truth values.

Hence $p \vee q \equiv q \vee p$.

(ii) From columns (3) and (4), we can see that for all possible truth values of p and q, $p \wedge q$ and $q \wedge p$ have the same truth values. Hence $p \wedge q \equiv q \wedge p$.

Example 2 : Prove the following associative laws :

(i) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(ii) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.

Solution :

(i) Here there are 3 statements p, q and r. For each statement there are two possibilities for truth values T and F. We have in all $2^3 = 8$ possible cases. They are shown below in a tabular form.

1	2	3	4	5	6	7
p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

From this table we can say that columns (5) and (7) are identical. So, we can say that, $(p \vee q) \vee r \equiv p \vee (q \vee r)$.

(ii) Here there are 3 statements p, q and r. For each statement there are two possibilities for truth values T and F. We have in all $2^3 = 8$ possible cases. They are shown below in a tabular form.

1	2	3	4	5	6	7
p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	F	T	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

From this table, we can say that columns (5) and (7) are identical. So, we can say that $(p \wedge q) \wedge r = p \wedge (q \wedge r)$.

Example 3 : Prove the following De-Morgan's Laws :

(i) $\sim (p \vee q) = (\sim p) \wedge (\sim q)$

(ii) $\sim (p \wedge q) = (\sim p) \vee (\sim q)$

Solution : We shall prepare first the following truth table :

1	2	3	4	5	6	7	8	9	10
p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$(\sim p) \wedge (\sim q)$	$p \wedge q$	$\sim(p \wedge q)$	$(\sim p) \vee (\sim q)$
T	T	T	F	F	F	F	T	F	F
T	F	T	F	F	T	F	F	T	T
F	T	T	F	T	F	F	F	T	T
F	F	F	T	T	T	T	F	T	T

(i) From the above truth table, columns (4) and (7) are identical.

Hence $\sim (p \vee q) = (\sim p) \wedge (\sim q)$

(ii) From the above truth table, columns (9) and (10) are identical

Hence $\sim (p \wedge q) = (\sim p) \vee (\sim q)$

Example 4 : Prove the following distributive laws :

(i) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

(ii) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

Solution : For proving the required, we have to first prepare the following truth table (containing columns 1 to 13) :

1	2	3	4	5	6	7
p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$
T	T	T	T	T	T	T
T	T	F	T	T	T	F
T	F	T	T	T	F	T
T	F	F	F	F	F	F
F	T	T	T	F	F	F
F	T	F	T	F	F	F
F	F	T	T	F	F	F
F	F	F	F	F	F	F

8	9	10	11	12	13
$(p \wedge q) \vee (p \wedge r)$	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T
T	F	T	T	T	T
T	F	T	T	T	T
F	F	T	T	T	T
F	T	T	T	T	T
F	F	F	T	F	F
F	F	F	F	T	F
F	F	F	F	F	F

(i) From columns (5) and (8), $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

(ii) From columns (10) and (13), $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

Example 5 : Prove the following laws of absorption :*

(i) $p \wedge (p \vee q) = p$ (ii) $p \vee (p \wedge q) = p$

Solution : For proving the required, we have to first prepare the following truth table :

1	2	3	4	5	6
p	q	$p \vee q$	$p \wedge (p \vee q)$	$p \wedge q$	$p \vee (p \wedge q)$
T	T	T	T	T	T
T	F	T	T	F	T
F	T	T	F	F	F
F	F	F	F	F	F

(i) From the above table, (1) and (4) are identical columns.

$\therefore p = p \wedge (p \vee q)$

(ii) From the above table (1) and (6) are identical columns.

$\therefore p = p \vee (p \wedge q)$

* Using truth table prove the following :

(i) $P \vee (P \wedge Q) = P$ (ii) $P \wedge (P \vee Q) = P$ *(April, 2010)*

Example 6 : If t and c denote tautology and contradiction respectively and p is a statement then prove that

(i) $p \vee t = t$ (ii) $p \wedge c = c$ (iii) $p \wedge (\sim p) = c$

Solution : We shall prepare the following truth table for proving the required.

1	2	3	4	5	6	7
p	$\sim p$	t	c	$p \wedge (\sim p)$	$p \vee t$	$p \wedge c$
T	F	T	F	F	T	F
F	T	T	F	F	T	F

(i) From the above table, columns (3) and (6) are identical.

$\therefore t = p \vee t \qquad \Rightarrow p \vee t = t$

(ii) From the above table, columns (4) and (7) are identical.

$\therefore c = p \wedge c \qquad \Rightarrow p \wedge c = c$

(iii) From the above table, columns (4) and (5) are identical.

$\therefore c = p \wedge (\sim p) \qquad \Rightarrow p \wedge (\sim p) = c$

10. Universal quantifier and Existential quantifier :

The symbol $\forall$ (for all or for every) is called a **universal quantifier.**

The symbol $\exists$ (there exists) is called an **existential quantifier**.

e.g. "For all $x \in R$, $x^2 \geq 0$" ..... (A)

This statement A can be written as

"$\forall x \in R$, $x^2 \geq 0$" ..... (A_1)

"There exists $x \in N$ such that $x + 3 = 10$" ..... (B)

This statement B can be written as

"$\exists x \in N$ such that $x + 3 = 10$" ..... (B_1)

11. Negation of quantified statements :

$\sim (\forall x, p) = \exists x (\sim p)$ (De-Morgan)

$\sim (\exists x, p) = \forall x (\sim p)$ (De-Morgan)

e.g. The negation of " $\forall x \in R$, $x + 3 = 3 + x$" is ' '$\exists\, x \in R$ such that $x + 3 \neq 3 + x$".

12. Implication :

Let p and q are statements.

A statement of the form "if p then q" is called an **implication,** and is written as $p \Rightarrow q$ (read as "p implies q" or "p only if q"). Here p is called **antecedent** and q is called **consequent.**

In the implication $p \Rightarrow q$, p is called a **sufficient condition** for q and q is called a **necessary condition** for p.

e.g. p is "A ball is thrown upwards"

q is "It comes on the ground".

"if p then q" means $p \Rightarrow q$

It is to be noted that :

(1) $p \Rightarrow q \equiv (\sim p) \vee q$

(2) $p \Rightarrow q \equiv [(\sim q) \Rightarrow (\sim p)]$

(3) $\sim (p \Rightarrow q) \equiv p \wedge (\sim q)$

For the truth values T of p and F of q, the truth value of $p \Rightarrow q$ is F. In other possibilities, for the truth values of p and q, the truth value of $p \Rightarrow q$ is T.

The truth table for an implication is :

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

13. Double implication :

Let p and q are statements.

A statement of the form "p if and only if q" is called a **double implication** and is written as $p \Leftrightarrow q$. $p \Leftrightarrow q$ is a conjunction of $p \Rightarrow q$ and $q \Rightarrow p$.

Thus $p \Leftrightarrow q$ is true if and only if p and q have the same truth values that is p and q are equivalent statements. Hence we may say that $(p \Leftrightarrow q) \equiv (p \equiv q)$.

Truth table for double implication is :

1	2	3	4	5
p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Ill-1 : "$ab = 0$ if and only if $a = 0$ or $b = 0$" is an example of double implication.

III-2 : "Two triangles are equilateral if and only if they are equiangular" is an example of double implication.

14. Valid (logical) Argument :

The implication that statement S follows from statements $S_1, S_2, S_3, \ldots, S_n$ is called an argument. Thus, the statement $S_1 \wedge S_2 \wedge S_3 \wedge \ldots \wedge S_n \Rightarrow S$ is an argument. In symbols argument is represented by $(S_1, S_2, S_3, \ldots, S_n; S)$. In this argument $S_1, S_2, \ldots, S_n$ is called hypothesis and S is called conclusion. Such an argument is said to be logically valid if the conclusion S is true whenever the hypothesis statements $S_1, S_2, \ldots, S_n$ are all true. Not all arguments are logical. To test the logical validity of an argument $(S_1, S_2, \ldots, S_n; S)$, we must form a truth table. The rows in which all statements $S_1, S_2, S_3, \ldots, S_n$ have truth value T in the truth table are called **critical rows.*** If the conclusion S has truth value T in every critical row of the truth table of an argument $(S_1, S_2, \ldots, S_n; S)$ then we say that the argument is logically valid.

Example 7 : Examine the logical validity of the following argument :

Hypothesis : $S_1 : p \vee q$, $S_2 : \sim p$, Conclusion : $S : q$

		Hypothesis			Conclusion
		S_1	S_2		S
p	q	$p \vee q$	$\sim p$		q
T	T	T	F		T
T	F	T	F		F
F	T	$\boxed{T}$	$\boxed{T}$	critical row	T
F	F	F	T		F

The third row is the critical row, since both $\sim p$ and $p \vee q$ are true. In this critical row, the conclusion q is also true. So this is a logically valid argument.

Example 8 : Prove that the argument in the following example is not logically valid.

Hypothesis : $S_1 : p \wedge (\sim q) \Rightarrow r$,

$S_2 : p \vee q$ and

$S_3 : q \Rightarrow p$

Conclusion : $S : r$.

* Define Critical raw : (March 2015)

					Hypothesis				Conclusion
					S_1	S_2	S_3		S
p	q	r	$\sim q$	$p \wedge (\sim q)$	$p \wedge (\sim q) \Rightarrow r$	$p \vee q$	$q \Rightarrow p$		r
T	T	T	F	F	T	T	T	c.row*	T
T	T	F	F	F	T	T	T	c.row	F
T	F	T	T	T	T	T	T	c.row	T
T	F	F	T	T	F	T	T		F
F	T	T	F	F	T	T	F		T
F	T	F	F	F	T	T	F		F
F	F	T	T	F	T	F	T		T
F	F	F	T	F	T	F	T		F

* C . row = Critical row

The conclusion is not true in all critical rows. So the argument is not logically valid.

Example 9 : Prove the validity of the following argument :

(Oct.,Nov / 2009)

Hypothesis : $S_1 : p \Rightarrow q$, $S_2 : q \Rightarrow r$, Conclusion : $S : p \Rightarrow r$ (that is, implication obeys the law of transitivity.)

			Hypothesis			Conclusion
			S_1	S_2		S
p	q	r	$p \Rightarrow q$	$q \Rightarrow r$		$p \Rightarrow r$
T	T	T	T	T	critical row	T
T	T	F	T	F		F
T	F	T	F	T		T
T	F	F	F	T		F
F	T	T	T	T	critical row	T
F	T	F	T	F		T
F	F	T	T	T	critical row	T
F	F	F	T	T	critical row	T

The conclusion is true in all critical rows. Hence, the argument is logically valid.

15. Second method of checking the logical validity of an argument :

For the hypothesis S_1, S_2, S_3,..., S_n and conclusion S, if $(S_1 \wedge S_2 \wedge \ldots \wedge S_n) \Rightarrow S$ is a tautology, then the argument is logically valid. Here, if all the statements in the hypothesis are true, their conjunction is also true. Also the conclusion is also true, so the implication is also true.

Example 10 : Is the following argument logically valid ?

Hypothesis : $S_1 : p \Rightarrow q$, $S_2 : p$; Conclusion : $S : q$

Consider the implication $[(p \Rightarrow q) \wedge p] \Rightarrow q$

$$\begin{aligned}
[\{(\sim p) \vee q\} \wedge p] \Rightarrow q &= [\{(\sim p) \wedge p\} \vee (q \wedge p)] \Rightarrow q \\
&= [c \vee (q \wedge p)] \Rightarrow q \\
&= (q \wedge p) \Rightarrow q \\
&= [\sim (q \wedge p)] \vee q \\
&= [(\sim p) \vee (\sim q)] \vee q \\
&= (\sim p) \vee [(\sim q) \vee q] \\
&= (\sim p) \vee t \\
&= t
\end{aligned}$$

Hence, the argument is logically valid.

Example 11 : Examine the validity of the argument.

Hypothesis : $S_1 : p \Rightarrow q$, $S_2 : p \Rightarrow r$, Conclusion : $S : p \Rightarrow (q \wedge r)$

Now $[(p \Rightarrow q) \wedge (p \Rightarrow r)] \Rightarrow [p \Rightarrow (q \wedge r)]$

$= \quad [(\sim p) \vee q] \wedge [(\sim p) \vee r] \Rightarrow [p \Rightarrow (q \wedge r)]$

$= \quad [(\sim p) \vee (q \wedge r)] \Rightarrow [p \Rightarrow (q \wedge r)]$ (Distributive law)

$= \quad [p \Rightarrow (q \wedge r)] \Rightarrow [p \Rightarrow (q \wedge r)]$

$= \quad \sim [p \Rightarrow (q \wedge r)] \vee [p \Rightarrow (q \wedge r)]$

$= \quad t$

$\therefore$ The argument is logically valid.

Example 12 : Which of the following are statements ?

(1) $\sqrt{2}$ is rational number.

(2) $5 \times 7 = 57$.

(3) May you live long.

(4) For real number x, $x^2 \geq 0$.

(5) Where are you going ?

Answer :

(1) "$\sqrt{2}$ is a rational number". Is a statement and its truth value is F.

(2) "$5 \times 7 = 57$" is a false statement.

(3) "May you live long" is not a statement.

(4) "For real number $x, x^2 \geq 0$" is a statement and its truth value is T.
(5) "Where any you going ?" is not a statement. It is a question sentence.

Example 13 : Give the negations of the following :
(1) $\forall x \in z, x^2 \in N$ (2) $\forall x \in A, x \in B$ (3) $\exists x \in N, x^2 = 4$.

Answer :
(1) The negation of "$\forall x \in Z, x^2 \in N$" is "$\exists x \in z, x^2 \notin N$".
(2) The negation of "$\forall x \in A, x \in B$" is "$\exists x \in A, x \notin B$".
(3) The negation of "$\exists x \in N, x^2 = 4$" is "$\forall x \in N, x^2 \neq 4$".

Example 14 : Using De-Morgan's laws give the negations of the following :
(1) He is a doctor and he is a Mathematician.
(2) $x \notin A$ and $x \in B$.

Answer : (1) Let p : He is a doctor.
q : He is a mathematician.
We use De-Morgan's law,
$\sim (p \wedge q) = \sim p \vee \sim q$
$\therefore$ The required negation :
He is not a doctor or he is not a mathematician.
(2) Let $p : x \notin A$ and $q : x \in B$
$\therefore \sim p : x \in A$ and $\sim q : x \notin B$
We use De Morgan's Law,
$\sim (p \wedge q) = (\sim p) \vee (\sim q)$
$\therefore$ The required negation : $x \in A$ or $x \notin B$.

Example 15 : Prove that if $p \vee q = p \vee r$ and $p \wedge q = p \wedge r$ then $q = r$.
Deduce that the negation of a given statement is unique.

Answer :
$q = q \wedge (q \vee p)$ (Law of absorption)
$= q \wedge (p \vee q)$ (Commutative law)
$= q \wedge (p \vee r)$ (Given)
$= (q \wedge p) \vee (q \wedge r)$ (Distributive law)
$= (r \wedge p) \vee (r \wedge q)$ (Given and Commutative law)
$= r \wedge (p \vee r)$ (Distributive law)
$= r \wedge (r \vee p)$ (Given)
$= r \wedge (p \vee r)$ (Commutative law)
$= r$ (Law of absorption)

Let q and r be two negations of p, if possible. Then
$p \wedge q = c$, $p \wedge r = c$;
$p \vee q = t$, $p \vee r = t$;
$\therefore$ $p \wedge q = p \wedge r$ and $p \vee q = p \vee r$.

$\therefore \quad q = r$

Hence, the negation of p is unique.

Example 16 : There are two ad-boards in front of two restaurants.

One says, "cheap food is not healthy food."

Another says, "Healthy food is not cheap."

Actually both mean the same thing. How ?

Answer : Let p : food is healthy.

q : food is cheap

$\therefore$ Ad-board in front of one restaurant is $q \Rightarrow \sim p$

Ad-board in front of another restaurant is $p \Rightarrow \sim q$.

We shall construction the truth table as follows :

(1)	(2)	(3)	(4)	(5)	(6)
p	q	$\sim p$	$\sim q$	$q \Rightarrow \sim p$	$p \Rightarrow \sim q$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

Columns (5) and (6) are identical.

$\therefore (q \Rightarrow \sim p) \equiv (p \Rightarrow \sim q)$

$\therefore$ Meanings of both the ad-boards are the same.

Example 17 : If $p \Rightarrow q$ is true, determine truth value of $(\sim p) \vee (p \Leftrightarrow q)$.

Answer :

(1)	(2)	(3)	(4)	(5)	(6)
p	q	$p \Rightarrow q$	$\sim p$	$p \Leftrightarrow q$	$\sim p \vee (p \Leftrightarrow q)$
T	T	T	F	T	T
T	F	F	F	F	F
F	T	T	T	F	T
F	F	T	T	T	T

Columns (3) and (6), when $p \Rightarrow q$ is true then $\sim p \vee (p \Leftrightarrow q)$ is also true.

$\therefore$ Truth value of $\sim p \vee (p \Leftrightarrow q)$ is T.

EXERCISE

1. Which of the following sentences are statements ?
   (1) What a beautiful solar eclipse ?
   (2) Every triangle is an equilateral triangle.
   (3) Please, stand up.
   (4) Every quadrilateral is a square.
   [Ans. : (2) and (4) are statements (1) and (3) are not the statements]
2. Prepare truth tables for the following statements :
   (1) $(p \wedge q) \vee r$ (2) $(p \vee q) \wedge r$
   (3) $(\sim p) \vee q$ (4) $p \wedge (\sim q)$
   (4) $[(\sim p) \vee (\sim q)] \wedge r$.
3. Prove the law of absorption :
   $p \vee (p \wedge q) = p$
4. Give the negations of the following : (Using De Morgan's laws).
   (1) $3 + 6 = 9$ and $3.6 = 18$
   (2) $x^2 \geq 0$ and $x \in N$
   (3) $x \in A$ and $x \in B$
   [Ans. : (1) $3 + 6 \neq 9$ or $3.6 \neq 18$ (2) $x^2 < 0$ or $x \notin N$
   (3) $x \notin A$ or $x \notin B$]
5. Give the negations of the following statements.
   (1) $\exists x \in N, x + 7 < 0$ (2) Every triangle is an equiangular triangle.
   (3) $\forall x \in z, x^2 < 0$ (4) Every set is a non empty set.
   [Ans. : (1) $\forall x \in N, x + 7 \nless 0$
   (2) There is a triangle which is not equiangular.
   (3) $\exists x \in z, x^2 \nless 0$
   (4) There is a set which is not non-empty.]
6. Write the following as implications :
   (1) The square of an even integer is even.
   (2) The product of two negative numbers is positive.
   (3) The square of a prime number is not prime.
   [Ans. : (1) If an integer is even, its square is even.
   (2) If two numbers are negative, their product is positive.
   (3) If a number is prime, its square is not a prime.]
7. Prepare truth tables for the following :
   (1) $(p \wedge q) \Rightarrow r$ (2) $[(\sim p) \vee q] \Rightarrow \sim q$.
8. Give the equivalent implications of the following :
   (1) $ab = 0 \Rightarrow a = 0$ or $b = 0$
   (2) $x \in A \cap B \Rightarrow x \in A$ and $x \in B$
   [Ans. : (1) $a \neq 0$ and $b \neq 0 \Rightarrow ab \neq 0$
   (2) $x \notin A$ or $x \notin B \Rightarrow x \notin A \cap B$]

9. Show that the implications $p \Rightarrow q$ and $(\sim p) \Rightarrow (\sim q)$ are not equivalent.

10. Using truth tables show that the following pair of Statements S_1 and S_2 are equivalent statements.
(1) $S_1 : (p \wedge q) \vee p$
$S_2 : p$
(2) $S_1 : \sim [p \vee \{(\sim p) \wedge (\sim q)\}]$
$S_2 : \sim p \wedge q$
(3) $S_1 : p \vee [\sim (q \wedge r)]$
$S_2 : [p \vee (\sim q)] \vee (\sim r)$.

11. Prove that the following propositions are tautologies
(1) $p \wedge q \Rightarrow p$
(2) $[(\sim q) \Rightarrow (\sim p)] \Rightarrow (p \Rightarrow q)$
(3) $(p \wedge q) \Rightarrow (p \Rightarrow q)$

12. Using truth tables check the validity of the arguments.
(1) Hypothesis $S_1 : p \Rightarrow (\sim q)$, $S_2 : r \Rightarrow q$, $S_3 : r$
Conclusion : $S : \sim p$ [Ans. : Valid]
(2) Hypothesis $S_1 : p \Rightarrow (\sim q)$, $S_2 = - p$
Conclusion : $S : q$ [Ans. : Not valid]
(3) Hypothesis $S_1 : p \Rightarrow q$, $S_2 : \sim p$
Conclusion : $S : q$ [Ans. : Not valid]

13. Without using truth table (i.e. by second method of checking logical validity of an argument), prove that the following arguments are logically valid.
(1) Hypothesis : $S_1 : p \wedge q$, $S_2 : p$
Conclusion : $S : q$
(2) Hypothesis : $S_1 : p \Rightarrow q$, $S_2 : q \Rightarrow r$, $S_3 : p$
Conclusion : $S : r$
(3) Hypothesis : $S_1 : p \wedge q$, $S_2 : \sim p$
Conclusion : $S : q$

14. Show that :
(1) $\{p \Rightarrow (q \Rightarrow r)\} \Rightarrow \{(p \Rightarrow q) \Rightarrow (p \Rightarrow r)\}$ is a tautology.
(2) $\sim (p \wedge q) \vee (\sim p \wedge q) \vee p$ is a tautology.

15. If I study then I will not fail in Mathematics. If I donot play cricket. then I will study. But I failed in Mathematics.
There fore I must have played cricket.
Prove that the above argument is logically valid.

University Questions

1. **Answer the following :** **(April 2010)**
   (1) Define with illustration :
   (i) Statement
   (ii) Logical connection AND and OR.
   (2) Using truth table prove the following :
   (i) $P \vee (P \wedge Q) = P$
   (ii) $P \wedge (P \vee Q) = P$
2. **Answer the following :** **(Oct./Nov. 2009)**
   (1) By using truth table, show that :
   $(p \Rightarrow q) \wedge (q \Rightarrow r) = p \Rightarrow r$
   (2) Prove that $(p \Rightarrow q) \wedge (r \Rightarrow s) \wedge (p \vee r) = q \vee s.$
3. Prove that the argument in the following example is not logically valid Hypothesis : $S_1 : p \vee (\sim q) \rightarrow r$ Conclusion : $S : r$
   $S_2 : p \vee q$
   $S_3 : q \rightarrow p$ (Dec., 2015)
4. Using truth table, prove that (Dec., 2015)
   (i) $(p \rightarrow q) = [(\sim q) \rightarrow (\sim p)]$
   (ii) $\sim(p \rightarrow q) = p \wedge (\sim q)$
5. Using truth table, prove that
   $(p \rightarrow q) \wedge (p \rightarrow r) = p \rightarrow (q \wedge r)$ (March 2015)
6. Using truth table, prove that (March 2015)
   $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

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