4 BOOLEAN ALGEBRA

: SYLLABUS :

Definition and examples of boolean algebra. Boolean functions. Representation and minimization of Boolean functions. Design example using Boolean algebra.

1. Introduction :

In the chapters on Mathematical logic and set theory we came across the two operations (e.g. $\wedge$ and $\vee$ in mathematical logic and $\cap$ and $\cup$ in set theory) along with the complementation [e.g. ~ (negation) in mathematical logic and " ' " (complementation) in set theory] which obey cirtain laws as follows :

(1) Commutative laws
(2) Associative laws
(3) Distributive laws
(4) Existence of Identity elements
(5) Existence of complement etc.

George Boole discovered an important tool for the study and application of mathematical "logic-boolean algebra". He developed the principles of logic in a symbolic form. The relation of this algebra of statements to logic is of the same nature as the relation of common algebra to arithmetic. This algebra is the boolean algebra. It is very useful in mathematical logic. C. E. Shannon observed that boolean algebra is very useful also in the analysis of electric circuits used in the construction of computer chips. In the last few decades 'boolean algebra' became an important tool in the construction and analysis of electronic computers.

2. Definition of Boolean algebra :

A non empty set B, together with two binary operations + and • defined on it, is said to be a boolean algebra if the following laws (1 to 5) hold :

(1) Commutative Laws : $\forall\, x, y \in B$, $x + y = y + x$ and $x \cdot y = y \cdot x$.

(2) Associative Laws : $\forall\, x, y, z \in B$, $(x + y) + z = x + (y + z)$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(3) Distributive Laws : $\forall\, x, y, z \in B$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ and $x + (y \cdot z) = (x + y) \cdot (x + z)$.

(4) Existence of Identity Elements : $\exists\, 0 \in B$ and $\exists\, 1 \in B$ such that $x + 0 = x$ and $x \cdot 1 = x$, $\forall\, x \in B$.

(5) Existence of Complement : For each $x \in B$, $\exists\, x' \in B$ such that $x + x' = 1$ and $x \cdot x' = 0$.

This boolean algebra is denoted by $(B, +, \cdot, ', 0, 1)$. 0 is called the zero element, 1 the unit element and x' the complement of x.

We should keep in mind, that the symbols + and • used here are not the ones used for ordinary addition and multiplication of numbers. They are simply notations for binary operations satisfying certain rules.

3. Examples of Boolean algebra :

We now take some examples of Boolean algebra.

Example 1 : Let P(X) be the power set of the set X. Define +, • and ' on it as follows :

$A + B = A \cup B$, $A \cdot B = A \cap B$, $A' = X - A$

we can see that $(P(X), +, \cdot, ', \phi, X)$ is a boolean algebra.

The commutative, associative and distributive laws have been proved already in chapter 1. Taking $0 = \phi$ and $1 = X$, we get

$A + 0 = A \cup \phi = A$, $A \cdot X = A \cap X = A$

$\therefore$ ϕ and X are respectively, the zero and unit elements.

Also $A + A' = A \cup (X - A) = X$, $A \cdot A' = A \cap (X - A) = \phi$

$\therefore$ $X - A = A'$ is the complement of A.

$(P(X), \cup, \cap, -, \phi, X) = (P(X), +, \cdot, ', \phi, X)$ is a Boolean algebra.

Example 2 : Let L be the set of statements. Define +, • and ' on it as follows :

$p + q = p \vee q$, $p \cdot q = p \wedge q$, $p' = \sim p$

we can see that $(L, +, \cdot, ', c, t)$ is a Boolean algebra.

The commutative, associative and distributive laws have been proved already in chapter 13. Taking $0 = c$, and $1 = t$,

$p + 0 = p \vee c = p$, $p \cdot 1 = p \wedge t = p$

$p + p' = p \vee (\sim p) = t = 1$, $p \cdot p' = p \wedge (\sim p) = c = 0$

$\therefore$ The algebra of statements $(L, \vee, \wedge, \sim, c, t)$ is a boolean algebra.

Example 3 : Let B = (0, 1) and +, • and ' be defined on it as follows ;

+	0	1
0	0	1
1	1	1

•	0	1
0	0	0
1	0	1

and $0' = 1$, $1' = 0$

Prove that (B, +, •, ', 0, 1) is a Boolean algebra.

Clearly, 0 is the zero element and 1 the unit element. Other laws can be seen directly from the tables. (This is called binary boolean algebra.)

Example 4 : Let $D_6 = \{1, 2, 3, 6\}$. $\forall a, b \in D_6$, define +, • and ' by $a + b$ = lcm of a and b, $a \cdot b$ = gcd of a and b, $a' = \frac{6}{a}$.

Show that (D_6, +, •, ', 1, 6) is a boolean algebra.

Solution :

The following are the tables for +, • and '.

+	1	2	3	6
1	1	2	3	6
2	2	2	6	6
3	3	6	3	6
6	6	6	6	6

•	1	2	3	6
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
6	1	2	3	6

'	1	2	3	6
	6	3	2	1

In the first two tables, each entry is one of 1, 2, 3 or 6. Hence, + and • are binary operations. The symmetry about the principal diagonal indicates that the commutative laws hold.

We verify the associative laws for 1, 2 and 3.

$1 + (2 + 3) = 1 + 6 = 6$, $\quad (1 + 2) + 3 = 2 + 3 = 6$

and $(1 \cdot 2) \cdot 3 = 1 \cdot 3 =$, $1 \cdot (2 \cdot 3) = 1 \cdot 1 = 1$

It may be verified similarly for other cases.

To check the distributive laws, consider 2, 3, 6

$2 + 3 \cdot 6 = 2 + 3 = 6$, $\quad (2 + 3) \cdot (2 + 6) = 6 \cdot 6 = 6$ and

$2 \cdot (3 + 6) = 2 \cdot 6 = 2$, $\quad 2 \cdot 3 + 2 \cdot 6 = 1 + 2 = 2$

It may be verified similarly for other cases.

For the zero and unit elements, we check

$1 + 1 = 1, 1 + 2 = 2, 1 + 3 = 3, 1 + 6 = 6$

$1 \cdot 6 = 1, 2 \cdot 6 = 2, 3 \cdot 6 = 3, 6 \cdot 6 = 6$

∴ 1 is the zero element and 6 the unit element.

Also, $2 + 3 = 6$, $2 \cdot 3 = 1$, $1 + 6 = 6$, $1 \cdot 6 = 1$ gives $1' = 6$, $6' = 1$, $2' = 3$, $3' = 2$.

∴ (D_6, +, •, ', 1, 6) is a boolean algebra.

Example : 5 Let $D_8 = \{1, 2, 4, 8\}$. Define +, • and ' on D_8 by

$x + y$ = l.c.m. of x and y,

$x \cdot y$ = g.c.d. of x and y,

$$x' = \frac{8}{x}.$$

Verify that $(D_8, +, \cdot, ', 1, 8)$ is not boolean algebra.

Solution : The following are the tables for +, • and '.

+	1	2	4	8
1	1	2	4	8
2	2	2	4	8
4	4	4	4	8
8	8	8	8	8

•	1	2	4	8
1	1	1	1	1
2	1	2	2	2
4	1	2	4	4
8	1	2	4	8

'	1	2	4	8
	8	4	2	1

From the tables it is clear that $\forall\, x, y \in D_8$, $x + y \in D_8$ and $x \cdot y \in D_8$.

$\therefore$ + and • are binary operations.

The symmetry about the principal diagonal of first two tables indicates the **commutative laws** hold.

We verify **associative laws** for 1, 2, 4.

$(1 + 2) + 4 = 2 + 4 = 4 : 1 + (2 + 4) = 1 + 4 = 4$

$\therefore (1 + 2) + 4 = 1 + (2 + 4)$

$(1 \cdot 2) \cdot 4 = 1 \cdot 4 = 1;\ 1 \cdot (2 \cdot 4) = 1 \cdot 2 = 1$

$\therefore (1 \cdot 2) \cdot 4 = 1 \cdot (2 \cdot 4)$.

It may be verified similarly for other case.

We verify **distributive laws** for 2, 4, 8.

$2 \cdot (4 + 8) = 2 \cdot 8 = 2\ ;\ (2 \cdot 4) + (2 \cdot 8) = 2 + 2 = 2$

$\therefore 2 \cdot (4 + 8) = 2 \cdot 4 + 2 \cdot 8$

$2 + 4 \cdot 8 = 2 + 4 = 4\ ;\ (2 + 4) \cdot (2 + 8) = 4 \cdot 8 = 4$

$\therefore 2 + 4 \cdot 8 = (2 + 4) \cdot (2 + 8)$

It may be verified for other case

For the zero and unit elements we check :

$1 + 1 = 1,\ 1 + 2 = 2,\ 1 + 4 = 4,\ 1 + 8 = 8$

$\therefore$ 1 is the zero element

Now $1 \cdot 8 = 1,\ 2 \cdot 8 = 2,\ 4 \cdot 8 = 4,\ 8 \cdot 8 = 8$

$\therefore$ 8 is the unit element of D_8

Now $1 + 1' = 1 + 8 = 8$ & $1 \cdot 1' = 1 \cdot 8 = 1$

$\therefore$ For $x = 1$, $x + x' = 8$ (unit element) and $x \cdot x' = 1$ (zero element)

From the above table $2' = 4$

But $2 + 2' = 2 + 4 = 4 \neq 8$ (unit element of D_8)

Also, $2 \cdot 2' = 2 \cdot 4 = 2 \neq 1$ (zero element)

∴ Complement of 2 is not 4.

∴ 2 does not have a complement

∴ $(D_8, +, \cdot, ', 1, 8)$ is not a Boolean algebra.

4. Principle of Duality :

First we shall take the definition of a dual statement and then we shall take the principle of duality.

4.1 Dual statement :*

The dual of any statement in a boolean algebra is the statement obtained by interchanging the operations + and •, and simultaneously interchanging the elements 0 and 1 in the original statement.

e.g. (i) The dual statement of $x + x' = 1$ is $x \cdot x' = 0$.

(ii) The dual statement of

$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ is

$x + (y \cdot z) = (x + y) \cdot (x + z)$

(iii) The dual statement of

$x + 0 = x$ is

$x \cdot 1 = x$

4.2 Principle of Duality & its uses :

In a Boolean algebra, the dual of a true statement is also true statement.

We shall see this principle in the proofs of the following two theorems :

Theorem - 1 : In a boolean algebra the zero element and the unit element are unique.

Proof : (1) Let us suppose that there are two zero elements 0' and 0", $0' \in B$, $0'' \in B$.

$\therefore x + 0' = x, x + 0'' = x \ \forall \ x \in B$ ..... (A)

Taking $x = 0''$ in the first statement of A, are get

$0'' + 0' = 0''$ ..... (B)

Taking $x = 0'$ in the second statement of A, are get

$0' + 0'' = 0'$ ..... (C)

Since commutative law is true, we can say that

$0'' + 0' = 0' + 0''$ ..... (D)

i.e. L.H.S. of (B) & (C) are equal ∴ R.H.S. of (B) & (C) are equal

$\therefore 0'' = 0'$

Hence the zero element is unique.

* Explain Duality in Boolean Algebra *(April, 2010; March, 2015)*

(2) Let us suppose that there are two unit elements 1' and 1'', 1' ∈ B, 1'' ∈ B.

$$\therefore x \cdot 1' = x,\ x \cdot 1'' = x \ \forall\ x \in B \qquad \text{..... (I)}$$

Taking $x = 1''$ in the first statement of (I), we get

$$1'' \cdot 1' = 1'' \qquad \text{..... (II)}$$

Taking $x = 1'$ in the second statement of (I), we get

$$1' \cdot 1'' = 1' \qquad \text{..... (III)}$$

Since commutative law is true we can say that

$$1'' \cdot 1' = 1' \cdot 1'' \qquad \text{..... (IV)}$$

$\therefore \qquad 1'' = 1'$ [By (II) & (III)]

Hence the unit element is unique.

Theorem - 2 : In a boolean algebra, the complement of an element is unique.

Proof : Let $x \in$ B and x has two complements y, z in B.

$$\therefore x + y = 1,\ x \cdot y = 0 \qquad \text{..... (i)}$$

$$x + z = 1,\ x \cdot z = 0 \qquad \text{..... (ii)}$$

Now $y = y + 0$ ($\therefore$ 0 is the identity for +)

$= y + (x \cdot z)$ (By second statement of (ii))

$= (y + x) \cdot (y + z)$ (By distributive law)

$= 1 \cdot (y + z)$ (By first statement of (i))

$= (x + z) \cdot (y + z)$ (By first statement of (ii))

$= z + (x \cdot y)$ (By distributive law)

$= z + 0$ (By second statement of (i))

$= z$ ($\therefore$ 0 is the identity for +)

$\therefore \qquad y = z$

$\therefore$ Complement of x is unique.

Note : The complement of x is denoted by x'.

Example 6 : In a boolean algebra B, prove that $x + x = x$ and $x \cdot x = x \ \forall\ x \in$ B.

(i.e. **Idempotent law**)

Solution : (1) To prove $x + x = x$

$x = x + 0$ (0 is the identity for +)

$= x + (x \cdot x')$ (x' is the complement of x)

$= (x + x) \cdot (x + x')$ (Distributive law)

$= (x + x) \cdot (1)$ (x' is the complement of x)

$= x + x$ (1 is the unit element)

$\therefore\ x + x = x$

(2) To prove $x \cdot x = x$

$x = x \cdot 1$ (1 is the identity element for $\cdot$)

$= x \cdot (x + x')$ (x' is the complement of x)

$= x \cdot x + x \cdot x'$ (Distributive law)

$= x \cdot x + 0 = x \cdot x$ (x' is the complement of x)

$\therefore x \cdot x = x$

[N.B. : (1) After proving $x + x = x$ we can apply the principle of duality, which gives $x \cdot x = x$.

(2) After proving $x \cdot x = x$ we can apply the principle of duality, which $x + x = x$.]

Example 7 : In a boolean algebra B, prove that

$x + 1 = 1$ and $x \cdot 0 = 0 \;\; \forall x \in$ B.

Solution : (1) To prove $x + 1 = 1$.

$1 = x + x'$ (x' is the complement of x)

$\therefore \; 1 = x + (x' \cdot 1)$ (1 is the unit element)

$\therefore \; 1 = (x + x') \cdot (x + 1)$ (Distributive law)

$\therefore \; 1 = 1 \cdot (x + 1)$ ($x + x' = 1$)

$\therefore \; 1 = x + 1$ (1 is the unit element)

$\therefore \; x + 1 = 1$

(2) To prove $x \cdot 0 = 0$

$0 = x \cdot x'$ (x' is the complement of x)

$= x \cdot (x' + 0)$

$= (x \cdot x') + (x \cdot 0)$

$= 0 + (x \cdot 0)$

$= x \cdot 0$

$\therefore \quad x \cdot 0 = 0$

Note that this proof (ii) can be obtained by taking the dual at every step, in the proof of the first part.

Also note that by using the principle of duality for the result $x + 1 = 1$ we can immediately say that $x \cdot 0 = 0$.

Example 8 : Prove that in a boolean algebra B, $x + (x \cdot y) = x$ and $x \cdot (x + y) = x \;\; \forall x, y \in$ B

Solution : (1) To prove $x + (x \cdot y) = x$

Now $x = x \cdot 1$ (1 is the unit element)

$\therefore \; x = x \cdot (y + 1)$ (By Ex. 7)

$\therefore \; x = (x \cdot y) + (x \cdot 1)$ (Distributive law)

$\therefore \; x = (x \cdot y) + x$ (1 is the unit element)

$\therefore \quad x = x + (x \cdot y)$ (Commutative law)

(2) To prove $x \cdot (x + y) = x$

$$\begin{aligned} x &= x + 0 && \text{(0 is the zero element)} \\ &= x + (y \cdot 0) && \text{(By Ex. 7)} \\ &= (x + y) \cdot (x + 0) && \text{(Distributive law)} \\ &= (x + y) \cdot x && \text{(0 is the zero element)} \\ &= x \cdot (x + y) && \text{(Commutative law)} \end{aligned}$$

$\therefore \quad x \cdot (x + y) = x$

Example 9 : In a boolean algebra show that $0' = 1$ and $1' = 0$.

Solution : Again, $x + 1 = 1, \quad \forall x \in B \Rightarrow 0 + 1 = 1$

$x \cdot 0 = 0, \ \forall x \in B \Rightarrow 1 \cdot 0 = 0 \Rightarrow 0 \cdot 1 = 0$

$\therefore 0' = 1$ and $1' = 0$

Example 10 : In a boolean algebra B, show that for any $x \in B$, $(x')' = x$.

Solution : For $x \in B$,

$x + x' = 1$ and $x \cdot x' = 0 \quad \Rightarrow x' + x = 1$ and $x' \cdot x = 0$

But $x' + (x')' = 1$ and $x' \cdot (x')' = 0$

$\therefore x = (x')'$ ($\because$ the complement of x' is unique)

Example 11 : For every x, y in a boolean algebra B.

$(x + y)' = x' \cdot y'$ and $(x \cdot y)' = x' + y'$ (Demorgan's laws)

Solution : Now

$$\begin{aligned} (x + y) \cdot (x' \cdot y') &= ((x + y) \cdot x') \cdot y' \\ &= [(x \cdot x') + (y \cdot x')] \cdot y' \\ &= (0 + (y \cdot x')) \cdot y' \\ &= (y \cdot x') \cdot y' \\ &= (y \cdot y') \cdot x' \\ &= 0 \cdot x' = 0 \end{aligned}$$

$\therefore \quad (x + y) \cdot (x' \cdot y') = 0$ ..... (i)

Also

$$\begin{aligned} (x + y) + (x' \cdot y') &= x + (y + (x' \cdot y')) \\ &= x + [(y + x') \cdot (y + y')] \\ &= x + [(y + y') \cdot (y + x')] \\ &= x + (1 \cdot (y + x')) \\ &= x + (y + x') = x + (x' + y) \\ &= (x + x') + y \\ &= 1 + y \\ &= 1 \end{aligned}$$

$\therefore \quad (x + y) + (x' \cdot y') = 1$ .....(ii)

From results (i) and (ii), we can say that $x' \cdot y'$ is a complement of $(x + y)$.

Also, the complement is unique.

$\therefore \quad (x + y)' = x' \cdot y'$

Applying the principle of duality to the above rule, we get De Morgan's second rule, $(x \cdot y)' = x' + y'$.

Example 12 : In a boolean algebra, prove that
$(x + y)' \cdot (x' + y') = x' \cdot y' \ \forall \ x, y \in B$.

Solution :

$$\begin{aligned}
\text{LHS} &= (x + y)' \cdot (x' + y') \\
&= (x' \cdot y') \cdot (x' + y') && \text{(De-Morgan's law)} \\
&= (x' \cdot y') \cdot x' + (x' \cdot y') \cdot y' && \text{(Distributive law)} \\
&= x' \cdot (x' \cdot y') + x' \cdot (y' \cdot y') \\
& && \text{(Commutative law \& associative law)} \\
&= (x' \cdot x') \cdot y' + x' \cdot y' && (\therefore\ x \cdot x = x \text{ \& associative law}) \\
&= (x' \cdot y') + (x' \cdot y') \\
&= x' \cdot y' = \text{R.H.S.} && (\therefore\ x + x = x)
\end{aligned}$$

Hence $(x + y)' \cdot (x' + y') = x' \cdot y'$

5. Boolean function :

We know that $f(x) = 3x^2 + 5x + 2$, $x \in R$ is a real valued **function** of x.

We also know that $3x^2 + 5x + 2$ is called an **expression** where x is any $x \in R$. Here R is the set of real values. Similar to this when x is an element in B (the boolean algebra we can define an expression known as boolean expression and also a function which is called a boolean function. Now we shall take the discussion of boolean expression and boolean function.

Boolean Expressions :* Let $(B, +, \cdot, ', 0, 1)$ be a boolean algebra and $x_1, x_2, x_3, \ldots, x_n$ any elements in B.

A boolean expression is defined recursively as follows :

(1) $0, 1, x_1, x_2, x_3, \ldots, x_n$ are boolean expressions

(2) If x and y are boolean expressions, then $x + y$, $x \cdot y$ and x' are boolean expressions.

A boolean expression in $x_1, x_2, \ldots, x_n$ is denoted by $X(x_1, x_2, x_3, \ldots, x_n)$.

For example, if $x_1, x_2, x_3, x_4 \in B$ then $(x_1 \cdot x_2 \cdot x_3) + x_2' + (x_3' \cdot x_1' \cdot x_4)$ is a boolean expression.

If a boolean expression contains both + and $\cdot$ but brackets are not mentioned it is a tradition that $\cdot$ precedes over +

Value of a Boolean Expression#**:** Let $X(x_1, x_2, x_3, \ldots, x_n)$ be a boolean expression and $(a_1, a_2, a_3, \ldots, a_n) \in B^n$. Then $X(a_1, a_2, a_3, \ldots a_n)$ is said to be the value of the boolean expression for the n-tuple $(a_1, a_2, a_3, \ldots, a_n)$.

For example, let $X(x_1, x_2, x_3) = x_1 + x_2 + (x_1' \cdot x_3)$

Then the value of this boolean expression for $(1, 1, 1) \in B^3$ is,

Define value of a Boolean Expression with example (Dec. 2015)

$X(1, 1, 1) = 1 + 1 + (1' \cdot 1) = 1 + 1 + (0 \cdot 1) = 1 + 1 + 0 = 1$

Boolean Function : Let $X(x_1, x_2, x_3, \ldots, x_n)$ be a boolean expression. Then $f : B^n \to B$, $f(x_1, x_2, x_3, \ldots, x_n) = X(x_1, x_2, x_3, \ldots, x_n)$ is called a boolean function.

So $f: \{0, 1\}^2 \to \{0, 1\}$, $f(x_1, x_2) = (x_1 \cdot x_2') + x_2$ is a boolean function.

Here $f(0, 0) = (0 \cdot 0') + 0 = (0 \cdot 1) + 0 = 0 + 0 = 0$

$f(1, 0) = (1 \cdot 0') + 0 = (1 \cdot 1) + 0 = 1 + 0 = 1$

$f(0, 1) = (0 \cdot 1') + 1 = (0 \cdot 0) + 1 = 0 + 1 = 1$ and

$f(1, 1) = (1 \cdot 1') + 1 = (1 \cdot 0) + 1 = 0 + 1 = 1$

From the above, we can say that for different values of x_1 and x_2 we get different values of $f(x_1, x_2)$. The values of x_1 and x_2 are called **input** and the corresponding values of $f(x_1, x_2)$ are called **output.** Using these values of input and output we can prepare the table known as input-output table for a boolean function. For the above boolean function the input / output table will be as follows.

Input (x_1, x_2)		Output $f(x_1, x_2)$
x_1	x_2	$x_1 \cdot x_2' + x_2$
0	0	0
1	0	1
0	1	1
1	1	1

($\therefore$ $0' = 1$ and $1' = 0$)

Now we shall take some examples :

Example 13 : Let B = {0, 1}. Prepare an input/output table for the boolean function. *(March, 2015)*

$f : B^2 \to B$, $f(x_1, x_2) = x_1 \cdot x_2'$.

Now $f(1, 1) = 1 \cdot 1' = 1 \cdot 0 = 0,$

$f(1, 0) = 1 \cdot 0' = 1 \cdot 1 = 1,$

$f(0, 1) = 0 \cdot 1' = 0 \cdot 0 = 0,$

$f(0, 0) = 0 \cdot 0' = 0 \cdot 1 = 0$

$\therefore$ The input/output table will be as follows :

Input		Output
x_1	x_2	$x_1 \cdot x_2'$
0	0	0
0	1	0
1	0	1
1	1	0

6. Getting a boolean expression and boolean function from a given input/output table.

We shall try to understand the techniques by taking the following examples :

Example 14 : Prove that a function whose input/output bit-value table is as given below, is a boolean function.

Input			Output
x_1	x_2	x_3	$f(x_1, x_2, x_3)$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	0

rows with output 1.

We have to find a boolean expression such that $f(x_1, x_2, x_3) = X(x_1, x_2, x_3)$

For this, the method is as follows :

Select the rows in which output is 1. In each of these rows, for each x_i, if its value is 1, select x_i; if the value is 0, select x_i'. Combine these x_is or x_i's by applying $\cdot$. Add all these terms by applying +.

Here, in the first row, all three have value 1. So the term selected is $x_1 \cdot x_2 \cdot x_3$.

In the next row with output 1; x_1, x_2, x_3 have values 1, 0, 0 respectively. Hence, the corresponding term obtained is $x_1 \cdot x_2' \cdot x_3'$.

Similarly, the third term so obtained is $x_1' \cdot x_2 \cdot x'_3$.

$\therefore f(x_1, x_2, x_3) = (x_1 \cdot x_2 \cdot x_3) + (x_1 \cdot x_2' \cdot x_3') + (x_1' \cdot x_2 \cdot x_3')$.

This is a boolean expression. It can be checked that its bits-value table is the same as the given one.

Here, a term such as $x_1 \cdot x_2 \cdot x_3$ is called a minterm.

Hence, $f(x_1, x_2, x_3) = X(x_1, x_2, x_3)$

$$= (x_1 \cdot x_2 \cdot x_3) + (x_1 \cdot x_2' \cdot x_3') + (x_1' \cdot x_2 \cdot x_3')$$

This expression, using the minterms is called the canonical form as a sum of product.

Example 15 : Write down the boolean expression and the boolean function from the given input/output table :

Input			Output
x_1	x_2	x_3	$f(x_1, x_2, x_3)$
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	1

From the first row with output 1, we get the minterm $x_1 \cdot x_2 \cdot x_3$, from the second row with output 1, we get the minterm $x_1 \cdot x_2 \cdot x_3'$, from the third row with output 1, we get the minterm $x_1 \cdot x_2' \cdot x_3$, and from the last row with output 1, we get the minterm $x_1' \cdot x_2' \cdot x_3'$.

We get the boolean expression

$X(x_1, x_2, x_3) = (x_1 \cdot x_2 \cdot x_3) + (x_1 \cdot x_2 \cdot x_3') + (x_1 \cdot x_2' \cdot x_3) + (x_1' \cdot x_2' \cdot x_3')$, and the boolean function is

$$\begin{aligned} f(x_1, x_2, x_3) &= X(x_1, x_2, x_3) \\ &= (x_1 \cdot x_2 \cdot x_3) + (x_1 \cdot x_2 \cdot x_3') + (x_1 \cdot x_2' \cdot x_3) \\ &\quad + (x_1' \cdot x_2' \cdot x_3') \end{aligned}$$

Example 16 : For the element x, y of a boolean algebra, prove that $x \cdot y' = 0 \Leftrightarrow x \cdot y = x$.

Solution :

(1) Let $x \cdot y' = 0$ • we want to prove $x \cdot y = x$

$$\begin{aligned} \text{L.H.S.} &= x \cdot y \\ &= x \cdot y + 0 \\ &= x \cdot y + x \cdot y' && (\because x \cdot y' = 0 \text{ - given}) \\ &= x \cdot (y + y') && \text{(Distributive law)} \\ &= x \cdot 1 \\ &= x \\ &= \text{R.H.S.} && \therefore x \cdot y' = 0 \Rightarrow x \cdot y = x \quad \text{..... (1)} \end{aligned}$$

(2) Let $x \cdot y = x$ we want prove $x \cdot y' = 0$.

$$\begin{aligned} \text{L.H.S.} &= x \cdot y' \\ &= (x \cdot y) \cdot y' && (\because x = x \cdot y \text{ – given}) \\ &= x \cdot (y \cdot y') && \text{(Associative law)} \end{aligned}$$

$= x \cdot 0$

$= 0$

$=$ R.H.S. $\quad \therefore x \cdot y = x \Rightarrow x \cdot y' = 0$ ..... (2)

$\therefore$ From (1) & (2),

$x \cdot y' = 0 \Leftrightarrow x \cdot y = x.$

Example 17 : Prove that :

$x \cdot z = y \cdot z$ and $x \cdot z' = y \cdot z' \Rightarrow x = y.$

Solution : L.H.S. $= x = x \cdot 1$ $\quad$ ($\because$ 1 is unit element)

$= x \cdot (z + z')$ $\quad$ ($\because z + z' = 1$)

$= (x \cdot z) + (x \cdot z')$ $\quad$ (by dist. law)

$= (y \cdot z) + (y \cdot z')$ $\quad$ (Given)

$= y \cdot (z + z')$ $\quad$ (by dist. law)

$= y \cdot 1$ $\quad$ ($\because z + z' = 1$)

$= y =$ R.H.S.

$\therefore \quad x \cdot z = y.\, z$ and $x \cdot z' = y \cdot z' \Rightarrow x = y$

Example 18 : D_{21} {1, 3, 7, 21}, $\forall\ x, y \in D_{21}$

$x + y$ = LCM of x, y

$x \cdot y$ = GCD of x, y

$$x' = \frac{21}{x}$$

Show that D_{21} is a Boolean Algebra *(April 2010)*

Solution :

Table of +

+	1	3	7	21
1	1	3	7	21
3	3	3	21	21
7	7	21	7	21
21	21	21	21	21

Table of •

•	1	3	7	21
1	1	1	1	1
3	1	3	1	3
7	1	1	7	7
21	1	3	7	21

'	1	3	7	21
	21	7	3	1

It is obvious from the first two tables that $\forall\ x, y \in D_{21}$, $x + y \in D_{21}$ and $x \cdot y \in D_{21}$

$\therefore$ + and • are binary operations.....................(1)

From the first two tables it is obvious that there is a symmetry about the principal diagonal which indicates that the commutative laws hold ....(2)

We shall verify the associative laws for 1, 3, & 7.

$1 + (3 + 7) = 1 + 21 = 21, \qquad (1 + 3) + 7 = 3 + 7 = 21$

and $1.\ (3.7) = 1.1 = 1, \qquad (1.3).7 = 1.7 = 1$

It may be verified similarly for other cases ....(3)

To check the distributive laws, consider 3, 7, 21.

$3 + 7.21 = 3 + 7 = 21$, $(3 + 7) \bullet (3 + 21) = 21 \bullet 21 = 21$

$3 \bullet (7 + 21) = 3.21 = 3$, $3.7 + 3.21 = 1 + 3 = 3$.

It may be verified for other cases ......(4)

For the zero and unit elements, we check

$1 + 1 = 1$, $1 + 3 = 3$, $1 + 7 = 7$, $1 + 21 = 21$; (1 is zero)

$1.21 = 1$, $3.\ 21 = 3$, $7.21 = 7$, $21.21 = 21$ (21 is unit)

$\therefore$ 1 is the zero element and 21 is the unit element ....(5)

Also $3 + 7 = 21$, $3.7 = 1$; $1 + 21 = 21$, $1.21 = 1$ gives

$1' = 21$, $21' = 1$, $3' = 7$, $7' = 3$

$\therefore$ There is an existence of a complement

$\therefore$ $(D_{21}, +, \bullet, ', 1, 21)$ is a Boolean algebra.

Example 19 : Simplify the Boolean Expression

$x + x' \bullet (x + y) + y\ .\ z$ *(April, 2010)*

Solution : $x + x' \bullet (x + y) + y \bullet z.$

$= 1 \bullet (x + y) + y \bullet z$

$= (x + y) + y \bullet z$

$= x + (y.1 + y \bullet z)$ $(\because y = y\ .\ 1)$

$= x + y \bullet (1 + z)$

$= x + y \bullet 1$ $(\because 1 + z = 1)$

$= x + y.$

which is the simplified form.

Example 20 : Show that D_9 is not a Boolean Algebra where

D_9 = {division of 9} and

$\forall\ x, y \in D_9$

$x + y$ = L.C.M. of x & y

$x \bullet y$ = G.C.D. of x and y

$x' = \dfrac{9}{x}.$ *(April, 2010; March, 2015)*

Solution : $D_9 = \{1, 3, 9\}$

Table of +

+	1	3	9
1	1	3	9
3	3	3	9
9	9	9	9

Table of •

•	1	3	9
1	1	1	1
3	1	3	3
9	1	3	9

'	1	3	9
	9	3	1

It is obvious from the first two tables that

$\forall\, x, y \in D_9,\ x + y \in D_9$ and $x \cdot y \in D_9$

$\therefore$ + and $\cdot$ are binary operations ......(1)

From the first two tables there is a symmetry about principal diagonal which indicates that the commutative laws hold ....(2)

We shall verify the associative law for 1, 3, 9

$(1 + 3) + 9 = 3 + 9 = 9,\ \ 1 + (3 + 9) = 1 + 9 = 9$ &

$(1 \cdot 3) \cdot 9 = 1 \cdot 9 = 1,\ 1. (3 \cdot 9) = 1 \cdot 3 = 1$

Hence associative laws hold .....(3)

$1 \cdot (3 + 9) = 1 \cdot 9 = 1$

$1 \cdot 3 + 1 \cdot 9 = 1 + 1 = 1$

$\therefore 1 \cdot (3 + 9) = 1 \cdot 3 + 1 \cdot 9$

$1 + (3 \cdot 9) = 1 + 3 = 3$

$(1 + 3) \cdot (1 + 9) = 3 \cdot 9 = 3$

$\therefore 1 + 3 \cdot 9 = (1 + 3) \cdot (1 + 9)$

$\therefore$ Distributive laws hold ....(4)

Here $\left.\begin{array}{l} 1 + 1 = 1 \\ 1 + 3 = 3 \\ 1 + 9 = 9 \end{array}\right\} \Rightarrow 1$ is the zero element

$\left.\begin{array}{l} 1 \cdot 9 = 1 \\ 3 \cdot 9 = 3 \\ 9 \cdot 9 = 9 \end{array}\right\} \Rightarrow 9$ is the unit element

$\therefore$ zero and unit elements exist .....(5)

For complementations

$1 + 9 = 9,\ 1 \cdot 9 = 1$ $\qquad \therefore 1' = 9$

$9 + 1 = 9,\ 9 \cdot 1 = 1$ $\qquad \therefore 9' = 9$

$3' + 3 = 3,\ 3 \cdot 3 = 3 \neq 9$ $\qquad \therefore 3'$ does not exist.

$\therefore \{D_9, +\cdot, ', 1, 9\}$ is not a Boolean algebra. ($\because 3' = 3$ is not possible)

Example 21 : Find the product sum canonical firm of $f(x_1\ x_2)$ $= x_1 \cdot x_2 + x_1' \cdot x_2 + x_1 \cdot x_2'$ *(April, 2010; Dec., 2015)*

Solution :

$f(x_1 \cdot x_2)$ = Sum of product given. Taking its dual form we shall have, Product sum in the form :

$f(x_1 x_2) = (x'_1 + x'_2) \bullet (x_1 + x'_2) \bullet (x'_1 + x_2)$

I	II	III	IV – Dual of Column III
x_1	x_2	$x_1 \bullet x_2 + x'_1 \bullet x_2 + x_1 \bullet x'_2$	$(x'_1 + x'_2) \bullet (x_1 + x'_2) \bullet (x'_1 + x_2)$
1	1	$1 + 0 + 0 = 1$	$0.1.1 = 0$
1	0	$0 + 0 + 1 = 1$	$1.1.0 = 0$
0	1	$0 + 1 + 0 = 1$	$1.0.1 = 0$
0	0	$0 + 0 + 0 = 0$	$1.1.1 = 1$

$f(x_1, x_2) = (x'_1 + x'_2) \bullet (x_1 + x'_2) \bullet (x'_1 + x_2)$
which is the product of sum.

EXERCISE

1. Let D_9 is the set of positive divisors of 9. Define +, •, ' on D_9 as follows :
   $a + b$ = l.c.m. of a and b
   $a \bullet b$ = g.c.d. of a and b and
   $$a' = \frac{9}{a}.$$
   Prove that (D_9, +, •, ', 1, 9) is not a boolean algebra.
   (l.c.m. = least common multiple and g.c.d. = greatest common divisor)
2. B = $\{a, b\}$. Show that
   (P(B), $\cup$, $\cap$, ', ϕ, $\{a, b\}$) is a boolean algebra.
3. Let $D_{10} = \{1, 2, 5, 10\}$, $x + y$ = l.c.m. of x and y,
   $x \bullet y'$ = g.c.d. of x and y,
   $$x' = \frac{10}{x}$$
   Show that (D_{10}, +, •, ', 1, 10) is a boolean algebra.

For the elements x, y, z of • boolean algebra, prove the following statements : (Exs. 4 to14)

4. $(x + y) \bullet (x + y) = x + (x \bullet y) + y$
5. $x + y = x + z,\ x' + y = x' + z \Rightarrow y = z$
6. $x \bullet y' = 0 \Leftrightarrow x \bullet y = x$
7. $x + y = 0 \Leftrightarrow x = y = 0$
8. $x = 0 \Leftrightarrow y = (x \bullet y') + (x' \bullet y),\ \forall y$
9. $x + (x' \bullet y) = x + y$

10. $x \cdot (x' + y) = x \cdot y$

11. $(x \cdot y) + (x \cdot y') = x$

12. $x = y \Leftrightarrow (x \cdot y') + (x' \cdot y) = 0$

13. $(x + y) \cdot (x' + z) = (x \cdot z) + (x' \cdot y) + (y \cdot z)$

14. $(x \cdot y)' = x' + y'$

15. Find the bits value of $((x_1 \cdot x_2') + x_3) \cdot x_1'$ for

(1) $x_1 = 0, x_2 = 0, x_3 = 1$ (2) $x_1 = 1, x_2 = 1, x_3 = 1$

[Ans. : (1) 1 (2) 0]

(Dec., 2015)

16. Construct the input/output table for :

(1) $f(x_1, x_2) = x_1' \cdot x_2$ (2) $f(x_1, x_2) = (x_1 \cdot x_2)' + x_2$

(3) $f(x_1, x_2, x_3) = (x_1 \cdot x_2') \cdot x_3$

[Ans. : (1)

x_1	x_2	x_1'	$x_1' \cdot x_2$
1	1	0	0
1	0	0	0
0	1	1	1
0	0	1	0

(2)

x_1	x_2	$x_1 \cdot x_2$	$(x_1 \cdot x_2)'$	$(x_1 \cdot x_2)' + x_2$
1	1	1	0	1
1	0	0	1	1
0	1	0	1	1
0	0	0	1	1

(3)

x_1	x_2	x_3	x_2'	$x_1 \cdot x_2'$	$(x_1 \cdot x_2') \cdot x_3$
1	1	1	0	0	0
1	1	0	0	0	0
1	0	1	1	1	1
1	0	0	1	1	0
0	1	1	0	0	0
0	1	0	0	0	0
0	0	1	1	0	0
0	0	0	1	0	0

]

17. Obtain the boolean expression and the boolean function from the given input / output table :

Input			Output
x_1	x_2	x_3	s
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	1

[Ans. : $f(x_1, x_2, x_3) = X(x_1, x_2, x_3) = (x_1 \cdot x_2 \cdot x_3) + (x_1 \cdot x_2 \cdot x_3')$ $+ (x_1 \cdot x_2' \cdot x_3) + (x_1' \cdot x_2' \cdot x_3')$]

18. Show that the following boolean expressions are equivalent. Express them in the simplest form as a sum of products :

(1) $(x + y) \cdot (x' + z) \cdot (y + z)$

(2) $(x \cdot z) + (x' \cdot y) + (y \cdot z)$

(3) $(x + y) \cdot (x' + z)$

(4) $(x \cdot z) + (x' \cdot y)$

[Ans. : Each expression is $xyz + xy'z + x'yz + x'yz'$]

19. Let $n \in$ N and B = $\{a \mid n$ is divisible by $a\}$.

Let $a + b$ = lcm of a and b, $a \cdot b$ = gcd of a and b and $a' = \frac{n}{a}$.

Prove that (B, +, •, ', 1, n) is a boolean algebra if, and only if, n is not divisible by square of any prime. i.e. n is squarefree.

University Questions

1. **Answer the following :** **(April 2010)**

(1) Explain duality in Boolean Algebra.

(2) Show that D_{21} is a Boolean Algebra, *(Dec., 2015)*

Where $D_{21} = \{1, 3, 7, 21\}$ and

$\forall\ x, y, \in D_{21}$,

$x + y$ = L.C.M. of x, y

$x \cdot y$ = G.C.D. of x, y

$x' = \frac{21}{x}$

(3) Simplify the Boolean Expression.
$x + x' \bullet (x + y) + y \bullet z$

[**Ans.** : $x + y$]

(4) Show that D_9 is not a Boolean Algebra where D_9 = {divisors of 9} and $\forall x, y \in D_9$ *(March, 2015)*
$x + y$ = L.C.M. of x and y
$x \bullet y$ = G.C.D. of x and y

$x' = \frac{9}{x}$. *(Dec., 2015)*

(5) Find the product sum canonical form of
$f(x_1, x_2) = x_1 \bullet x_2 + x'_1 \bullet x_2 + x_1 \bullet x'_2$

[**Ans.** : $(x'_1 + x'_2) \bullet (x_1 + x'_2) \bullet (x'_1 + x_2)$]

(Oct./Nov. 2009)

2. Answer the following :

(1) Define Boolean Algebra.

(2) Show that D_6 = {Divisor of 6} is a Boolean Algebra where $\forall a, b, \in D_6$
$a + b$ = L.C.M. of a, b
$a \bullet b$ = G.C.D. of a, b

and $a' = \frac{6}{a}$.

(3) Express $f(x_1, x_2, x_3) = (x_1 \bullet x_2) + (x_1 \bullet x_3) + (x_2 \,.\, x_3)$
As a sum of product in three variables

[**Ans.** : $(x'_1 + x'_2) \bullet (x'_1 + x'_3) \bullet (x'_2 + x'_3)$]

(4) Show that D_8 is not a Boolean Algebra where D_8 (Divisor of 8) $\forall$ $a, b, \in D_8$,
$a + b$ = L.C.M. of a, b
$a \bullet b$ = G.C.D. of a, b

$a' = \frac{8}{a}$

(5) In a Boolean algebra prove that
$(x + y)' \bullet (x' + y') = x' \bullet y'$; $\forall$ $x, y \in$ B

■ ■ ■