2 FUNCTIONS

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: SYLLABUS :

Functions

1. Meaning and Definition of a Function :

We are familiar with the word function. The demand of a commodity depends upon its price, hence we say that demand is a function of price. Similarly the price of railway ticket is a function of distance. Generally by function we understand the relationship between two variables. In this chapter we shall try to understand the exact mathematical meaning of a function.

Suppose there are 20 students in a class and all of them have different roll numbers. The roll numbers of three students-Mahesh, Tapan and Himanshu are respectively 7, 12 and 14. If we denote the names of these three students by a set A and the roll numbers of all the students of the class by a set B, then

A = {Mahesh, Tapan, Himanshu}

B = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20}

Here each student of set A i.e. each element of set A can be associated uniquely with element of set B by some definite relationship. In this case the relationship is student and his roll number.

i.e, Roll number of Mahesh = 7

Roll number of Tapan = 12

Roll number of Himanshu = 14

This type of relationship is known as a function.

Suppose : A = {1,2,3} and B = {1,4,9} are two given sets. Here each member of set A is associated with one and only one member of set B by some definite relationship. It is seen that the square of each element of set A is in set B.This type of relationship is called a function. The function is generally denoted by f, and it is expressed as $f : A \to B$ i.e. f is a function from A to B.

Definition of a function :

If A and B are two non empty sets and if each member of set A is associated with one and only one member of set B by some relationship or correspondence, then that relationship or correspondence is said to be a function from A to B. If f is such a correspondence then symbolically the function can be represented by $f : A \to B$ i.e. f is a function from A to B.

2. Domain, Co-domain and Range of a Function :*

If $f : A \to B$ is a function then set A is called domain of the function and it is denoted by D_f. The set B is called Co-domain of function f. Each element of domain should be associated uniquely with an element of co-domain by some correspondence f. In otherwords each element of the domain must have an unique image in co-domain. The functions can also be respresented by symbols f, g, h, t, etc.

The image of each element x of the domain obtained by the relationship f can also be denoted by $f(x)$. If $2 \in A$ then its image is represented by $f(2)$. The set obtained of all the images of the elements of the domain by the relationship f is called the Range of the function f, and it is denoted by $f(A)$.

i.e. $f(A) = \{f(x) / x \in A\}$

The set of the images of all the elements of A i.e. the range of the function f is a subset of B. So $f(A) \subset B$. The range is denoted by R_f. It should be clearly noted that **the range of a function is a subset of its co-domain or co-domain itself.**

This will be clear by few illustrations.

Illustration 1 : Give the domain co-domain and range of the function $f : A \to B$; A = {1, 2, 3}; B = {1, 2, 3, 4, 5, 6, 7}, $f(x) = 2x$

Ans. :

Domain	$D_f = \{1, 2, 3\}$
Co-domain	$B = \{1, 2, 3, 4, 5, 6, 7\}$
Range	$R_f = \{f(1), f(2), f(3)\}$

* Define Domain and Range of a function. (Dec. 2007, Sept./Oct. 2009)

* Define Range of the function (Oct./Nov. 2009 – New Course)

$= \{2(1), 2(2), 2(3)\}$
$= \{2, 4, 6\}$

Range is a subset of Co-domain B.

Illustration 2 : Determine domain, co-domain and range of $f : A \to N$; $f(x) = 3x + 1$, where $A = \{1, 2, 3, 4\}$

Ans. :

Here, Domain $D_f = \{1, 2, 3, 4\}$

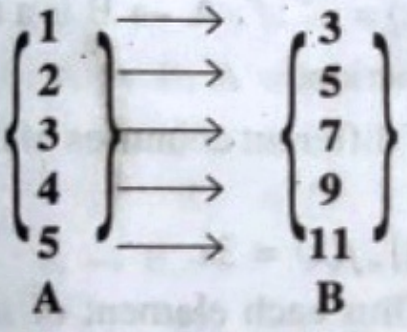
Co-domain $= N$

Range $R_f = \{f(1), f(2), f(3), f(4)\}$
$= \{3(1) + 1; 3(2) + 1; 3(3) + 1; 3(4) + 1\}$
$= \{4, 7, 10, 13\}$

It is clear that R_f is a subset of Co-domain N i.e. $R_f \subset N$.

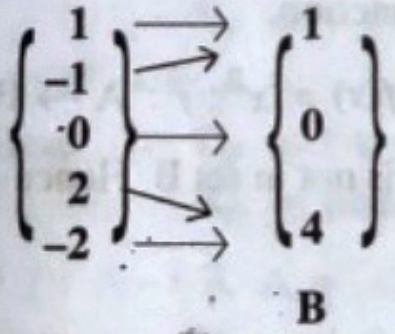
Illustration 3 : A few relationships are shown below. Determine wheather they are functions or not.

i)



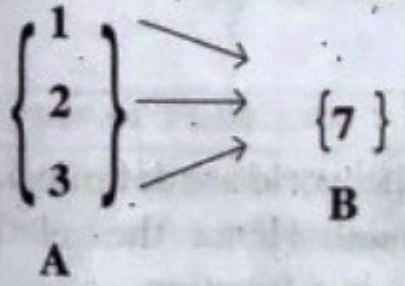
Here each element of set A is associated uniquely with element of set B by some relationship. So $A \to B$ is a function.

ii)



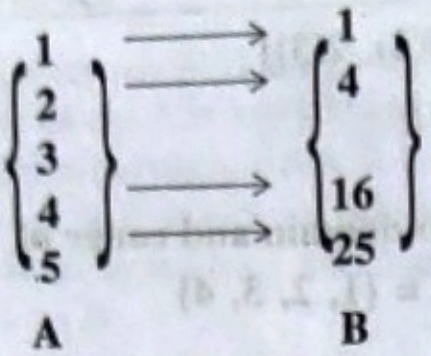
Here each element of set A is uniquely associated with element of set B. So $A \to B$ is a function.

iii)



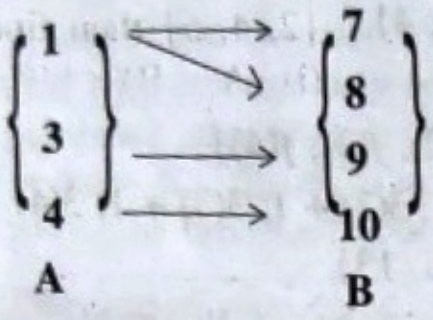
Here each element of set A is associated with one and only one element of set B. So $A \to B$ is a function.

(iv)



Here the image of element 3 of A is not in set B. So $A \to B$ is not a function.

(v)



Here element 1 of A is associated with two elements 7 and 8 of set B. Hence the association is not unique. So $A \to B$ is not a function.

Illustration 4 : State whether the following statements are true or false.

(i) $f: A \to B$, $A = \{1, 2, 3\}$, $B = \{3, 6, 9, 12\}$, $f(x) = 3x$ is a function.

(ii) $A = \{-1, 0, 1, 2\}$, $B = \{1, 4, 9\}$, $f(x) = x^2$; $f: A \to B$ is a function.

(iii) $f: N \to N$, $f(x) = 5x + 2$ is a function.

(iv) The relationship between the set of different countries of the world and their capitals is a function.

Ans. : (i) $A = \{1, 2, 3\}$; $B = \{3, 6, 9, 12\}$, $f(x) = 3x$.

Here $f(1) = 3$, $f(2) = 6$ and $f(3) = 9$. Thus each element of set A has unique image in set B. So $f: A \to B$ is a function.

(ii) $A = \{-1, 0, 1, 2\}$; $B = \{1, 4, 9\}$, $f(x) = x^2$; $f: A \to B$

Here for $0 \in A$, $f(0) = (0)^2 = 0$ which is not in set B. Hence $A \to B$ is not a function.

(iii) $f: N \to N$; $f(x) = 5x + 2$

Here $f(1) = 7$, $f(2) = 12$, $f(3) = 17$, ... Thus for all $x \in N$ the images 7, 12, 17.... are in set N.

$\therefore$ $f: A \to B$ is a function.

(iv) The capitals of different countries of the world are definite, i.e. each country is uniquely associated with its capital. Hence the relationship between different countries and their capitals is a function.

Illustration 5 : Give domain and range of the following functions.

(i) $f : A \to B$, $A = \{x / x \in N, 1 \le x \le 4\}$, $f(x) = 5x + 2$

Ans. : Here $D_f = \{1, 2, 3, 4\}$

$R_f = \{f(1), f(2), f(3), f(4)\}$

$= \{5(1) + 2; 5(2) + 2; 5(3) + 2; 5(4) + 2\}$

$= \{7, 12, 17, 22\}$

(ii) $g : N \to N$: $g(x) = x^2 + 1$, $x \in N$, $x < 5$

Ans. : Here $D_f = \{1, 2, 3, 4\}$

$R_f = \{g(1), g(2), g(3), g(4)\}$

$= \{2, 5, 10, 17\}$

(iii) $f : Z \to Z$, $f(x) = x + 2$, $x \in Z$

Ans. : Here $D_f = Z$

$R_f = \{..., -3, -2, -1, 0, 1, 2, 3 ...\}$

$= Z$

(iv) $f : A \to B$, $A = \{x / x \in N, 2 \le x < 7)$, $f(x) = 2x + 1$

Ans. : Here $D_f = \{2, 3, 4, 5, 6\}$

$R_f = \{5, 7, 9, 11, 13\}$

(v) $h : A \to B$, $A = (x / x \in Z, -3 < x < 3\}$, $h(x) = x + 1$

Ans. : Here $D_f = \{-2, -1, 0, 1, 2\}$

$R_f = \{-1, 0, 1, 2, 3\}$

(vi) $t : A \to B$; $A = (x / x \in N, 3 < x < 7\}$, $t(x) = 4x$

Ans. : Here $D_f = \{4, 5, 6\}$

$R_f = \{16, 20, 24\}$

(vii) $f : A \to B$, $A = \{x / x \in Z, -2 \le x < 2\}$, $f(t) = t^2 + 2$

Ans. : Here $D_f = \{-2, -1, 0, 1\}$

$R_f = \{6, 3, 2, 3\}$

$= \{2, 3, 6\}$

3. Types of Functions :

The following are main types of functions.

(i) One-one function (ii) Many-one function (iii) Constant function.

(i) One-one function :*#

Suppose $f: A \to B$ is a function. If for any two different elements of set A the images are different in set B, then f is said to be One-one function, i.e. for $f: A \to B$, if $a_1 \neq a_2$; for $a_1, a_2 \in A$, $f(a_1) \neq f(a_2)$ then f is said to be one-one function.

If A is a set of names of the students of a class and B is a set of roll numbers of the students of that class then the correspondence between the names of the students and their roll numbers is a one-one function. Similarly if $f: A \to B$ $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5, 6\}$, $f(x) = x + 2$

$f(1) = 3, f(2) = 4, f(3) = 5$

Here for different elements of set A, the images are different in set B. Hence $f: A \to B$ is a one-one function.

(ii) Many-one function :**

Suppose $f: A \to B$ is a function. If for any two different elements of set A, the images in the co-domain are same then $f: A \to B$ is said to be Many-one function.

i.e. for $f: A \to B$ $a_1 \neq a_2$; $a_1, a_2 \in A$, $f(a_1) = f(a_2)$ then f is said to be a many-one function.

e.g. $f(x) = x^2, x \in Z$;

$f(-1) = 1$ and $f(1) = 1$

Thus for two different elements of Z we get the same image. So $f(x) = x^2$, $x \in Z$ is a many-one function.

(iii) Constant function :*

Suppose $f: A \to B$ is a function. If for all the elements of set A, the image is the same in the co-domain, then f is said to be a constant function. Here the co-domain is of one element only.

e.g. $f: N \to N, f(x) = 5$

Then $f(1) = 5, f(2) = 5, f(3) = 5, \ldots$

$\therefore f$ is a constant function

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* Define : One – one function. (Nov.Dec. 2008)

** Define : Many – one function (March/April 2007)

*** Define constant function (Nov./Dec. 2009, April 2007)

\# When do you say a function is one to one ? (Dec. 2015)

4. Equal Functions :*#

If two functions f and g are defined on the same domain and for each element a of the domain, $f(a) = g(a)$, then f and g are said to be equal functions and we write $f = g$.

i.e. if

$f : A \to B$ and $g : A \to C$ are two functions and for each $a \in A$, $f(a) = g(a)$ then f and g are said to be equal functions. Both the functions must be defined on the same domain is a very essential condition.

If $f : A \to B$, $f(x) = x^2$ and

$g : A \to C$, $g(x) = 4x - 3$ and

$A = \{1, 3\}$ then

$f(1) = 1^2 = 1$; $g(1) = 4(1) - 3 = 1$

$f(3) = 3^2 = 9$; $g(3) = 4(3) - 3 = 9$

Thus for each element of A, the images obtained by functions f and g are same hence $f = g$.

5. Real Functions :

If for a function $f : A \to B$, the domain A of the function is a set of real numbers or its subset, then f is said to be a function of a real variable. Here each $a \in A$, is a real number i.e. $A \subset R$. Moreover if the range R_f of the function is also a set of real numbers or its subset, then f is said to be a real function. i.e. for a real function $A \subset R$ and $B \subset R$.

e.g. $f : N \to R$, for $x \in N$ the domain is a set of natural numbers N. Also the range R_f of f is a set of real numbers R, hence $f : N \to R$ is a real function of a real variable.

5.1 Usful real functions :

Demand :* Demand of a commodity can be defined as the quantity bought at a particular price at a particular place and within a specified period of time.

The demand of a commodity depends upon its price. Demand Function : The relationship between demand and price is called demand function. d = f(p) represents demand function.

Supply function : The relationship between supply S and price p is called a supply function.

* Define equal functions : (Nov. 2008, March-April 2007, 2009, 2010)

** What is demand ? (Nov./Dec., - 2008, 2012)

* When do you say that two functions are equal ? (March 2014)

$S = f(p)$ represents a supply function.

Cost Function : The total cost C in producing x units of an item can be given by $C = f(x)$ which is a cost function.

Revenue Function : The total amount received i.e. the revenue R depends upon number of units sold x. Hence $R = f(x)$ is revenue function. If the demand function is of the form $p = f(x)$ then the total revenue function can be given by

Total revenue = number of units sold $\times$ price per unit

$$\therefore \quad R(x) = x \cdot p$$

$$\therefore \quad = x \cdot f(x)$$

Profit function : Revenue function – Cost function is denoted by $P(x)$. $P(x)$ is called the profit function.

Fixed Cost and Variable Cost :* The cost which does not depend upon the number of units of production is called fixed cost.

The cost which depend on the number of units produced is called variable cost.

The cost function is the total of fixed cost and the variable cost.

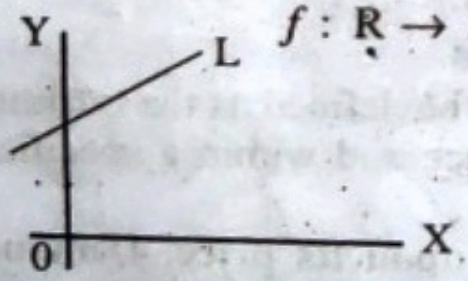
Break - even point : It is the point at which the profit function $P(x) = 0$. i.e. for break-even point, $R(x) = C(x)$.

Demand, supply, revenue etc represent real situations hence they are real functions.

6. Different Functions and their Graphs :

A function whose domain and range are sets of real numbers or their subsets is called real function of real variables. The following are some familiar types of real functions.

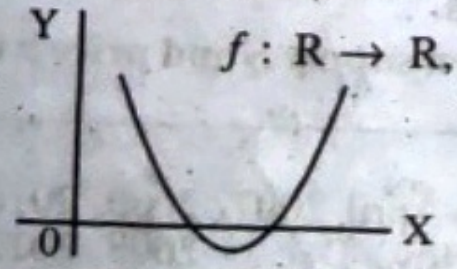
(1) Linear function



$f : R \to R,$

$f(x) = ax + b$ is a linear function. The graph of this function is a straight line. This type of function is called a linear function.

(2) Quadratic function



$f : R \to R,$

$f(x) = ax^2 + bx + c, (a \neq 0)$ is quadratic function. The graph of this function is a parabola.

What is fixed cost and variable cost ? (Nov. Dec, 2009)

(3) Polynomial function*

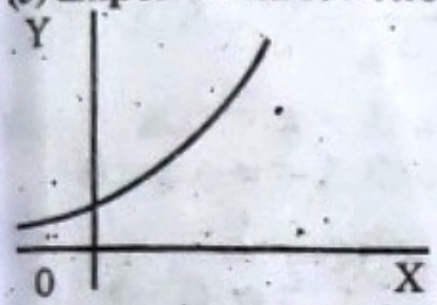
$$f : R \to R,\ f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$$

This function is called a polynomial function of degree n. By putting different values 3,4,5,of n, we get different functions. All these functions are known as polynomial functions. For different values of n, the nature of the graphs of the functions are different.

(4) Rational function

$f(x) = \dfrac{g(x)}{h(x)}$ where $h(x) \neq 0$ is called a rational function.

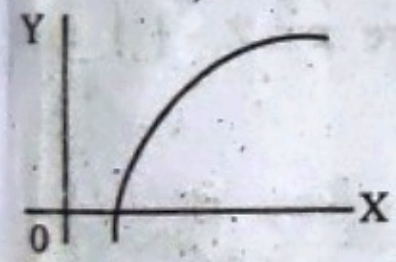
(5) Exponential function



$f : R \to R;\ f(x) = a^x,\ (a > 0 \text{ and } a \neq 1)$

This function is called exponential function and the graph of this function is as shown in the figure.

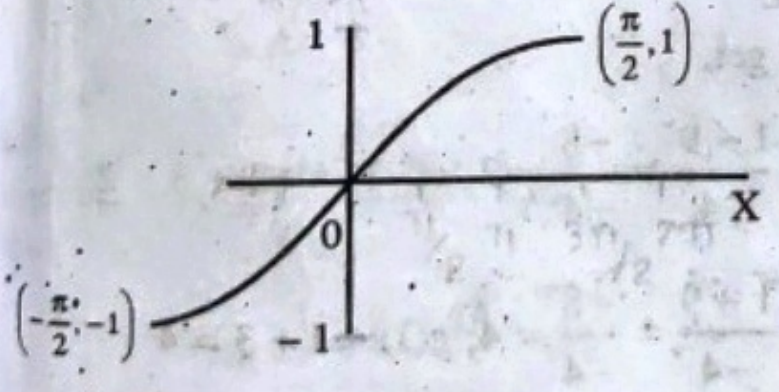
(6) Logarithmic function



$f : R^+ \to R,\ f(x) = \log_a x \quad (a \neq 1, a > 0)$

This type of function is called a logarithmic function and its graph is as shown in the figure.

(7) Trigonometric function



$y = \sin x$ is a trigonometrical function

Illustration 6 : $f : N \to N$ and $f(x) = 5x - 2$. If the range of function f is {3, 8, 13} find the domain of f.

* Define Polynomial function (Nov.Dec. 2008 (New)

Ans. : Here $f : N \to N$, $f(x) = 5x - 2$ and the range of function f is $\{3, 8, 13\}$

Now,

$$f(x) = 3 \quad \therefore \quad 5x - 2 = 3$$
$$\therefore \quad 5x = 3 + 2$$
$$\therefore \quad 5x = 5$$
$$\therefore \quad x = 1$$
$$f(x) = 8 \quad \therefore \quad 5x - 2 = 8$$
$$\therefore \quad 5x = 8 + 2$$
$$\therefore \quad x = \frac{10}{5} = 2$$
$$f(x) = 13 \quad \therefore \quad 5x - 2 = 13$$
$$\therefore \quad 5x = 13 + 2$$
$$\therefore \quad x = \frac{15}{5} = 3$$

The domain of the function is $D_f = \{1, 2, 3\}$.

Illustration 7 : If $f(x) = \dfrac{x^2 - 9}{x - 3}$ **where** $x \in Z - \{3\}$ **and** $g(x) = x + 3, x \in Z$. **Is** $f(x) = g(x)$ **?**

Ans. : $$f(x) = \frac{x^2 - 9}{x - 3}, x \in Z - \{3\}$$

$$g(x) = x + 3, \ x \in Z$$

Taking $x = -1, 1, 2$ we get.

$$f(-1) = \frac{(-1)^2 - 9}{-1 - 3} = \frac{1 - 9}{-4} = \frac{-8}{-4} = 2; \ g(-1) = -1 + 3 = 2$$

$$f(1) = \frac{(-1)^2 - 9}{-1 - 3} = \frac{1 - 9}{-4} = \frac{-8}{-4} = 4; \ g(1) = 1 + 3 = 4$$

$$f(2) = \frac{(2)^2 - 9}{2 - 3} = \frac{4 - 9}{-1} = \frac{-5}{-1} = 5; \ g(2) = 2 + 3 = 5$$

Here for each $x \in Z - \{3\}$, $f(x) = g(x)$

However the functions are not equal, as they are not defined on the same domain. The domain of f is $Z - \{3\}$ and the domain of g is Z. i.e. $f(3)$ is not defined while $g(3) = 6$, hence $f \neq g$

Illustration 8 : Examine whether the following functions are equal.

$$f(x) = \frac{x^2 - 9x + 14}{x - 2}, \; x \in Z - \{2\},$$

and $g(x) = x - 7$, where $x \in Z - \{2\}$.

Ans. :

$$f(x) = \frac{x^2 - 9x + 14}{x - 2}; \quad x \in Z - \{2\}$$

$$g(x) = x - 7; \; x \in Z - \{2\}$$

Taking $x = -2, 0, 1$ we get

$$f(-2) = \frac{(-2)^2 - 9(-2) + 14}{-2 - 2}, \quad g(-2) = -2 - 7$$

$$= \frac{4 + 18 + 14}{-4} = \frac{36}{-4} \qquad = -9$$

$$= -9$$

$$f(0) = \frac{(0)^2 - 9(0) + 14}{0 - 2}; \qquad g(0) = 0 - 7$$

$$= \frac{14}{-2} = -7 \qquad = -7$$

$$f(1) = \frac{(1)^2 - 9(1) + 14}{1 - 2}; \qquad g(1) = 1 - 7$$

$$= \frac{1 - 9 + 14}{-1} \qquad = -6$$

$$= \frac{15 - 9}{-1} = -6$$

Here for each $x \in Z - \{2\}$ we get $f(x) = g(x)$. Also both the functions are defined on the same domain. Therefore the functions are equal i.e. $f = g$.

Illustration 9 : If $f(x) = \frac{1}{x+1}$ prove that $f(-x) - f(x) = \frac{2x}{1-x^2}$.

Ans. :

$$f(x) = \frac{1}{x+1}$$

$$f(-x) = \frac{1}{-x+1} = \frac{1}{1-x}$$

$$\text{L.H.S.} = f(-x) - f(x)$$

$$= \frac{1}{1-x} - \frac{1}{1+x}$$

$$= \frac{(1+x)-(1-x)}{(1-x)(1+x)}$$

$$= \frac{1+x-1+x}{1-x^2}$$

$$= \frac{2x}{1-x^2}$$

$$= \text{R.H.S.}$$

$$\therefore \quad f(-x) - f(x) = \frac{2x}{1-x^2}.$$

Illustration 10 : If $f(x) = x(x+1)(2x+1)$, prove that $f(x) - f(x-1) = 6x^2$.

Ans. :

$$f(x) = x(x+1)(2x+1)$$

$$= (x^2 + x)(2x+1)$$

$$= x^2(2x+1) + x(2x+1)$$

$$= 2x^3 + x^2 + 2x^2 + x$$

$$= 2x^3 + 3x^2 + x \quad \ldots(1)$$

Now, $f(x-1) = (x-1)(x-1+1)\{2(x-1)+1)\}$

$$= (x-1)(x)(2x-2+1)$$

$$= (x^2 - x)(2x-1)$$

$$= x^2(2x-1) - x(2x-1)$$

$$= 2x^3 - x^2 - 2x^2 + x$$

$$= 2x^3 - 3x^2 + x \quad \ldots (2)$$

Using results (1) and (2),

$$\text{L.H.S.} = f(x) - f(x-1)$$

$$= 2x^3 + 3x^2 + x - (2x^3 - 3x^2 + x)$$

$$= 2x^3 + 3x^2 + x - 2x^3 + 3x^2 - x$$

$$= 6x^2$$

$$= \text{R.H.S.}$$

$$\therefore \; f(x) - f(x-1) = 6x^2$$

Illustration 11 : If $y = f(x) = \dfrac{ax+b}{cx-a}$ prove that $x = f(y)$.

Ans. : Here $y = \dfrac{ax+b}{cx-a}$

$$\therefore \quad y(cx - a) = ax + b$$

$$\therefore \quad cyx - ay = ax + b$$

$$\therefore \quad cyx - ax = ay + b$$

$$\therefore \quad x(cy - a) = ay + b$$

$$\therefore \quad x = \frac{ay+b}{cy-a}$$

$$\therefore \quad = f(y)$$

Illustration 12 : The cost of manufacturing 1000 pens is Rs. 17,000 and that of manufacturing 1,500 pens is Rs. 22,000. If the relationship between the number of units produced and its cost is linear, find the relation and also find the cost of manufacturing 2000 pens.

Ans : Suppose the linear relationship between the number of pens x and its cost y is $y = ax + b$

When $x = 1000$, $y = 17000$

When $x = 1500$, $y = 22000$

$$\therefore \quad 17000 = 1000a + b$$

$$22000 = 1500a + b$$

$$- \qquad - \qquad -$$

$$\therefore \quad -5000 = -500a$$

$$\therefore \quad a = 10$$

Now, $17000 = 1000\,(10) + b$

$\therefore \quad b = 7000$

$\therefore$ Linear relationship is

$$y = 10x + 7000$$

When $x = 2000$

$$y = 10(2000) + 7000$$
$$= 27000$$

Illustration 13 : It is observed that a quadratic function fits the data points (1, 9) (2, 14), (3, 23). Find the quadratic function and estimate y when $x = 4$.

Ans : Suppose the quadratic function is $y = ax^2 + bx + c$

when $x = 1$, $y = 9$

when $x = 2$, $y = 14$

when $x = 3$, $y = 23$

$\therefore \quad 9 = a + b + c$ ..... (i)

$14 = 4a + 2b + c$ ..... (ii)

$23 = 9a + 3b + c$ ..... (iii)

From equations (ii) and (i)

$$\begin{array}{r} 4a + 2b + c = 14 \\ a + b + c = 9 \\ - \quad - \quad - \quad - \\ \hline 3a + b = 5 \end{array}$$

From equations (iii) and (ii)

$$\begin{array}{r} 9a + 3b + c = 23 \\ 4a + 2b + c = 14 \\ - \quad - \quad - \quad - \\ \hline 5a + b = 9 \end{array}$$

$\therefore \quad 3a + b = 5$

$5a + b = 9$

$- \quad - \quad -$

$-2a = -4$

$\therefore \quad a = 2$

Now, $3a + b = 5$

$\therefore \quad 6 + b = 5$

$\therefore \quad b = -1$

And, $a + b + c = 9$

$\therefore \quad 2 - 1 + c = 9$

$\therefore \quad c = 8$

$\therefore$ Quadratic function is $y = 2x^2 - x + 8$

when $x = 4$

$$y = 2(4)^2 - 4 + 8 = 36$$

Illustration 14 : The demand function of bicycle is $d = f(p) = \sqrt{5600 - 4p}$. Find demand for price Rs. 1000. At what price of bicycle the demand will be 20 ?

Ans. : $d = f(p) = \sqrt{5600 - 4p}$

when p = Rs 1000, the demand of bicycle

$$d = \sqrt{5600 - 4(1000)}$$
$$= \sqrt{5600 - 4000}$$
$$= \sqrt{1600}$$
$$= 40$$

For d = 20, the price of a bicycle,

$$20 = \sqrt{5600 - 4p}$$
$$\therefore \quad 400 = 5600 - 4p$$
$$\therefore \quad 4p = 5600 - 400$$
$$\therefore \quad 4p = 5200$$
$$\therefore \quad p = 1300 \text{ Rs.}$$

Illustration 15 : The demand function of sugar in the market is $d = f(p) = 1605 - 5p^2$. Find demand of sugar for price Rs. 5, 6 and 8 per kg. respectively. At what price the demand of sugar will be zero ?

Ans. : The demand function is

$d = f(p) = 1605 - 5p^2$

When $p = 5$, $d = f(5) = 1605 - 5(5)^2$
$= 1605 - 125 = 1480$ kg

When $p = 6$, $d = f(6) = 1605 - 5(6)^2$
$= 1605 - (5 \times 36) = 1605 - 180$
$= 1425$ kg

When $p = 8$, $d = f(8) = 1605 - 5(8)^2$
$= 1605 - (5 \times 64) = 1605 - 320$
$= 1285$ kg

When the demand of sugar is zero

i.e. $d = 0$

$0 = f(p) = 1605 - 5p^2$

$\therefore \quad 0 = 1605 - 5p^2$

$\therefore \quad 5p^2 = 1605$

$\therefore \quad p^2 = \frac{1605}{5} = 321$

$\therefore \quad p = \sqrt{321} = 17.92$

Hence when the price of sugar is Rs. 17.92 per kg. its demand will be zero.

Illustration 16 : If the revenue function of a commodity is $R = 125x - \frac{3x^2}{2}$ find the revenue for demand $x = 20$.

Ans. : $R = 125x - \frac{3x^2}{2}$

When $x = 20$

$R = 125(20) - \frac{3(20)^2}{2}$

$= 2500 - 600$

$= 1900$

Illustration 17 : If the cost function of a commodity is $C = 1200 - 45x + 2x^2$, find the total cost for producing 25 units.

Ans. : $C = 1200 - 45x + 2x^2$

when $x = 25$

$C = 1200 - (45 \times 25) + 2(25)^2$

$= 1200 - 1125 + 1250$

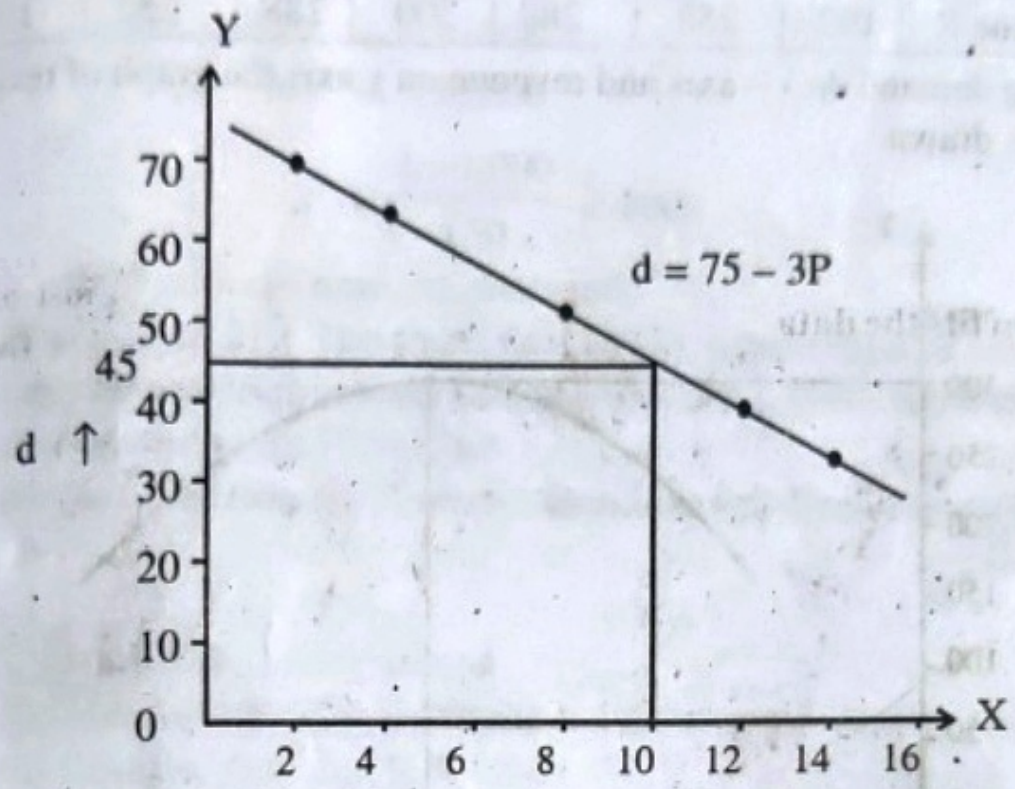
$= 1325.$

Illustration 18 : The demand function of a commodity is $d = 75 - 3p$. Draw the demand curve and find the demand when price is Rs. 10.

Ans. : For different values of price p, we shall obtain the different values of demand d.

p (price)	2	5	8	12	15
d (demand)	69	60	51	39	30

Taking price p on x axis and demand d on y axis the graph is drawn



When price is Rs. 10, the demand is 45 units as seen from the graph.

Illustration 19 : The demand function of a commodity is $x = 60 - 3p$. Find the revenue function and draw the revenue curve. At what price of the commodity the revenue will be maximum ? Find the maximum revenue and the price corresponding to it.

Ans. : The demand function is

$$x = 60 - 3p$$

$$\therefore \quad 3p = 60 - x$$

$$\therefore \quad p = 20 - \frac{x}{3}$$

Now revenue $\quad R = x \cdot p$

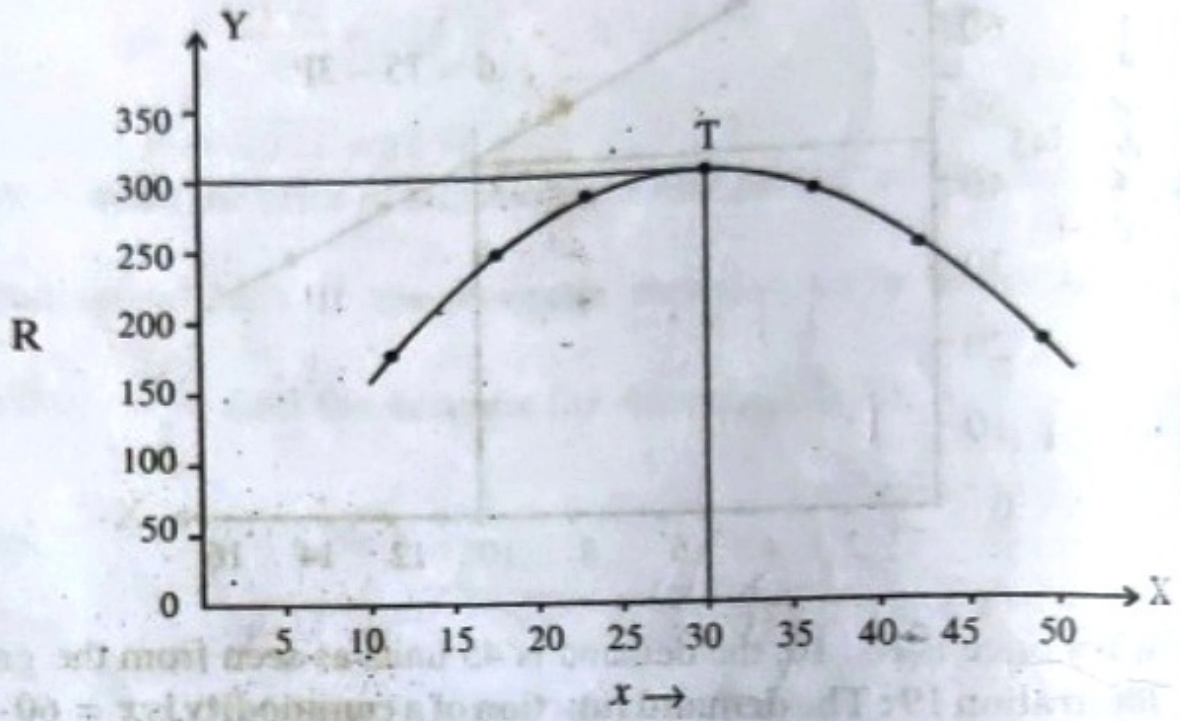
$$\therefore \quad R = x\left(20 - \frac{x}{3}\right)$$

$$\therefore \quad R = 20x - \frac{x^2}{3}$$

For different values of x, the values of R are obtained and tabulated as follows :

Demand x	12	18	24	30	36	42	48
Revenue R	192	252	288	300	288	252	192

Taking demand on x – axis and revenue on y axis the graph of revenue function is drawn.



The maximum revenue is obtained at the highest point T of the curve. From the graph the maximum revenue is R = 300 units and the demand for maximum revenue is $x = 30$.

Now, $\quad R = x \cdot p$

$$\therefore \quad p = \frac{R}{x} = \frac{300}{30} = 10$$

At maximum revenue the price is Rs. 10.

Illustration 20 : The fixed cost of a factory manufacturing pressure cookers is Rs. 1,50,000 and the variable cost per cooker is Rs. 200. If the selling price of a cooker is Rs. 350, find the number of cookers to be produced for no profit – no loss.

Ans. : Suppose for no profit – no loss x cookers are produced

$\therefore$ Total cost of production = $1,50,000 + 200x$

Revenue R = $350x$

For no profit – no loss,

Total revenue = Total cost

$\therefore \quad 350x = 1,50,000 + 200x$

$\therefore \quad 350x - 200x = 1,50,000$

$\therefore \quad 150x = 1,50,000$

$$\therefore \quad x = \frac{1,50,000}{150} = 1000$$

$\therefore$ 1000 cookers must be produced.

Illustration 21 : The fixed cost in the production of transistors is Rs 2,00,000 and the variable cost per unit is Rs. 1000. If the selling price of a transistor is Rs 1500, find.

(i) Cost function (ii) Revenue function (iii) Break–even point

(Dec., 2015)

Ans. :

If x transistors are produced

(i) Cost function $C = 2,00,000 + 1000x$

(ii) Revenue function $R = 1500x$

(iii) For breakeven point

Total revenue = Total cost

$\therefore \quad 1500x = 2,00,000 + 1000x$

$\therefore \quad 500x = 2,00,000$

$$\therefore \quad x = \frac{2,00,000}{500} = 400$$

Hence for breakeven, 400 transistors should be manufactured.

Illustration 22 : $f(x) = 10^x, x \in R$ and $g(x) = \log_{10}x, x \in R^+$. Verify the validity of the followings :

(i) $f(x + y) = f(x) f(y)$

(ii) $g(x + y) = g(x) + g(y)$

Ans. : (i) $f(x) = 10^x, x \in R$

L.H.S. $= f(x + y)$

$\therefore \quad f(x + y) = 10^{x+y}$

$= 10^x.10^y$

$\therefore \quad = f(x). f(y)$

$\therefore \quad f(x + y) = f(x).f(y)$

$\therefore$ Statement is valid

(ii) $g(x) = \log_{10}{}^{x}$, $x \in R^+$

L.H.S $g(x + y) = \log_{10}(x + y)$

$\neq \log_{10}{}^{x} + \log_{10}{}^{y}$

$\neq g(x).g(y)$

$\therefore$ The statement is not valid.

EXERCISE

1. Explain the meaning of a function. Give its definition.
2. State different types of functions with illustrations.
3. What are equal functions ?
4. What is a real function. Give illustrations.
5. Fill up the gaps :
   (i) $f(x) = 2x^2 + 3x - 1 \therefore f(-1) =$ .....
   (ii) $f(x) = x^2$ and $g(x) = 7x - 6$, $x \in \{1, 6\} \therefore f$ and g are .... functions
   (iii) $g(x) = 1$, $x \in N$ is a ..... function.
   (iv) If $f(x) = x^2 + 2^x$, $f(2) - f(0) =$ .....
   (v) $f(x) = x^2$, $x \in Z - \{0\}$ is a ..... function.
   (vi) $g(x) = \dfrac{x^3 - 1}{x^3 + 1} \therefore g(1) =$ .....
   (vii) $f: N \to N, f(x) = 3x, -1, x \in \{1, 2, 3\}$, then the range of f is ....
   (viii) $f(x) = x^2 - 3x \therefore f(1) - f(-1) =$.....
   (ix) $g(x) = 5x + 2$, $x \in N$ and $g(x) = 17$ then $x =$ .....
   (x) $f(x) = x^2 - x \therefore f(x + 1) - f(x) =$.....

   [Ans. : (i) –2 (ii) equal (iii) constant (iv) 7 (v) many–one (vi) 0 (vii) {2, 5, 8} (viii) –6 (ix) 3 (x) 2x
6. Determine the types of the following functions :
   (i) $g : N \to Z$, $g(x) = 3x - x^2$, $x \in N$.
   (ii) $f : N \to N$, $f(x) = 2x^2 + 3$; $x \in N$.
   (iii) $f : Z - \{2\} \to Z$; $f(x) = \dfrac{x^2 + x - 6}{x - 2}$, $x \in Z$.
   (iv) $f : R \to R$, $f(x) = 5$, $x \in R$.

(v) $h : R \to R$, $h(t) = t^2 + 3t - 12$.

[**Ans.** : (i) many–one (ii) one–one (iii) one–one (iv) constant (v) many–one]

7. $f : A \to B$, $f(x) = 2x - 3$ and $R_f = \{-3, 1, 0\}$. Fnd the domain A of f. [**Ans.** : $A = \left\{0, 2, \frac{3}{2}\right\}$]

8. $f(x) = 1 - 4x$ and $x \in \left\{-\frac{1}{2}, 0, 2\right\}$, find the range of the function.

[**Ans.** $\{3, 1, -7\}$]

9. If $f(x) = x^2$; $g(x) = 5x - 6$ and $x \in \{2, 3\}$, prove that $f = g$.

10. $f : Z - \{-1\} \to Z$, $f(x) = \dfrac{x^3 + 1}{x + 1}$, $x \in Z - \{-1\}$ and

$g : Z \to Z$, $g(x) = x^2 - x + 1$, $x \in Z$; examine whether the functions are equal.

[**Ans.** No, as they are not defined on the same domain].

11. $f(x) = 2x^2 - 1$, $g(x) = 2x - 1$, $x \in \{0, 1, 2\}$. Are the functions equal ? [**Ans.** : No].

12. If $f(x) = 1 - x - x^2$ find $f(1)$, $f(-1)$ and $f(-2)$.

[**Ans.** : $f(1) = -1$, $f(-1) = 1$, $f(-2) = -1$]

13. If $f(x) = 2x^2 - 2x + 3$ find $f(1)$, $f(0)$, $f(3)$ and $f(-2)$.

[**Ans.** : 3, 3, 15, 15]

14. If $f(x) = 1 - \dfrac{1}{1 - x^2}$ find $f(2)$, $f(-2)$ and $f(0)$.

[**Ans.** : $\frac{4}{3}, \frac{4}{3}, 0$]

15. If $g(x) = \dfrac{x(x-2)}{x-1}$ find $g(0) + g(-1) + g(3) + g(2)$

[**Ans.** : 0]

16. If $f(x) = \dfrac{x^2 - 2x + 7}{x^3 + 1}$, $x \in R - \{-1\}$, find $f(1) + f(0)$.

[**Ans.** : 10]

17. If $f(x) = x^2 + 4x + 5$ and $g(x) = 2x + 1$, prove that $f(1) - 2g(2) = 0$.

18. If the domain of $f(x) = \dfrac{x+3}{x+4}$ is $\{0, 2, 4, 6\}$ find R_f

[Ans. : $\{3/4, 5/6, 7/8, 9/10\}$]

19. If domain of $h(x) = \sqrt{x^2+3}$ is $\{1, 2, 3, 4\}$ find R_f.

[Ans. : $\{2, \sqrt{7}, \sqrt{12}, \sqrt{19}\}$]

20. If $R_f = \{3, 8, 13, 18\}$ of $f(x) = 5x - 2$ find its D_f.

[Ans. : $\{1, 2, 3, 4\}$]

21. If $f(x) = \dfrac{x^2 - x}{x+3}$ find $\dfrac{f(1) + f(2)}{f(-2) + f(0)}$. [Ans. : $\dfrac{1}{15}$]

22. If $f(x) = x^2 (x-1)^2$, $x \in R$, prove that $f(x+1) - f(x) = 4x^3$.

23. $f : N \to N$, $f(n) = \dfrac{n^2 (n+1)^2}{4}$ find $f(n) - f(n-1)$.

[Ans. : n^3]

24. If $f(x) = 5x^2 + 7x + 2$, find the value of x when $f(x) = 110$

[Ans. : $x = 4$, $x = -\frac{27}{5}$]

25. If $f(x) = \dfrac{x^2+1}{x^3 - x + 1}$, find $f(1) - f(0)$. [Ans. : 1]

26. If $f(x) = \dfrac{1}{x} + \dfrac{2}{x-3}$ $x \in R - \{0, 3\}$ find $f(1)$, $f(2)$, $f\left(\frac{1}{3}\right)$ and $f(-3)$.

[Ans. : $f(1) = 0$, $f(2) = -\frac{3}{2}$, $f(\frac{1}{3}) = \frac{9}{4}$, $f(-3) = -\frac{2}{3}$]

27. If $f(x) = \dfrac{1}{x}$, $x \in Z - \{-1, 0, 1\}$, Prove that

$$f(x+1) - f(x-1) = \frac{2}{1-x^2}.$$

28. If $f(x) = x^3 - 2x + \dfrac{1}{x}$ prove that $f(x) + f(-x) = 0$.

29. If $f(x) = x^3$, $g(x) = 3x^2 - 2x$, $x \in \{0, 1, 2\}$ prove that $f = g$.

30. The demand function of a transistor is $d = f(p) = \sqrt{4850 - 5p}$ If the price of a transistor is Rs. 250, find the demand. [Ans. : 60]

If the demand function is $d = f(p) = 200 - 3p^2$, find the demand when $p = 5$.

[Ans. : 125]

The supply function of a commodity is $S = f(p) = 5p^2 + 10p - 120$, find the supply at price Rs. 10.

[Ans. : 480]

If the demand function is $x = \frac{50 - 2p}{3}$, find the revenue function. Also find the revenue when the demand is of 10 units.

[Ans. : $R = \frac{50x - 3x^2}{2}$, R = 100 unit]

A firm manufactures x pressure cookers at a total cost of $\frac{x^2}{30} + 13x + 5000$. Find the cost of manufacturing 30 pressure cookers.

[Ans. : Rs. 5420]

The demand function of a commodity is $d = 25 - \frac{p}{4}$. Draw the demand curve and from it, find the demand when the price is 40 and the price when the demand is of 20 units.

[Ans. : $d = 15$, $p = 20$]

The supply function of a commodity is $S = 5p - 1$. Draw the supply curve and determine the supply when the price is Rs. 12.

[Ans. : 59 units]

The expenditure of a hostal depends upon the number of students. When there are 40 students the expenditure is Rs. 50,000 and for 0 students the expenditure is Rs. 62,000. If the relationship between the number of students and hostel expenditure is linear, ind the relationship and also find the expenditure when there are 0 students in the hostel.

[Ans. : $y = 1200x + 2000$, 74,000]

The life expentancy of males in 1987 in a country is 70 years. In 962 it was 60 years. Assuming the life expentancy to be a linear unction of time, make a prediction of the life expentancy of males n that country in 1997.

[Ans. : $E = 0.4t + 60$, Where t = year − 1962, 74 years]

39. It is observed that a quadratic function $y = ax^2 + bx + c$ fits the points (– 1, 8), (1, 4) and (2, 5) find the constants a, b and c and estimate y when $x = 4$.

[**Ans. :** $a = 1, b = -2, c = 5$; 13]

0. A shopkeeper earns Rs. 380 in the first week Rs. 660 in the second week and Rs. 860 in the third week. On plotting the points (1,380), (2,660) and (3,860) the shopkeeper feels that a quadratic function may fit the data. Find the quadratic function that fits the data. Also estimate the earning of the fourth week.

[**Ans. :** $y = -40x^2 + 400x + 20$; 980]

1. The fixed cost of a factory is Rs. 60,000 and the variable cost per unit of production is Rs 50. If the selling price per unit is Rs. 100 find breakeven point. [**Ans. :** 1200 units]

2. The fixed cost of a factory producing a particular type of toys is Rs. 7050 and the variable cost per toy is Rs. 20. If the selling price per toy is Rs. 50, find the number of toys to be produced for no profit – no loss ? [**Ans. :** 235 toys]

3. The cost function of an item is $C(x) = 4x + 770$ and the selling price per item is Rs. 15. Find breakeven point. If the profit is Rs. 1100, find the number of units produced. [**Ans. :** 70, 170]

4. The fixed cost of a factory producing buckets is Rs. 2500 and the variable cost per bucket is Rs 11. The tax on each bucket is Re 1 and the selling price per bucket is Rs 22. Find the number of buckets to be produced for no profit – no loss. If 300 buckets are produced, find the profit. If the loss is Rs. 500, find the number of buckets produced.

[**Ans. :** 250; profit Rs. 500; 200 buckets]

5. A text book publisher finds that the production cost of each book is Rs. 30 and the fixed cost is Rs. 15,000. If each book can be sold for Rs. 45, then determine.

(i) The cost function

(ii) The revenue function

(iii) The break-even point

[**Ans. :** (i) $c(x) = 30x + 15000$ (ii) $R(x) = 45x$, (iii) Break even point 1000 books.]

46. The total cost and the total revenue of a company that produces and sells x units of a particular product are respectively.
$C(x) = 5x + 350$; $R(x) = 50x - x^2$
Find
(i) Break even values
(ii) The values of x that produce profit.
(iii) The values of x that produce loss
[Ans. : (i) $x = 10$ or 35 (ii) when $10 < x < 35$ (iii) $x < 10$ and $x > 35$]

47. If $f(x) = x(x-1)(2x-1)$ then prove that
$f(x+1) - f(x+2) = -6(x+1)^2$.

University Questions
Theory Questions & Problems

1. Define equal function. *(Nov./Dec. 2008, April 2009)*
2. Define Range of the function. *(Oct./Nov. 2009)*
3. Define One – one function. *(Dec. 2015, March 2015)*
4. What is fixed Cost and variable cost ? *(Nov/Dec. 2008)*
5. Define constant function, real function and equal function with illustration. *(March/April 2007)*

Problems

1. Let $f : N \to N$ and $f(x) = \dfrac{x^2(x+1)^2}{4}$ then find $f(x) - f(x-1)$. *(April 2010)*
[Ans. : x^3]

2. A magazine publisher finds that the variable cost of each magazine is Rs. 40 and the fixed cost is Rs. 18,000. If each book can be sold for Rs. 50 then determine the break even point. *(April 2010)*
[Ans. : $x = 1800$]

3. If $f(x) = \dfrac{1}{x}$, $x \in 2 - \{-1, 0, 1\}$ then prove that $f(x+1) - f(x-1) = \dfrac{2}{1-x^2}$ *(Dec, 2015; April 2010)*

4. It is observed that a quadratic function $y = ax^2 + bx + c$ fits the points $(-1, 8)$, $(1, 4)$ and $(2, 5)$, then find the constants a, b and c. *(Dec. 2015; April 2010)*
[Ans. : $a = 1$, $b = -2$, and $c = 5$]

5. If $f(t) = t^2$ and $g(t) = 3t^2 - 2t$, $t \in \{0, 1, 2\}$ then verify that $f(t) = g(t)$. *(April, 2010)*

6. If the demand function is $d = f(p) = 2000 - 3p^2$ then find the demand when $p = 25$ and find p when demand is 4. *(April, 2010)*
[**Ans. :** (i) $d = 125$ (ii) $p = 25.79$]

7. Find the domain and range of f, where $f(x) = \sqrt{q - x^2}$, $x \in R$
(March/April 2007, April 2009, Dec. 2007)
[**Ans. :** (i) Domain = $\{x / -3 \le x \le 3, x \in R\}$
(ii) Range = $\{y / 0 \le y \le 3, y \in R\}$]

8. If $f(x) = \dfrac{x^2 - x}{x + 3}$ then find $\dfrac{f(0) + f(-2)}{f(1) + f(3)}$
(April 2009, Nov./Dec. 2008)
[**Ans.:** 6]

9. Examine whether following functions are equal.
$f : R^+ \to R^+; f(x) = \sqrt{x^2}$
$g : R^+ \to R^+; g(x) = |x|$
(Nov./Dec. 2008, April 2009)
[**Ans. :** Yes, $f = g$]

10. The demand function of bicycle is
$d = f(p) = \sqrt{5600 - 4p}$. Find demand for price Rs. 1000.
At what price of bicycle, the demand will be 20 ? *(April, 2009)*
[**Ans. :** (i) Demand = 40 (ii) Price = Rs. 1300]

11. If $f(x) = 5x^2 + 7x + 2$ then find x for $f(x) = 110$.
(Oct./Nov. 2009, Dec. 2007)
[**Ans. :** $x = 4$ or $x = -\dfrac{27}{5}$]

12. If $f(x) = x^3 - 2x + \dfrac{1}{x}$ then find $f(x) + f(-x)$.
(March 2015; Dec. 2015; Oct./Nov., 2009)
[**Ans.:** 0]

13. The fixed cost of a factory producing particular types of bag is Rs. 9000 and the variable cost per bag is Rs. 110. If the selling price per bag is Rs. 240 then find profit function.
(Oct./Nov. 2009; March, 2015)
[**Ans. :** $p(x) = 130x - 9000$]

14. If $f(x) = x^2 + 4x + 5$ and $g(x) = 2x + 1$ then prove that $f(1) - 2\,g(2) = 0$ *(Oct./Nov. 2009 Nov.Dec. '08; Dec., 2015)*

15. If $f(x) = \dfrac{x^2 + x}{x - 3}$ then find $\dfrac{f(2) + f(-1)}{f(-1) + f(-2)}$ *(Oct./Nov. 2009)* **[Ans. : 15]**

16. The total cost and total Revenue functions are given as $C(x) = 5x + 350$ and $R(x) = 50x - x^2$ then find the profit for $x = 10$. *(Oct./Nov. 2009)* **[Ans. : Profit = 0]**

17. If $f(x) = \dfrac{x(x+1)}{x-2}$; then find $f(-1) + f(0) + f(-3)$. *(Sept./Oct., 2009)* **[Ans. : $\dfrac{-6}{5}$]**

18. If $f(x) = 2x^2 - 1$, $g(x) = 2x - 1$, $x \in \{0, 1, 2\}$ then are the functions f and g equal ? *(Sept/Oct. 2009; Dec., 2015)* **[Ans. : f and g are not equal]**

19. $f : N \to N$ and $f(x) = 5x - 2$, if the range of function f is $\{3, 8, 13\}$ then find the domain of f. *(Sept. Oct. 2009, Nov./Dec. '08)* **[Ans. : $D_f = \{1, 2, 3\}$]**

20. If $f(x) = \dfrac{1}{x+1}$ then (i) prove that $f(-x) - f(x) = \dfrac{2x}{1 - x^2}$ *(Sept./Oct. 2009, March/April, 2007)*

21. If the cost function of a commodity is $C = 1200 - 45x + 2x^2$ then find the total cost for producing 25 units. *(Sept.Oct. 2009)* **[Ans. : 1325]**

22. If $f(x)\ x^2\ (x - 1)^2$, $x \in R$ then prove that $f(x + 1) - f(x) = 4x^3$ *(March/April 2007; Nov./Dec. 2008)*

23. The fixed cost in the production of monitor is Rs. 4,00,000 and the variable cost per unit is Rs. 2,000. If the selling price is Rs. 3,000 the find
    (i) Cost function
    (ii) Revenue function

(iii) Break even point. *(Nov./Dec. 2008)*

[Ans. : (i) $C(x) = 4,00,000 + 2000x$ (ii) $R(x) = 3000x$ (iii) Break – even point is at $x = 400$]

24. The demand function of a commodity is $d = 25 - \frac{P}{4}$ Find the demand d when price P is 40 and the price P when demand d is of 20 units. *(March/April 2007)*

[Ans. : (i) $d = 15$ (ii) $P = 20$]

25. If $f(x) = x(x+1)(2x+1)$ then prove that $f(x) - f(x-1) = 6x^2$ *(Dec. 2007)*

26. If $f(x) = \frac{x^2 - x}{x+3}$ then find $\frac{f(1) + f(2)}{f(-2) + f(0)}$ *(Dec. 2007)*

[Ans. : $\frac{1}{15}$]

27. If $y = f(x) = \frac{ax + b}{cx - a}$ then prove that $x = f(y)$ *(Dec. 2007)*

Objective Questions

1. Define equal functions. *(April, 2010)*

2. If $f(x) = 2x^2 + 3x - 1$ then find $f(2) - f(1)$ *(April, 2010)*

[Ans. : 9]

3. If $g(x) = 5x + 4$ and $g(x) = 24$ then find x. *(April, 2009)*

[Ans. : $x = 4$]

4. Define constant function. *(Oct./Nov., 2009)*

5. If $f(x) = x^2 - 2x$ then find $f(x+2) - f(x+1)$. *(Oct./Nov. 2009)*

[Ans. : $2x + 1$]

6. Define Domain and Range of a function. *(Sept./Oct. 2009, Dec. 2007)*

7. What is demand ? *(Nov./Dec. 2008)*

8. Define polynomial function *(Nov./Dec. 2008)*

9. Define : Many – one function. *(March/April 2007)*

10. If $f(x) = x^2 - x$ then find $f(x+1) - f(x)$. *(Dec. 2007)*

[Ans. : $2x$]

11. If $f(x) = 4x + 7$ and $f(x) = 3$ then find x. *(Sept/Oct. 2009)*

[Ans. : $x = -1$]

12. Find $f(-1) + f(2)$ for $f(x) = x^2 - x + 1$ *(March 2015)*

[Ans. : 6]