Date
Page

Name ⇒ SHUBHAM
Roll No ⇒ 87
class ⇒ BCA 3rd Sem
Notes ⇒ Mathematics - III

Date 01-08/2022

Date
Page 1

Mathematics

Differential Equation is an equation which involves diff. defficient or differentials.

for e.g. $\frac{dy}{dx} = 2x+2$

Two types of D.E

1. ordinary 2. Pertial

✡ ordinary Diff. equation : An O. D.E is that in which all the Diff. Coefficients are w.r.t single independent Variable

e.g. $\frac{dy}{dx} + x + 1 = 0$

✡ order of D.E. is the order of highest differential Coefficient occuring in it.

$$\left(\frac{dy}{dx}\right)^2 + 7x + 5 = 0$$

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2 = 0$$

order = 2 , degree = 1.

Date
Page 2

✿ Degree of D.E. ⇒ It is the Degree of highest differential coefficient when the equation has been made from free fractions and radicals as far as the diff. coeff. are concerned.

e.g.>
$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 2 = 0$$

order = 2
Degree = 1

Q>
$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/2}}{\frac{d^2y}{dx^2}} = 2$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/2} = 2\frac{d^2y}{dx^2}$$

Squaring both side.

$$\left[\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/2}\right]^2 = 4\left[\frac{d^2y}{dx^2}\right]^2$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^5 = 4\left[\frac{d^2y}{dx^2}\right]^2$$

Date
Page 3

order = 2, Degree = 2

Q> Linear and Non Linear Diff. Equation.

Sol A linear Eq. is said to be linear if the unknown function and all of its derivative occuring in the equation occur only in the first degree, and not multiple together.

$$\frac{dy}{dx} + \sin x = 0 \quad \text{linear}$$

$$\left[\frac{d^2y}{dx^2}\right]^2 + x^2\left(\frac{dy}{dx}\right)^3 = 0 \quad \text{Non linear.}$$

Solution of differential equation

Q> $y = 3\cos(\log x) + 4\sin(\log x)$ — (1)

Solution of differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

Sol diff equation (1) both side w.r.t. x

$$\frac{dy}{dx} = 3\left[-\sin(\log x)\frac{1}{x}\right] + 4\cos\log x\,\frac{1}{x}$$

Page 4

$$x \cdot \frac{dy}{dx} = -3\sin(\log x) + 4\cos(\log x)$$

diff once again

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx}(1) = -3\cos(\log x)\frac{1}{x} + 4\left[-\sin(\log x)\right]\frac{1}{x}$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -\left[3\cos(\log x) + 4\sin(\log x)\right]$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

Q) $y = ae^{x} + be^{-x} + x^2$ — (1)

Solution of diff. equation

$$\frac{d^2y}{dx^2} - y + x^2 - 2 = 0 \quad -(2)$$

Sol diff equation (1) both side w.r.t. x

$$\frac{dy}{dx} = ae^{x} - be^{-x} + 2x$$

Page 5

differentiate once again

$$\frac{d^2y}{dx^2} = ae^{x} + be^{-x} + 2 \quad - \text{☆}$$

Put the value of ☆ and y in (2)

$$ae^{x} + be^{-x} + 2 - \left[ae^{x} + be^{-x} + x^2\right] + x^2 - 2 = 0$$

$$\cancel{ae^{x}} + \cancel{be^{-x}} + \cancel{2} - \cancel{ae^{x}} - \cancel{be^{-x}} - \cancel{x^2} + \cancel{x^2} - \cancel{2} = 0$$

Prove that

$$\frac{d^2y}{dx^2} - y + x^2 - 2 = 0$$

Q) $y = a\cos(x+b)$

Diff w.r.t x both side

$$\frac{dy}{dx} = -a\sin(x+b)$$

$$\frac{d^2y}{dx^2} = -\left[a\cos x + b\right]$$

$$\frac{d^2y}{dx^2} = -\left[y\right]$$

$$\frac{d^2y}{dx^2} + y = 0$$

H.W

Date
Page 6

Q) $y = e^x [a \cos x + b \sin x]$ — (1)

Sol

diff both side w.r.t. x

$$\frac{dy}{dx} = e^x [a(-\sin x) + b \cos x] + e^x [a \cos x + b \sin x]$$

$$\frac{dy}{dx} = e^x [-a \sin x + b \cos x] + y$$

$$\frac{dy}{dx} - y = e^x [-a \sin x + b \cos x] \quad \text{— (2)}$$

again diff both side w.r.t x

$$\cancel{\frac{d^2y}{dx^2} = e^x [-a \cos x - b \sin x] + e^x [-a \cos x - b \sin x] \pm \frac{dy}{dx}}$$

$$\cancel{= -e^x [a \cos x + b \sin x] + e^x [a \cos}$$

$$\frac{d^2y}{dx^2} = e^x [-a \cos x - b \sin x] + e^x [-a \sin x + b \cos x] + \frac{dy}{dx}$$

$$= -e^x [+a \cos x + b \sin x] + e^x [-a \sin x + b \cos x] + \frac{dy}{dx}$$

from (1) and (2)

Date
Page 7

$$\frac{d^2y}{dx^2} = -y + \frac{dy}{dx} - y + \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = -2y + 2\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} + 2y - 2\frac{dy}{dx} = 0$$

Ques: Determine the diff. eq will represent the family of circle on x axis and radius unity.

Sol

let Centre on x-axis be $(a, 0)$

Also radius $= 1$

the eq. of the family of circle is

$(x-a)^2 + (y-0)^2 = 1$ or $(x-a)^2 + y^2 = 1$

$2(x-a) + 2y \frac{dy}{dx} = 0$ or $x - a = -y\frac{dy}{dx}$

Putting the value of $x-a$ in (1), we get

$\left(-y \frac{dy}{dx}\right)^2 + y^2 = 1$ or $y^2 \left[\frac{dy}{dx}\right]^2 + y^2 = 1$

which is required diff. equation

Page 8

Q) Variables

$$\frac{dy}{dx} = \frac{x}{y}$$

$$y\,dy = x\,dx$$

Q) $$\frac{dy}{dx} = \frac{3x^2}{1+y^2}$$

Sol Saprating the variable

$$(1+y^2)\,dy = 3x^2\,dx$$

Integrate both side

$$\int (1+y^2)\,dy = 3\int x^2\,dx$$

$$\int dy + \int y^2 dy = 3\int x^2 dx$$

$$y + \frac{y^3}{2+1} = 3\,\frac{x^3}{3}$$

$$\frac{3y + y^3}{3} = x^3$$

$$3y + y^3 = 3x^3$$

$$3y + y^3 - 3x^3 = 0$$

Page 9

Q) $$x\cdot\cos y\,dy = (x e^x \log x + e^x)\,dx$$

Sol $$\cos y\,dy = \frac{(x e^x \log x + e^x)}{x}\,dx$$

$$\int \cos y\,dy = \int \left(e^x \log x + \frac{e^x}{x}\right) dx$$

$$\sin y = \int \underset{II}{e^x}\,\underset{I}{\log x}\,dx + \int \underset{II}{e^x}\,\underset{I}{x^{-1}}\,dx$$

$$\sin y = \log x \int e^x dx + \int \left[\frac{d\log x}{dx}\cdot\int e^x dx\right] dx + x^{-1}\int e^x dx$$

$$+ \int \left[\frac{d\,x^{-1}}{dx}\cdot\int e^x dx\right] dx$$

$$\sin y = \log x\; e^x +$$

Q) $$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

Sol $$\frac{dy}{dx} = e^x\cdot e^{-y} + x^2 e^{-y}$$

$$\frac{dy}{dx} = e^{-y}(e^x + x^2)$$

$$\frac{dy}{e^{-y}} = (e^x + x^2)\,dx$$

$$e^y\,dy = (e^x + x^2)\,dx$$

$$\int e^y dy = \int (e^x)\,dy + \int x^2 dx$$

$$e^y = e^x + \frac{x^{2+1}}{3} = \frac{3e^x + x^3}{3}$$

$$\boxed{3e^y - 3e^x - x^3 = 0}$$

Date
Page 10

Q. Solve the diff equation

$$\operatorname{cosec} x \log y \frac{dy}{dx} + x^2 y^2 = 0$$

Sol

$$\operatorname{cosec} x \log y \frac{dy}{dx} = -x^2 y^2$$

$$y^{-2} \log y \, dy = -x^2 \frac{1}{\operatorname{cosec} x} dx$$

Integrate both side

$$\int \underset{\mathrm{I}}{\log y} \, \underset{\mathrm{II}}{y^{-2}} \, dy = -\int \underset{\mathrm{I}}{x^2} \, \underset{\mathrm{II}}{\sin x} \, dx$$

$$\int y^{-2} = \frac{y^{-1}}{-1}$$

$$\log y \int y^{-2} dy - \int \left[\frac{1}{y} \times \frac{y^{-1}}{-1} \right] dy = -\left\{ x^2 \int \sin x \, dx - \int [\cos x \times \sin x] \, dx \right.$$

$$-\frac{\log y}{y} + (-y^{-1}) = -\left\{ x^2(-\cos x) + \int \underset{\mathrm{I}}{2x} \, \underset{\mathrm{II}}{(-\cos x)} \, dx \right.$$

$$\cancel{-\frac{\log y}{y} - y^{-1} = -\{ x^2 \cos y - [\cos x(-\cos x)}$$

Date
Page 11

$$-\frac{\log y}{y} - y^{-1} = -\left\{ x^2(-\cos x) + \left[2x(\sin x) - \int [2 \times \sin x] \, dx \right] \right.$$

$$\cancel{= -\{ -x^2 \cos x + [+ 2x \cos x + 2 \sin x] \}}$$

$$\cancel{= -\{ -x^2 \cos x + 2x \cos x - 2 \sin x \}}$$

$$\cancel{-\frac{\log y}{y} - y^{-1} = x^2 \cos x - 2x \cos x + 2 \sin x}$$

$$= -\left\{ x^2(-\cos x) + \left[2x \sin x - 2\int \sin x \, dx \right] \right\}$$

$$= x^2 \cos x - 2x \sin x + 2 \cos x$$

$$-\frac{\log y}{y} - y^{-1} = x^2 \cos x - 2x \sin x + 2 \cos x$$

(H.W) Date Page 12

Q1) $\frac{dy}{dx} = x^5 + x^2 - \frac{2}{x}$

Sol $dy = \left(x^5 + x^2 - \frac{2}{x}\right) dx$

Integrate both side

$\int dy = \int x^5 dx + \int x^2 dx - 2\int x^{-1} dx$

$y = \frac{x^6}{6} + \frac{x^3}{3} - 2 \times \ln|x|$

$y = \frac{x^6}{6} + \frac{x^3}{3} - 2\ln|x|$

$\frac{1}{x}$

$\overline{x^2}$

Q2) $\frac{dy}{dx} = e^{x+y} + x^2 e^y$

Sol $\frac{dy}{dx} = e^x \cdot e^y + x^2 e^y$

$\frac{dy}{dx} = e^y\left(e^x + x^2\right)$

$dy\, e^{-y} = \left(e^x + x^2\right) dx$

Integrate both side

$\int e^{-y} dy = \int \left(e^x + x^2\right) dx$

let

$-y = u$

Date Page 13

diff both side

$\frac{d(-y)}{dy} dy = \frac{du}{du} du$

$-dy = du \Rightarrow dy = -du$

$-\int e^u \cdot du = \int e^x dx + \int x^2 dx$

$-e^u = e^x + \frac{x^3}{3}$

$e^x + \frac{x^3}{3} + e^u = 0$

Put $u = -y$

$e^x + \frac{x^3}{3} + e^{-y} = 0$

3Q) $\frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos y}$

Sol $dy = \frac{1 - \cos x}{1 + \cos x} dx$

$(1 + \cos y)dy = (1 - \cos x)dx$

Integrate both side

$\int dy + \int \cos y\, dy = \int dx - \int \cos x\, dx$

$y + \sin y = x - \sin x$

$x + y + \sin x + \sin y = 0$

Date
Page 14

4 Q) $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$

Sol

$\frac{dy}{dx} = \frac{2x\log x + x}{\sin y + y\cos y}$

$(\sin y + y\cos y)\,dy = (2x\log x + x)\,dx$

$\int \sin y\,dy + \int \underset{I}{y}\,\underset{II}{\cos y}\,dy = 2\int \underset{I}{\log x}\,\underset{II}{x} + \int x\,dx$

$-\cos y + y\sin x - \int [(1)\sin y]\,dy = 2\left[\log x \frac{x^2}{2} - \int \left[\frac{1}{x} \times \frac{x^2}{2}\right] dx\right]$

$-\cos y + y\sin x + \cos y = 2\left[\log x \frac{x^2}{2} - \frac{x^2}{2\times 2}\right]$

$-\cos y + y\sin x + \cos y = \log x\, x^2 - \frac{x^2}{2}$

5 Q) $\frac{dy}{dx} = e^{x+y} + x^2 e^y$ subject the condition is $x = 0,\ y = 0$

Sol

$\frac{dy}{dx} = e^x \cdot e^y + x^2 e^y$

$\frac{dy}{dx} = e^y [e^x + x^2]$

Date
Page 15

$\frac{dy}{e^y} = [e^x + x^2]\,dx$

Integration both side

$\Rightarrow \int \frac{dy}{e^y} = \int (e^x + x^2)\,dx$

$\Rightarrow -e^{-y} = e^x + \frac{x^3}{3} + c$

$\Rightarrow e^x + e^{-y} + \frac{x^3}{3} + c = 0$

6 Q) $\frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$

Sol

we know that

$\cos x = 1 - 2\sin^2 \frac{x}{2}$

or $\cos x = 2\cos^2 \frac{x}{2} - 1$

so $\frac{dy}{dx} = \frac{\cancel{1} - \cancel{1} + 2\sin^2 \frac{x}{2}}{\cancel{1} + 2\cos^2 \frac{x}{2} - \cancel{1}}$

$\frac{dy}{dx} = \frac{\cancel{2}\sin^2 \frac{x}{2}}{\cancel{2}\cos^2 \frac{x}{2}}$

$\frac{dy}{dx} = \tan^2 \frac{x}{2}$

$\cos x = 1 - 2\sin^2 \frac{x}{2}$

$\cos x = 2\cos^2 \frac{x}{2} - 1$

$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

$\tan^2 x + 1 = \sec^2 x$

$\tan^2 x = \sec^2 x - 1$

Date
Page 16

$$dy = \tan^2 \frac{x}{2}\, dx$$

but we know that

$$\tan^2 x = \sec^2 x - 1$$

$$dy = \left(\sec^2 \frac{x}{2} - 1\right) dx$$

Integrate both side

$$\int dy = \int \sec^2 \frac{x}{2}\, dx - \int dx$$

$$y = \tan \frac{x}{2} - x$$

Q5> $\frac{dy}{dx} = e^{x+y} + x^2 e^y$ Subject the condition when $x = 0,\ y = 0$

Sol

$$\frac{dy}{dx} = e^x . e^y + x^2 . e^y$$

$$\frac{dy}{dx} = e^y \left(e^x + x^2 \right]$$

$$\frac{dy}{e^y} = \left[e^x + x^2 \right) dx$$

Integrate both side

Date
Page 17

$$\int \frac{dy}{e^y} = \int (e^x + x^2)\, dx$$

$$-e^{-y} = e^x + \frac{x^3}{3} + c \quad - \text{(I)}$$

$$-1 = 1 + 0 + c$$

$$c = -1 - 1$$

$$c = -2$$

Put in (I), we get

$$-e^{-y} = e^x + \frac{x^3}{3} - 2$$

$$\Rightarrow \boxed{e^x + e^{-y} + \frac{x^3}{3} - 2 = 0}$$

Chapter – 2 / Unit – I

Linear equation with Constant Coefficient

Step 3> From the roots of the A.E. write down the corresponding part of the Complete Solution (C.S) as follows:

Roots of A.F.	
(a)(i) One real and different root m_1	$c_1 e^{m_1 x}$
(ii) Two real and equal roots m_1, m_1	$e^{m_1 x}(c_1 + c_2 x)$
(iii) Three real and equal roots m_1, m_1, m_1	$e^{m_1 x}(c_1 + c_2 x + c_3 x^2)$
b) (i) one pair of complex and different roots $\alpha \pm i\beta$	$e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$
(ii) Two pairs of complex and equal roots $\alpha \pm i\beta$, $\alpha \pm i\beta$. and so on	$e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]$

Q1> $P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} - P_n y = Q$

Sol

S.F $\Rightarrow$ $P_0 D^n + P_1 D^{n-1} - P_n = Q$

A.E $P_0 D^n + P_1 D^{n-1} - P_n = 0$

if root is $= (1, 1)$

then C.S $y = e^x (c_1 + c_2 x)$

if root is $= (1, 3)$

then C.S $y = c_1 e^x + c_2 e^{3x}$

if root is $= \pm i$

then C.S $y = e^x (c_1 \cos x + c_2 \sin x)$

Q1> Solve $\frac{d^2 y}{dx^2} - 7\frac{db}{dx} + 12y = 0$

Sol

S.F $(D^2 - 7D + 12)y = 0$

A.E

$D^2 - 7D + 12 = 0$

$D^2 - 3D - 4D + 12 = 0$

$D(D-3) - 4(D-3) = 0$

$(D-4)(D-3) = 0$

$D = 3, 4$

C.S $\Rightarrow$ $y = c_1 e^{3x} + c_2 e^{4x}$

Complete Solution

Q) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

Sol

S.F
$(D^2 + 4D + 13)y = 0$

A.F
$D^2 + 4D + 13 = 0$

$d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-4 \pm \sqrt{16 - 4\times13}}{2}$

$= \frac{-4 \pm \sqrt{36}\, i}{2}$

$= \frac{-4 \pm 6i}{2} = \overset{\alpha}{-2} \pm \overset{\beta}{3} i$

C.S
$y = e^{-2x}(C_1 \cos 3x + C_2 \sin 3x)$

$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$

Q) $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 0$

Sol
S.F $(D^3 - 4D^2 + 5D - 2)y = 0$

A.E
$D^3 - 4D^2 + 5D - 2 = 0$

$(D-1)\ (D^2 - 3D + 2) = 0$

$(D-1)\ (D-2)(D-1) = 0$

$D = 1, 1, 2$

C.S
$y = e^{x}(C_1 + C_2 x) + C_3 e^{2x}$

$1, -1$

$1 - 4 + 5 - 2 = 0$

$6 - 6 = 0$

$0 = 0$

$D = 1$

$D - 1 = 0$

1	1	-4	5	-2
	0	1	-3	2
	1	-3	2	0

H.W

Q) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 40y = 0$

Sol

S.F $(D^2 + 3D - 40)y = 0$

2	40
2	20
2	10
5	5
	1

A.E. $D^2 + 3D - 40 = 0$

$D^2 - 5D + 8D - 40 = 0$

$D(D-5) + 8(D-5) = 0$

$(D+8)\ (D-5) = 0$

$D = 5, -8$

C.S
$y = C_1 e^{5x} + C_2 e^{-8x}$

Q2> $$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

Sol

SF $(D^2 + 6D + 9)y = 0$

$3 | 9$, $3 | 3$, 1

A.E $D^2 + 6D + 9 = 0$

$D^2 + 3D + 3D + 9 = 0$

$D(D+3) + 3(D+3) = 0$

$D = +3, +3$

C.S.

$$y = e^{+3x}(c_1 + c_2 x)$$

Q3> $$\frac{d^2y}{dx^2} + \cancel{a}\, a^2 y = 0$$

Sol

S.F $(D^2 + 0D + a^2)y = 0$

A.E $D^2 + 0D + a^2 = 0$

$\sqrt{D^2} = \sqrt{-a^2}$

$D \times D = -a \times a$

$D = -a, a$

C.S. $y = c_1 e^{-ax} + c_2 e^{ax}$

Q4> $$\frac{d^2y}{dx^2} + y = 0$$ and Given $y = 2$, $x = 0$ and $y = -2$, $x = \pi/2$

Ans $\cancel{D^2 + D = 0}$

$D(D+1) = 0$

$D = 0, -1$

The Given eq: $\frac{d^2y}{dx^2} + y = 0$

S.F. $(D^2 + 1)y = 0$

A.E. is $D^2 + 1 = 0$

$D^2 = -1$

$D = \pm i$

$y = e^{0x}(c_1 \cos x + c_2 \sin x)$ — (1)

Put $y = 2$ and $x = 0$

$2 = e^0(c_1 \cos 0 + c_2 \sin 0)$

$\boxed{c_1 = 2}$ — (2)

Put $y = -2$ and $x = \pi/2$

$$-2 = C_1 \cos\frac{\pi}{2} + C_2 \sin\frac{\pi}{2}$$

$$-2 = 0 + C_2$$

$$\boxed{C_2 = -2} \quad — \text{(III)}$$

Put (II) and (III) in (I)

$$\boxed{y = 2\cos x - 2\sin x}$$

Q) ~~Putting value of~~

:-> **Rule 1** :- Rule to evaluate $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$

Put $D = a$

Note :- $f(a) = 0$ (case of failure)

then $\frac{1}{f(D)} e^{ax} = x \frac{1}{\frac{d}{dD}[f(D)]} e^{ax}$

Ques:- Solve $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 3y = e^{2x}$

Sol:-

S.F. $= [4D^2 + 4D - 3]\, y = e^{2x}$

A.E. $= 4D^2 + 4D - 3 = 0$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -4 \pm \sqrt{4^2 - 4 \times 4 \times -3}$$

$$= \frac{-4 \pm \sqrt{16 - (-48)}}{8}$$

$$= \frac{-4 \pm \sqrt{16 + 48}}{8} = \frac{-4 \pm \sqrt{64}}{8}$$

$$= \frac{-4 \pm 8}{8}$$

$$= \frac{-4 - 8}{8}, \frac{-4 + 8}{8}$$

$$= -\frac{\cancel{12}^{\,3}}{\cancel{8}_{\,2}}, \frac{4}{8}\ \frac{1}{2}$$

$$= -\frac{3}{2}, \frac{1}{2}$$

C.F. : $C_1 e^{1/2\, x} + C_2 e^{-3/2\, x}$

P.I. $= \frac{1}{4D^2 + 4D - 3} \times e^{2x}$

$$= \frac{1}{4(2)^2 + 4 \times 2 - 3} \times e^{2x}$$

$$= \frac{1}{4(4) + 8 - 3} \times e^{2x} = \frac{1}{16 + 8 - 3} \times e^{2x}$$

$$= \frac{1}{21} \times e^{2x}$$

C.S : $C_1 e^{\frac{1}{2} x} + C_2 e^{-3/2\, x} + \frac{1}{21} \times e^{2x}$

Ques: $\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = 2$

Sol

S.F. : $(D^4 - D^2)\, y = 2$

Date
Page

A.E.: $D^4 - D^2 = 0$

$\therefore \quad D^2(D^2 - 1) = 0$

$= \quad D^2 = 0, 1$

$= \quad D = 0, 0, -1, 1$

$\therefore \quad C.F = e^{0x}(C_1 + C_2 x) + C_3 e^{-x} + C_4 e^{x}$
$= (C_1 + C_2 x)e^{0x} + C_3 e^{x} + C_4 e^{-x}$

P.I. : $\dfrac{1}{D^4 - D^2} \times 2 \times e^{0x}$

$= 2 \dfrac{1}{0 - 0} e^{0x}$

Case of failure

$\therefore$ P.I. $= 2 \dfrac{1}{D^4 - D^2} e^{0x} = 2x \dfrac{1}{4D^3 - 2D} e^{0x} = 2x \dfrac{1}{0} e^{0x}$

$\therefore$ the case fails again

So P.I. $= 2x \dfrac{1}{4D^3 - 2D} e^{0x} = 2x^2 \dfrac{1}{12D^2 - 2} e^{0x}$

$= 2x^2 \dfrac{1}{D - 2} e^{0x}$

$= 2x^2 \dfrac{1}{-2} = -x^2$

$\therefore$ C.S. is $y = C_1 + C_2 x + C_3 e^{-x} + C_4 e^{x} - x^2$

H.W

Date
Page

Ques:- $\dfrac{d^3y}{dx^3} - y = (e^x + 1)^2$

Sol:-

S.F. : $(D^3 - 1)y = e^{2x} + 2e^x + 1$

A.E. : $D^3 - 1 = 0$

$(D - 1)(D^2 + D + 1) = 0$

$\therefore \quad D = 1, \dfrac{-1 \pm \sqrt{1 - 4}}{2} = 1$

$= \dfrac{-1 \pm \sqrt{-3}}{2} = 1$

$= \dfrac{-1 \pm i\sqrt{3}}{2}$

$\therefore \quad D = 1, -\dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}$

$\therefore$ C.F. $= C_1 e^x + e^{-\frac{x}{2}}\left(C_2 \cos \dfrac{\sqrt{3}}{2} x + C_3 \sin \dfrac{\sqrt{3}}{2} x\right)$

P.I. $= \dfrac{1}{D^3 - 1}(e^{2x} + 2e^x + 1) = \dfrac{1}{D^3 - 1} e^{2x} + \dfrac{1}{D^3 - 1} 2e^x +$
$\dfrac{1}{D^3 - 1} \quad - (1)$

Now $\dfrac{1}{D^3 - 1} e^{2x} = \dfrac{1}{2^3 - 1} e^{2x} = \dfrac{1}{7} e^{2x}$

$\dfrac{1}{D^3 - 1} 2e^x = 2 \cdot \dfrac{1}{1^3 - 1} e^x = 2 \cdot \dfrac{1}{0} e^x$

$\therefore$ the rule fails

$\therefore \quad \dfrac{1}{D^3 - 1} 2e^x = 2x \dfrac{1}{3D^2} e^x = \dfrac{2x}{3} e^x = \dfrac{2\cancel{x}e^3}{3}$

Date
Page

$$\frac{1}{D^3-1}(1) = \frac{1}{D^3-1} e^{0x} = \frac{1}{D^3-1} \cdot 1 = -1$$

$\therefore$ from eq^n / P.I. $= \frac{1}{7} e^{2x} + \frac{2}{3} x e^{x} - 1$

$\therefore$ C.S is $y = c_1 e^{x} + e^{-\frac{x}{2}} \left[c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x \right) \frac{1}{7} e^{2x} + \frac{2}{3} x e^{x} - 1$

H.W

Ques $\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 4y = e^{-2x}$

Sol S.F. $= (D^2 + 4D + 4) y = e^{-2x}$

A.E. $= D^2 + 4D + 4 = 0$

$D^2 + 2D + 2D + 4 = 0$

$D(D+2) + 2(D+2) = 0$

$(D+2)(D+2) = 0$

$D = -2, -2$

C.F. $= (c_1 + c_2 x) e^{-2x}$

P.I. $= \frac{1}{D^2 + 4D + 4} \cdot e^{-2x}$

$= \frac{1}{(-2)^2 + 4 \times (-2) + 4} e^{-2x} = \frac{1}{4 + (-8) + 4} e^{-2x}$

$= \frac{1}{8-8} e^{-2x} = \frac{1}{0} e^{-2x}$

[Case of failure]

Date
Page

$$x \frac{1}{2D + 4 + 0} \cdot e^{-2x} = x \frac{1}{2(-2) + 4 + 0} e^{-2x}$$

$$x \frac{1}{-4+4} e^{-2x} = x \frac{1}{0} e^{-2x}$$

$$= x^2 \frac{1}{2+0+0} e^{-2x} = x^2 \frac{1}{2} e^{-2x}$$

H.W

Ques $\frac{d^3y}{dx^3} - \frac{3dy}{dx} + 2y = e^{2x}$

Sol S.F. $= (D^3 - 3D + 2) y = e^{2x}$

A.E. $D^3 - 3D + 2 = 0$

Put $D = 1$

$(1)^3 - 3 \times 1 + 2 = 0$

$1 - 3 + 2 = 0$

$-2 + 2 = 0$

$(D-1) = 0$

Now

$$\begin{array}{c|cccc} & 1 & 0 & -3 & 2 \\ 1 & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

$(D-1)(D^2 + D - 2) = 0$

$(D-1) = 0$ $\quad D^2 + D - 2 = 0$

$D = 0 + 1$ $\quad D^2 + 2D - D - 2 = 0$

$D = 1$

$D(D+2) - 1\ (D+2) = 0$

$(D+2)\ (D-1) = 0$

$D+2 = 0 \qquad D-1 = 0$

$D = 0-2 \qquad D = 0+1$

$D = -2 \qquad D = 1$

C.F $= (C_1 + C_2)\, e^{1x} + C_3\, e^{-2x}$

P.I. $= \dfrac{1}{D^3 - 3D + 2} \cdot e^{2x}$

$= \dfrac{1}{(2)^3 - 3\times 2 + 2} \cdot e^{2x}$

$= \dfrac{1}{8 - 6 + 2} \cdot e^{2x} = \dfrac{1}{4}\, e^{2x}$

C.S. = C.F + P.I.

$(C_1 + C_2)\, e^{1x} + C_3\, e^{-2x} + \dfrac{1}{4}\, e^{2x}$

Rule ⇒ 2

Rule to evaluate $\dfrac{1}{f(D^2)} \cos ax$, $\dfrac{1}{f(D^2)} \sin ax$

$= \dfrac{1}{f(-(a)^2)} \cos ax$

$D^2 = (-a^2)$

Q. $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - y = \sin 2x$

S.F $= (D^3 + D^2 - D - 1)\, y = \sin 2x$

A.E $= (D^3 + D^2 - D - 1) = 0$

$D^2(D+1) - 1(D+1) = 0$

$(D^2 - 1)(D+1) = 0$

$D = 1, -1, -1,$

C.F

$= C_1 e^{+x} + (C_2 + C_3 x)\, e^{-x}$

P.I $= \dfrac{1}{D^3 + D^2 - D - 1} \sin 2x$

$= \dfrac{1}{D \cdot D^2 + D^2 - D - 1} \sin 2x$

$\Rightarrow \dfrac{1}{-4D-4-D-1} \sin 2x$

$= \dfrac{1}{-5D-5} \sin 2x = \dfrac{1}{-5(D+1)} \dfrac{(D-1)}{(D-1)} \sin 2x$

$\Rightarrow \dfrac{D-1}{-5(D^2-1)} \sin 2x \Rightarrow \dfrac{D-1}{-5(-4-1)} \sin 2x$

$\Rightarrow \dfrac{D-1}{-5(-5)} \sin 2x = \dfrac{D-1}{25} \sin 2x$

$\Rightarrow \dfrac{1}{25}[D \sin 2x - 1 \sin 2x]$

$= \dfrac{1}{25}[2\cos 2x - \sin 2x]$

C.S = C.F + P.I

$= c_1 e^{x} + (c_2 + c_3 x)e^{-x} + \dfrac{1}{25}[2\cos 2x - \sin 2x]$

Q) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = \sin 2x$

S.F $(D^2 + D + 1)y = \sin 2x$

A.E $D^2 + D + 1 = 0$

$= \dfrac{-b \pm \sqrt{b^2 - 4ab}}{2a}$

$D(D+1)+1=0$
$D(D+1)=-1$
$D=-1$
$D=-1$
#

$\dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm \sqrt{-3}}{2} = \dfrac{-1 \pm \sqrt{3}i}{2}$

$= \dfrac{1 \pm \sqrt{3}i}{2} \quad -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2} i$

C.F $= e^{-\frac{1}{2}x}\left[C_1 \cos \dfrac{\sqrt{3}}{2}x + C_2 \sin \dfrac{\sqrt{3}}{2}x\right]$

P.I $= \dfrac{1}{D^2+D+1} \sin 2x = \dfrac{1}{-4+D+1} \sin 2x$ put $D^2 = -2^2$

$\Rightarrow \dfrac{1}{D-3} \sin 2x = \dfrac{D+3}{D^2-9} \sin 2x$

$= \dfrac{D+3}{-4-9} = -\dfrac{(1)D+3}{13} \sin 2x$

$= -\dfrac{1}{13}[D \sin 2x + 3 \sin 2x]$

$= \dfrac{1}{-13}[2\cos 2x + 3 \sin 2x]$

S.S is $y = e^{-\frac{1}{2}x}\left[C_1 \cos \dfrac{\sqrt{3}}{2}x + C_2 \sin \dfrac{\sqrt{3}}{2}x\right] + \left(-\dfrac{1}{13}\right)[2\cos 2x + 3 \sin 2x]$

Rule 3

$$\frac{1}{f(D)} x^m$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$$

Q) Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x$

S.F

$$(D^2 - 3D + 2)y = 0$$

A.E

$$D^2 - 3D + 2 = 0$$

$$D^2 - D - 2D + 2 = 0$$

$$D(D-1) - 2(D-1) = 0$$

$$D = 1, 2$$

C.F

$$C_1 e^{1x} + C_2 e^{2x}$$

P.I

$$\frac{1}{D^2 - 3D + 2} x$$

$$= \frac{1}{2 - 3D + D^2} x = \frac{1}{2\left(1 - \frac{3D + D^2}{2}\right)} x$$

$$= \frac{1}{2}\left[\frac{1}{\left[1 - \left(\frac{3D}{2} - D^2\right)\right]}\right] x$$

$$= \frac{1}{2}\left\{\left[1 - \left(\frac{3D}{2} - D^2\right)\right]^{-1}\right] x$$

$$= \frac{1}{2}\left[1 + (-1)\left[-\left(\frac{3D}{2} - D^2\right)\right]\right] x$$

$$= \frac{1}{2}\left[1 + \frac{3D}{2} - D^2\right] x \qquad -D^2 x \quad (1) \quad (0)$$

$$= \frac{1}{2}\left[x + \frac{3D}{2} x\right]$$

$$= \frac{1}{2}\left[x + \frac{3}{2}\right]$$

$\therefore$ G.S $= C_1 e^{x} + C_2 e^{2x} + \frac{x}{2} + \frac{3}{4}$

Q) $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} - 2y = x^2$

S.F $\Rightarrow (D^3 - 3D - 2)y = x^2$

A.E $\Rightarrow D^3 - 3D - 2 = 0$

Put $D = -1$

$-1 + 3 - 2 = 0$

one Root is -1

$(D+1)(D$

Date
Page

$(D+1)(D^2 - D - 2) = 0$

$$\begin{array}{r|rrrr} & 1 & 0 & -3 & -2 \\ -1 & & -1 & +1 & +2 \\ \hline & 1D^2 & -1D & -2 & 0 \end{array}$$

$\Rightarrow (D+1)[D^2 + D - 2D - 2] = 0$

$$\begin{array}{r|r} 2 & -2 \\ \hline -1 & -1 \end{array}$$

$(D+1)[D(D+1) - 2(D+1)] = 0$

$(D+1)[(D-2)(D+1)] = 0$

$D = -1, -1, 2$

C.F $\Rightarrow e^{-1x}(c_1 + c_2 x) + c_3 e^{2x}$

P.I. $\dfrac{1}{D^3 - 3D - 2} x^2$

$= \dfrac{1}{-2 - 3D + D^3} x^2$

$= -\dfrac{1}{2}\left[\dfrac{1}{1 + \frac{3D}{2} - \frac{D^3}{2}}\right] x^2$

$= -\dfrac{1}{2}\left[\dfrac{1}{1 + \left(\frac{3D}{2} - \frac{D^3}{2}\right)}\right] x^2$

$= -\dfrac{1}{2}\left[\left[1 + \dfrac{3D - D^3}{2}\right]^{-1}\right] x^2$

Date
Page

$= \dfrac{-1}{2}\left[1 + (-1)\left(\dfrac{3D}{2} - \dfrac{D^3}{2}\right)\right] x^2$

$= \dfrac{-1}{2}\left[1 - \dfrac{3D}{2} + \dfrac{D^2}{2}\right] x^2$

$= \dfrac{-1}{2}\left[x - \dfrac{3D}{2} x^2 + \dfrac{D^2}{2} x^2\right]$

$= \dfrac{-1}{2}\left[x - \dfrac{3}{2} x + \dfrac{1}{2} \times \cancel{2x} \times (1)\right]$

$= \dfrac{-1}{2}\left(x - 3x + 1\right)$

$D^2 x$

$2x$

$2x$

(1)

$-\dfrac{3D}{2} x^2$

$-\dfrac{3}{2} x(x)$

$= -3(x)$

e^{ax} e^{ax} $e^{ax}x^n$
e^{ax} sin a

Rule 4

evaluate $\frac{1}{f(D)}(e^{ax}v)$

$\Rightarrow e^{ax}\frac{1}{f(D+a)}v$

where v is e^{ax} or $\sin ax$ or x^m

Q) $\frac{d^2y}{dx^2}+y = xe^{2x}$

Sol

S.F

$(D^2+1)y = xe^{2x}$

A.E

$D^2+1=0$

$=\pm i$

$=0\pm i$

C.F.

$e^{0x}(C_1\cos x + C_2\sin(-x))$

P.I

$= \frac{1}{D^2+1}xe^{2x}$ $\qquad D^2$

$(D+a)^2$

$(D+2)^2$

$= e^{2x}\frac{1}{(D+2)^2+1}x$

$= e^{2x}\frac{1}{D^2+4+4D+1}x$

$= e^{2x}\frac{1}{5+4D+D^2}x$

$= e^{2x}\frac{1}{5\left(1+\frac{4D+D^2}{5}\right)}x$

$= \frac{e^{2x}}{5}\left(1+\frac{4D+D^2}{5}\right)^{-1}x$

$= \frac{e^{2x}}{5}\left[1+(-1)\left(\frac{4D+D^2}{5}\right)+\cdots\right]x$

$= \frac{e^{2x}}{5}\left[x-\frac{4}{5}\right]$ $\qquad -\frac{4D}{5}-\frac{D^2}{5}$

C.S = C.F + P.I

$= e^{0x}(C_1\cos x + C_2\sin(-x)) + \frac{e^{2x}}{5}\left[x-\frac{4}{5}\right]$

H.W

$\frac{d^4y}{dx^2} - y = e^x\cos x$

Sol

S.F

$(D^4-1)y = e^x\cos x$

A.E $\quad D^4-1=0$

$(D^2)^2-((1)^2)^2=0$

$(D^2+1)(D^2-1)=0$

$(D^2+1)(D+1)(D-1)=0$

$D=-1, 1, 0\pm 1i$

C.F.

$$= C_1 e^{-x} + C_2 e^{2x} + e^{0x}(C_3 \cos x + C_4 \sin(-x))$$

P.I.

$$\frac{1}{D^4 - 1} e^x \cos x$$

$$e^x \frac{1}{(D+1)^4 - 1} \cos x$$

$$e^x \frac{1}{[(D+1)^2 (D+1)^2] - 1} \cos x$$

$$e^x \frac{1}{[(D^2+1+2D)(D^2+1+2D)] - 1} \cos x$$

$$e^x \frac{1}{D^4 + D^2 + 2D^3 + D^2 + 1 + 2D + 2D^3 + 2D + 4D^2} \cos x$$

$$e^x \frac{1}{[D^4 + 6D^2 + 4D^3 + 4D + 1] - 1} \cos x$$

$$e^x \frac{1}{D^4 + 6D^3 + 6D^2 + 4D} \cos x$$

$$e^x \frac{1}{-1 - 6D - 6 + 4D} \cos x$$

$$e^x \frac{1}{-2D - 7} \cos x$$

$$-e^x \frac{1\,(2D-7)}{(2D+7)(2D-7)} \cos x$$

$$-e^x \frac{(2D-7)}{4D^2 - 49} \cos x$$

$$-e^x \frac{(2D-7)}{-4-49} \cos x$$

$$\frac{e^x}{53}[2D \cos x - 7 \cos x]$$

$$\frac{e^x}{53}[-\sin x \cdot 2 - 7 \cos x]$$

C.S = C.F + P.I.

$$= C_1 e^{-x} + C_2 e^{2x} + e^{0x}[(C_3 \cos x + C_4 \sin(-x)] + \frac{e^x}{53}[-\sin x \cdot 2 - 7 \cos x]$$

Q) $$\frac{d^2y}{dx^2} - \frac{4\,dy}{dx} + 3y = e^x \cos 2x + \cos 3x$$

Sol

S.F $(D^2 - 4D + 3)y = e^x \cos 2x + \cos 3x$

A.E

$$(D^2 - 4D + 3) = 0$$

$$D^2 - 1D - 3D + 3 = 0$$

$$D(D-1) - 3(D-1) = 0$$

$$(D-3)(D-1) = 0$$

$$D = 1, 3$$

C.F

$= C_1 e^{1x} + C_2 e^{3x}$

PI

$$\frac{1}{D^2-4D+3}\left[e^x \cos 2x + \cos 3x\right]$$

$$\frac{1}{D^2-4D+3} e^x \cos 2x + \frac{1}{D^2-4D+3} \cos 3x$$

$$\frac{1}{(D+1)^2-4(D+1)+3} e^x \cos 2x + \frac{1}{-9-4D+3} \cos 3x$$

$$\frac{1}{D^2+1+2D-4D-4+3} e^x \cos 2x + \frac{1}{-(4D+6)} \cos 3x$$

$$e^x \frac{1}{D^2-2D} \cos 2x - \frac{(4D-6)}{16D^2-36} \cos 3x$$

$$= e^x \frac{1}{-4-2D} \cos 2x + \frac{1}{180}(4D-6) \cos 3x$$

$$= -\frac{e^x(2D-4)}{4D^2-16} \cos 2x + \frac{1}{180}(4D-6)\cos 3x$$

$$= -\frac{e^x(2D-4)}{-16-16} \cos 2x + \frac{1}{180}(4D-6)\cos 3x$$

$$= \frac{e^x}{32}\left[2D\cos 2x - 4\cos 2x\right] + \frac{1}{180}\left[4D\cos 3x - 6\cos 3x\right]$$

16, 9, 144, 36, 180

$-153-36$

153, 36, 189

$4+2D$

$(2D+4)(2D-)$

$$= \frac{e^x}{32}\left[4(-\sin 2x) - 4\cos 2x\right] + \frac{1}{180}\left[12(-\sin 3x) - 6\cos 3x\right]$$

C.S = C.F + P.I

$$= C_1 e^x + C_2 e^{3x} + \frac{e^x}{32}\left[-\sin 2x \cdot 4 - 4\cos 2x\right] + \frac{1}{180}\left[-\sin 3x \cdot 12 - 6\cos 3x\right]$$

Q) $(D^3 - D^2 - 9D + 9)y = e^x(x^2 + \cos 2x)$

Sol

A.E

$D^3 - D^2 - 9D + 9 = 0$

	1	-1	-9	9
1		1	0	-9
	1	0	-9	0

$(1) - (1)^2 - 9(1) + 9$

$1 - 1 - 9 + 9$

$10 - 10$

$(D+1)(D^2-9) = 0$

$D^2 = 9$

$D = \pm\sqrt{9} = \pm 3$

$D = +3, -3, -1$

C.S $= C_1 e^{3} + C_2 e^{-3x} + C_3 e^{-1x}$

P.I

$$\frac{1}{D^3-D^2-9D+9} e^x(x^2 + \cos 2x)$$

$$\frac{1}{D^3-D^2-9D+9}$$

$$= \frac{1}{(D+1)^3 - (D+1)^2 - 9(D+1) + 9} e^x (x^2 + \cos 2x)$$

$(D+1)^2 (D+1)$

$$= e^x \frac{1}{[(D)^2 + (1) + 2D](D+1) - [D^2 + (1) + 2D] - 9D - 9 + 9} (x^2 + \cos 2x)$$

D^2

$$= \frac{e^x (x^2 + \cos 2x)}{[D^3 + D + 2D^2 + D + 1 + 2D] - D^2 - 1 - 2D - 9D}$$

$$= \frac{e^x (x^2 + \cos 2x)}{[D^3 + D + 2D^2 + D + 1 + 2D - D^2 - 1 - 2D - 9D]}$$

$$= \frac{e^x (x^2 + \cos 2x)}{[D^3 + D^2 - 9D + 2D]}$$

$$= e^x \frac{(x^2 + \cos 2x)}{(D^3 + D^2 - 7D)}$$

$$= e^x \left[\frac{1}{D^3 + D^2 - 7D} x^2 + \frac{1}{D^3 + D^2 - 7D} \cos 2x \right]$$

$$= e^x \left[\frac{1}{-7D + D^2 + D^3} x^2 + \frac{1}{-8 - 4 - 7D} \cos 2x \right]$$

$$= e^x \left[\frac{1}{7} \left(\frac{1}{-D + \frac{D^2}{2} + \frac{D^3}{2}} \right) x^2 + \frac{1}{-12 - 7D} \cos 2x \right]$$

Date ____
Page ____

Rule 5^{th}

$$\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V + \frac{d}{dD}\left(\frac{1}{f(D)}\right)V$$

Q) $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = x\sin x$

Sol

S.F

$$(D^2+7D+12)y = x\sin x$$

A.E

$$D^2+7D+12 = 0$$

$$D = -3, -4$$

C.F

$$C_1e^{-3x} + C_2e^{-4x}$$

P.I.

$$= \frac{1}{D^2+7D+12}x\sin x$$

$$= x\frac{1}{D^2+7D+12}\sin x + \left(\frac{1}{D^2+7D+12}\right)'v$$

$$= x\frac{1}{(-1)^2+7D+12}\sin x + \frac{-[2D+7]}{(D^2+7D+12)^2}\sin x$$

$$= x\frac{1}{7D+11}\sin x + \frac{-(2D+7)}{(7D+11)^2}$$

Date ____
Page ____

$$= x\frac{1}{7D+11}\sin x + \frac{-(2D+7)}{49D^2+121+154D}\sin x$$

$$= x\frac{1}{7D+11}\sin x - \frac{(2D+7)}{154D+72}\sin x$$

$$= x\frac{(7D-11)}{49D^2-121}\sin x - \frac{(2D+7)(154D-72)}{(154D+72)(154D-72)}\sin x$$

$$= x\frac{(7D-11)}{-49-121}\sin x + \frac{[308D^2+934D-504]}{23716+5184}$$

$$= -\frac{x}{170}(7D-11)\sin x + \frac{[934D-812]}{28900}\sin x$$

$$= -\frac{x}{170}(7D\sin x - 11\sin x) + \frac{1}{28900}[934D\sin x - 812\sin x]$$

$$= -\frac{x}{17}[7\cos x - 11\sin x] + \frac{1}{28900}[934\cos x - 812\sin x]$$

C.S = C.F + P.I

$$= C_1e^{-3x} + C_2e^{-4x} + \frac{1}{28900}[934\cos x - 812\sin x] - \frac{x}{17}(7\cos x - 11\sin x]$$

14
11

14
14

154

144
49

75

121
49

72

121
49

170

(iv) $\frac{1}{f(D)} e^{ax} v = e^{x} \frac{1}{f(D+a)} v$

H.W

Q

$$\frac{d^2y}{dx^2} + \frac{3dy}{dx} + 2y = x e^{x} \sin x$$

Sol

S.F

$$(D^2 + 3D + 2)y = x \, \underbrace{e^{x} \sin x}_{v}$$

A.E

$$D^2 + 3D + 2 = 0$$

$$D^2 + D + 2D + 2 = 0$$

$$D(D+1) + 2(D+1) = 0$$

$$D = -1, -2$$

$$C.F = C_1 e^{-1x} + C_2 e^{-2x}$$

P.I.

$$\frac{1}{D^2 + 3D + 2} x e^{x} \sin x$$

$$\cancel{x \frac{1}{D^2+3D+2} e^{x} \sin x + \frac{d}{dD}\left(\frac{1}{D^2+3D+2}\right) e^{x} \sin x}$$

$$P.I = e^{x} \frac{1}{(D+1)^2 + 3(D+1) + 2} x \sin x$$

$$= e^{x} \frac{1}{D^2 + 1 + 2D + 3D + 3 + 2} x \sin x$$

$$= e^{x} \frac{1}{D^2 + 5D + 6} x \sin x$$

$e^{mx}(C_1 + C_2 x)$

$$e^{x}\left[x\left(\frac{1}{D^2+5D+6} \sin x\right) + \frac{d}{dD}\left(\frac{1}{D^2+5D+6}\right) \sin x\right]$$

$$e^{x}\left[x \frac{1}{-1+5D+6} \sin x + \frac{-[2D+5)}{(D^2+5D+6)^2} \sin x\right]$$

$$e^{x}\left[x \frac{1}{5D+5} \sin x - \frac{[2D+5]}{(-1+5D+6)^2} \sin x\right]$$

$$e^{x}\left[\frac{x}{5} \frac{(D-1)}{D^2-1} \sin x - \frac{[2D+5]}{50D} \sin x\right]$$

$$e^{x}\left[\frac{x}{10}(\cos x - \sin x) - \frac{1}{50} \frac{2D^2+5D}{D^2} \sin x\right]$$

$$e^{x}\left(\frac{x}{10}(\sin x - \cos x) + \frac{1}{50}(-2+5D) \sin x\right]$$

$$e^{x}\left(\frac{x}{10}(\sin x - \cos x) + \frac{1}{50}(5D \sin x - 2 \sin x)\right]$$

C.S

$$C_1 e^{-1x} + C_2 e^{-2x} + e^{x}\left(\frac{x}{10}(\sin x - \cos x) + \frac{1}{50}(5D \sin x - 2 \sin x)\right)$$

$\underline{5D+5}$

$25D^2 + 25 + 50D$

$-25 + 25$

$25(D^2 + 1 + 2D)$

$(D^2 + 2D + 1)$

$(\cdot 2D)$

$50D$

Date ______
Page ______

H.W
$$x^3\frac{d^3y}{dx^3} + 6x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 4y = \log x^2$$

Sol

S.F
$$(x^3D^3 + 6x^2D^2 + 4xD - 4)y = \log x^2$$

Put $x = e^z$

$z = \log x$

$xD = \theta$

$x^2D^2 = \theta(\theta-1) \Rightarrow x^3D^3 = \theta(\theta-1)(\theta-2)$

A.F.
$$[\theta(\theta-1)(\theta-2) + 6\theta(\theta-1) + 4\theta - 4]y$$
$$= z^2$$
$$((\theta^2-\theta)(\theta-2) + 6\theta^2 - 6\theta + 4\theta - 4]y = z^2$$
$$[\theta^3 - 2\theta^2 - \theta^2 + 2\theta + 6\theta^2 - 6\theta + 4\theta - 4]y = z^2$$
$$[\theta^3 + 3\theta^2 - 4]y = z^2$$

$\theta = 1$
$(\theta-1)$
$1 + 3 - 4$
$4 - 4 = 0$

$$(\theta-1)\quad(\theta^2 + 4\theta + 4) = 0$$
$$(\theta-1)\quad(\theta^2 + 2\theta + 2\theta + 4) = 0$$
$$\theta(\theta+2) + 2(\theta+2) = 0$$
$$(\theta+2)(\theta+2) = 0$$
$$\theta = -2, -2, +1$$

1 | 1 3 0 -4
  | 1 4 4
  | 1 4 4 0

Date ______
Page ______

$$C.F = C_1e^z + (C_2 + C_3z)e^{-2z}$$

$$P.I. = \frac{1}{\theta^3 + 3\theta^2 - 4}(z^2)$$
$$= \frac{1}{-4 + 3\theta^2 + \theta^3}(z^2)$$
$$= \frac{1}{-4\left(1 - \frac{3\theta^2}{4} + \frac{\theta^3}{4}\right)}z^2$$
$$= -\frac{1}{4}\left[1 + \left(-\frac{3\theta^2}{4} + \frac{\theta^3}{4}\right)\right]^{-1}z^2$$
$$= -\frac{1}{4}\left[1 + (-1)\left(\left(-\frac{3\theta^2}{4}\right) + \frac{\theta^3}{4}\right)\right]z^2$$
$$= -\frac{1}{4}\left(1 + (-1)\left(\frac{\theta^3}{4} - \frac{3\theta^2}{4}\right)\right)z^2$$
$$= -\frac{1}{4}\left(1 - \frac{\theta^3}{4} + \frac{3\theta^2}{4}\right]z^2$$
$$= -\frac{1}{4}\left(1 + \frac{3\theta^2 - \theta^3}{4}\right]z^2$$
$$= -\frac{1}{4}\left(1 + \frac{3\theta^2}{4}\right)z^2$$
$$= -\frac{1}{4}\left(z^2 + \frac{3\theta^2}{4}z^2\right)$$
$$= -\frac{1}{4}\left((\log x)^2 + \frac{3}{2}\right)$$

$$G.S = c_1 x + (c_2 + c_3 \log x) x^{-2} - \frac{1}{4}\left[(\log x)^2 + \frac{3}{2}\right]$$

COMPLEX NUMBERS

Complex number

$(a+ib)$

$\therefore$ Complex number is in form $a+ib$

Conjugate of Complex number

if $z = a+ib$

then its Conjugate $= a-ib$

$$\bar{z} = a - ib$$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

modulus of Complex number

if $z = x+iy$

$$|z| = \sqrt{x^2+y^2}$$

Inverse of Complex number

$$\frac{1}{z} = z^{-1}$$

Q) Simplify $2i^2 + 6i^3 + 3i^{16} - 6i^{19} + 4i^{25}$

Sol

$2i^2 + 6i^3 + 3(i^4)^4 - 6(i^3)^6 i + 4(i^4)^6 i$

$2(-1) + 6(-i) + 3(1)^4 - 6(-i)^6 i + 4(1) i$

$-2 - 6i + 3 - 6i^6 + 4i$

$-2 - \cancel{6i} + 3 + \cancel{6i} + 4i = i + i + 1$

$i^4 \quad i^2$

$1 \quad (-1)$

Q) Convert $i^9 + i^{19}$ is in the form $a+ib$

Sol

$$i^9 + i^{19}$$

$$i^8 \cdot i + i^{18} \cdot i = (i^2)^4 \cdot i + (i^2)^9 \cdot i$$

$$= (-1)^4 \cdot i + (-1)^9 \cdot i$$

$$= i - i = 0 + 0i$$

Q) express $(5-3i)^3$ is in the form $a+ib$

Sol

$$125 - 27i^3 + 3 \times 25 \times (-3i) + 3 \times 9 i^2 \times 5$$

$$= 125 - 27i^3 - 225i + 135i^2$$

$$= 125 + 27i - 225i - 135$$

$$= 10 - 198i$$

Q) find the Conjugate of $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$.

Sol

$$\Rightarrow \frac{6+9i-4i-6i^2}{2+4i-i-2i^2} = \frac{6+9i-4i+6}{2+4i-i+2}$$

$$= \frac{12+5i}{4+3i} = \frac{(12+5i)}{(4+3i)} \times \frac{(4-3i)}{(4-3i)}$$

$$= \frac{48-36i+20i-15i^2}{16-12i+12i-9i^2}$$

$$= \frac{48-16i+15}{16+9} = \frac{63-16i}{25} = \frac{63}{25} - \frac{16}{25}i$$

Q) find the modulus

Sol

$$\frac{(1+i)}{(1-i)} - \frac{(1-i)}{(1+i)}$$

S. $$\frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)} = \frac{(1)^2+i^2+2i-[(1)^2+i^2-2i]}{(1-i)(1+i)}$$

$$= \frac{\cancel{1}-\cancel{1}+2i-\cancel{1}+\cancel{1}+2i}{(1)^2-(i)^2} = \frac{\cancel{4}i}{\cancel{2}} = 2i$$

$$z = 0 + 2i$$

$$|z| = \sqrt{(0)^2+(2)^2} = \sqrt{4} = 2$$

Squre root of Complex number

Q) find the Squre root $7-24i$

Sol let $\sqrt{7-24i} = x - iy$ — (1)

Squaring both side

$$7-24i = x^2 - y^2 - 2ixy$$

$$7-24i = (x^2-y^2) - 2ixy$$

equating real and imaginary part

$$7 = x^2 - y^2 \text{ — (2)} \qquad -24 = -2xy$$

$$24 = 2xy \text{ — (3)}$$

$$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$$

from (2) and (3)

$$= (7)^2 + (24)^2$$

$$= 49 + 576 = 625$$

$$x^2 + y^2 = 25 \text{ — (4)}$$

Date
Page

adding 3 and 4

$$x^2 - y^2 + x^2 + y^2 = 25 + 7$$
$$2x^2 = 32$$
$$3x^2 = 2 \times 4$$
$$x = \pm 4$$

Sub 3 and 4

$$2y^2 = 18$$
$$y^2 = 9$$
$$y = \pm 3$$

x and y Put in 1

$$\sqrt{7 - 24i} = \pm(4 - 3i)$$

Q> find the Squre root of $7 + 24i$

Sol

let $\sqrt{7 + 24i} = x + iy$

Squaring both side

$$7 + 24i = x^2 + i^2y^2 + 2xiy$$
$$7 + 24i = (x^2 - y^2) + 2xyi$$
$$7 = x^2 - y^2 \quad -(2)$$
$$24 = 2xy \quad -(3)$$

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$$

from 2 and 3

$$= 49 + (24)^2$$
$$= 49 + 576 = 625$$
$$x^2 + y^2 = 25 \quad -(4)$$

adding 2 and 4

$$x^2 + y^2 + x^2 - y^2 = 7 + 25$$
$$2x^2 = 32$$

Date
Page

$x = \pm 4$ but ~~$x \neq -ve$~~

So $x = \pm 4$

sub 2 and 4

$$x^2 + y^2 - x^2 + y^2 = 25 - 7$$

$$2y^2 = 18$$
$$y^2 = 9$$
$$y = \pm 3$$

$$\sqrt{7 + 24i} = \pm(4 + i3)$$

Q> Cuberoot of Unity

Sol Cuberoot of Unity are $1, \omega, \omega^2$

or $1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$

Note

Sum of Cube root of Unity

$$1 + \omega + \omega^2 = 1 + \frac{(-1 + i\sqrt{3})}{2} + \frac{(-1 - i\sqrt{3})}{2}$$
$$= \frac{2 - 1 + i\sqrt{3}}{2} + \frac{(-1 - i\sqrt{3})}{2}$$
$$= \frac{\cancel{2} \cancel{-1} + i\sqrt{3} - 1 - i\sqrt{3}}{2} = 0$$

Date
Page

(2) multiplication

$$1 \times \left(\frac{-1+i\sqrt{3}}{2}\right) \times \left(\frac{-1-i\sqrt{3}}{2}\right)$$

$$= 1 \times \frac{1+i\sqrt{3}-i\sqrt{3}+3}{4} = \frac{4}{4}$$

Q> find the cube root of -27

Sol

let $x = (-27)^{\frac{1}{3}}$

taking qube on both side

$x^3 = (-27)$

taking all term left side

$x^3 + 27 = 0$

$x^3 + (3)^3 = 0$

$(x+3)(x^2+9-3x) = 0$

$x = -3$

$x^2 - 3x + 9 = 0$

$\frac{3 \pm 3\sqrt{3}i}{2} = \frac{3(1 \pm \sqrt{3}i)}{2}$

~~$= \frac{3 \pm \sqrt{27}i}{3} = 3($~~

3	27
3	9
3	3
	1

$3\sqrt{3}$

$x^3 + \cancel{9x} - \cancel{3x^2}$

$+ \cancel{3x^2} + 27 - \cancel{9x}$

$x^3 + 27$

Date
Page

Q> if $1, \omega, \omega^2$ are the qube root of unity then show that

$$(3+3\omega+5\omega^2)^6 - (2+6\omega+2\omega^2)^3 = 0$$

Sol

Taking left hand side

$$= [(3+3\omega+3\omega^2)+2\omega^2]^6 - (4\omega+(2+2\omega+2\omega^2))^3 = 0$$

$$= (3[(1+\omega+\omega^2)+2\omega^2])^6 - [4\omega+2(1+\omega+\omega^2)]^3 = 0$$

$$= (2\omega^2)^6 - 64\omega^3 = 0$$

$$= 64 - 64 = 0$$

Q> show that $\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} = \omega$

Sol

taking L.H.S

$$\frac{(a+b\omega+c\omega^2) \times \omega}{(b+c\omega+a\omega^2) \times \omega^2}$$

$$\frac{\omega(a+b\omega+c\omega^2)}{(b\omega+c\omega^2+a(1))}$$

$= \omega =$ R.H.S.

Q if

$x = a + b$
$y = a\omega + b\omega^2$
$z = a\omega^2 + b\omega$

then

$x^3 + y^3 + z^3 = 3(a^3 + b^3)$

$(a+b)^3 + (a\omega + b\omega^2)^3 + (a\omega^2 + b\omega)^3$

$= a^3 + b^3 + 3a^2b + 3ab^2 + a^3 + b^3 + 3a^2b\omega + 3a\omega^2b^2$
$+ a^3 + b^3 + 3a^2\omega^2 b + 3ab^2\omega$

$= a^3 + a^3 + a^3 + b^3 + b^3 + b^3 + 3a^2b + 3ab^2$
$+ 3a^2b\omega + 3a\omega^2b^2 + 3a^2\omega^2 b + 3ab^2\omega$

$= 3a^3 + 3b^3 + 3a^2b(1 + \omega + \omega^2) + 3ab^2(1 + \omega^2 + \omega)$
$= 3a^3 + 3b^3 + 0 + 0$

$= 3(a^3 + b^3)$

Proved

⇒ Polar quadirates ⇒ $r(\cos\theta + i\sin\theta)$

- Standard form of Complex number.

$z = x + iy$
$x + iy = r(\cos\theta + i\sin\theta)$
$x = r\cos\theta$ — (1) $\quad \cos\theta = \frac{x}{r}$
$y = r\sin\theta$ — (2) $\quad \sin\theta = y/r$

Squaring and adding (1) and (2)

$r^2\cos^2\theta + r^2\sin^2\theta = x^2 + y^2$
$r = \sqrt{x^2 + y^2}$
$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \qquad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$

$\frac{\sin\theta}{\cos\theta} = y/x = \frac{y}{r} \times \frac{r}{x}$

$\tan\theta = y/x$
$\theta = \tan^{-1}\frac{y}{x}$

Q) find the Principal value of $-2 - 2i$

Sol let

$-2 - 2i = r(\cos\theta + i\sin\theta)$

$-2 = r\cos\theta$
$-2 = r\sin\theta$

Squaring and adding

$4 + 4 = r^2$
$8 = r^2 \Rightarrow r = \pm 2\sqrt{2}$
$r = 2\sqrt{2}$

$\cos\theta = \frac{-2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}}$

$\sin\theta = \frac{-2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4} - \pi \qquad \theta = \frac{\pi - 4\pi}{4} = \frac{-3\pi}{4}$

Q> Conver the Comple $\frac{-16}{1+i\sqrt{3}}$ int Polar form

Sol

$$z = \frac{-16}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}}$$

$$= \frac{-16+16i\sqrt{3}}{(1)^2 - i^2 3}$$

$$= \frac{-16+16\sqrt{3}i}{1+3}$$

$$= \frac{4(-4+4\sqrt{3}i)}{4}$$

$$= -4+4\sqrt{3}i$$

$$-4+4\sqrt{3}i = r(\cos\theta + i\sin\theta)$$

$$-4 = r\cos\theta$$
$$4\sqrt{3} = r\sin\theta$$

$$16 + 16\times 3 = r^2$$
$$64 = r^2$$
$$r = 8$$

$$\cos\theta = \frac{-4}{8} \Rightarrow -\frac{1}{2}$$

$$\sin\theta = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$$

$$\pi - \theta = \pi - \frac{\pi}{3}$$

$$= \frac{3\pi - \pi}{3} = \frac{2\pi}{3}$$

$$\Rightarrow x+iy = r\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$\Rightarrow -4+4\sqrt{3}i = 8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Q> Show that the points $1+i$, $-3+9i$, $3-3i$ lies on a straight line

Sol

let $z_1 = 1+i$ $z_2 = -3+9i$ $z_3 = 3-3i$

let A, B, C, represen the point z_1, z_2, z_3 respectively.

$$|AB| = |z_2 - z_1| = \sqrt{(-3-1)^2 + (9-1)^2}$$
$$= \sqrt{16+64}$$
$$= \sqrt{80} = 4\sqrt{5}$$

$$|BC| = |z_3 - z_2| = \sqrt{(3+3)^2 + (-3-9)^2}$$
$$= \sqrt{(6)^2 + (-12)^2}$$
$$= \sqrt{36+144}$$
$$= 6\sqrt{5}$$

$$|CA| = |z_1 - z_3| = \sqrt{(1-3)^2 + (1+3)^2}$$

$$= \sqrt{(2)^2 + (4)^2} = \sqrt{20} = 2\sqrt{5}$$

$$|AB| + |CA| = |BC|$$

$$4\sqrt{5} + 2\sqrt{5} = 6\sqrt{5}$$

$\therefore$ A B C are Colinear and A lies between B and C

$\pi r^2 = 5$

Date
Page

Q5 find the center and radius of Circel.

Sol

$$\left|\frac{z-i}{z+i}\right| = 2$$

$$\frac{|z-i|}{|z+i|} = 2$$

$$|z-i| = 2|z+i|$$

$$|x+iy-i| = 2|x+iy+i|$$

$$|x+i(y-1)| = 2|x+i(y+1)|$$

$$\sqrt{x^2+(y-1)^2} = 2\sqrt{x^2+(y+1)^2}$$

Squaring both side

$$x^2+(y-1)^2 = 4(x^2+(y+1)^2)$$

$$x^2+y^2+1-2y = 4x^2+4[y^2+1+2y]$$

$$x^2+y^2-2y+1 = 4x^2+4y^2+4+8y$$

$$x^2-4x^2+y^2-4y^2-2y-8y+1-4=0$$

$$3x^2+3y^2+10y+3=0$$

$$x^2+y^2+\frac{10y}{3}+1=0$$

Centre $(-g, -f)$

$\left[0, -\frac{5}{3}\right)$

$$r = \sqrt{(g)^2+(-f)^2-g}$$

$$= \sqrt{0+\frac{25}{9}-1}$$

De Moivres theorem

1) $\frac{1}{cis\theta} = cis(-\theta) \quad cis\theta = \cos\theta + i\sin\theta$

2) $cis\,\theta_1 . \sin\theta_2 = cis(\theta_1 + \theta_2)$

3) $\frac{cis\,\theta_1}{cis\,\theta_2} = cis\theta_1 - cis\theta_2$

~~Qs $z_1 = 1+i \quad z_2 = 4+4i \quad z_3 = 4+8i \quad z_4 = 1+5i$~~

Qs $\left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \cos 8\theta + i\sin 8\theta$

Taking L.H.S

$\left(\frac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^4 = \left(\frac{(\cos\theta + i\sin\theta)^4}{(\sin\theta + i\cos\theta)^4}\right)$

$\frac{(\cos\theta + i\sin\theta)^4}{\left(\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right)^4}$

$= \frac{\cos 4\theta + i\sin 4\theta}{\cos(2\pi - 4\theta) + i\sin(2\pi - 4\theta)}$

$= \frac{\cos 4\theta + i\sin 4\theta}{\cos 4\theta - i\sin 4\theta} = \frac{(\cos 4\theta + i\sin 4\theta)}{(\cos 4\theta - i\sin 4\theta)}$

$= (\cos 4\theta + i\sin 4\theta)(\cos(4\theta) + i\sin 4\theta)$

$= \cos(8\theta) + i\sin 8\theta$

Qs $\left(\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\phi}\right)^8$

Sol L.H.S

$\left[\frac{1+\sin\phi + i\sin\phi}{1+\sin\phi - i\cos\phi}\right]^8$

$$\left[\begin{array}{l} \sin^2\theta - i^2\cos^2\theta = 1 \\ \sin^2\theta - i^2\cos^2\theta = 1 \\ (\sin^2\theta + i\cos\theta)(\sin^2\theta - i\cos\theta) = 1 \end{array}\right]$$

$\left[\frac{(\sin\theta + i\cos\theta)(\sin\theta - i\cos\theta) + (\sin\phi + i\cos\phi)}{1+\sin\phi - i\cos\phi}\right]$

$\left[(\sin\theta + i\cos\phi)\left[\frac{\sin\theta - i\cos\theta + 1}{1+\sin\theta - i\cos\theta}\right]\right]^8$

$\left(\frac{\sin\phi + i\cos\phi}{1}\right)^8$

$i^8\left(\frac{\sin\phi}{i} + \cos\phi\right)^8 = \left(\cos\phi + \frac{\sin\theta}{i}\right)^8$

$\Rightarrow \left(\cos\phi + i^{-1}\sin\theta\right)^8$

$\Rightarrow \cos 8\phi + i\sin 8\phi$

hw

Date ____ Page ____

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Q) $\left(\dfrac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right)^{10}$

$\Rightarrow \dfrac{(\cos\theta + i\sin\theta)^{10}}{(\sin\theta + i\cos\theta)^{10}} = \dfrac{\cos 10\theta + i\sin 10\theta}{\left(\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right)^{10}}$

$= \dfrac{\cos 10\theta + i\sin 10\theta}{[\cos(5\pi - 10\theta) + i\sin(5\pi - 10\theta)]}$

$= \dfrac{\cos 10\theta + i\sin 10\theta}{\cos(\pi - 10\theta) + i\sin(\pi - 10\theta)} = \dfrac{\cos 10\theta + i\sin 10\theta}{-\cos 10\theta + i\sin 10\theta}$

$= -\dfrac{(\cos 10\theta + i\sin 10\theta)}{(\cos 10\theta - i\sin 10\theta)}$

$\Rightarrow -(\cos 10\theta + i\sin 10\theta)(\cos 10\theta - i\sin 10\theta)^{-1}$

$\Rightarrow -(\cos 10\theta + i\sin 10\theta)(\cos 10\theta + i\sin 10\theta)$

$\Rightarrow -[\cos(10\theta + 10\theta) + i\sin(10\theta + 10\theta)]$

$\Rightarrow \boxed{-[\cos 20\theta + i\sin 20\theta]}$ Ans

Date ____ Page ____

Q) if $(\cos 2\theta + i\sin 2\theta) = p$, $(\cos 2\phi + i\sin 2\phi) = q$

prove

$2\cos(\phi - \theta) = \sqrt{\dfrac{p}{q}} + \sqrt{\dfrac{q}{p}}$

Sol $p = (\cos 2\theta + i\sin 2\theta)$, $q = (\cos 2\phi + i\sin 2\phi)$

$\therefore \dfrac{p}{q} = \dfrac{(\cos 2\theta + i\sin 2\theta)}{(\cos 2\phi + i\sin 2\phi)}$

$= (\cos 2\theta + i\sin 2\theta)(\cos 2\phi + i\sin 2\phi)^{-1}$

$= (\cos 2\theta + i\sin 2\theta)\,[\cos(-2\phi) + i\sin(-2\phi)]$

$= \cos(2\theta - 2\phi) + i\sin(2\theta - 2\phi)$

$\therefore \sqrt{\dfrac{p}{q}} = [\cos(2\theta - 2\phi) + i\sin(2\theta - 2\phi)]^{\frac{1}{2}}$

or $\sqrt{\dfrac{p}{q}} = \cos(\theta - \phi) + i\sin(\theta - \phi)$ — (I)

also $\sqrt{\dfrac{q}{p}} = \left(\dfrac{p}{q}\right)^{-1/2} = \cos(2\theta - 2\phi) + i\sin(2\theta - 2\phi)$

$\therefore \sqrt{\dfrac{q}{p}} = \cos(\theta - \phi) - i\sin(\theta - \phi)$ — II

$\therefore$ Adding I and II, we get

$\sqrt{\dfrac{p}{q}} + \sqrt{\dfrac{q}{p}} = 2\cos(\theta - \phi)$ Ans

11/01/22

Date
Page

$x^2+2x+2=0$

Q1) $\dfrac{(x+\alpha)^n - (x+\beta)^n}{\alpha-\beta} = \dfrac{\sin n\phi}{\sin^n\phi}$

where $x+1 = \cot\phi$

$\dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{2 \pm \sqrt{2-4}}{2a} = 1 \pm i$

let $\alpha = 1+i$, $\beta = 1-i$

$x+\alpha = x+1+i$, $x+\beta = x+1-i$

$x+\alpha = \cot\phi + i$, $x+\beta = \cos\phi - i$

$x+\alpha = \dfrac{\cos\phi}{\sin\phi} + i$, $x+\beta = \dfrac{\cos\phi}{\sin\phi} - i$

$x+\alpha = \dfrac{\cos\phi + i\sin\phi}{\sin\phi}$, $x+\beta = \dfrac{\cos\phi - i\sin\phi}{\sin\phi}$

$(x+\alpha)^n - (x+\beta)^n = \left(\dfrac{\cos\phi + i\sin\phi}{\sin\phi}\right)^n - \left(\dfrac{\cos\phi - i\sin\phi}{\sin\phi}\right)^n$

$= \dfrac{\cos n\phi + i\sin n\phi}{(\sin\phi)^n} + \dfrac{-\cos n\phi + i\sin n\phi}{(\sin\phi)^n}$

$= \dfrac{2i\sin n\phi}{(\sin\phi)^n}$

Date
Page

also $\alpha-\beta = 1+i-1+i$
$= 2i$

$= \dfrac{\cancel{2i}\sin n\phi}{\dfrac{\sin^n\phi}{\cancel{2i}}} = \dfrac{\sin n\phi}{\sin^n\phi}$

Q) If $x+\dfrac{1}{x} = 2\cos\theta$

P.T. That $x^n + \dfrac{1}{x^n} = 2\cos n\theta$

Q) P.T. that

$(1+i)^n + (1-i)^n = 2^{n/2+1}\cos n\pi/4$

$1+i = r(\cos\theta + i\sin\theta)$ — (1)

$r\cos\theta = 1$, $r = \dfrac{1}{\cos\theta}$

$r\sin\theta = 1$, $r = \dfrac{1}{\sin\theta}$

$\cos\theta = \dfrac{1}{\sqrt{2}}$,

$\sin\theta = \dfrac{1}{\sqrt{2}}$

$2 = r^2$

$\sqrt{2} = r$

$r = \sqrt{2}$

$1+i = \sqrt{2}\left[\cos\left(\dfrac{\pi}{4}\right) - i\sin\dfrac{\pi}{4}\right]$

$= \left(r\cos\left(\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)\right)^n + \left(r\cos\left(\dfrac{\pi}{4} - i\sin\dfrac{\pi}{4}\right)\right)^n$

$\left(x - \dfrac{1}{x}\right)$

$$= (\sqrt{2})^n \left(\cos n\frac{\pi}{4} + i\sin n\frac{\pi}{4}\right) + (\sqrt{2})^n \left(\cos n\frac{\pi}{4} - i\sin n\frac{\pi}{4}\right)$$

H.W

Q> $(\sqrt{3}+i)^n + (\sqrt{3}-i)^n = 2^{n+1}\cos n\pi/6$

Q> $\sin\alpha + \sin\beta + \sin\gamma + \cos\alpha + \cos\beta + \cos\gamma = 0$

Prove that $\Rightarrow \cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

$\Rightarrow \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0$

Sol let

$a = \cos\alpha + i\sin\alpha$

$b = \cos\beta + i\sin\beta$

$c = \cos\gamma + i\sin\gamma$

$\therefore a+b+c = \cos\alpha + i\sin\alpha + \cos\beta + i\sin\beta + \cos\gamma + i\sin\gamma$

$a+b+c = \cos\alpha + \cos\beta + \cos\gamma + i\sin\alpha + i\sin\beta + i\sin\gamma = 0$ — (1)

$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \cos(-\gamma) + i\sin(-\gamma) + \cos(-\beta) + i\sin(-\beta) + \cos(-\gamma) + i\sin(-\gamma)$

$\cancel{= \cos\gamma + i\sin\gamma +}$

$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \cos\gamma + \cos\beta + \cos\gamma - i\sin\alpha - i\sin\beta - i\sin\gamma = 0$

$\Rightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{b+a}{ab} + \frac{1}{c} = \frac{cb+ca+ab}{abc}$

$cb + ca + ab = 0$ (2)

$(a^2+b^2+c^2)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

$= a^2+b^2+c^2+2(ab+bc+ca)$

$= a^2+b^2+c^2 = 0$

let a b c values

$= \cos 2\alpha + \cos 2\beta + \cos 2\gamma + i\sin 2\alpha + i\sin 2\beta + i\sin 2\gamma$

$\Rightarrow \cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$

$\Rightarrow i\sin 2\alpha + i\sin 2\beta + i\sin 2\gamma = 0$

Root of Complex Number :-

Note :

$$1 = \cos 0 + i\sin 0$$
$$-1 = \cos\pi + i\sin\pi$$
$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$
$$-i = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$$

Q1> Evaluate : (i) $(1)^{1/3}$

Sol

$(\cos 0 + i\sin 0)^{1/3}$ [we : know]

$\Rightarrow \left[\cos(2n\pi + 0) + i\sin(2n\pi + 0)\right]^{1/3}$

$\Rightarrow \left[\cos 2n\pi + i\sin 2n\pi\right]^{\frac{1}{3}}$

$\Rightarrow$ using De - moivre's

$$\cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}$$

where $n = 0, 1, 2$

when $n = 0, 1, 2$

$\cos 0 + i\sin 0$, $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

$1 + i0$, $\cos\pi - \frac{\pi}{3} + i\sin\pi - \frac{\pi}{3}$, $\cos\pi + \frac{\pi}{3} + i\sin\pi + \frac{\pi}{3}$

$\boxed{1,\ -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ -\frac{1}{2} - i\frac{\sqrt{3}}{2}}$ Ans

H.W

(ii) $(1)^{\frac{1}{4}}$

we know

$(1)^{\frac{1}{4}} = (\cos 0 + i\sin 0)^{1/4}$

$= \left(\cos(2n\pi + 0) + i\sin(2n\pi + 0)\right)^{1/4}$

$(1)^{1/4} = (\cos 2n\pi + i\sin 2n\pi)^{1/4}$

Using De - moivre's Theorem:

$$\cos\frac{2n\pi}{4} + i\sin\frac{2n\pi}{4}$$

$\Rightarrow \cos\frac{n\pi}{2} + i\sin n\frac{\pi}{2}$

where $n = 0, 1, 2, 3,$

$= \cos 0 + i\sin 0$, $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$, $\cos\pi + i\sin\pi$,

$\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$

$\Rightarrow 1 + i0$, $0 + i1$, $-1 + i0$, $\cos\pi + \frac{\pi}{2}$, $\sin\left(\pi + \frac{\pi}{2}\right)$

$\Rightarrow 1 + i0$, $0 + i$, -1, $0 - i$ Ans

H.W (ii) find the value of $(-1)^{\frac{1}{4}}$

Sol we know:

$(-1)^{1/4} = (\cos\pi + i\sin\pi)^{1/4}$

$= \left(\cos(2n\pi + \pi) + \sin(2n\pi + \pi)\right)^{1/4}$

$= \left(\cos(2n+1)\pi + \sin(2n+1)\pi\right)^{1/4}$

Date ______
Page ______

Using De - marvel's theorem:

$$\Rightarrow \cos\frac{(2n+1)}{4}\pi + i\sin\frac{(2n+1)}{4}\pi$$

when $n = 0, 1, 2, 3$

$$\Rightarrow \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}, \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}, \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4},$$

$$\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$$

$$\Rightarrow \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \cos\pi - \frac{\pi}{4} + i\sin\pi - \frac{\pi}{4}, \cos\pi + \frac{\pi}{4}$$

$$+ i\sin\pi + \frac{\pi}{4}, \cos 2\pi - \frac{\pi}{4} + i\sin 2\pi - \frac{\pi}{4}$$

$$\Rightarrow \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

Q ~~linear~~ Cong

Linear Congurence

$$ax \equiv b \pmod m$$

Conguerent and inconguerent solution

the integers wich satisfies a given linear congruence mod m and blongs to the same resue ~~cal~~ class are are conguerent solution other wise its called inconguerent solutio

(1)

for example $4x \equiv 7 \pmod 5$ | $5 \mid 4x-7$

Here $x = 0, 1, 2, 3, 4$ out of five
but $x = 3$

$\therefore$ only 3 has its incongurent sol.

for exampl $2x \equiv 5 \pmod 8$ | $8 \mid 2x-5$

$x = 0, 1, 2 \ldots, 7$

$\therefore$ this equation has not sol.

for exampl $8x \equiv 10 \pmod 6$ | $6 \mid 8x-10$

$x = 0, 1, \ldots 5$

but $x = 2, 5$ | 2 3

$\therefore$ there are 2 incongurent sol.

Date
Page

note :-> $x \equiv x_1 + \frac{m}{d} t \pmod{m}$

Q) How many solution of this (≡) Congruences

$15x \equiv 25 \pmod{35}$

Sol

$ax \equiv b \pmod{m}$

$a = 15$
$b = 25$
$m = 35$

$\therefore = (15, 35) = 5$

$5 \mid 25$ because b divisible d

$\therefore$ there are 5 incongruent solution (mod 35)

H.W

Q) $15x \equiv 24 \pmod{35}$

Sol

$a = 15$
$b = 24$
$m = 35$
$= (15, 35) = 5$

$d \nmid b = 5 \nmid 24$ $\therefore$ there are no solution.

HW

Q) $20x \equiv 4 \pmod{30}$

Sol

$a = 20$, $b = 4$, $m = 30$
$= (20, 30) = 10 = d$

Date
Page

but $d \nmid b$ => there is no incongruent solutions modulo 30.

Q) Solve $18x \equiv 30 \pmod{42}$ — (1)

$a = 18$
$b = 30$
$m = 42$
$GCD = (18, 42) = 6$

$6 \mid 30$

$\therefore$ ~~first~~ (1) has 6 incongruent solution

$\frac{18x}{6} \equiv \frac{30}{6} \left(\bmod \frac{42}{6}\right)$ — (2)

$3x \equiv 5 \pmod{7}$

$x = 0, 1, 2, 3, 4, 5, 6$

$\therefore x = 4$ is the solution of (2)

$\therefore$ 6 solution are given by

$x \equiv 4 + \frac{42}{6} p \pmod{42}$

$\because 0 \le p \le 5$

Put $p = 0, 1, 2, 3, 4, 5$

$7 \mid 3x - 5$

$\dot{6}$

$21 - 5$

$\equiv 4 + 14$
$\equiv 18 \pmod{42}$
$25 \pmod{42}$
$\equiv 4 + 28 \pmod{42}$
$\equiv 32 \pmod{42}$
$\equiv 39 \pmod{42}$

Date ____
Page ____

1) $x \equiv 4 + 0 \pmod{42}$
2) $x \equiv 18 \pmod{42}$
3) $x \equiv 25 \pmod{42}$
4) $x \equiv 32 \pmod{42}$
5) $x \equiv 39 \pmod{42}$
6

H.W. Q) $9x \equiv 21 \pmod{30}$

H.W. Q) $140x \equiv 133 \pmod{301}$

Q) solve $3x - 7y \equiv 11 \pmod{13}$ - ①

$= (3, 7, 13) = 1$

$1 / 11$

now from ①

$3x \equiv 11 \pmod{13} + 7y$

$3x \equiv 11 + 7y \pmod{13}$

$3x \equiv (11 + 13) + (7y - 13y) \pmod{13}$

$3x \equiv (24) + (-6y) \pmod{13}$

$x = 8 - 2y \pmod{13}$

$y = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$

$y = 0 \Rightarrow x \equiv 8 \pmod{13}$

$y = 1 \Rightarrow x \equiv 6 \pmod{13}$

Date ____
Page ____

$y = 2 \Rightarrow x \equiv 4 \pmod{13}$

$y = 3 \Rightarrow x \equiv 2 \pmod{13}$

$y = 4 \Rightarrow x \equiv 0 \pmod{13}$

$y = 5 \Rightarrow x \equiv -2 \pmod{13} = 11 \pmod{13}$

$y = 6 \Rightarrow x \equiv -4 \pmod{13} = 9 \pmod{13}$

$y = 7 \Rightarrow x \equiv -6 \pmod{13} = 7 \pmod{13}$

$y = 8 \Rightarrow x \equiv -8 \pmod{13} \equiv 5 \pmod{13}$

$y = 9 \Rightarrow x \equiv -10 \pmod{13} = 3 \pmod{13}$

$y = 10 \Rightarrow x \equiv -12 \pmod{13} = 1 \pmod{13}$

$y = 11 \Rightarrow x \equiv -14 \pmod{13} = 12 \pmod{13}$

$y = 12 \Rightarrow x \equiv -16 \pmod{13} = 10 \pmod{13}$

$y = 0$, $x = 8$	$y = 1$, $x = 6$	$y = 2$, $x = 4$	$y = 3$, $x = 2$	$y = 4$, $x = 0$	$y = 5$, $x = 11$
$y = 6$, $x = 9$	$y = 7$, $x = 7$	$y = 8$, $x = 5$	$y = 9$, $x = 3$	$y = 10$, $x = 1$	$y = 11$, $x = 12$
$y = 12$, $x = 10$					

H.W $2x + 7y \equiv 5 \pmod{12}$

H.W $70x \equiv 168 \pmod{122}$

H.W chines remainder theorem state and prove

---

3) $9x \equiv 21 \pmod{30}$ — ①

$a = 9 \quad b = 21 \quad m = 30$

$(9, 30) = 3 = d$

$d \mid b = 3 \mid 21$

$\therefore$ ① has 3 incongruent solution.

$\frac{9x}{3} = \frac{21}{3} \left(\bmod \frac{30}{3}\right)$

$3x \equiv 7 \pmod{10}$ — ②

$x = 0, 1, 2, 3, \cancel{4, 5, 6, 7, 8, 9}$

$\therefore x = 9$ is the solution of ②

$x \equiv x_1 + \frac{m}{d} r \pmod{m}$

$x \equiv 9 + \frac{30}{3} r \pmod{30}$

$10 \mid 3x - 7$

9

4) Sol $140x \equiv 133 \pmod{301}$ — ①

$a = 140 \quad b = 133 \quad m = 301$

$(140, 301) = 7$

$7 \mid 133 \quad \therefore$ ① has 7 incongruent solutions

now ①

$\frac{\cancel{140}^{20}x}{7} = \frac{133}{7} \left(\bmod \frac{301}{7}\right)$

$20x = 19 \pmod{43}$

$(20, 43) = 1$

$1 \mid 19$

$43 = 20 \times 1 + 3$

$20 = 3 \times 6 + 2$

$3 = 2 \times 1 + 1$

$1 = 3 - 2 \times 1$

$= 3 - (20 - 3) \times 6 = 3 \times 7 - 20$

$= (43 - 20 \times 2)$

Date ______
Page ______

$= -20 \times 11 = 1 - 43 \times 7 = 11 \pmod{43}$

$20 \times (-11 \times 19) \equiv 19 \bmod 43$

$x \equiv -285 \bmod 45$

$x \equiv -285 + 7 \times 43 \pmod{43}$

$x \equiv 16 \bmod 43$

7 Solution $\quad 16 + \frac{301}{7} \bmod 301$

$= 16, 59, 102, \ldots \bmod 301$

Q5 $\quad 2x + 7y \equiv 5 \pmod{12}$ — (1)

Sol $\quad (2, 7, 12) = 1$

$1 / 5 \quad \therefore$ (1) has solutions

$2x \equiv \cancel{7y}\ 5 - 7y \pmod{12}$

$2x \equiv (5 + 13) - (7y + 13y) \pmod{12}$ $\quad \equiv -1 \pmod{12}$ $\quad 11 \pmod{12}$

$2x \equiv (18) - (20y) \pmod{12}$

$x \equiv 9 - 10y \pmod{12}$

Date ______
Page ______

$y = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$

$y = 0 \Rightarrow x \equiv 9 \pmod{2}$

$y = 1 \Rightarrow x \equiv -1 \pmod{12} \equiv 11 \pmod{12}$

$y = 2 \Rightarrow x \equiv -11 \pmod{12} \equiv 1 \pmod{12}$

$y = 3 \Rightarrow x \equiv -21 \pmod{12} \equiv 3 \pmod{12}$

$y = 4 \Rightarrow$

$y = 5 \Rightarrow$

$y = 6 \Rightarrow$

$y = 7 \Rightarrow$

$y = 8 \Rightarrow$

$y = 9 \Rightarrow$

$y = 10 \Rightarrow$

$y = 11 \Rightarrow$

Date
Page

Chinese Remainder Theorem

Statement. If the positive integers m_i, $i = 1, 2, 3, \ldots n$ are relatively prime in pairs and a_i, $i = 1, 2, 3, \ldots, n$ are any integers, then the congruences $x \equiv a_i \pmod{m_i}$ $i = 1, 2, 3, \ldots n$ have one and only one common solution congruent mod $(m_1 m_2 \ldots m_n)$

Proof. The given congruences are

$$x \equiv a_i \pmod{m_i}, \quad i = 1, 2, 3, \ldots, n$$

i.e.

$$\left.\begin{aligned} x &\equiv a_1 \pmod{m_1} \\ x &\equiv a_2 \pmod{m_2} \\ x &\equiv a_3 \pmod{m_3} \\ &\ldots\ldots \\ x &\equiv a_n \pmod{m_n} \end{aligned}\right\} \quad \ldots (1)$$

where $m_1, m_2, m_3, \ldots, m_n$ are relatively prime in pairs

Define a number m as

$$m = m_1 m_2 m_3 \ldots m_n$$

let $M_1, M_2, M_3, \ldots, M_n$ be defined as

$$m = m_1 M_1 = m_2 M_2 = m_3 M_3 = \ldots = m_n M_n$$

Such that $M_1 = m_2 m_3 \ldots m_n$, etc.

Consider the n congruences given by

$$M_i x = 1 \pmod{m_i} \quad \ldots (2)$$

since $(M_i, m_i) = 1$

$\therefore$ each of the congruences (2) has exactly one solution $x \equiv x_i \pmod{m_i}$

$$\left.\begin{aligned} M_1 x_1 &\equiv 1 \pmod{m_1} \\ M_2 x_2 &\equiv 1 \pmod{m_2} \\ M_3 x_3 &\equiv 1 \pmod{m_3} \\ &\ldots\ldots \\ M_n x_n &\equiv 1 \pmod{m_n} \end{aligned}\right\} \quad \ldots (3)$$

Date
Page

Consider $x = M_1 x_1 a_1 + M_2 x_2 a_2 + \ldots + M_n x_n a_n \quad \ldots (4)$

we will show that this value of x given by (4) satisfies each congruence of (1)...

$$x = M_1 x_1 a_1 + M_2 x_2 a_2 + \ldots + M_n x_n a_n$$

will be solution of first congruence of (1)

if $x = M_1 x_1 a_1 + M_2 x_2 a_2 + \ldots + M_n x_n a_n \equiv a_1 \pmod{m_1}$

i.e. if $M_1 x_1 a_1 \equiv a_1 \pmod{m_1}$

[$\because$ $M_2, M_3 \ldots M_n$ are all multiples of m_1]

i.e if $a_1 \equiv a_1 \pmod{m_1}$ [$\because$ $M_1 x_1 \equiv 1 \pmod{m_1}$ due to (3)]

which is true.

similarly we can prove that value of x given by (4) satisfies each of the remaining congruences in (1).

$\therefore$ $x = M_1 x_1 a_1 + M_2 x_2 a_2 + \ldots + M_n x_n a_n$

is a solution of given congruences.

Uniqueness

If y is another solution of given congruences then

$$\begin{aligned} y &\equiv a_1 \pmod{m_1} \\ y &\equiv x \pmod{m_1} \quad [\because x \equiv a_1 \pmod{m_1}] \\ y &\equiv x \pmod{m_2} \\ y &\equiv x \pmod{m_3} \\ &\ldots\ldots \\ y &\equiv x \pmod{m_n} \end{aligned}$$

$\because$ $m_1, m_2, m_3, \ldots, m_n$ are relative prime is pairs

$\therefore$ $y \equiv x \bmod (m_1 m_2 m_3 \ldots m_n)$

$\therefore$ $y \equiv x \pmod{m}$

$\therefore$ Solution of given congruences is unique

Cor. Since $y \equiv x \pmod{m}$

$\therefore$ $y = x + Km$

where K is an integer.

1) Solve: $x \equiv 1 \pmod 3$, $x \equiv 2 \pmod 5$, $x \equiv 3 \pmod 7$

Sol: Here $(3, 5) = (5, 7) = (3, 7) = 1$

$\therefore$ Congruences
$$x \equiv 1 \pmod 3 \quad - (1)$$
$$x \equiv 2 \pmod 5 \quad - (2)$$
$$x \equiv 3 \pmod 7 \quad - (3)$$

have a simultaneous solution which is unique mod $(3 . 5 . 7) =$ mod (105).

from (1), $3 \mid x-1$ i.e $x - 1 = 3t$ or $x = 1 + 3t$, $t \in Z$

it satisfies (2)

if $1 + 3t \equiv 2 \pmod 5$

$3 . t \equiv 1 \pmod 5$

clearly $t \equiv 2 \pmod 5$

$\Rightarrow 5 \mid t - 2$ i.e $t - 2 = 5K$ or $t = 2 + 5K, K \in Z$

$\therefore x = 1 + 3(2 + 5K) = 7 + 15K$

it satisfies (3) if $7 + 15K \equiv 3 \pmod 7$

if $15K \equiv -4 \pmod 7$

clearly $K \equiv 3 \pmod 7$

$\Rightarrow 7 \mid K - 3$ i.e. $K - 3 = 7l$ or $K = 3 + 7l$, $l \in Z$

thus $x = 7 + 15(3 + 7l) = 52 + 105l$

$x \equiv 52 \pmod{105}$

which is required solution.

2) $x \equiv 5 \pmod 6$, $x \equiv 4 \pmod{11}$, $x \equiv 3 \pmod{17}$

Sol: Here $(6, 11) = (11, 17) = (6, 17) = 1$

$\therefore$ Congruences
$$x \equiv 5 \pmod 6$$
$$x \equiv 4 \pmod{11}$$
$$x \equiv 3 \pmod{17}$$

have a simultaneous solution which is unique modulo $(6 . 11 . 7) =$ mod (1122)

from (1), $6 \mid x - 5$ i.e $x - 5 = 6t$ or $x = 5 + 6t$, $t \in Z$

it satisfies (2)

if $5 + 6t \equiv 4 \pmod{11}$

$6t \equiv -1 \pmod{11}$

clearly $t \equiv 9 \pmod{11}$

$\Rightarrow 11 \mid t - 9$ i.e. $t - 9 = 11K$ or $t = 9 + 11K$, $K \in Z$

So $x = 5 + 6(9 + 11K) = 59 + 66K$

It satisfies (3) if $59 + 66K \equiv 3 \pmod{17}$

if $66K \equiv -56 \pmod{17}$

clearly $K \equiv 11 \pmod{17}$

$\Rightarrow 17 \mid K - 11$ i.e. $K - 11 = 17l$ or $K = 11 + 17l$, $l \in Z$

Thus $x = 59 + 66(11 + 17l) = 785 + 1122l$

i.e. $x \equiv 785 \pmod{1122}$

which is required solution.

3) Solve $17x \equiv 9 \pmod{276}$

Sol: $276 = 3, 4, 23$ and $(3, 4) = (4, 23) = (23, 3) = 1$

$\therefore$ Congruences
$$17x \equiv 9 \pmod 3$$
$$17x \equiv 9 \pmod 4$$
$$17x \equiv 9 \pmod{23}$$

have a simultaneous solution which is unique modulo $(3, 4, 23) =$ mod 276.

Here from (1), $x \equiv 0 \pmod 3 \Rightarrow 3 \mid x$. i.e $x = 3t$, $t \in Z$

It satisfies (2)

if $17(3t) \equiv 9 \pmod 4$

$51t \equiv 9 \pmod 4$

clearly $t \equiv 3 \pmod 4$

$\Rightarrow 4 \mid t - 3$ i.e. $t = 3 + 4K$, $K \in Z$

$x = 3(3 + 4K) = 9 + 12K$

It satisfies (3) if $17(9 + 12K) \equiv 9 \pmod{23}$

if ℓ $204\,K \equiv -144 \pmod{23}$

clearly $K \equiv 2 \pmod{23}$

$\Rightarrow 23 \mid K-2$ i.e. $K = 2 + 23\ell,\ \ell \in \mathbb{Z}$

Thus $x = 9 + 12(2 + 23\ell) = 33 + 276\ell$

$x \equiv 33 \pmod{276}$ which is required solution of $17x \equiv 9 \pmod{276}$.

FERMAT THEOREM AND WILSON'S THEOREM

if a be any integer, then $a^p \equiv a \pmod{p}$

or

if $(a, p) = 1$, then $a^{p-1} \equiv 1 \pmod{p}$

Proof. we know that

$$(A+B)^p = A^p + {}^pC_1 A^{p-1}B + {}^pC_2 A^{p-2}B^2 + \ldots {}^pC_{p-1} A B^{p-1} + B^p$$

$$\therefore (A+B)^p \equiv (A^p + B^p) \pmod{p}$$

$[\because {}^pC_1, {}^pC_2, \ldots\ldots, {}^pC_{p-1}$ are all multiples of $p]$

Again $(A+B+C)^p \equiv [(A+B)^p + C^p] \pmod{p}$ $[\because$ of $(1)]$

$\Rightarrow (A+B+C)^p \equiv (A^p + B^p + C^p) \pmod{p}$

and so on

Ultimately,

$(A+B+C+\ldots$ upto a terms$)^p \equiv (A^p + B^p + C^p +$ upto a terms$) \pmod{p}$

Put $A = B = C = \ldots\ldots = 1$

$\therefore (1+1+1\ldots$ upto a terms$)^p \equiv (1+1+1+\ldots$ upto a terms$) \pmod{p}$

$\therefore a^p \equiv a \pmod{p}$

By restricted Cancellation law.

$$a^{p-1} \equiv 1 \pmod p$$

Another Proof. Consider $p-1$ integers

$a, 2a, 3a, \ldots, (p-1)a$ where $a \in \mathbb{Z}$

These integers are mutually incongruent and coprime to p

($\because la \equiv ma \pmod p$, $1 \le l \le m \le p \Rightarrow l \equiv m \pmod p$ as $(a, p) = 1$ which is impossible)

So that the above integers must be congruent modulo p to $1, 2, \ldots, p-1$ in some order

i.e. $ka \equiv j \pmod p$ for $1 \le k, j \le p-1$

Multiplying all these congruences, we get

$a, 2a, \ldots, (p-1)a \equiv 1, 2, \ldots (p-1) \pmod p$

i.e $a^{p-1} \lfloor p-1 \equiv \lfloor p-1 \pmod p$

$\Rightarrow a^{p-1} \equiv \pmod p$ $\qquad (\because p \nmid \lfloor p-1)$

$\Rightarrow a^p \equiv a \pmod p$

Hence the result.

1) Show that $6^{52}-1$ is divisible by 53.

Sol Here number is $6^{52}-1$ i.e. $a=6$, $p=53$

g.c.d. $(6, 53) = 1$

$\therefore$ by fermat's little theorem $6^{52} \equiv 1 \pmod{53}$

$[\because a^{p-1} \equiv 1 \pmod p]$

$\Rightarrow 6^{52}-1 \equiv 0 \pmod{53}$

i.e. $6^{52}-1$ is divisible by 53.

2) Show that $2^{100} \equiv 1 \pmod{11}$ and $2^{105} \equiv 10 \pmod{11}$

Sol By fermat's little theorem

$2^{10} \equiv 1 \pmod{11}$ is $(2, 11) = 1$

$\Rightarrow (2^{10})^{10} \equiv 1^{10} \pmod{11}$

$2^{100} \equiv 1 \pmod{11}$

further $2^{105} = 2^{100} \times 2^5$

$\equiv 1 \times 32 \pmod{11}$

$\equiv 10 \pmod{11}$ $\qquad (\because 32 \equiv 10 \pmod{11})$

3) If g.c.d $(a, 35) = 1$, then prove that $a^{12} \equiv 1 \pmod{35}$

Sol Since g.c.d $(a, 35) = 1$

$\Rightarrow$ g.c.d. $(a, 35) = 1$

$\Rightarrow$ g.c.d $(a, 5) = 1$ and g.c.d $(a, 7) = 1$ as $35 = 7 \times 5$

$\therefore$ by fermat's little theorem

$$a^4 \equiv 1 \pmod{5} \text{ and } a^6 \equiv 1 \pmod{7}$$

$$(a^4)^3 \equiv 1 \pmod{5} \text{ and } (a^6)^2 \equiv 1 \pmod{7}$$

$$a^{12} \equiv 1 \pmod{5} \text{ and } a^{12} \equiv 1 \pmod{7}$$

$\Rightarrow$ $a^{12} \equiv 1 \pmod{5 \times 7}$ as g.c.d. $(5, 7) = 1$

$$a^{12} \equiv 1 \pmod{35}$$

Q.> find the least non negative reside of $11^{470} \pmod{37}$

Sol By fermat's little theorem.

$11^{36} \equiv 1 \pmod{37}$ as g.c.d. $(11, 37) = 1$ and 37 is a prime

$\Rightarrow$ $(11^{36})^{13} \equiv 1^{13} \pmod{37}$ $\quad (\because 470 = 36 \times 13 + 2)$

$$11^{468} \equiv 1 \pmod{37}$$

$$11^{470} = 11^{468} \times 11^2 \equiv 1 \times 121 \pmod{37} = 10 \pmod{37}$$

i.e $11^{470} \equiv 10 \pmod{37}$

$\therefore$ 10 is the least non-negative residue of 11^{470} modulo 37.

GALOIS FIELDS

I) Properties of addition operations (+)

1) closure property

$a + b \in F \ \forall \ a, b \in F$

2) Commutative Law

$a + b = b + a \ \forall \ a, b \in F$

3) Associative law

$(a + b) + c = a + (b + c) \ \forall a, b, c \in F$

4) Additive Identity

There exists $0 \in F$ called the zero-element of F, such that

$$a + 0 = a = 0 + a \ \forall \ a \in F$$

The element 0 is called the additive identity of F.

5) Additive Inverse

for each $a \in F$, there exists $-a \in F$ such that

$$a + (-a) = 0 = (-a) + a \ \forall \ a \in F$$

The element $-a$ is called the additive inverse of a.

II Properties of Multiplication operation (·)

1) closure Property
$a \cdot b \in F \; \forall a, b \in F$

2) Commutative Property
$a \cdot b = b \cdot a \; \forall a, b \in F$

3) Associative law
$(a \cdot b) \cdot c = a \cdot (b \cdot c) \; \forall a, b, c \in F$

4) Multiplicative identity

There exists $1 \in F$, called unity of F, such that
$a \cdot 1 = a = 1 \cdot a \; \forall a \in F$
The element 1 is called the multiplication identity

5) Multiplicative inverse

for each $a (\neq 0) \in F$, there exists $a^{-1} \in F$ such that
$a \cdot a^{-1} = 1 = a^{-1} \cdot a$

The element a^{-1} is called the multiplicative inverse of a.

III Distributive property

i) $a \cdot (b + c) = a \cdot b + a \cdot c; \; \forall a, b, c \in F$

ii) $(b + c) \cdot a = b \cdot a + c \cdot a; \; \forall a, b, c \in F$

It is denoted by $(F, +, \cdot)$

Q) write down the composition table for the following.

i) GF(2) (ii) GF(3)

i) $GF(2) = \{0, 1\}$, Here $P = 2$

+	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

(ii) GF(3), Here $P = 3$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

(a) Addition of Polynomial over GF(2)

$f(x) = a_0 + a_1 x + a_2 x^2 \ldots$
$g(x) = b_0 + b_1 x + b_2 x^2 \ldots$

$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2$

b) subtraction of polynomial over GF(2)

In this polynomial we have $-1 = 1$

Eg: Let $a(x) = 1 + x + x^3 + x^5$
$b(x) = 1 + x^2 + x^3 + x^4 + x^7$ Then find

(i) $a(x) + b(x)$ (ii) $a(x) - b(x)$

Ans i) $a(x) + b(x) = (1+1) + x + x^2 + (1+1)x^3 + x^4 + x^5 + x^7$
$= x + x^2 + x^4 + x^5 + x^7$

(ii) $a(x) - b(x) = (1-1) + x - x^2 + (x-1)x^3 + x^4 + x^5 \ldots$
$(1+1) + x + x^2 + (1+1)x^3 + x^4 + x^5 + x^7$

(c) Multiplication of polynomial over $GF(2)$

Let $f(x) = a_0 + a_1 x + a_2 x^2 \ldots$
$g(x) = b_0 + b_1 x + b_2 x^2 \ldots$
$f(x) g(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2$
$+ a_1 b_1 + a_2 b_0) x^2$

D.