Applied Machine Learning

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SE ZG568 / SS ZG568,
Applied Machine Learning
Lecture No. 4 [09- Feb-2025]

Recap

Basics of Linear Algebra,

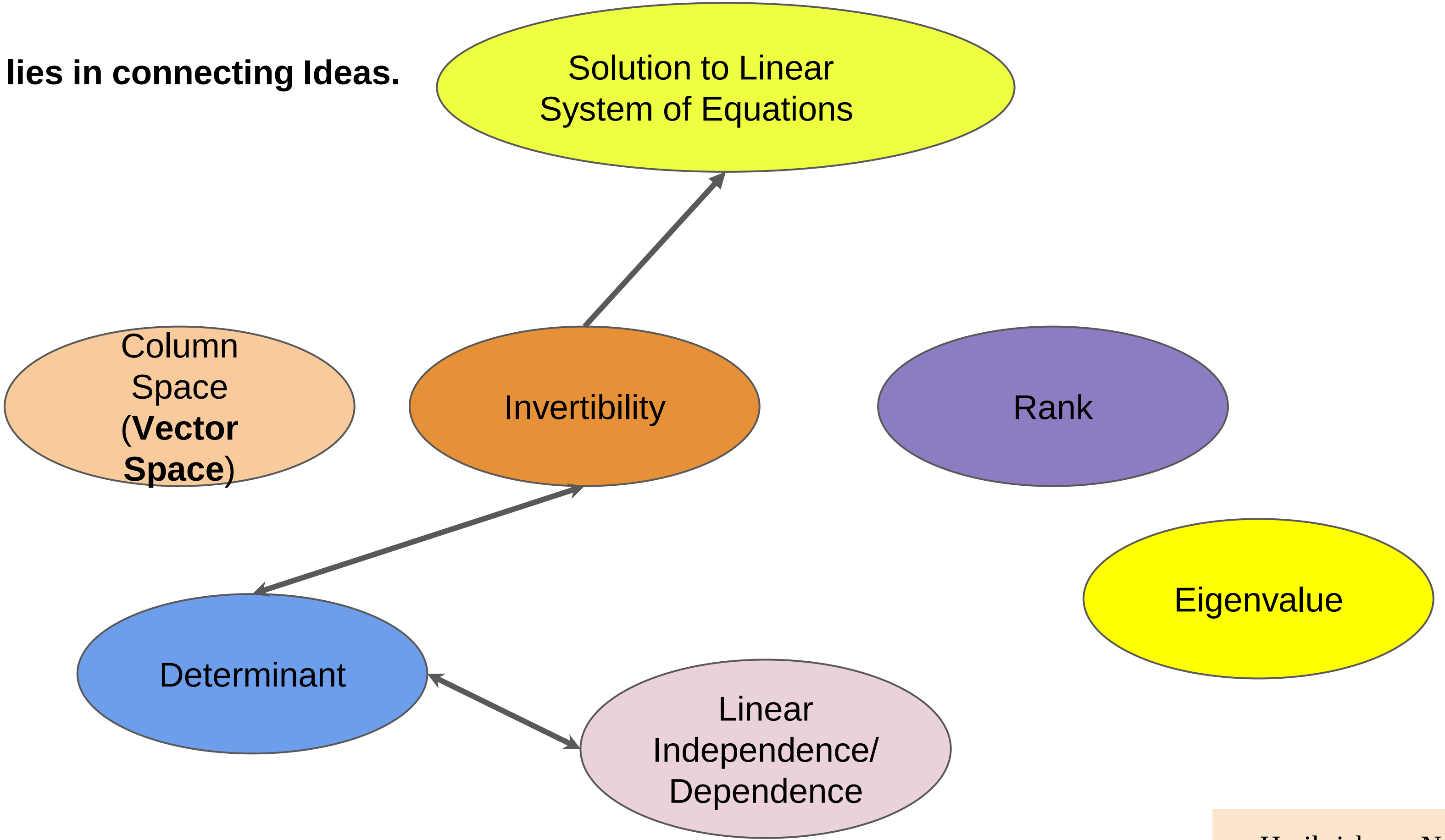
Row Picture, Col Picture, Algebraic Way

Solution to System of Linear Equations

Inverse of a Matrix

Linear Regression, PCA

Beauty lies in connecting Ideas.



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Orthogonal and Orthonormal Matrix

Two vectors are orthogonal when their dot product is zero

Orthogonal vectors	Orthonormal vectors

$$\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

L2 - norm = $\sqrt{2}$

$\vec{a}\cdot\vec{b} = 0$

$\Rightarrow \|a\|_2 \|b\|_2 \cos\theta = 0$

$\cos\theta = 0$ ✓

$\theta = 90°$ ✓

$\boxed{\|a\|_2, \|b\|_2 \neq 0}$

$a = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ $b = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$

L2 norm of $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \sqrt{1^2+1^2} = \sqrt{2}$

$90°$

$\vec{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} =$ L2 norm of $\vec{a} = \sqrt{1^2+2^2} = \sqrt{5}$

$\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$ $\sqrt{\left(\frac{1}{\sqrt{5}}\right)^2+\left(\frac{2}{\sqrt{5}}\right)^2} = \sqrt{5/5} = 1$

$\vec{a} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \vec{b} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

L2 norm of $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$$\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = 0$$

$\sqrt{\left(1/\sqrt{2}\right)^2 + \left(-1/\sqrt{2}\right)^2}$ $= \sqrt{\frac{1}{2}+\frac{1}{2}} = 1$

L2 - norm = 1

$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$a^T b = a \cdot b$

$\Rightarrow \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = 1\times(-2) + 2\times 1 = 0$

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Orthonormal Matrix

$O(N^3)$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{array}{cc} \text{col}_1 & \text{col}_2 \\ \end{array}$$
$A = \begin{bmatrix} a\ 1/\sqrt{2} & 1/\sqrt{2}\ b \\ c\ -1/\sqrt{2} & 1/\sqrt{2}\ d \end{bmatrix}$

$\text{col}_1 \cdot \text{col}_2 = 0$

$\|\text{col}_1\|_2 = 1, \quad \|\text{col}_2\|_2 = 1$

$$\begin{array}{cccc} \text{col}_1 & \text{col}_2 & \cdots & \text{col}_n \end{array}$$
$$\begin{bmatrix} | & | & \cdots & | \\ | & | & & | \end{bmatrix}$$

for $i \neq j$
$\text{col}_i \cdot \text{col}_j = 0$
for all i from 1 to N
$\|\text{col}_i\|_2 = 1$

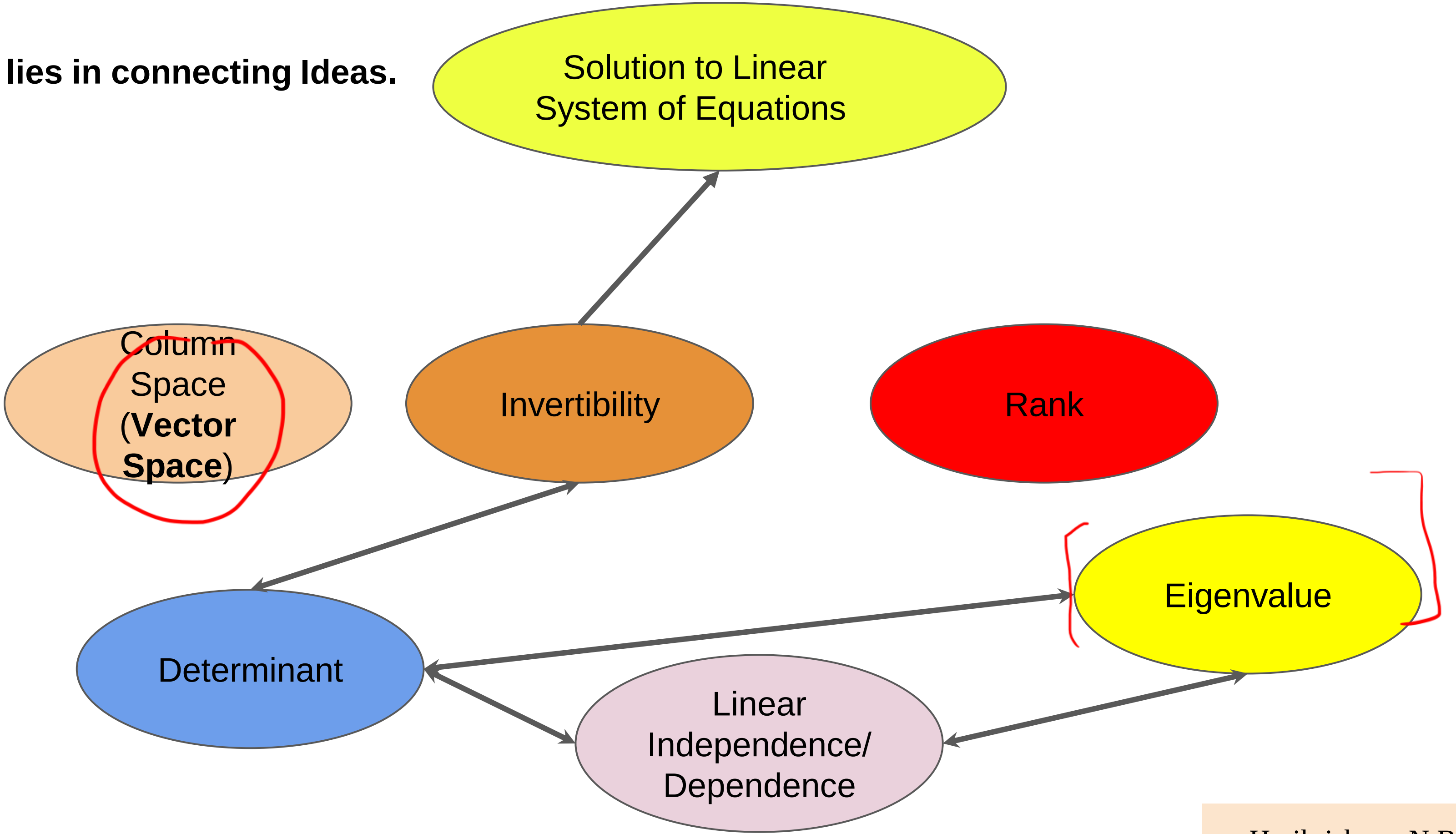
$|A| = (ad - bc)$

$\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}}$

$\frac{1}{2} + \frac{1}{2} = 1$

$$\begin{array}{cc} \text{col}_1 & \text{col}_2 \end{array}$$
$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

Beauty lies in connecting Ideas.



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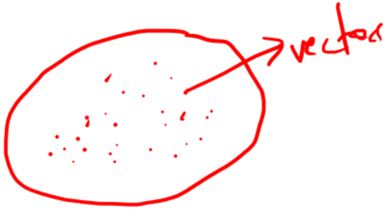
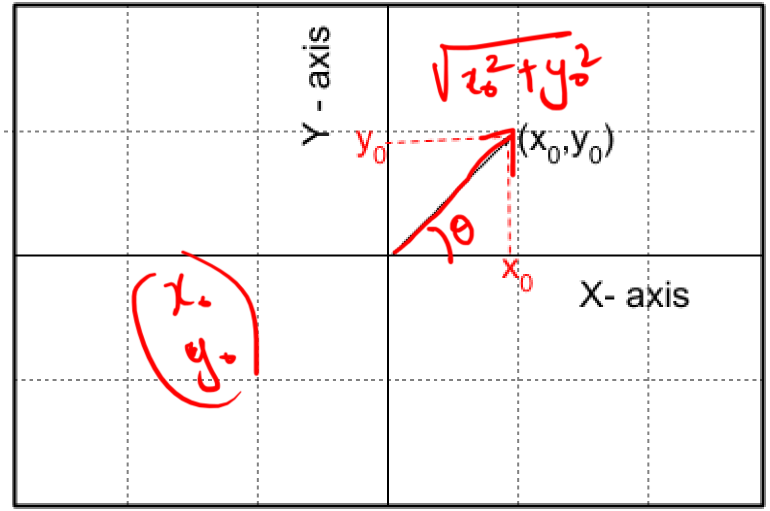
Third Iteration

Vectors - Different Understanding



Physicists



Computer Scientist

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Mathematicians

Vector space is a **collection of objects**(it can be anything) called vectors which satisfies mainly two important properties:

1. **closed under vector addition** ✓
2. **closed under scalar multiplication.** ✓

Vector Space - Coffee Space

Coffee space = {coffee$_1$, coffee$_2$, ... coffee$_1$, coffee$_N$ ... }

$\mathbb{R}$ = {... 1.2, ... 2.5 }

1.2 + 2.5 = 3.7

Coffee Space - In Coffee space we have different kinds of coffee with varying strength. Now we will understand the vector space properties with this metaphor.

$\mathbb{R}^2 = \{ \binom{x}{y}$ such that $x \in \mathbb{R}, y \in \mathbb{R} \}$

$x \in \mathbb{R}$, y, $a \in \mathbb{R}^2$, $a = \binom{a_x}{a_y}$

Closed under Vector Addition

$\vec{b} = -\vec{a}$, $b \in \mathbb{R}^2$, $b = \binom{b_x}{b_y}$

$\vec{a} = \binom{2}{3}$, $\vec{b} = \binom{-2}{-3}$

$\mathbb{R}^2$ and remove the vector $\binom{0}{0}$

Closed under Scalar Multiplication

$\mathbb{R}^2 = \{ \binom{x}{y}$ such $x, y \in \mathbb{R} \}$

$a + b = \binom{0}{0}$

$c \binom{x}{y}$ where $c \in \mathbb{R}$

$\mathbb{R}^2$ is a valid space

Adding two coffee's will give you another coffee which is in the coffee space

Scaling a coffee will give a coffee which is in the coffee space

Vector Space

$\mathbb{R}\left\{\begin{pmatrix}-1\\-6\end{pmatrix}\begin{pmatrix}-2\\3\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}3\\0\end{pmatrix}\right\}$ $+\begin{pmatrix}2\\-3\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

$\mathbb{R}^2$ and remove $\begin{pmatrix}0\\0\end{pmatrix}$

- A real vector space is a set/collection of "*vectors*" together with the rules for vector addition and multiplication by real numbers.*

closed under

closed also under scalar

*Strang, Gilbert. *Linear Algebra and Its Applications*. Cengage Learning, 2017.

$\begin{pmatrix}3\\2\end{pmatrix} = 3\begin{pmatrix}1\\0\end{pmatrix} + 2\begin{pmatrix}0\\1\end{pmatrix}$ $\boxed{\alpha_i \begin{pmatrix}1\\0\end{pmatrix} + \beta_i \begin{pmatrix}0\\1\end{pmatrix} \quad \alpha_i, \beta_i \in \mathbb{R}}$

Dimension and Basis of a Vector Space

Span of $\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$

$\begin{pmatrix}5\\4\end{pmatrix} = \alpha_i \begin{pmatrix}1\\0\end{pmatrix} + \beta_i \begin{pmatrix}2\\0\end{pmatrix}$ $\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2\\0\end{pmatrix} \right\}$ $\dim(\mathbb{R}^2) = 2$ $\dim(\mathbb{R}^3) = 3$

Dimension of a Vector space - Every vector space has a dimension. Dimension is the number of basis vectors required to span the vector space.

$\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$ are col_1 col_2 linearly independent or linearly dependent

$col_2 = \alpha\, col_1$

Properties of Basis Vectors -

- Basis vectors has to be linearly independent.
- Basis vectors should span the vector space.

$\boxed{col_j = \sum_{i=1, i\neq j}^{n} \alpha_i\, col_i}$ → linearly Dependent

$\begin{array}{ccc} col_1 & col_2 & col_3 \end{array}$
$\begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$

are these col's linearly dependent or independent?

$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

$col_3 = col_1 + col_2$

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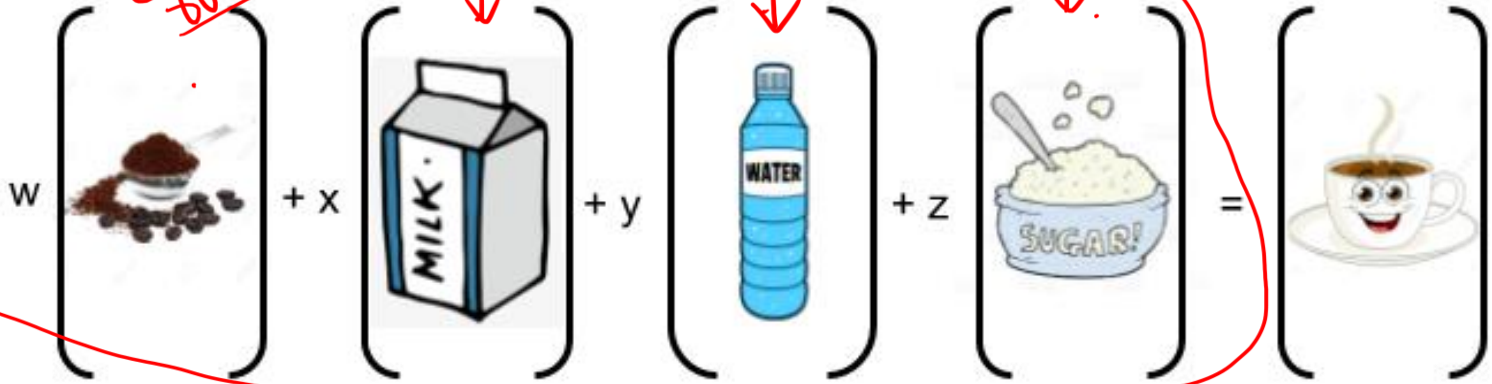
Dimension and Basis of a Coffee Space

- **Linear Independence**
- **Span the space**

{coffee powder, milk, water, sugar}

Coffee Space- Vector Space

coffee powder



Coffee powder, milk, water and sugar are the basis vectors. Since there are only 4 basis vectors then coffee space has a dimension of 4.

My Friend's Horrible Coffee

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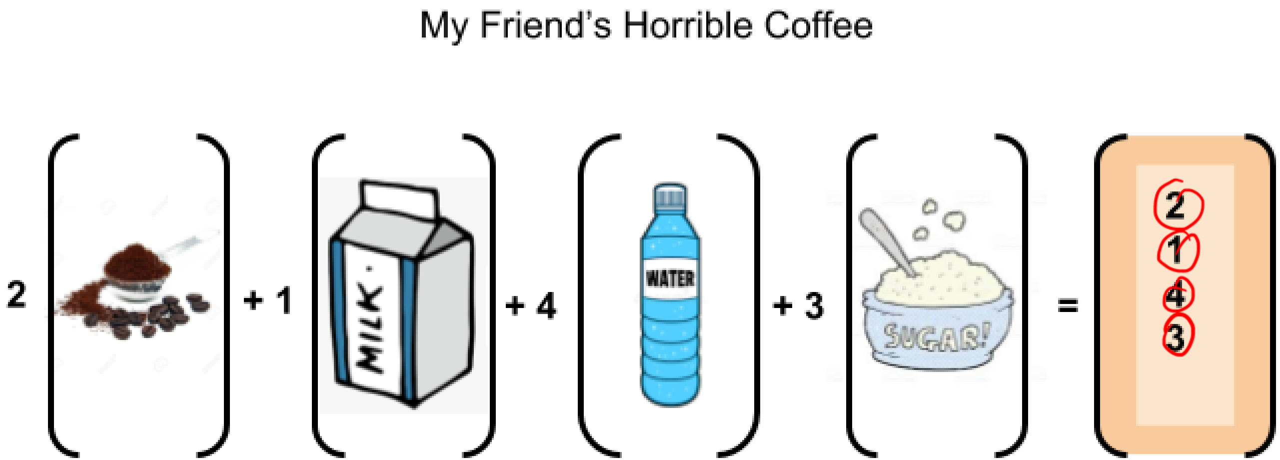
My Friend’s Horrible Coffee

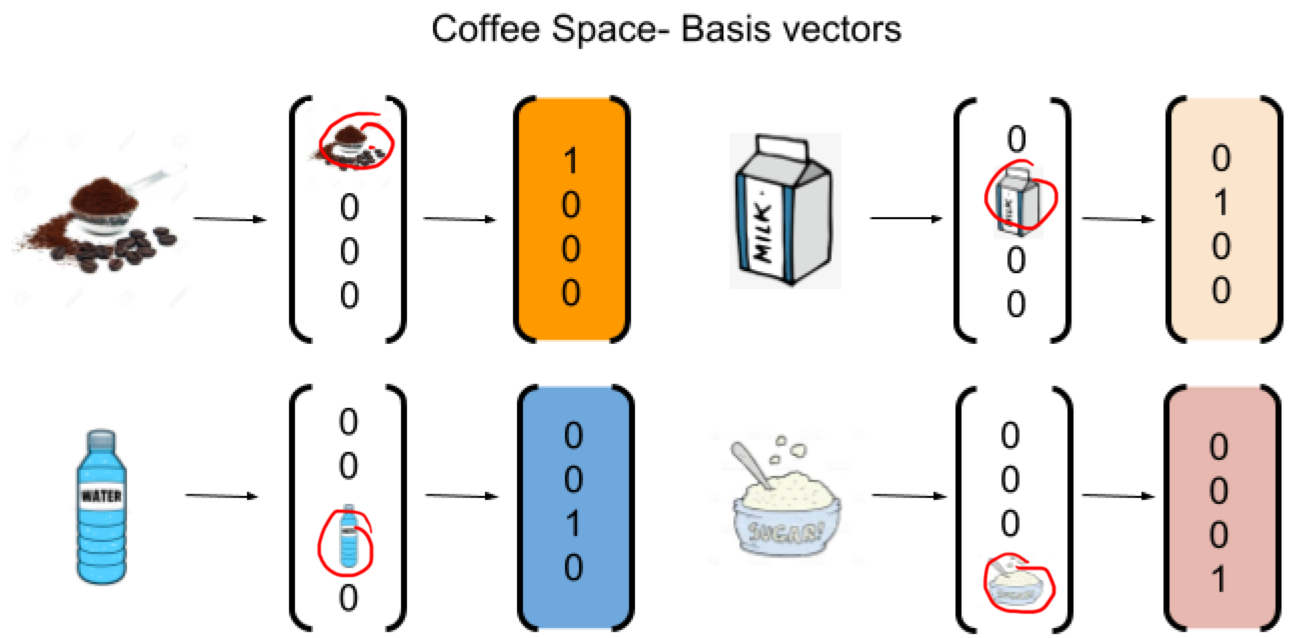


My Friend's Horrible Coffee

Col 1

$$2\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + 4\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\1\\4\\3\end{bmatrix}$$

Visualizing Coffee Space Basis Vectors



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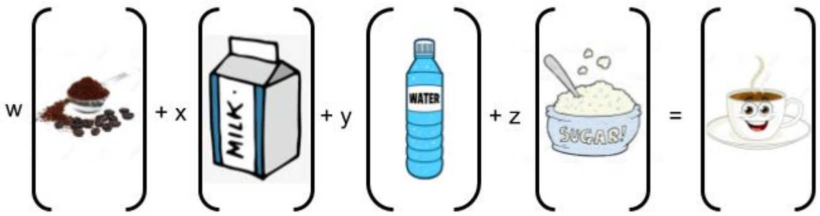
Matrix Multiplication - Visualization

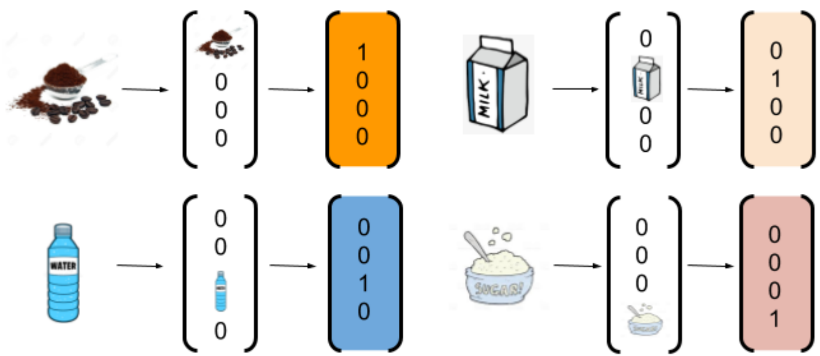
$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{bmatrix}5\\3\end{bmatrix} \Rightarrow x\begin{pmatrix}1\\0\end{pmatrix}+y\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}5\\3\end{pmatrix}$$

$$x+0y=5$$
$$0x+1y=3$$
$$x\begin{pmatrix}1\\0\end{pmatrix}+y\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}5\\3\end{pmatrix}$$

Coffee Space- Vector Space



Coffee Space- Basis vectors



Ax = b

$$w\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + x\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}5\\6\\7\\8\end{bmatrix}$$

Ax = b

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}w\\x\\y\\z\end{bmatrix} = \begin{bmatrix}5\\6\\7\\8\end{bmatrix}$$

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Column Space - Visualization

$C(A) \Rightarrow w\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + x\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + z\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ $w, x, y, z \in \mathbb{R}$

$\mathbb{R}^4$

$\dim(C(A)) = 4$

Ax = b

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}w\\x\\y\\z\end{bmatrix} = \begin{bmatrix}5\\6\\7\\8\end{bmatrix}$$

$\mathbb{R}^2$

$x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix} + z\begin{pmatrix}1\\1\end{pmatrix}$

$x, y, z \in \mathbb{R}^2$

$\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{pmatrix}9\\6\end{pmatrix}$

$\dim(C(A))$

Ax = b

$$w\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + x\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}5\\6\\7\\8\end{bmatrix}$$

Column Space of Matrix A - Column space of matrix A denoted as C(A) is the space spanned by the column vectors of A.

C(A)

$$span\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

Dimension of C(A) = 4. Since 4 linearly independent vectors are there in the columns of matrix A. These vectors act as the basis and span the entire R^4.

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Thinking

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \end{bmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \\ w \\ k \end{pmatrix}$$

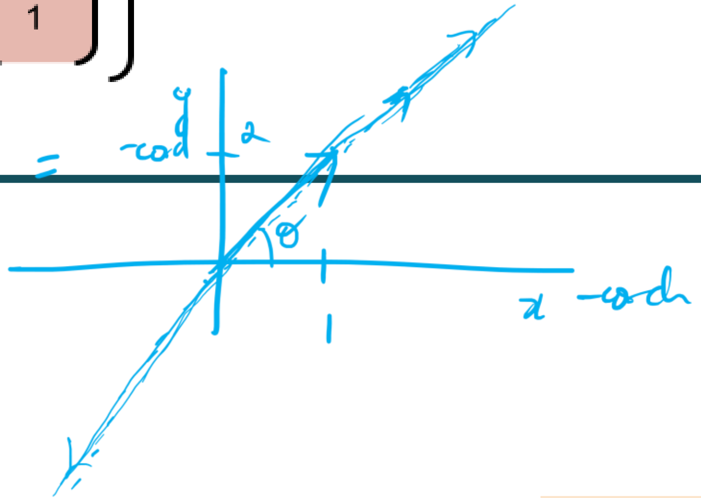
$C(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 5 \\ 10 \end{pmatrix} \right\}$

$$x\begin{pmatrix} 1 \\ 2 \end{pmatrix} + y\begin{pmatrix} 2 \\ 4 \end{pmatrix} + z\begin{pmatrix} 3 \\ 6 \end{pmatrix} + w\begin{pmatrix} 4 \\ 8 \end{pmatrix} + k\begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

C(A)

Why $span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ **can represent any point in R^4 ?**

$\dim(C(A)) =$



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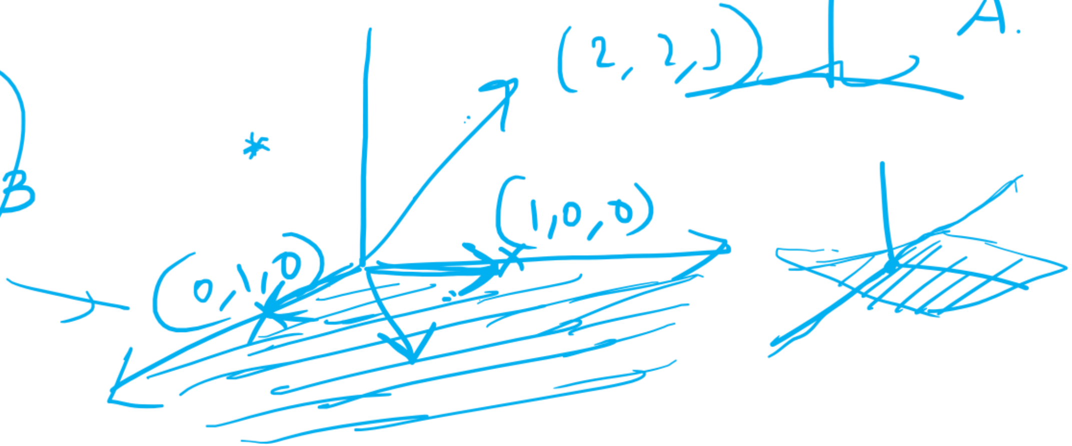
$Ax = b$ has a solution only b lies in the column space of A.

Can you see the Column Space?

$$\begin{array}{cc} col_1 & col_2 \end{array}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ \boxed{0} \end{bmatrix} \rightarrow B$$

Span$\{col_1, col_2\}$



$$w\begin{pmatrix} 1 \\ 0 \\ \boxed{0} \end{pmatrix} + z\begin{pmatrix} 0 \\ 1 \\ \boxed{0} \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} ?$$

$$\begin{pmatrix} 2 \\ 2 \\ 5 \end{pmatrix}$$

$$w\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

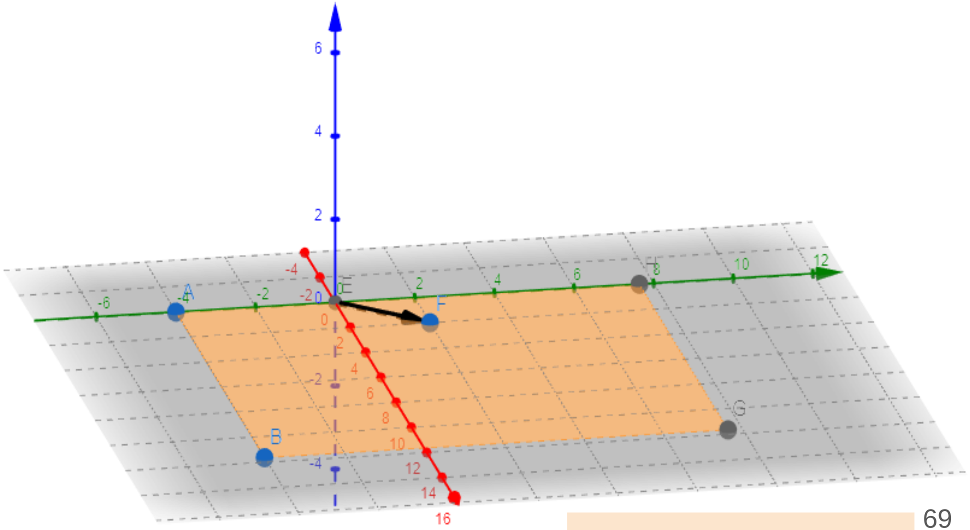
$$2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

Can you see the Column Space?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \rightarrow w \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

C(A)

$$span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \rightarrow$$



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Some Observations !!!

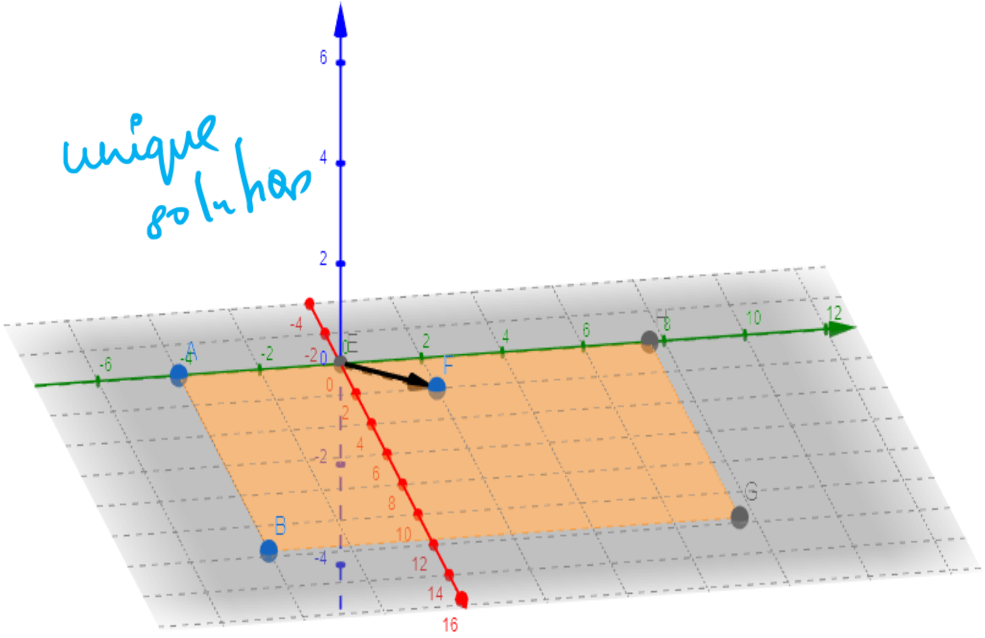
Linearly independent

$$\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}w\\x\end{bmatrix}=\begin{bmatrix}2\\2\\0\end{bmatrix}$$

C(A)

$$span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$$

unique solution



What is the dimension of Column space of Matrix A?

Will the basis vectors of C(A) span the entire 3-D space?

cols of A is linearly dependent & b lies in the col space of A then

What can you say about this?

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} w \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

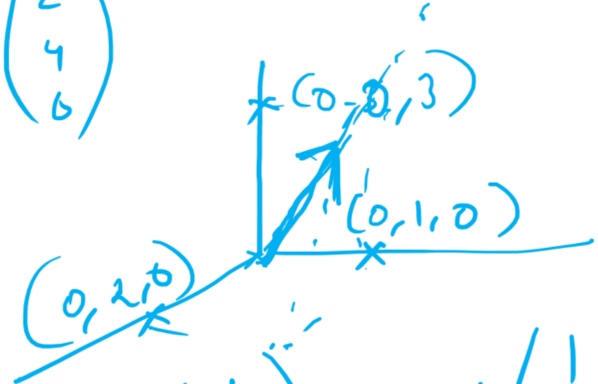
Identify the cols

$$2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

$C(A) = \text{span}\{col_1, col_2\}$

$\Rightarrow w\, col_1 + x\, col_2$ where $w, x \in \mathbb{R}$

$\dim(C(A)) = 1$



$$w\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 2x\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(1, 0)
(0, 1/2)
(2, $\cancel{-1}$)

$$\begin{pmatrix} 1/2 \\ 1 \\ 3/2 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1 \\ 3/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

What can you say about this?

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} w \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Downarrow$$

$$w \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

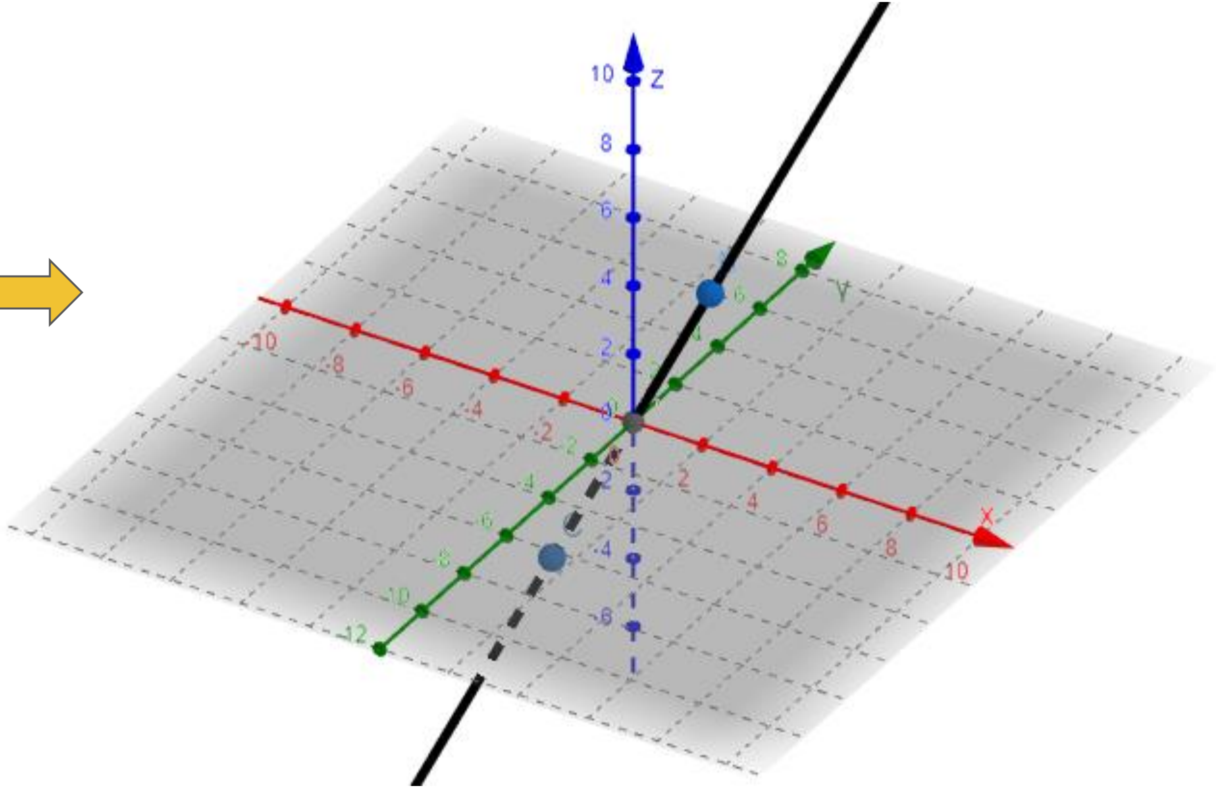
Column Space of Matrix A - Column space of matrix A denoted as C(A) is the space spanned by the column vectors of A.

$$C(A) = span \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Dimension of C(A) = 1. Here Column space is a line passing through origin.

Do you see a Subspace ?

$$C(A) = span\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$$



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Is there anything Mysterious ?

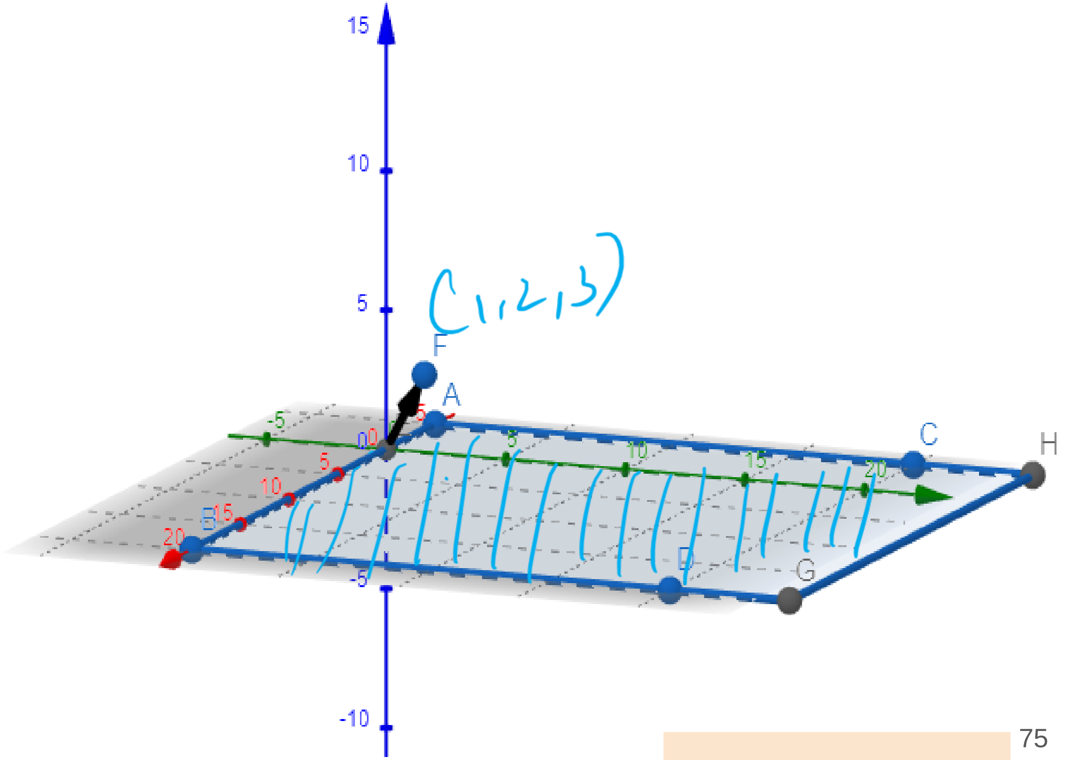
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Is there anything Mysterious ?

$$\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}w\\x\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\Downarrow$$

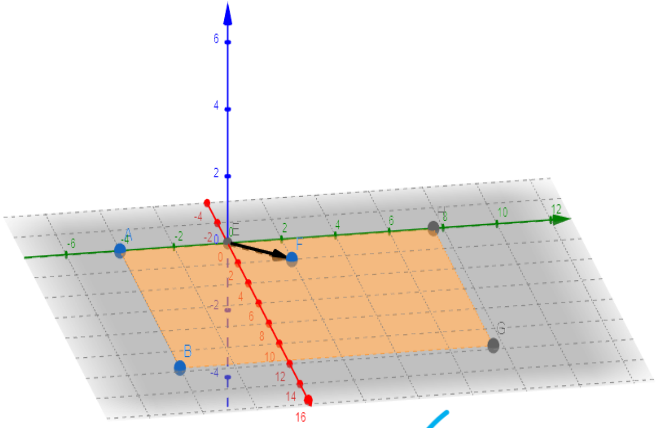
$$w\begin{bmatrix}1\\0\\0\end{bmatrix}+x\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

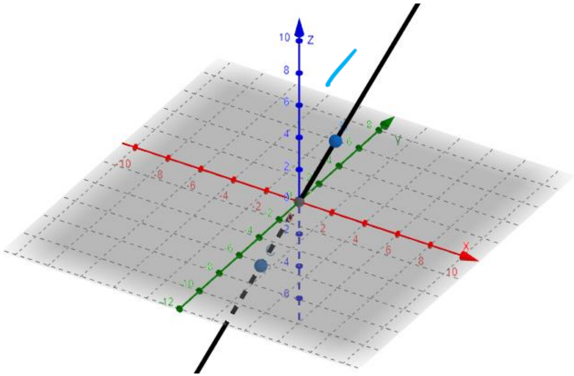


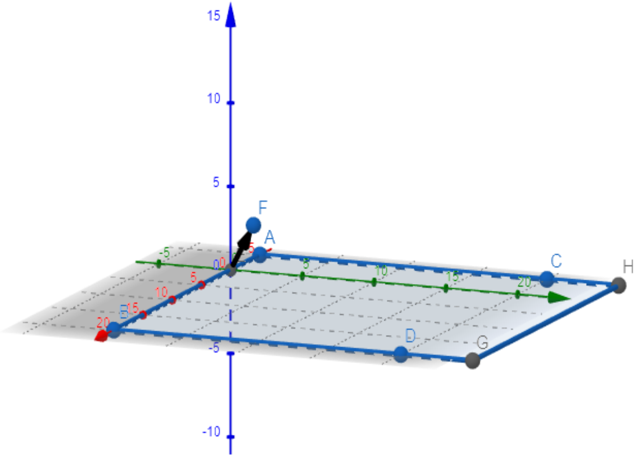
linearly depende

Solution to Ax = b

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} w \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$







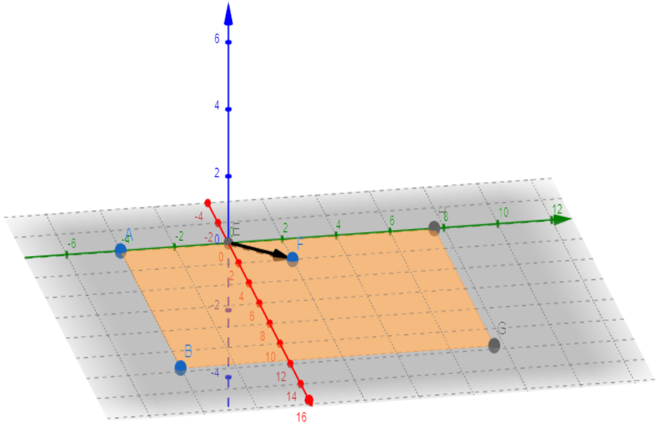
UNIQUE SOLUTION

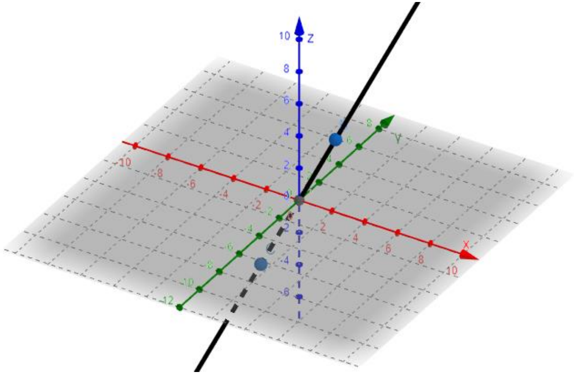
INFINITELY MANY SOLUTIONS

NO SOLUTION

So when does Ax = b have a Solution

Ax = b has solution when b lies in the column space of A or in other words b is a linear combination of column vectors of A.





UNIQUE SOLUTION

INFINITELY MANY SOLUTIONS

- For unique solution and infinitely many solutions b lies in the column space of A.

- In the case of NO solution b does not lie in the column space of A.

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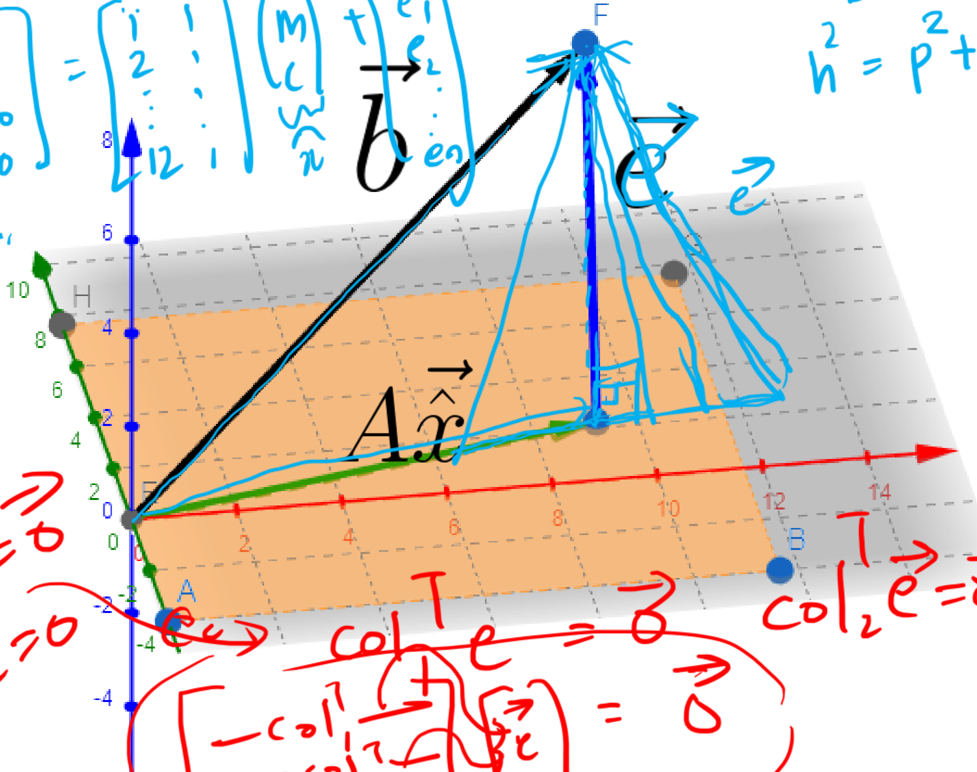
NO SOLUTION CASE 🙁

Can I find the best approximate solution ?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$A\vec{x} \neq \vec{b}$$



x_1 Jan 1 50
x_2 Feb 2 -30
3 100
4

unknown variables

$error^2 = p^2 + b^2$

$h^2 = p^2 + b^2$

$A\vec{x} + \vec{e} = \vec{b}$

$\vec{e}$ is orthogonal to the col space

$\vec{e}$ is orthogonal span

$\vec{e} \perp col$

$col_1 \cdot \vec{e} = 0$

$col_2 \cdot \vec{e} = 0$

$col^T \vec{e} = \vec{0}$

$col_2^T \vec{e} = \vec{0}$

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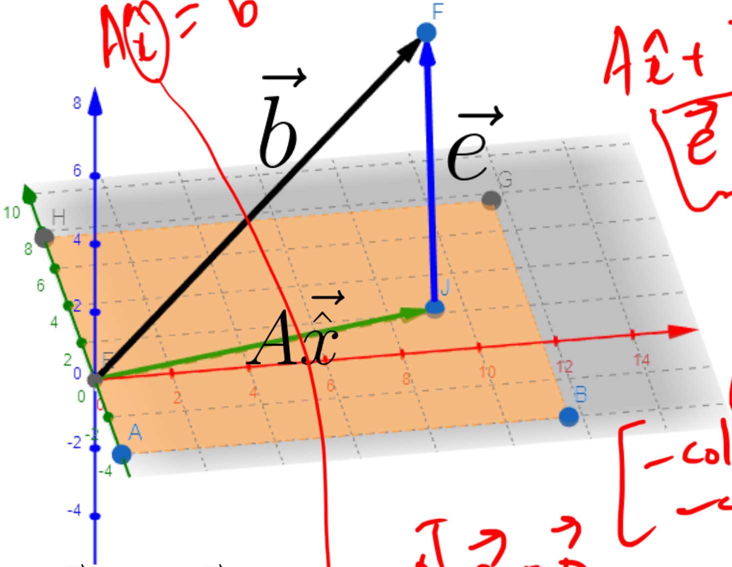
No solution case - Visualization

$A\hat{x} \neq \vec{b}$

$A\hat{x} = b$



$A\hat{x} + \vec{e} = \vec{b}$

$\boxed{\vec{e} = \vec{b} - A\hat{x}}$

$$A\vec{\hat{x}} + \vec{e} = \vec{b}$$

$B^{-1}B = I$

$(A^TA)^{-1}(A^TA) = I$

$$\begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} \vec{\hat{x}} \end{bmatrix} + \begin{bmatrix} \vec{e} \end{bmatrix} = \begin{bmatrix} \vec{b} \end{bmatrix}$$

$\begin{bmatrix} -\text{col}_1^T- \\ -\text{col}_2^T- \end{bmatrix} [\vec{e}] = \vec{0}$

$A^TA\hat{x} = A^T\vec{b}$

$$\boxed{A^T\vec{e} = \vec{0}}$$

$A^T\vec{e} = \vec{0}$

$A^T(\vec{b} - A\hat{x}) = \vec{0}$

$A^Tb - A^TA\hat{x} = \vec{0}$

$$\begin{bmatrix} -b_1^T- \\ -b_2^T- \end{bmatrix} \begin{bmatrix} \vec{e} \end{bmatrix} = \begin{bmatrix} \vec{0} \end{bmatrix}$$

$\hat{x} = (A^TA)^{-1}A^TA\hat{x} = (A^TA)^{-1}A^Tb$

$-A^TA\hat{x} = -A^T\vec{b}$

$$A\vec{\hat{x}} + \vec{e} = \vec{b}$$
$$\vec{e} = \vec{b} - A\vec{\hat{x}}$$
$$A^T\vec{e} = \vec{0}$$
$$A^T(\vec{b} - A\vec{\hat{x}}) = \vec{0}$$
$$A^T\vec{b} - A^TA\vec{\hat{x}} = \vec{0}$$
$$A^TA\vec{\hat{x}} = A^T\vec{b}$$
$$\vec{x} = (A^TA)^{-1}A^T\vec{b}$$

ORTHOGONAL

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NO SOLUTION CASE

Can I find the best approximate solution ?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad A\vec{x} \neq \vec{b}$$

$$A\vec{\hat{x}} + \vec{e} = \vec{b}$$

$$\vec{e} = \vec{b} - A\vec{\hat{x}}$$

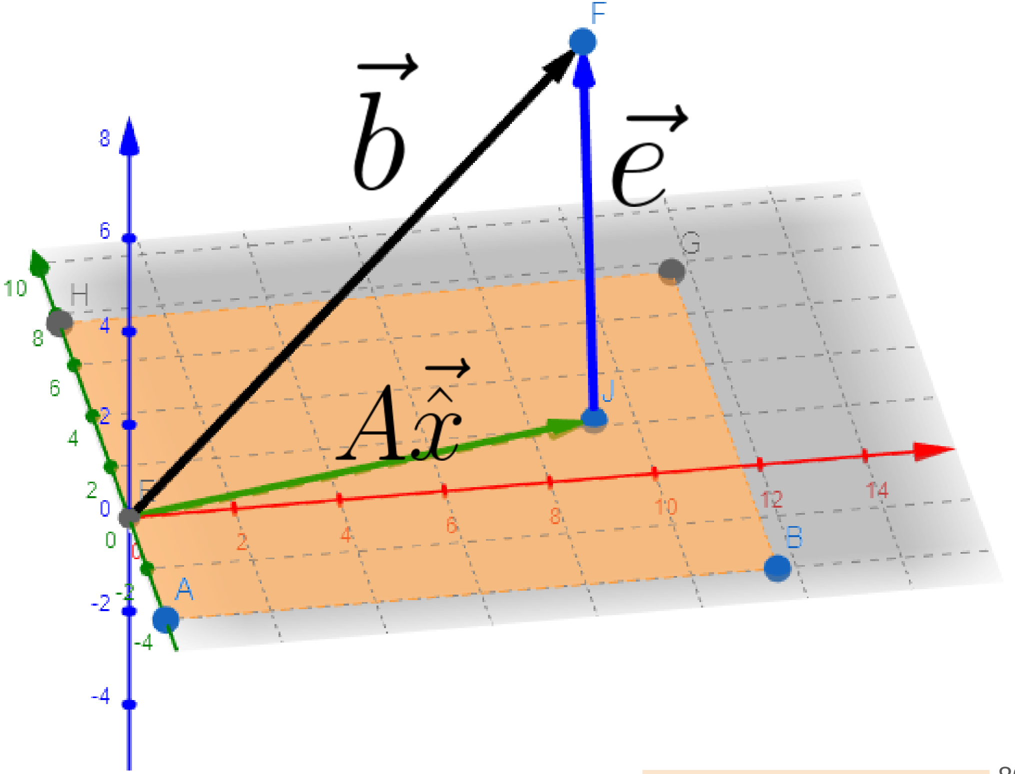
$$A^T\vec{e} = \vec{0}$$

$$A^T(\vec{b} - A\vec{\hat{x}}) = \vec{0}$$

$$A^T\vec{b} - A^T A\vec{\hat{x}} = \vec{0}$$

$$A^T A\vec{\hat{x}} = A^T\vec{b}$$

$$\vec{x} = (A^T A)^{-1} A^T\vec{b}$$



Linear Least Square Regression

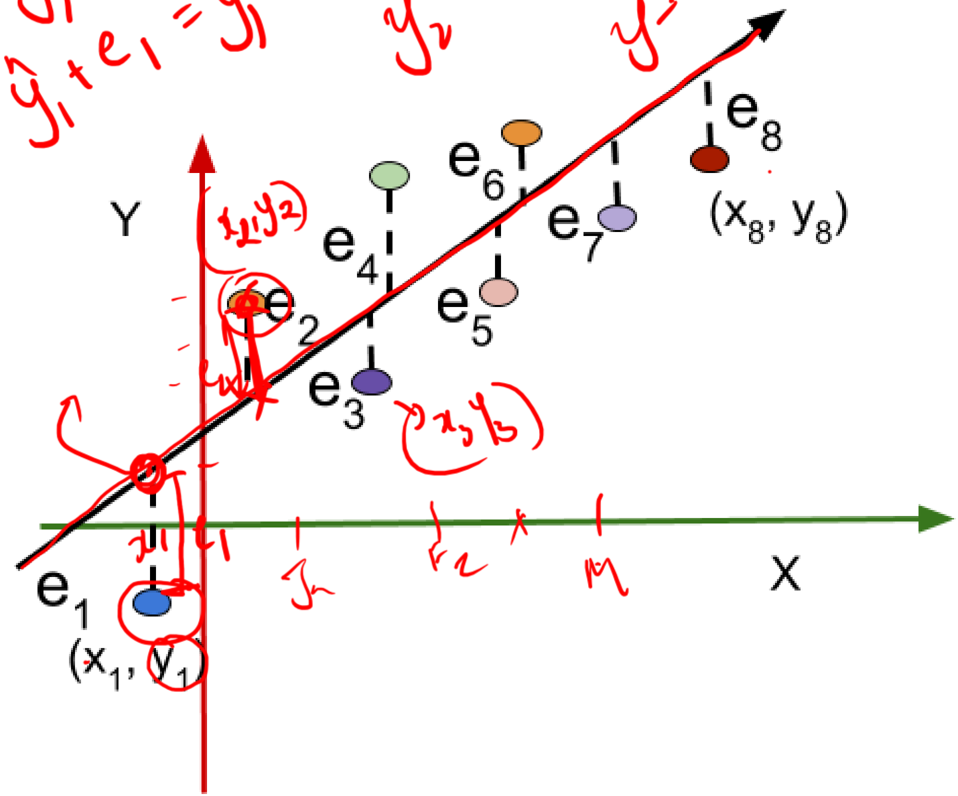
$y_2 = \hat{y}_2 + e_2$

$\hat{y}_1 = mx_1 + c$

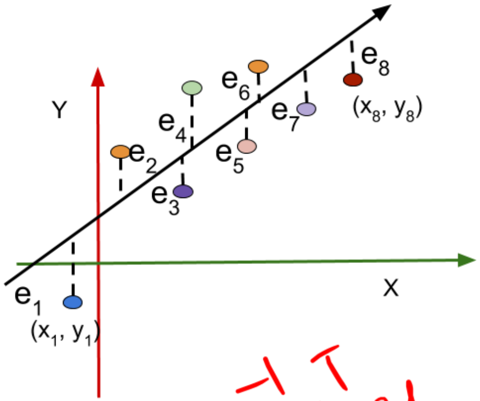
$\hat{y}_1 + e_1 = y_1$

$\hat{y}_2 = mx_2 + c$

$y = mx + c$



$$
\begin{aligned}
y_1 &= mx_1 + c + e_1 \\
y_2 &= mx_2 + c + e_2 \\
y_3 &= mx_3 + c + e_3 \\
y_4 &= mx_4 + c + e_4 \\
y_5 &= mx_5 + c + e_5 \\
y_6 &= mx_6 + c + e_6 \\
y_7 &= mx_7 + c + e_7 \\
y_8 &= mx_8 + c + e_8
\end{aligned}
$$



$$
\begin{aligned}
y_1 &= mx_1 + c + e_1\\
y_2 &= mx_2 + c + e_2\\
y_3 &= mx_3 + c + e_3\\
y_4 &= mx_4 + c + e_4\\
y_5 &= mx_5 + c + e_5\\
y_6 &= mx_6 + c + e_6\\
y_7 &= mx_7 + c + e_7\\
y_8 &= mx_8 + c + e_8
\end{aligned}
\Rightarrow
\begin{bmatrix} y_1\\ y_2\\ y_3\\ y_4\\ y_5\\ y_6\\ y_7\\ y_8 \end{bmatrix}
=
\underbrace{\begin{bmatrix} x_1 & 1\\ x_2 & 1\\ x_3 & 1\\ x_4 & 1\\ x_5 & 1\\ x_6 & 1\\ x_7 & 1\\ x_8 & 1 \end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} m\\ c \end{bmatrix}}_{\hat{x}}
+
\underbrace{\begin{bmatrix} e_1\\ e_2\\ e_3\\ e_4\\ e_5\\ e_6\\ e_7\\ e_8 \end{bmatrix}}_{\vec{e}}
$$

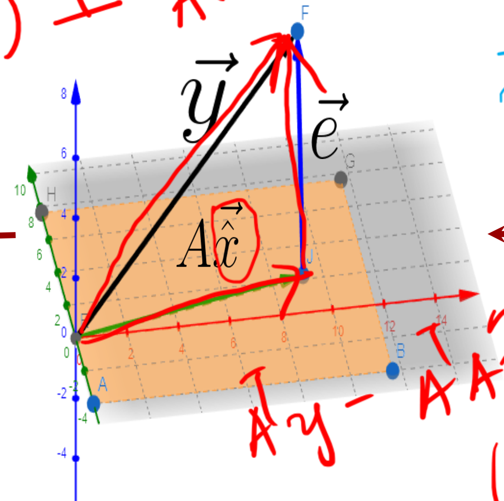
unknowns

$\hat{x} = (A^T A)^{-1} A^T y$

$\vec{e} = (y - A\hat{x}) \perp A^T$

$\vec{y} = A\hat{x} + \vec{e}$

$$\boxed{\vec{y} = A\hat{\vec{x}} + \vec{e}}$$



$$\boxed{\vec{x} = (A^T A)^{-1} A^T \vec{y}}$$

$A^T(y - A\hat{x}) = \vec{0}$

$A^T y - A^T A\hat{x} = \vec{0}$

$(A^T A)^{-1} A^T A\hat{x} = (A^T A)^{-1} A^T y$

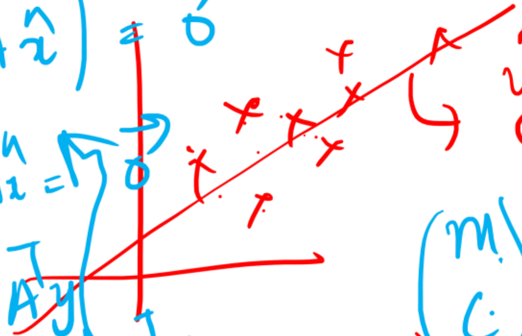
$\hat{x} = (A^T A)^{-1} A^T y$

$$A^T(y - A\hat{x}) = \vec{0}$$

$$A^Ty - A^TA\hat{x} = \vec{0}$$

$$A^TA\hat{x} = A^Ty$$

$$\hat{x} = (A^TA)^{-1}A^Ty$$



$$\hat{y} = \boxed{m}x + \boxed{c}$$

$$mx_i + c$$

$$\begin{pmatrix} m \\ c \end{pmatrix}$$

$$y_1 = \hat{y}_1 + e_1 = mx_1 + c + e_1$$

$$y_2 = \hat{y}_2 + e_2 = mx_2 + c + e_2$$

$$\vdots$$

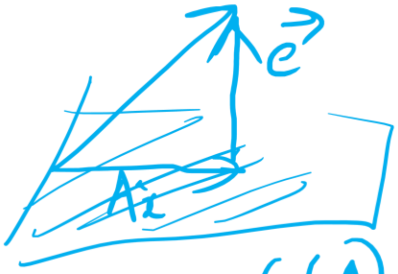
$$y_n = \hat{y}_n + e_n = mx_n + c + e_n$$

$$y \neq A\hat{x}$$

$$\boxed{\vec{y} = A\hat{x} + \vec{e}}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

$$\vec{y} = A\hat{x} + \vec{e}$$



$$\vec{e} \perp C(A)$$

$$A^T\vec{e} = \vec{0}$$

$$\begin{pmatrix} -\text{col}_1^T- \\ -\text{col}_2^T- \end{pmatrix} \begin{bmatrix} \vec{e} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{col}_1 \cdot \vec{e} = 0$$

$$\text{col}_1^T \vec{e} = 0$$

$$\text{col}_2^T \cdot \vec{e} = 0$$

$C(A) \perp \vec{e}$

$A^T = \begin{bmatrix} -a_1^T- \\ -a_2^T- \\ \vdots \\ -a_n^T- \end{bmatrix}$ $A^T\vec{e} = \vec{0}$

What happens in this case ? - Code it

$A = \begin{bmatrix} | & | & | & & | \\ a_1 & a_2 & a_3 & \cdots & a_n \\ | & | & | & & | \end{bmatrix}$

$a_i \cdot \vec{e} = \vec{0}$

$a_i^T e = 0$

$A^T \vec{e} =$

$\begin{bmatrix} -a_1^T- \\ -a_2^T- \\ \vdots \\ -a_n^T- \end{bmatrix} \begin{bmatrix} | \\ \vec{e} \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{bmatrix} = \begin{bmatrix} x_1 & x_1^2 & 1 \\ x_2 & x_2^2 & 1 \\ x_3 & x_3^2 & 1 \\ x_4 & x_4^2 & 1 \\ x_5 & x_5^2 & 1 \\ x_6 & x_6^2 & 1 \\ x_7 & x_7^2 & 1 \\ x_8 & x_8^2 & 1 \end{bmatrix} \begin{bmatrix} m \\ p \\ c \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \end{bmatrix}$$

$$\vec{y} = A\vec{x} + \vec{e}$$

What change you will observe in the graph?

What happens when you add more higher order terms like x^3, x^4.. x^n?

$$\boxed{\vec{x} = (A^TA)^{-1}A^T\vec{y}}$$

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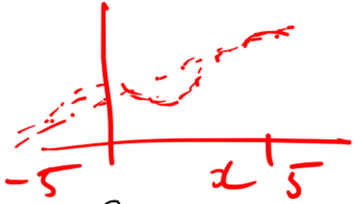
$$\boxed{y_{data} = x^3 + \sin x + \text{noise}}$$

(y_{data}) vs x

step 1



$$y_{data} = \begin{bmatrix} y_{data}(0) \\ y_{data}(1) \\ \vdots \\ y_{data}(n) \end{bmatrix} \qquad x = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(n) \end{bmatrix}$$

$$m x_0 + k x_0^2 + c$$

$$y_{data} = \underbrace{\begin{bmatrix} x_0 & x_0^2 & 1 \\ x_1 & x_1^2 & 1 \\ \vdots & \vdots & \vdots \\ x_n & x_n^2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ k \\ c \end{bmatrix}}_{\hat{z}} + \underbrace{\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}}_{e}$$

$$\hat{z} = (A^T A)^{-1} A^T y_{data}$$

$$\downarrow$$

$$\begin{pmatrix} m \\ c \end{pmatrix}$$

Applications of Least Squares in Signal Processing

- Linear/ Non-linear Prediction
- Denoising
- Deconvolution
- System Identification
- Estimating Missing Data

[Link to Ivan Selesnick's Tutorial]

LEAST SQUARES WITH EXAMPLES IN SIGNAL PROCESSING*

Ivan Selesnick

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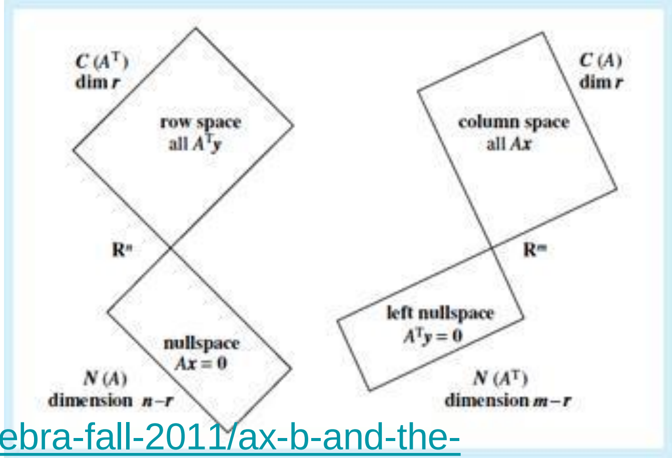
Four Fundamental Subspaces

- **Column Space**
- **Left Null Space**
- **Row Space**
- **Right Null Space**

Fundamental Theorem of Linear Algebra

- Column space and Row space both have dimension r (rank).
- The Right Null Space have dimension *n-r* and the left null space has dimension *m-r*.
- Right Null Space is the orthogonal complement of the row space.
- Left Null Space is the orthogonal complement of the column space



Source: https://ocw.aprende.org/courses/mathematics/18-06sc-linear-algebra-fall-2011/ax-b-and-the-four-subspaces/