BITS Pilani
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# Data Structures and Algorithms Design

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# CONTACT SESSION 16 -PLAN

| Contact Sessions(#) | List of Topic Title | Text/Ref Book/external resource |
|---|---|---|
| 16 | **Definition of P and NP classes and examples, Understanding NP-Completeness: CNF-SAT Cook-Levin theorem**<br><br>**Polynomial time reducibility: CNF-SAT and 3-SAT, Vertex Cover** | **T1: 13.1, 13.2, 13.3** |

# P and NP classes

| Polynomial Time | Exponential Time |
| --- | --- |
| Linear Search O(n) | 0/1 Knapsack O(2^n) |
| Binary Search O(log n) | Tower of Hanoi O(2^n) |
| Merge Sort O(nlogn) | TSP O(2^n) |
| Quicksort O(nlogn),O(n^2) | Sum of Subsets O(2^n) |
| Fractional KS O(nlogn) | |
| Task Scheduling O(n^2) | |
| Kruskal's MST O(mlogn) | |

# Decision and Optimization Problems

- **Decision Problem**: computational problem with intended output of “yes” or “no”, 1 or 0
- **Optimization Problem**: computational problem where we try to maximize or minimize some value.
- Introduce parameter k and ask if the optimal value for the problem is at most or at least k. Turn optimization into decision

# Decision and Optimization Problems

- Example: Hamiltonian circles:
- A Hamiltonian circle in an undirected graph is a simple circle that passes through every vertex exactly once.
- The **decision** problem is: Does a given undirected graph has a Hamiltonian circle?

# Decision and Optimization Problems



- Many problems will have decision and optimization versions
- **Traveling salesman problem**
- Optimization problem: Given a weighted graph, find Hamiltonian cycle of minimum weight.
- Decision problem: Given a weighted graph and an integer k, is there a Hamiltonian cycle with total weight at most k?

# Decision and Optimization Problems

- **Knapsack** : Suppose we have a knapsack of capacity W and n objects with weights w1, …, wn, and values v1, …, vn
- **Optimization problem**: Find the largest total value of any subset of the objects that fits in the knapsack (and find a subset that achieves the maximum value)
- **Decision problem**: Given k, is there a subset of the objects that fits in the knapsack and has total value at least k?

# Decision and Optimization Problems

- **Graph coloring**: assign a color to each vertex so that adjacent vertices are not assigned the same color. Chromatic number: the smallest number of colors needed to color G.
- We are given an undirected graph G = (V, E) to be colored.
- **Optimization problem**: Given G, determine the chromatic number .
- **Decision problem**: Given G and a positive integer k, is there a coloring of G using at most k colors? If so, G is said to be k-colorable

# Decision and Optimization Problems

- **Decision Problem**: computational problem with intended output of “yes” or “no”, 1 or 0

  Examples:

  - Does a text T contain a pattern P?
  - Does an instance of 0/1 Knapsack have a solution with benefit at least K?
  - Does a graph G have an MST with weight at most K?

# Polynomial-Time

- The class P consists of those problems that are solvable in polynomial time.
- More specifically, they are problems that can be solved in time $O(n^k)$ for some constant k, where n is the size of the input to the problem
- The key is that n is the **size of input**

# The Complexity Class NP

- NP is not the same as non-polynomial complexity/running time.
- NP does not stand for not polynomial.
- NP = Non-Deterministic polynomial time

# Deterministic

- In a deterministic computer we can determine in advance for every computational cycle, the output state by looking at the input state
- In **deterministic algorithm**, for a given particular input, the computer will always produce the same output going through the same states

# Deterministic



//input: an array A[1..N] and an element el that is in the array

//output: the position of el in A

```
search(el, A, i)
{
  if (A[i] = el) then return i
  else
     return search(el, A, i+1)
}
```

Complexity: O(N)

el = 9

7 5 3 8 9 3

# Non Deterministic

- In a nondeterministic computer we may have more than one output state.
- The computer "chooses" the correct one ("nondeterministic choice")
- For the same input, the compiler may produce different output in different runs.
- Non-deterministic algorithms can't solve the problem in polynomial time and can't determine what is the next step

# Non Deterministic

- Non-deterministic algorithms can show different behaviors for the same input on different execution and there is a degree of randomness to it.
- A **nondeterministic algorithm** for a problem **A** is a two-stage procedure. **In the first phase**, a procedure makes a guess about the possible solution for A. **In the second phase**, a procedure checks if the guessed solution is indeed a solution for A

# Non Deterministic

- **Search an element x on A[1:n] where n>=1, on successful search return j if a[j] is equals to x otherwise return 0.**

*j= choice(a, n)* → non deterministic

*if(A[j]==x) then*

*{*

*write(j);*

*success();*

*}*

*write(0); failure();* — O(1)

O(1) polynomial

Complexity: O(1)

# Non Deterministic

- **choice(X) :** chooses any value randomly from the set X.
- **failure() :** denotes the unsuccessful solution.
- **success() :** Solution is successful and current thread terminates.

# ND Sorting

1 2 3 4
3 9 4 2

## Non Deterministic Sorting

```
1.  Algorithm Nsort(A,N)
2.  // sort n positive numbers
3.  {
4.  For I = 1 to n do B[i] = 0 ; // initialize B
5.  For I = 1 to n do
6.  {
7.  J = choice(1,n);
8.  If B[j] != 0 then failure();
9.  B[j] = A[i];
10. }
11. For I = 1 to n-1 do // verify the order
12. If B[i] > B[i+1] then failure ();
13. Write (B);
14. Success();
15. }
```

ND

O(n)

- In the for loop of 5 to 10 each A[i] is assigned to position in B
- Line 7 non deterministically identifies this position
- Line 8 checks whether this position is already used or not !
- The order of numbers in B is some permutation of the initial order in A
- For loop of line no 11 and 12 verifies the B is in ascending order,
- Since there is always a set of choices at line no 7 for ascending order, this is a sorting algorithm of Complexity O(N)
- All deterministic sorting algorithms must have A complexity O(nlogn)

# The Complexity Class NP

- The class **NP** consists of all problems that can be solved in polynomial time by **nondeterministic algorithms**
- That is, both phase 1 and phase 2 run in polynomial time
- If A is a problem in P then A is a problem in NP because
  - Phase 1: use the polynomial algorithm that solves A
  - Phase 2: write a constant time procedure that always returns true

# The Complexity Class NP

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- Every decision problem solvable by a polynomial time deterministic algorithm is also solvable by a polynomial time nondeterministic algorithm.
- To see this, observe that any deterministic algorithm can be used as the checking stage of a nondeterministic algorithm.
- If I ∈ P, and A is any polynomial deterministic algorithm for I, we can obtain a polynomial nondeterministic algorithm for I merely by using A as the checking stage and ignoring the guess.
- Thus I ∈ P implies I ∈ NP

# The Complexity Class NP

- The key question is are there problems in NP that are not in P or is $\boxed{P = NP?}$
- We don't know the answer to the previous question
- We say that a non-deterministic algorithm A **accepts** a string x if there exists some sequence of choose operations that causes A to output "yes" on input x.
- NP is the complexity class consisting of all languages accepted by **polynomial-time non-deterministic** algorithms.

# The Complexity Class NP



- How to prove that a problem $A$ is in NP:

  Show that $A$ is in P

  OR

  Write a nondeterministic algorithm solving A that runs in polynomial time

# NP example

Problem: Decide if a graph has an MST of weight K

choice( )

Algorithm:

1. Non-deterministically choose a set T of n-1 edges
2. Test that T forms a spanning tree
3. Test that T has weight at most K

Analysis: Testing takes O(n+m) time, so this algorithm runs in polynomial time.

# The Complexity Class NP Alternate Definition

- We say that an algorithm B **verfies** the acceptance of a language L if and only if, for any x in L, there exists a certificate y such that B outputs “yes” on input (x,y).
- NP is the complexity class consisting of all languages verified by **polynomial-time** algorithms.

- We know: P is a subset of NP.
- Major open question: P=NP?
- Most researchers believe that P and NP are different

# Satisfiability Problem

- **A Boolean formula is *satisfiable* if there exists some assignment of the values 0 and 1 to its variables that causes it to evaluate to 1.**
- CNF – Conjunctive Normal Form. ANDing of clauses of ORs

$$(x_0 \vee x_2) \wedge (\neg x_0 \vee x_1) \wedge (x_0 \vee x_1 \vee \neg x_2) = 1$$

0 0 0

$$\left\{ \begin{array}{l} x_0 = 1 \\ x_1 = 0 \\ x_2 = 0 \end{array} \right\} - \text{choice}(\,)$$

# CNF satisfiability

- This problem is in *NP*. Nondeterministic algorithm:
  - Guess truth assignment
  - Check assignment to see if it satisfies CNF formula

  - It is easy to show that CNF-SAT is in NP
  - For, given a Boolean formula S, we can construct a simple nondeterministic algorithm that first "guesses" an assignment of Boolean values for the variables in S and then evaluates each clause of S in turn.
  - If all the clauses of S evaluate to 1, then S is satisfied; otherwise, it is not.

# CNF satisfiability

- Example:

  $(A \vee \neg B \vee \neg C) \wedge (\neg A \vee B) \wedge (\neg B \vee D \vee F) \wedge (F \vee \neg D)$

- Truth assignments:

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1. | 0 | 1 | 1 | 0 | 1 | 0 |
| 2. | 1 | 0 | 0 | 0 | 0 | 1 |
| 3. | 1 | 1 | 0 | 0 | 0 | 1 |

4. ... (how many more?)

Checking phase: $\Theta(n)$

# 3 SAT

P NP

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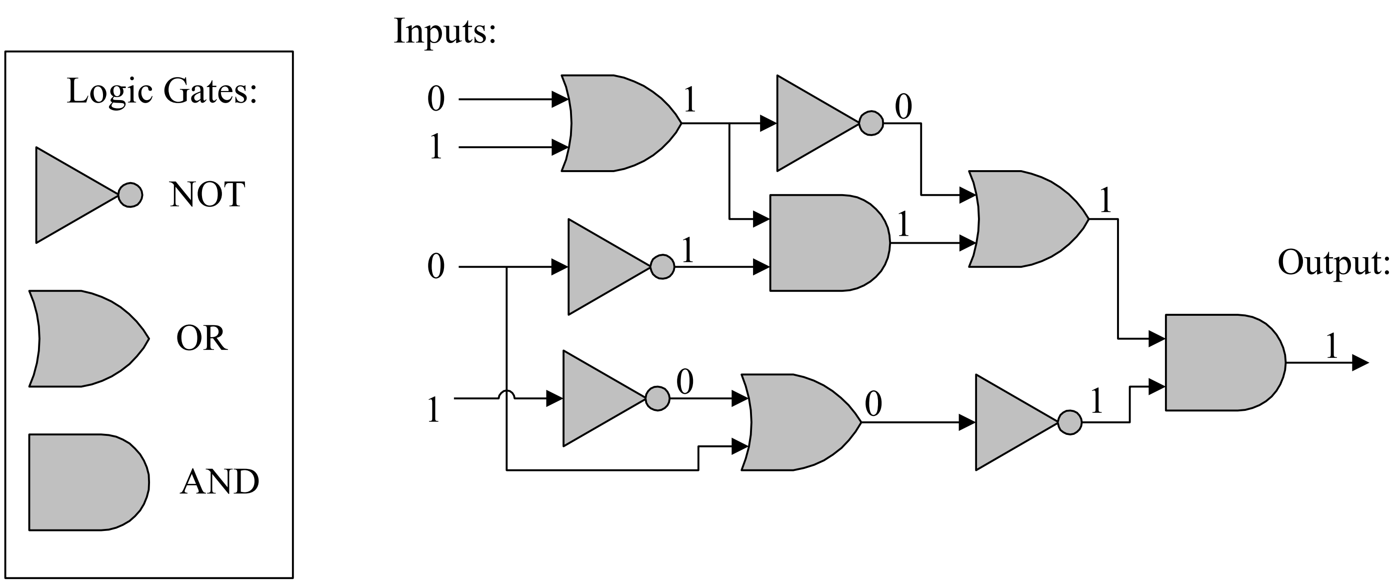
- Problem: Given a CNF where each clause has 3 variables, decide whether it is satisfiable or not.

- $(x_1 \vee x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_2) \wedge (\neg x_1 \vee x_2 \vee x_2)$

- 3SAT is NP Complete

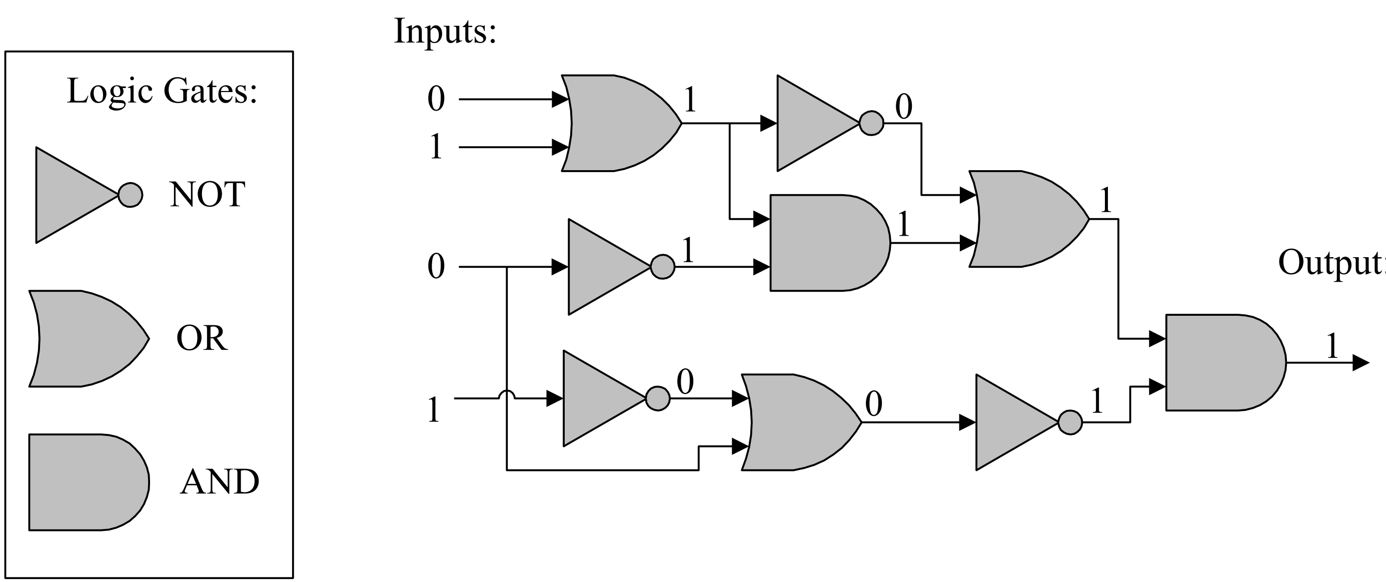
# An Interesting Problem

- A Boolean circuit is a circuit of AND, OR, and NOT gates; the CIRCUIT-SAT problem is to determine if there is an assignment of 0's and 1's to a circuit's inputs so that the circuit outputs 1.



# CIRCUIT-SAT is in NP

- Non-deterministically choose a set of inputs and the outcome of every gate, then test each gate's I/O.



# P And NP Summary

- **P** = set of problems that can be solved in polynomial time
  - Examples: Fractional Knapsack, …
- **NP** = set of problems for which a solution can be verified in polynomial time
  - Examples: 0/1 Knapsack,…, Hamiltonian Cycle, CNF SAT, 3-CNF SAT
- Clearly **P** $\subseteq$ **NP**
- Open question: Does **P** = **NP**?
  - Most suspect not
  - An August 2010 claim of proof that P $\neq$ NP, by Vinay Deolalikar, researcher at HP Labs, Palo Alto, has flaws

# Polynomial-Time Reducibility

- Language L is polynomial-time reducible to language M if there is a function computable in polynomial time that takes an input x of L and transforms it to an input f(x) of M, such that x is a member of L if and only if f(x) is a member of M.
- Shorthand, $L^{poly}M$ means L is polynomial-time reducible to M

# Polynomial-Time Reducibility

- A problem R can be *reduced* to another problem Q if any instance of R can be rephrased to an instance of Q, the solution to which provides a solution to the instance of R
  - This rephrasing is called a *transformation*
- Intuitively: If R reduces in polynomial time to Q, R is "no harder to solve" than Q
- Example: lcm(m, n) = m * n / gcd(m, n),

lcm(m,n) problem is reduced to gcd(m, n) problem

# Polynomial-Time Reducibility

- For example a Hamiltonian circuit problem is polynomially reducible to the decision version of the traveling salesman problem.
- Hamiltonian circuit problem(HCP): Does a given undirected graph have a Hamiltonian cycle?
- Traveling salesman problem(TSP): Given a weighted graph and an integer k, is there a Hamiltonian cycle with total weight at most k?

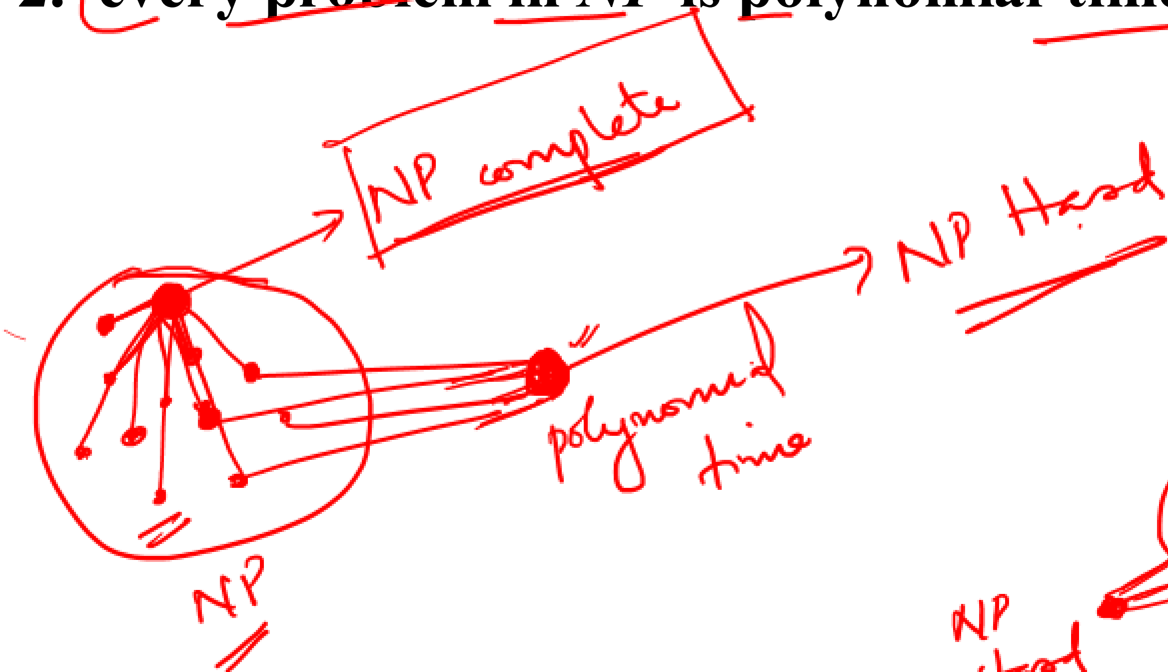
# NP Hard and NP complete

P
NP

- A decision problem *D* is *NP*-complete iff

***1. D ∈ NP***

**2. every problem in *NP* is polynomial-time reducible to *D***

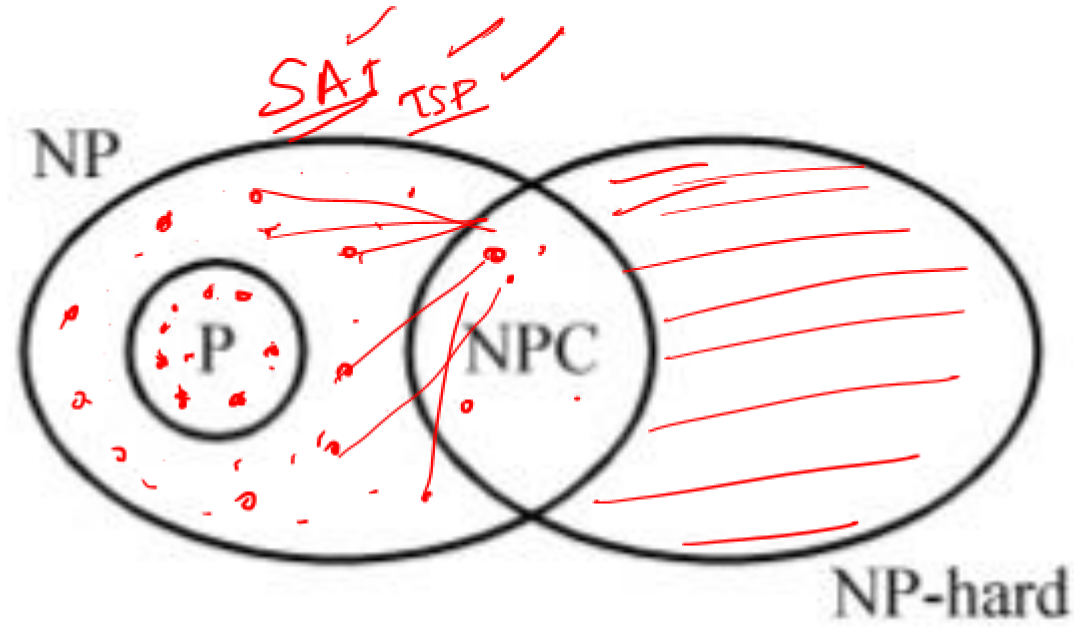



D CNF-Sat

# NP Hard and NP complete

- If R is *polynomial-time reducible* to Q, we denote this $R \leq_p Q$
- Definition of NP-Hard and NP-Complete:
  - If all problems $R \in$ **NP** are *polynomial-time* reducible to Q, then Q is *NP-Hard*
  - We say Q is *NP-Complete* if Q is NP-Hard and $Q \in$ **NP**
- If $R \leq_p Q$ and R is NP-Hard, Q is also NP-Hard

# NP Hard and NP complete



# NP Hard and NP complete

- Belief: P is a proper subset of NP.
- Implication: the NP-complete problems are the hardest in NP.
- Why: Because if we could solve an NP-complete problem in polynomial time, we could solve every problem in NP in polynomial time.
- That is, if an NP-complete problem is solvable in polynomial time, then P=NP.
- Since so many people have attempted without success to find polynomial-time solutions to NP-complete problems, showing your problem is NP-complete is equivalent to showing that a lot of smart people have worked on your problem and found no polynomial-time algorithm

# Cook's Theorem

## Cook's theorem

NP = P iff the satisfiability problem is a P problem.

- SAT is NP-complete.
- It is the first NP-complete problem.
- Every NP problem reduces to SAT.

**Stephen Arthur Cook**

# Amusing analogy
# (thanks to lecture notes at University of Utah)

- Students believe that every problem assigned to them is NP-complete in difficulty level, as they have to find the solutions. ☺

- Teaching Assistants, on the other hand, find that their job is only as hard as NP, as they only have to verify the student's answers.

- When some students confound the TAs, even verification becomes hard