Applied Machine Learning

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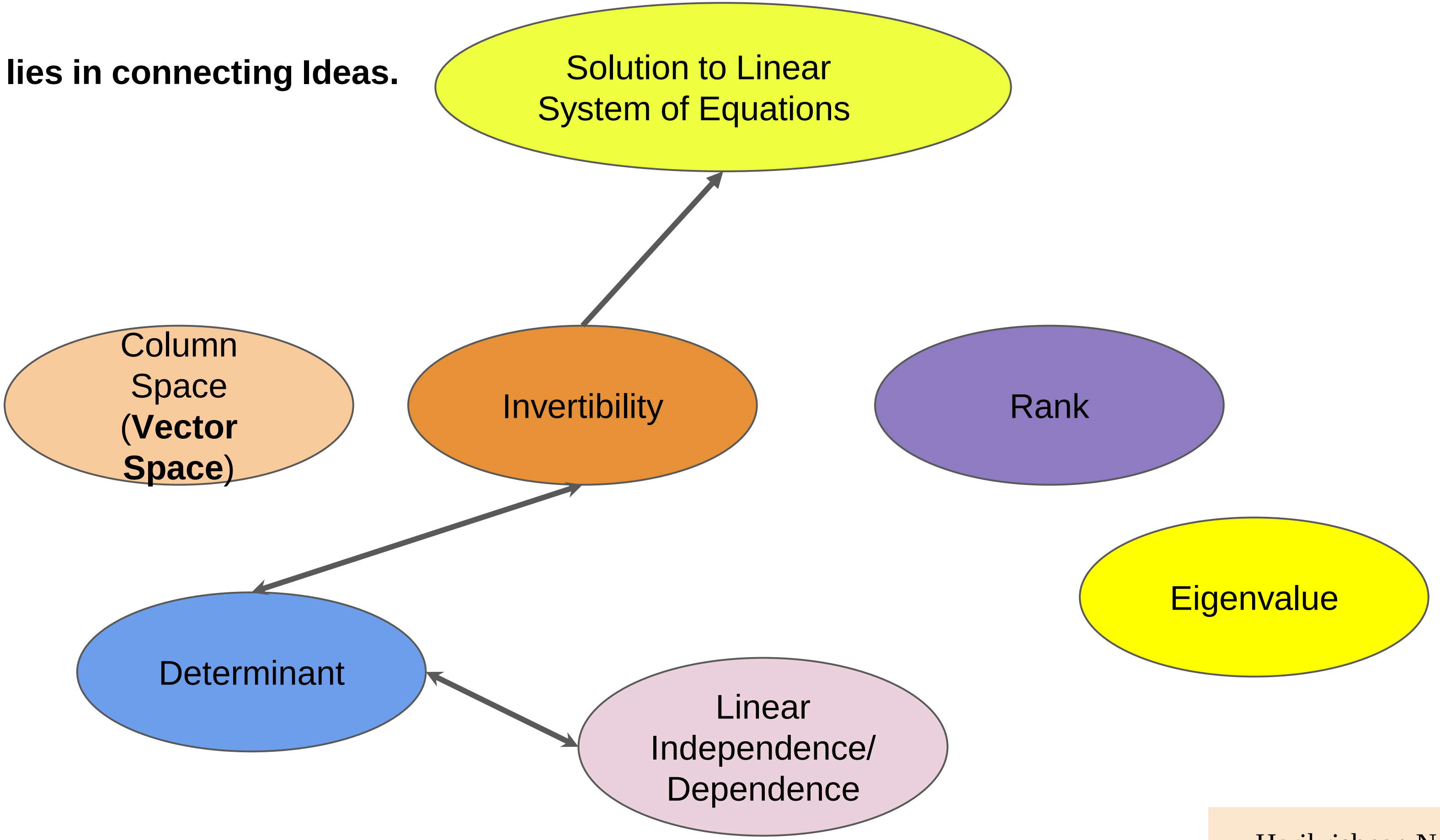
SE ZG568 / SS ZG568,
Applied Machine Learning
Lecture No. 5 [16- Feb-2025]

Recap

Basics of Linear Algebra,
Row Picture, Col Picture, Algebraic Way
Solution to System of Linear Equations
Inverse of a Matrix
Linear Regression,
PCA

Beauty lies in connecting Ideas.



Orthogonal and Orthonormal Matrix

Orthogonal vectors

$$\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

L2 - norm = $\sqrt{2}$

Orthonormal vectors

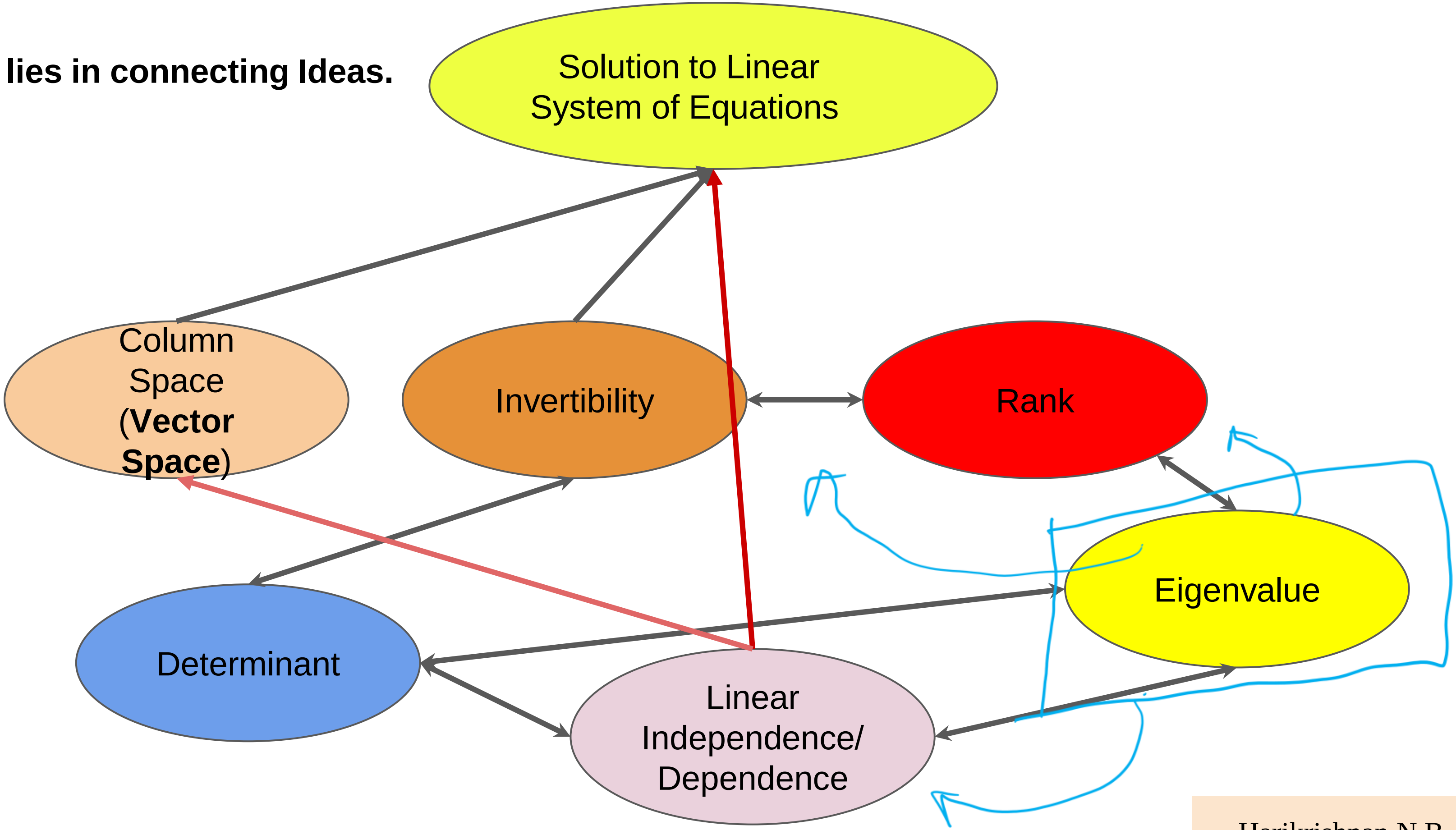
$$\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = 0$$

L2 - norm = 1

Orthonormal Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

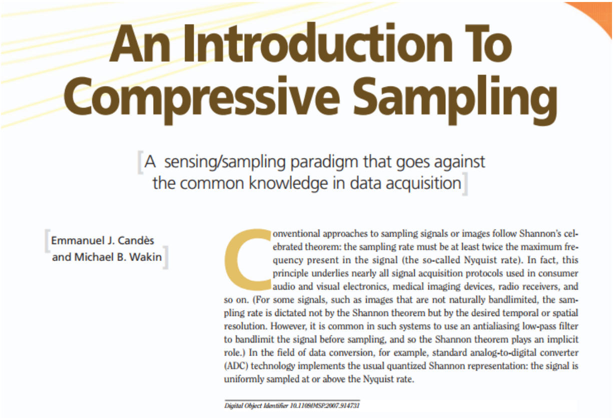
Beauty lies in connecting Ideas.



Do we know everything about *y*=*Ax* ?

Emmanuel Candès Terence Tao David Donoho Justin Romberg



An Introduction To Compressive Sampling

[A sensing/sampling paradigm that goes against the common knowledge in data acquisition]

[Emmanuel J. Candès and Michael B. Wakin]

Conventional approaches to sampling signals or images follow Shannon's celebrated theorem: the sampling rate must be at least twice the maximum frequency present in the signal (the so-called Nyquist rate). In fact, this principle underlies nearly all signal acquisition protocols used in consumer audio and visual electronics, medical imaging devices, radio receivers, and so on. (For some signals, such as images that are not naturally bandlimited, the sampling rate is dictated not by the Shannon theorem but by the desired temporal or spatial resolution. However, it is common in such systems to use an antialiasing low-pass filter to bandlimit the signal before sampling, and so the Shannon theorem plays an implicit role.) In the field of data conversion, for example, standard analog-to-digital converter (ADC) technology implements the usual quantized Shannon representation: the signal is uniformly sampled at or above the Nyquist rate.

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Near Optimal Signal Recovery From Random Projections: Universal Encoding Strategies?

Emmanuel Candes† and Terence Tao♯

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♯ Department of Mathematics, University of California, Los Angeles, CA 90095

Matrix Vector Multiplication as a Transformation

eigenvectors

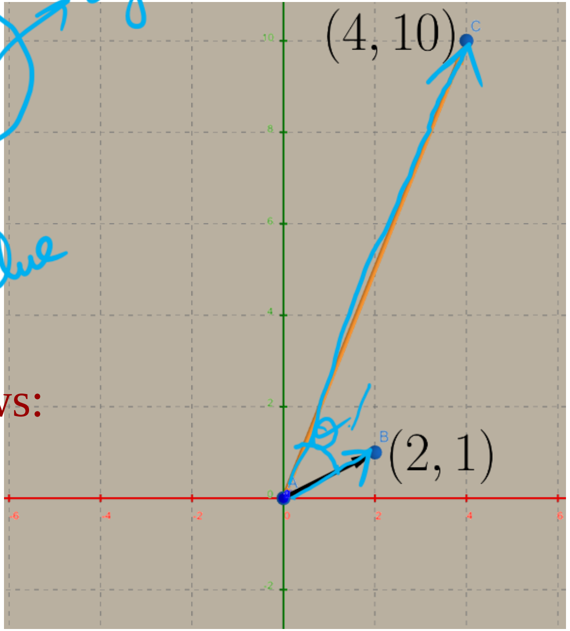
Intuition for Matrix vector multiplication for Square Matrices

$$\boxed{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$$

λ

$$\boxed{A\vec{x}_i = \lambda_i \vec{x}_i}$$

eigenvalue



Matrix(Square Matrix) vector multiplication can be seen as follows:

- Rotation ✓
- Stretching or Shrinking ✓

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} & \\ & \end{bmatrix}_{2\times 2}$$

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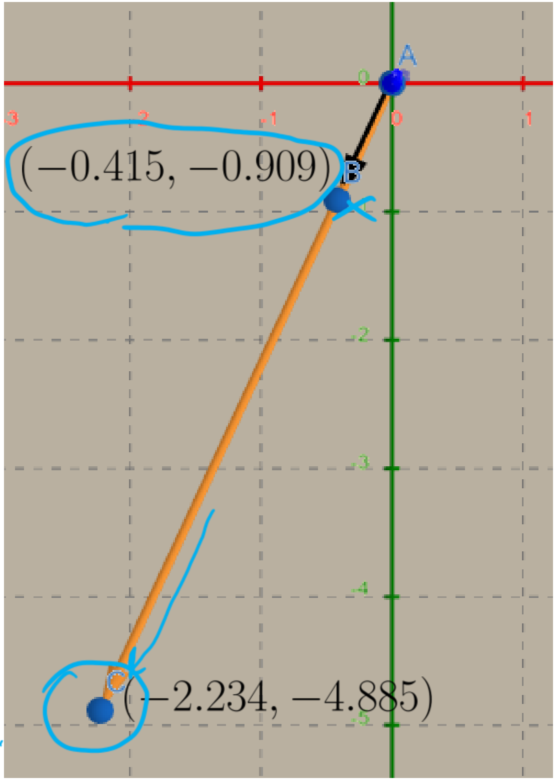
Special Vectors

A $\qquad \vec{x}$

$$\boxed{\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}}\begin{bmatrix}-0.415\\-0.909\end{bmatrix} = \begin{bmatrix}-2.234\\-4.885\end{bmatrix} = 5.372\begin{bmatrix}-0.415\\-0.909\end{bmatrix}$$

$$A\vec{x} \qquad\qquad \text{eigenvalue} \quad \lambda\vec{x}$$

$$\boxed{A\vec{x} = \lambda\vec{x}}$$



1. Direction of $\vec{x}$ is unchanged. (No rotation)
2. Only the magnitude is scaled by a factor λ
3. $\vec{x}$ - **eigenvector of matrix A**
4. **λ - eigenvalue of matrix A**

$$A = \begin{bmatrix} 5 & 3 \\ 0 & 2 \end{bmatrix}_{2\times 2}$$

Suppose $\vec{x}$ is an eigenvector of "A"

Size of $\vec{x}$ 2×1 $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, λ

$A\vec{x} \Rightarrow$

$$\begin{bmatrix} 5 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A\vec{x} = \lambda\vec{x}$$

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$\underbrace{\begin{bmatrix} 5 & 3 \\ 0 & 2 \end{bmatrix}_{2\times 2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}}_{} - \lambda\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} \\ \\ \end{bmatrix}_{2\times 1} - \lambda\begin{bmatrix} \\ \\ \end{bmatrix}_{2\times 1} = \begin{bmatrix} \\ \\ \end{bmatrix}_{2\times 1}$$

$$A\vec{x} = \lambda\vec{x}$$

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$\Rightarrow \underset{2\times 1}{\vec{x}}\,(\underset{2\times 2}{A - \lambda}) = \vec{0} \quad \times$$

$$(A - \lambda I)\,\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} A \end{bmatrix} - \begin{bmatrix} \lambda\cdot 1 & 0 \\ 0 & \lambda\cdot 1 \end{bmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$A\vec{x} - \underbrace{\begin{bmatrix} \lambda\cdot 1 & 0 \\ 0 & \lambda\cdot 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{} =$$

$$A\vec{x} - \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix} = A\vec{x} - \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{0}$$

$$A\vec{x} - \lambda\vec{x} = 0$$

$$3\cdot 2 + 4\cdot 2 \Rightarrow 2\cdot(3+4)$$

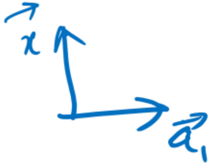
scale $= (3+4)\cdot 2$

$A - \lambda \nearrow$

$A = \begin{bmatrix} 5 & 3 \\ 0 & 2 \end{bmatrix}$

$\begin{bmatrix} - a_1^T - \\ - a_2^T - \end{bmatrix} \begin{bmatrix} \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$A\vec{x} = \lambda\vec{x}$$



$\boxed{a_1^T\vec{x} = 0}$, $\boxed{a_2^T\vec{x} = 0}$

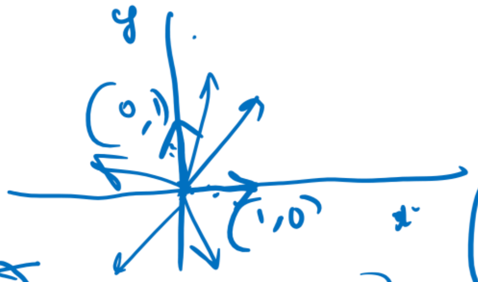
$A\vec{x} = \lambda\vec{x}$

$(A - \lambda I)\cdot\vec{x} = 0$

$(A - \lambda I)\cdot\vec{x} = \vec{0}$

non zero vector

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \end{bmatrix} = ?$

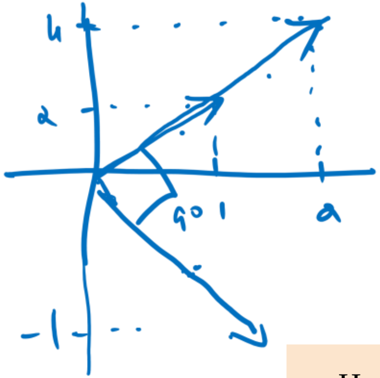


$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1\times\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$col_2 \Rightarrow 2\cdot col_1$



$$\boxed{(A-\lambda I)\cdot \vec{x} = \vec{0}}$$

$\downarrow$ Linearly Dependent

$$\boxed{\det(A-\lambda I) = 0}$$

$$A = \begin{bmatrix} 5 & 3 \\ 0 & 2 \end{bmatrix}$$

$$\det(A-\lambda I) = 0$$

$$\begin{bmatrix} 5-\lambda & 3 \\ 0 & 2-\lambda \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \det(A) = 0$$

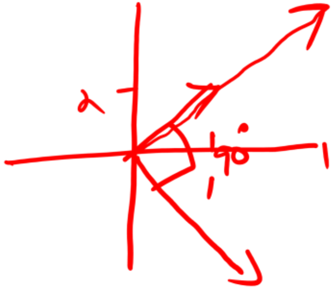
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Can you find a non zero $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ which satisfies the above equation

$$\boxed{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

$\downarrow$ Linearly Dependent

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$



$$\det\begin{pmatrix} \overset{a}{5-\lambda} & \overset{b}{3} \\ \overset{c}{0} & \overset{d}{2-\lambda} \end{pmatrix} = 0$$

$ad - bc$

$(5-\lambda)\times(2-\lambda) - 3\times 0 = 0$

$(5-\lambda)\times(2-\lambda) = 0$

$\lambda = 5$ or $\lambda = 2$

Fo $\lambda=5$, I want to find eigenvector

$(A-\lambda I)\cdot\vec{x} = \vec{0}$

$\boxed{\lambda=5}$

$$\begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix}\cdot\vec{x} = \vec{0}$$

$$\begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

Fo $\lambda=2$, I want to find eigenvector

$(A-\lambda I)\cdot\vec{x} = \vec{0}$

$A\vec{x}_1 = \lambda_1\vec{x}_1$

$A\vec{x}_2 = \lambda_2\vec{x}_2$

$\boxed{\text{Fo } \lambda=2}$

$$\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}\boxed{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$A\vec{x}$

$$\overset{A}{\begin{bmatrix} 5 & 3 \\ 0 & 2 \end{bmatrix}}\overset{\vec{x}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \overset{\lambda}{5}\overset{\vec{x}}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$\overset{A}{\begin{bmatrix} 5 & 3 \\ 0 & 2 \end{bmatrix}}\overset{\vec{x}}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \overset{\lambda}{2}\overset{\vec{x}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

Eigenvalues and Eigenvectors

- For an n x n square matrix A, there are ‘n’ eigenvalues and ‘n’ eigenvectors. Let x_1, x_2, x_3,....x_n be the ‘n’ eigenvectors and $\lambda_1, \lambda_2, \lambda_3, \ldots\lambda_n$, are the corresponding eigenvalues.

$$A\vec{x_1} = \lambda_1\vec{x_1}$$
$$A\vec{x_2} = \lambda_2\vec{x_2}$$
$$A\vec{x_3} = \lambda_3\vec{x_3}$$
$$.$$
$$.$$
$$A\vec{x_n} = \lambda_n\vec{x_n}$$

$$X = \begin{bmatrix} | & | & | & & | \\ \vec{x_1} & \vec{x_2} & \vec{x_3} & \cdots & \vec{x_n} \\ | & | & | & & | \end{bmatrix}$$

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$A = \begin{bmatrix} 5 & 3 \\ 0 & 2 \end{bmatrix}$ $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\boxed{\lambda_1 = 5}$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\boxed{\lambda_2 = 2}$

$\vec{x}, \vec{y}$ are eigenvectors of A (λ_1, λ_2)

$X = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$

$\begin{bmatrix} \begin{bmatrix} 5 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} & \begin{bmatrix} 5 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \end{bmatrix}$

$A_{2\times 2} X_{2\times 2} = (\quad)_{2\times 2}$

$\begin{bmatrix} 5 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_1 & y_2 \end{bmatrix} = \begin{bmatrix} 5x_1 + 3x_2 & 5y_1 + 3y_2 \\ 0x_1 + 2x_2 & 0y_1 + 2y_2 \end{bmatrix}$

$AX = \begin{bmatrix} A\vec{x} & A\vec{y} \end{bmatrix} = \begin{bmatrix} \lambda_1 \vec{x} & \lambda_2 \vec{y} \end{bmatrix} = \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

A is a square matrix of size 2×2. Let $\vec{x}_1$ and $\vec{x}_2$ are the eigenvectors of A with eigen value λ_1 and λ_2 respectively $(\lambda_1 \neq \lambda_2) \neq 0$

$$X = \begin{bmatrix} | & | \\ \vec{x}_1 & \vec{x}_2 \\ | & | \end{bmatrix}$$ stacking eigenvectors of A as cols

$$\underline{\underline{AX}} = \begin{bmatrix} | & | \\ A\vec{x}_1 & A\vec{x}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \lambda_1\vec{x}_1 & \lambda_2\vec{x}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{x}_1 & \vec{x}_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$A \begin{bmatrix} | & | \\ \vec{x}_1 & \vec{x}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{x}_1 & \vec{x}_2 \\ | & | \end{bmatrix} \overbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}^{\Lambda}$$

$$AX = X\Lambda$$

$$AXX^{-1} = X\Lambda X^{-1}$$

$$A = X\Lambda X^{-1}$$

Very Very Important Part

- For an n x n square matrix A, there are ‘n’ eigenvalues and ‘n’ eigenvectors. Let $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$,....$\vec{x}_n$ be the ‘n’ eigenvectors and $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$, are the corresponding eigenvalues.

$$AX = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} | & | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \cdots & \vec{x}_n \\ | & | & | & & | \end{bmatrix}$$

Spectral Decomposition

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} | & | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \cdots & \vec{x}_n \\ | & | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & | & & | \\ A\vec{x}_1 & A\vec{x}_2 & A\vec{x}_3 & \cdots & A\vec{x}_n \\ | & | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & | & & | \\ \lambda_1\vec{x}_1 & \lambda_2\vec{x}_2 & \lambda_3\vec{x}_3 & \cdots & \lambda_n\vec{x}_n \\ | & | & | & & | \end{bmatrix}$$

$$\begin{bmatrix} | & | & | & & | \\ \lambda_1\vec{x}_1 & \lambda_2\vec{x}_2 & \lambda_3\vec{x}_3 & \cdots & \lambda_n\vec{x}_n \\ | & | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \cdots & \vec{x}_n \\ | & | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

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Spectral Decomposition

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} | & | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \cdots & \vec{x}_n \\ | & | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 & \cdots & \vec{x}_n \\ | & | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$\boxed{AX = X\Lambda}$$

$$AXX^{-1} = X\Lambda X^{-1}$$

$$AI = X\Lambda X^{-1}$$

$$\boxed{A = X\Lambda X^{-1}}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

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Practical Challenges and Important Points

AA^{-1} & AA^{T}

$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

When can we apply $A = X\Lambda X^{-1}$?

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}_{2\times 2}$ $\quad A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- A should be a square matrix
- When A has ‘n’ linearly independent eigenvectors, then X^{-1} always exist.

What happens when A is Symmetric (A^T = A)?

$x_1 \perp x_2$

$x_1 \cdot x_2 = 0$

$\|x_1\|_2 = 1$ $\quad \|x_2\|_2 = 1$

A^{-1}

- The eigenvectors of a symmetric matrix A can be chosen as "ORTHONORMAL". So in this case X is orthonormal.
- For an ORTHONORMAL matrix X, the inverse is its transpose $X^{-1} = X^T$

$$A = X\Lambda X^{-1}$$
$$A = X\Lambda X^{T}$$

$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

- The eigenvectors of a symmetric matrix A can be chosen as ORTHONORMAL. So in this case X is orthonormal. Why?

$x_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}$

$\boxed{\text{Rule 1} \rightarrow \vec{x}_i \cdot \vec{x}_j = 0 \text{ for all } i \neq j}$

Rule 2 $\rightarrow \|x_i\|_2 = 1$ for $i = 1$ to N

$X^T X \Rightarrow \begin{bmatrix} - & x_1^T & - \\ - & x_2^T & - \\ - & x_3^T & - \\ & \vdots & \\ - & x_n^T & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} x_1^T x_1 & x_1^T x_2 & \cdots & x_1^T x_n \\ x_2^T x_1 & x_2^T x_2 & \cdots & x_2^T x_n \\ \vdots & & & \\ x_n^T x_1 & \cdots & & x_n^T x_n \end{bmatrix}$

$x_1^T x_1$

$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

The eigenvectors of a symmetric matrix can be chosen orthonormal

$A^T = A$, let x_i, x_j be the eigenvectors of A and their corresponding eigenvalues be λ_i and λ_j. Note that $\boxed{\lambda_i \neq \lambda_j}$

$\vec{x}_1 \quad \vec{x}_2$

$\begin{pmatrix} 2 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\|\vec{x}_1\|_2 = \sqrt{5}$

$A x_i = \lambda_i x_i$, $\quad \boxed{A x_j = \lambda_j x_j}$

$A^T x_i = \lambda_i x_i$

$x_j^T A = (A^T x_j)^T$

$\underset{m \times n}{B} \underset{n \times m}{C}$

$(BC)^T \to C^T B^T$

$(BC)^T = C^T B^T$

$(A^T x_j)^T = x_j^T (A^T)^T = x_j^T A$

$A x_i = \lambda_i x_i$

$x_j^T A x_i = \lambda_i x_j^T x_i$

$(A^T x_j)^T x_i = \lambda_i x_j^T x_i$

same

$\lambda_j x_j^T \cdot x_i = \lambda_i x_j^T x_i$

$\lambda_j x_j^T x_i - \lambda_i x_j^T x_i = 0$

$(\lambda_j - \lambda_i)(x_j^T x_i) = 0$

$\underset{m \times r}{K} \cdot \underset{r \times n}{Z}$

$(KZ)^T \to Z^T K^T$

$\begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

Practical Challenges and Important Points

When can we apply $A = X\Lambda X^{-1}$?

- A should be a square matrix
- When A has 'n' linearly independent eigenvectors, then X^{-1} always exist.

What happens when A is Symmetric ($A^T = A$)?

- The eigenvectors of a symmetric matrix A can be chosen as ORTHONORMAL. So in this case X is orthonormal.
- For an ORTHONORMAL matrix X, the inverse is its transpose $X^{-1} = X^T$

$$A = X\Lambda X^{-1}$$
$$A = X\Lambda X^T$$

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Practical Challenges

What if A is not a square matrix?

- We cannot apply Spectral Decomposition.

Don’t Worry!!!

Singular Value Decomposition works for any Matrix.