BITS Pilani
Hyderabad Campus

Data Structures and Algorithms Design ZG519

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SESSION 3 -PLAN

Online Sessions(#)	List of Topic Title	Text/Ref Book/external resource
3	Analyzing Recursive Algorithms: Recurrence relations, Iteration Method, Substitution Method, Recursion Tree, Master Method.	T1: 1.4, 2.1

Analyzing Recursive Algorithms

- Recursive calls:-A procedure P calling itself-calls to P are for solving sub problems of smaller size.
- Recursive procedure call should always define a ***base case***.
- Base case-small enough that it can be solved directly without using recursion.

10

5

3

2

1

Analyzing Recursive Algorithms

rM (A, 5)

- Algorithm recursiveMax(A,n)

  // Input : An array A storing n>=1 integers

  //Output: The maximum element in A

  if n = 1 then

  return A[0]

  return max{ recursiveMax(A,n-1),A[n-1]}

n = 5

A[0] A[1] A[2] A[3] A[4]

A[5] = {3, 4, 1, 5, 2}

return max (5, 2) => 5

return max { rM (A, 4), 2 }

return max { rM (A, 3), 5 }

return max { rM(A,2), 1 }

return max(rM(A,1), 4)

return max(4, 5)

return max(4, 1)

return (max(3, 4))

Analyzing Recursive Algorithms

- A ***recurrence*** is an equation or inequality that describes a function in terms of its value on smaller inputs.
- ***Recurrence equation***: defines mathematical statements that the running time of a recursive algorithm must satisfy
- Analysis of ***recursiveMax***
  - T(n)-Running time of algorithm on an input size n

$$T(n) = \begin{cases} 3 \quad \text{Const} & \text{if } n=1 \\ T(n-1) + 7 & \text{otherwise} \end{cases}$$

base case

$$T(n) = T(n-1) + \text{Const}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

Solving Recurrences

Solving recurrences : Iterative Method

Analyzing Recursive Algorithms-Iterative method

- **General Plan-Iterative Method**
  - Identify the parameter to be considered based on the size of the input.
  - Identify the basic operation in the algorithm
  - Obtain the number of times the basic operation is executed.
  - Obtain an initial condition-base case
  - Obtain a recurrence relation
  - Solve the recurrence relation and obtain the order of growth and express using asymptotic notations.

Analyzing Recursive Algorithms-RecursiveMax

- Analysis of ***recursiveMax***

  T(n)-Running time of algorithm on an input size n

$$T(n) = \begin{cases} 3 & \text{if } n=1 \\ T(n-1) + 7 & \text{otherwise} \end{cases}$$

```
Algorithm recursiveMax(A,n)
// Input : An array A storing  n>=1 integers
 //Output: The maximum element in A
 if  n = 1 then
 return A[0]
return max{ recursiveMax(A,n-1),A[n-1]}
```

$$\begin{aligned} T(n) &= T(n-1) + 7 \\ &= T(n-2) + 7 + 7 \\ &= T(n-2) + 14 \\ &= T(n-3) + 21 \\ &\vdots \\ &= T(n-i) + 7i \end{aligned}$$

$\underline{T(1)}$

$n - i = 1$

$i = n - 1$

when $i = n-1$

$$\begin{aligned} T(n) &= T(n-(n-1)) + 7(n-1) \\ &= T(1) + 7n - 7 \\ &= 3 + 7n - 7 \\ &= \underline{7n-4} \text{ is } \underline{O(n)} \end{aligned}$$

Analyzing Recursive Algorithms- Example **1:-Factorial of a number**

– ***Algorithm fact(n)***

//Purpose: Computes factorial of n

//Input: A positive integer n

//Output: factorial of n

If(n=0)

return 1

return n*fact(n-1)

$$T(n) = \begin{cases} 0 & n = 0 \\ T(n-1)+1 & \text{otherwise} \end{cases}$$

Analyzing Recursive Algorithms- Example **1:-Factorial of a number**

- **Analysis**
  - Parameter to be considered –n
  - Basic operation –Multiplication
  - T(n) = 0 if n=0

    1+T(n-1) Otherwise

Time taken to compute fact(n-1)

Time to multiply n*fact(n-1)

Analyzing Recursive Algorithms- Example **1:-Factorial of a number**

innovate achieve lead

- **Solve the recurrence**

$T(n) = T(n\text{-}1) + 1$

$[T(n\text{-}2) + 1] + 1 = \mathrm{T}(n\text{-}2) + 2$ substituted $T(n\text{-}2)$ for $T(n\text{-}1)$

$[T(n\text{-}3) + 1] + 2 = T(n\text{-}3) + 3$ substituted $T(n\text{-}3)$ for $T(n\text{-}2)$

.. a pattern evolves

$T(n) = 1+T(n-1)$

$=2+T(n-2)$

$=3+T(n-3)$

$=\ldots.$

$=i+T(n-i)$

When n=0 T(0)=0,No multiplications

When i=n, T(n) $=n+T(n-n)$

$=n+0$

$=n$ $\quad$ $\underline{\mathbf{T(n) \in \Theta(n)}}$

$T(0) = 0$

$n - i = 0$

$i = n$

$= O(n)$

Analyzing Recursive Algorithms-Example **2:-Tower of hanoi**

Step 1 – Move n-1 disks from **source** to **temp**

Step 2 – Move n^{th} disk from **source** to **dest**

Step 3 – Move n-1 disks from **temp** to **dest**

Algorithm Hanoi(n, source, dest, temp)

//Input: n :number of disks

//Output :All n disks on dest

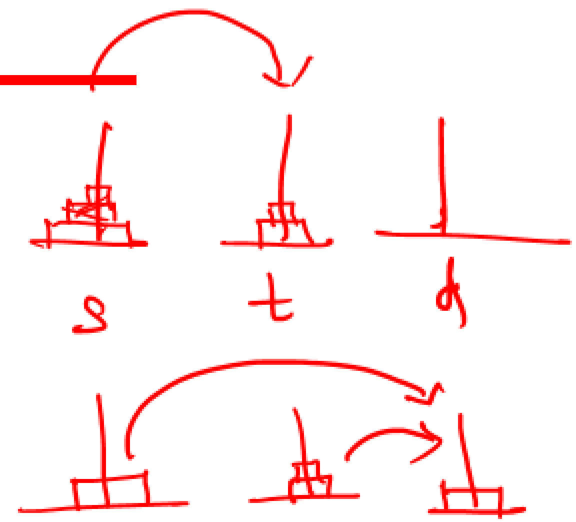
If disk = 1

move disk from source to dest

Hanoi(n - 1, source, temp, dest) // Step 1

move nth disk from source to dest // Step 2

Hanoi(n - 1, temp, dest, source) // Step 3



$T(1) = 1$

Analyzing Recursive Algorithms- Example **2:-Tower of hanoi**

1.Problem size is n, the number of discs

2. The basic operation is moving a disc from rod to another

3. Base case $M(1) = 1$

4. Recursive relation for moving n discs

$$\underset{s \to d}{M(n)} = \underset{s \to t}{M(n\text{-}1)} + \underset{s \to d}{1} + \underset{t \to d}{M(n\text{-}1)} = 2M(n\text{-}1) + 1$$

$$T(n) = \begin{cases} 1 & n = 1 \\ 2M(n-1)+1 & \text{otherwise} \end{cases}$$

Analyzing Recursive Algorithms-Example 2: Tower of hanoi

Solve using backward substitution

$M(n) = 2M(n\text{-}1) + 1$

$= 2[2M(n\text{-}2) + 1] + 1 = 2^2M(n\text{-}2) + 2+1$

$=2^2[2M(n\text{-}3) + 1] + 2+1$

$= 2^3M(n\text{-}3) + 2^2 + 2 + 1$

...

$M(n) = 2^iM(n\text{-}i) + \underbrace{2\text{^}i\text{-}1+2\text{^}i\text{-}2+....+2\text{^}3+2\text{^}2+2\text{^}1+2}$^0

$M(n) = 2^iM(n\text{-}i) + (2\text{^}i\text{-}1)/(2\text{-}1)$ It's a GP with a=1,r=2,n=i

$= 2^iM(n\text{-}i) + 2\text{^}i\text{-}1$

Analyzing Recursive Algorithms-Example 2:- Tower of hanoi

- When i=n-1

$$
\begin{aligned}
M(n) &= 2^{n-1} M(n-(n-1)) + 2^{n-1} - 1 \\
&= 2^{n-1} M(1) + 2^{n-1} - 1 \\
&= 2^{n-1} + 2^{n-1} - 1 \\
&= 2 * 2^{n-1} - 1 \\
&= 2 * (2^{n} / 2) - 1 \\
&= 2^{n} - 1
\end{aligned}
$$

$\underline{M(n) \in O(2^n)}$

- Time complexity is exponential
- More computattions even for smaller value of n
- Doesnt necessarily mean algorithm is poor
- Nature of the problem itself is computationally expensive.

Analyzing Recursive Algorithms-Example **3:Exercise**

ALGORITHM *BinRec(n)*

//Input: A positive decimal integer *n*

//Output: The number of binary digits in *n*'s binary representation

if *n* = 1 **return** 1

else return *BinRec(n/2)* + 1

Let us set up a recurrence and an initial condition for the number of additions *A(n)* made by the algorithm. The number of additions made in computing *BinRec(n/2)* is *A(n/2)*, plus one more addition is made by the algorithm to increase the returned value by 1. This leads to the recurrence

$A(n) = A(n/2) + 1$ for $n > 1$.

$A(1)=0$

Analyzing Recursive Algorithms- Example **3:Exercise**

Base condition A(1)= 0

$$A(n)=A(n/2)+1$$

The presence of $n/2$ in the function's argument makes the method of backward substitutions stumble on values of n that are not powers of 2. Therefore, the standard approach to solving such a recurrence is to solve it only for $n = 2^k$ and then take advantage of the theorem called the ***smoothness rule,*** which claims that under very broad assumptions the order of growth observed for $n = 2^k$ gives a correct answer about the order of growth for all values of n.

Solving recurrences : Recursion Tree

Solving recurrences : Recursion Tree

```
Void Test (int n)
{
        if(n>1)
        {
                for (i=1;i<n;i++)
                {
                        stmt;
                }
                Test(n/2);
                Test(n/2);
        }
}
```

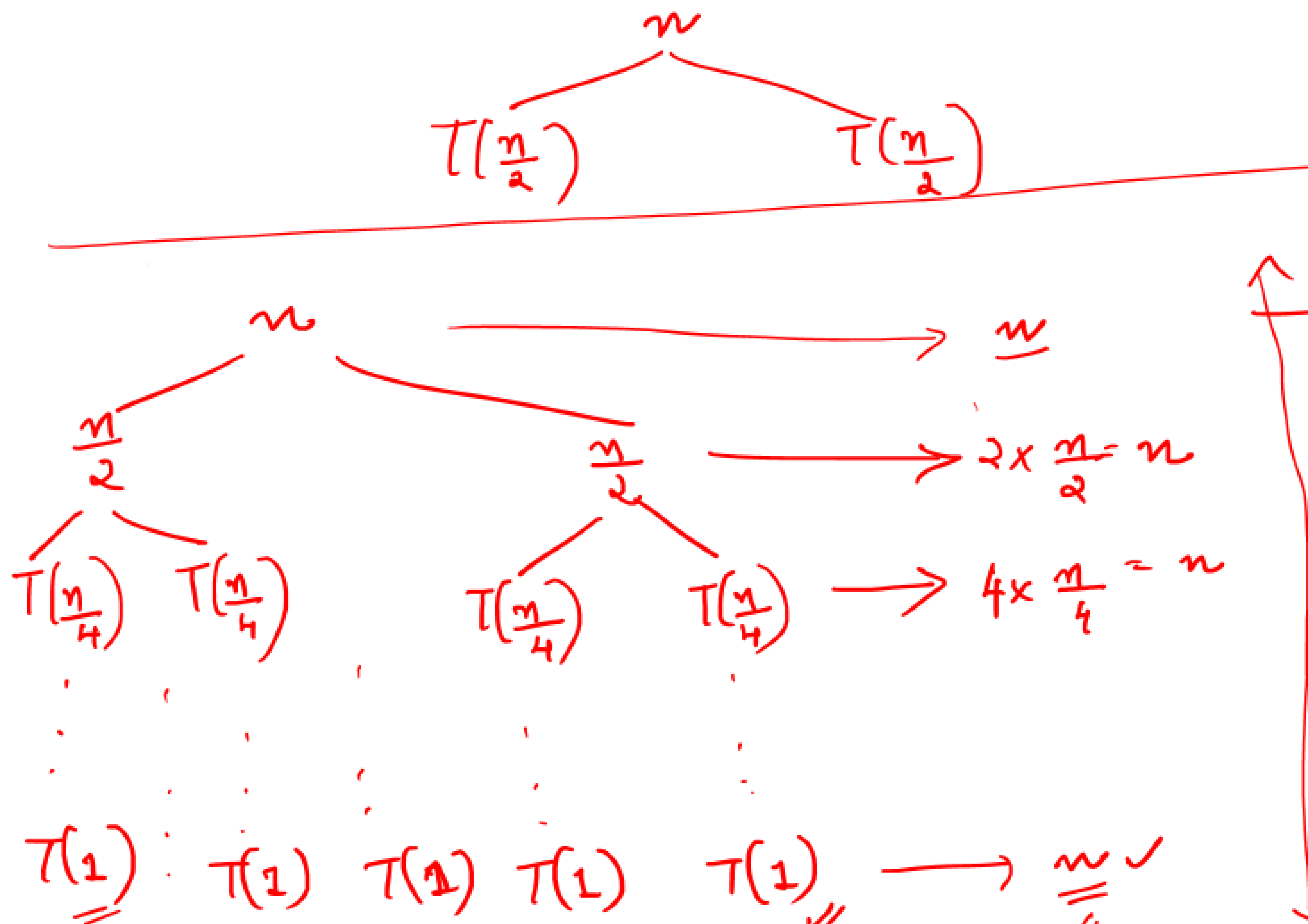
$$T(n) = \begin{cases} 0, & n=1 \\ 2T(n/2)+n & n>1 \end{cases}$$

$$T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + n$$

$$= 2T\left(\frac{n}{2}\right) + n$$

divide and Conquer

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$



$n \times \log_2 n$

$T(n) = n \log n$

level 0

level 1

h

Number of nodes at level $0 = 2^0 = 1$

$1 = 2^1 = 2$

$2 = 2^2 = 4$

$h = 2^h$

Size of subproblem at level $0 = n$

$1 = \frac{n}{2}$

$2 = n/2^2$

$3 = n/2^3$

$h = \frac{n}{2^h}$

At height h, the size becomes 1

ie $\frac{n}{2^h} = 1$

$n = 2^h$

$h = \log_2 n$

No: of nodes at level

$h = 2^h$

$= 2^{\log_2 n}$

$= n^{\log_2 2}$

$= n$

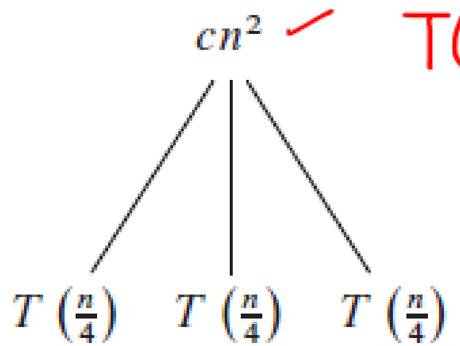
Solving Recurrences-Recursion Tree

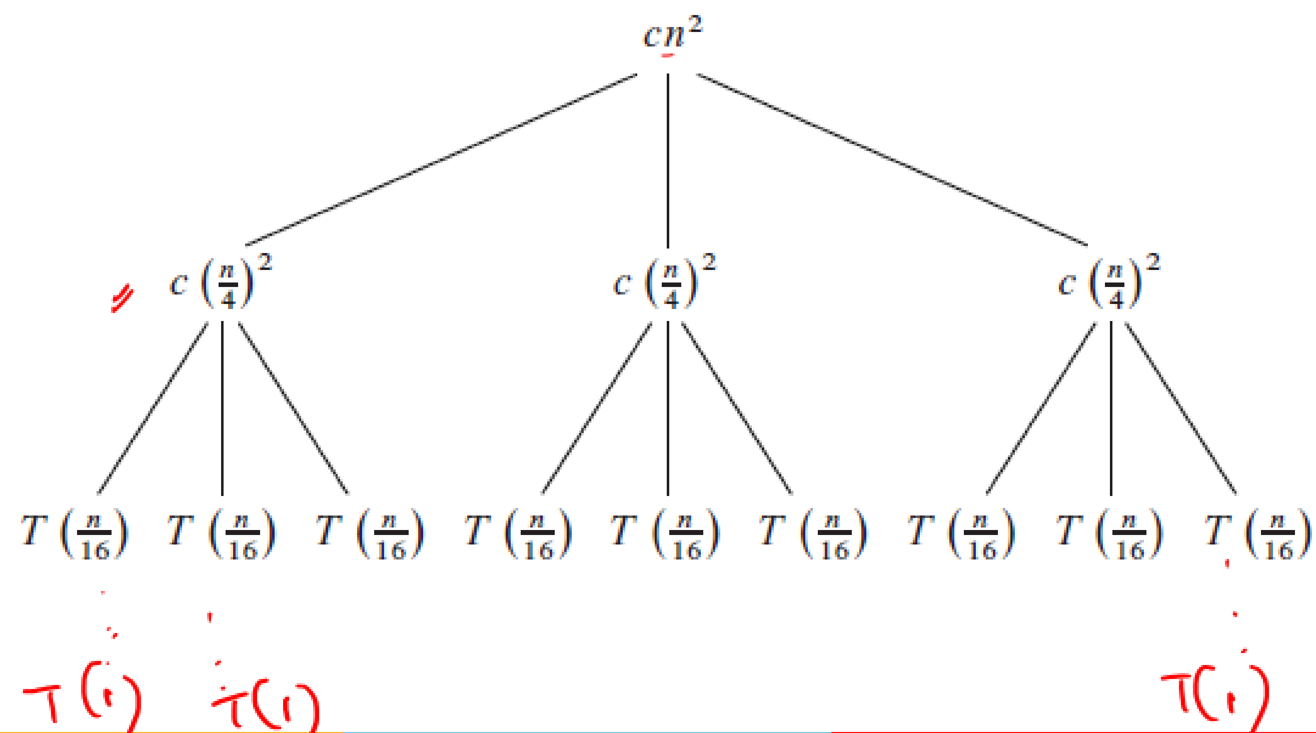
$O(n^2)$

$O(n^2)$

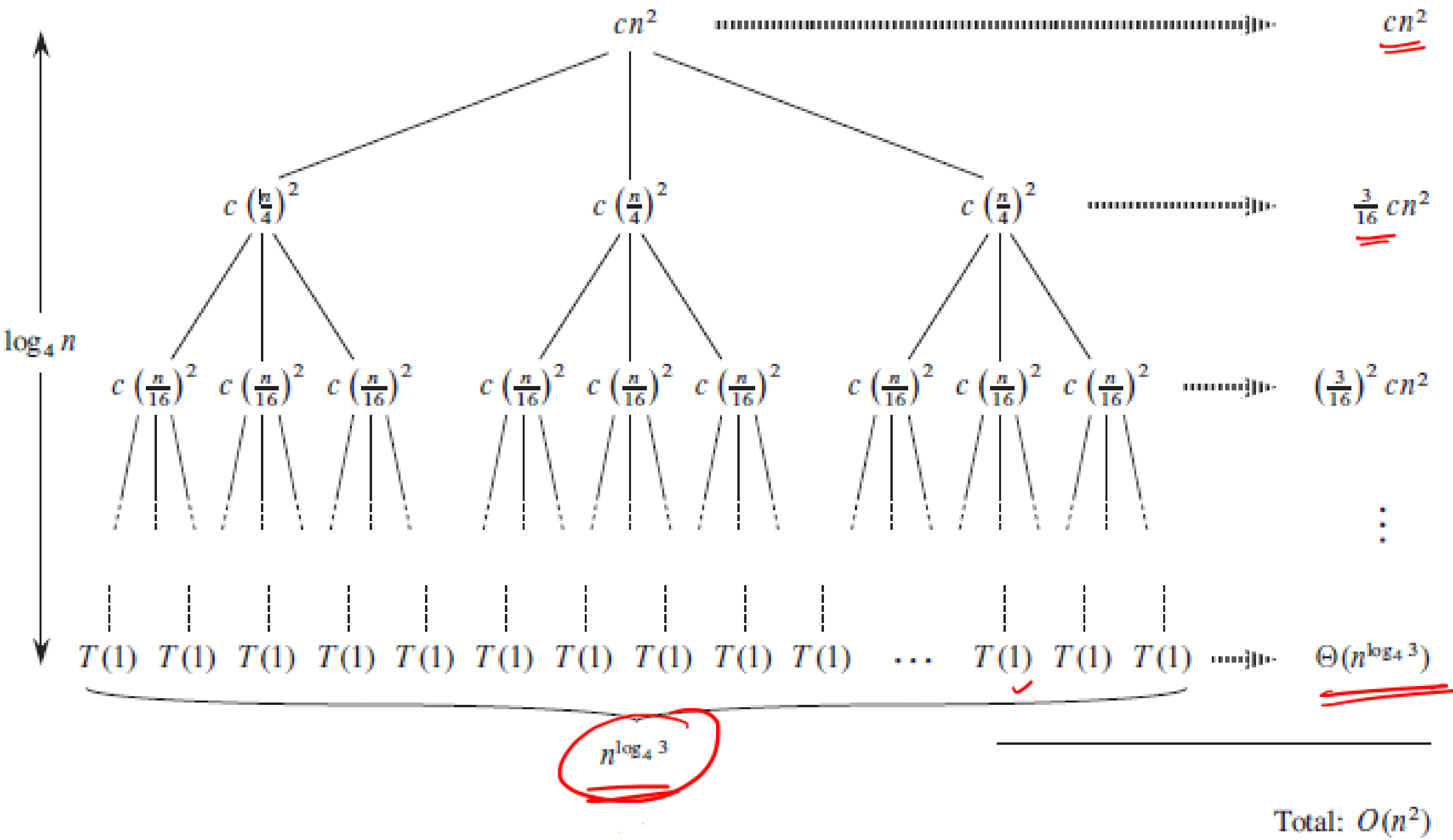
- Solve $T(n) = 3T(n/4) + cn^2$.

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$





Solving Recurrences-Recursion Tree



Number of nodes at level 0 $= 3^0 = 1$

$1 = 3^1 = 3$

$2 = 3^2 = 9$

$h = 3^h$

Size of subproblem at level 0 $= n$

$1 = \frac{n}{4}$

$2 = \frac{n}{4^2}$

$h = \frac{n}{4^h}$

No: of nodes at level $h = 3^h$

$= 3^{\log_4 n}$

$= n^{\log_4 3}$

$= n^{\log_b a}$

At height h, the size of the problem becomes 1

$\frac{n}{4^h} = 1$

$n = 4^h$

$h = \log_4 n = \log_b n$

Solving Recurrences-Recursion Tree

$$
\begin{aligned}
T(n) &= cn^2 + \frac{3}{16}cn^2 + \left(\frac{3}{16}\right)^2 cn^2 + \cdots + \left(\frac{3}{16}\right)^{\log_4 n-1} cn^2 + \Theta(n^{\log_4 3}) \\
&= \sum_{i=0}^{\log_4 n-1} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{(3/16)^{\log_4 n} - 1}{(3/16) - 1} cn^2 + \Theta(n^{\log_4 3})
\end{aligned}
$$

Geometric or exponential series

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1}$$

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{\log_4 n-1} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&< \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{1}{1-(3/16)} cn^2 + \Theta(n^{\log_4 3}) \\
&= \frac{16}{13} cn^2 + \Theta(n^{\log_4 3}) \\
&= O(n^2) .
\end{aligned}
$$

Solving recurrences: Master method
Ref: Textbook R2

- The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n)\,,$$

$\frac{n}{b}$

- where $a \geq 1$, $b > 1$, and f is asymptotically positive.
- (f(n)>0 for n>=n0)

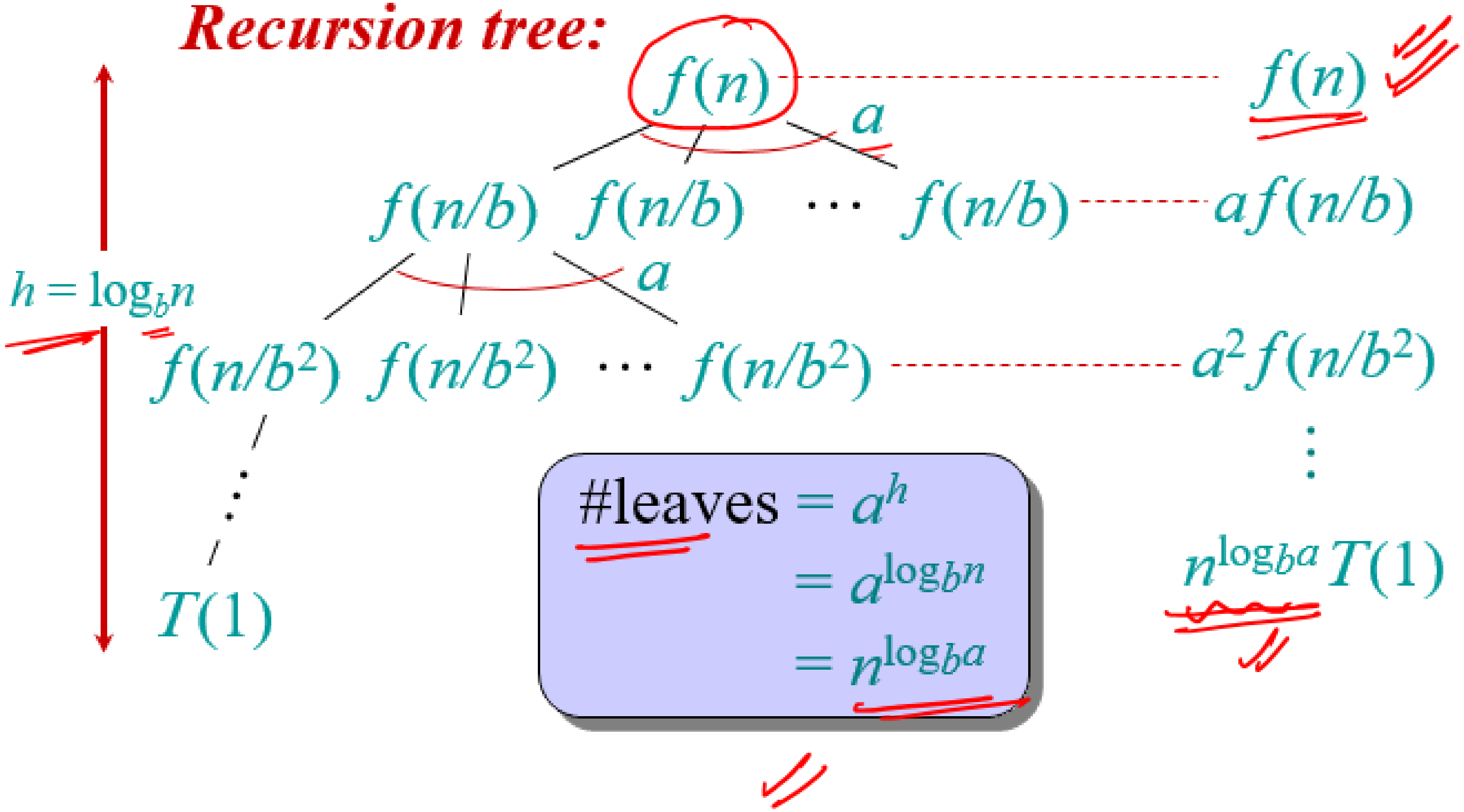
'a'

divide and Conquer

Idea of Master theorem

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

Recursion tree:



Solving recurrences: Master method
Ref: Textbook R2

$T(n) = aT(\frac{n}{b}) + f(n)$

Case 1:

If $\boxed{f(n)=O(n^{\log_b a-\varepsilon})}$*, for some constant* $\varepsilon > 0$*, then* $T(n)= \Theta(n^{\log_b a})$

f(n) grows polynomially slower than $n^{\log_b a}$

$n \quad n^2$

$\rightarrow \frac{n}{1} \quad \frac{n\log n}{1}$

Case 2:

If $f(n)=\Theta(n^{\log_b a})$*, then* $T(n)= \Theta(n^{\log_b a} \log n)$ $= \Theta(f(n) \log n)$

f(n) and $n^{\log_b a}$ grows at similar rates

Case 3:

If $f(n)=\Omega(n^{\log_b a+\varepsilon})$ *for some constant* $\varepsilon > 0$*, and if* $\boxed{af(n/b) \le cf(n)}$ *for some constant* $c < 1$ *and all sufficiently large n, then* $T(n)= \Theta(f(n))$

f(n) grows polynomially faster than $n^{\log_b a}$

Solving recurrences: Master method
Ref: Textbook R2

Case 2 : (Generalisation):

If there is a constant k >= 0, such that f(n) is $\Theta(n^{\log_b a} \log^k n)$, then T (n) is $\Theta(n^{\log_b a} \log^{k+1} n)$

Example:

$T(n)= 2T(n/2) + n \log n$

$a=2,b=2$ $f(n)=\boxed{n\log n}$

$n^{\log_b a} = \boxed{n}$

f(n) is asymptotically larger than $n^{\log_b a}$, But it is not polynomially larger.

So no standard case of master theorem applies.

It belongs to case 2 general case.

$f(n) = \Theta(n^{\log_b a} \log^k n) = \Theta(n^{\log_b a} \log^1 n)$

So $T(n)= \Theta(n \log^2 n)$

$n \log^1 n$

$\theta(n \log^2 n)$

Solving recurrences: Master method

$T(n) = aT(\frac{n}{b}) + f(n)$

$n^{\log_b a}$

$f(n)$

Example 1 : $T(n) = 2T(n/2) + n$

Sol:

Extract $a=2$, $b=2$ and $f(n) = n$

Determine $n^{\log_b a} = n^{\log_2 2} = n^1 = n$

Compare $n^{\log_b a} = n$

$f(n) = n$

Thus case 2: evenly distributed because

$f(n) = \theta(n)$

$T(n) = \theta\ (n^{\log_b a} \log(n))$

$= \theta\ (n^1 \log(n))$

$= \theta\ (n\log n)$

Solving recurrences: Master method

Example 2 : $T(n) = \overset{a}{9}\,T(n/\overset{b}{3}) + \overset{f(n)}{n}$

$n^2 \neq n$

a= 9 b=3 and f(n) =n

Determine $n^{\log_b a} = n^{\log_3 9} = n^2$

Compare: $n^{\log_b a} = n^2$

$f(n) = n$

Thus $\boxed{\text{case1}}$; (express f(n) in terms of $n^{\log_b a}$) because $f(n) = O(n^{2-\varepsilon})$

$T(n) = \theta\,(n^{\log_b a}) = \theta(n^2)$

Solving recurrences: Master method

Example 3 : $T(n) = 3T(n/4) + n\log n$

$f(n) = n\log n$

$f\left(\frac{n}{b}\right) = \frac{n}{b}\log\left(\frac{n}{b}\right)$

a= 3,

b=4,

f(n) = nlogn

Determine; $n^{\log_b a} = n^{\log_4 3}$ $\log_4 3 < 1$

Compare: $n^{\log_b a}$ and f(n)

$n^{\log_4 3} <= n\log n$ f(n) is asymptotically and polynomially larger

Thus case 3, but we have to check the reqularity condition!

The following should be true:

$af(n/b) <= cf(n)$ where $c<1$

$a(n/b)\log(n/b) <= cf(n)$

$=> 3(n/4)\log(n/4) <= c\, n\log n$

$3/4\, n\log(n/4) <= c\, .n\log n,$

this is true for c=3/4 Hence. $T(n) = \theta(n\log(n))$

Case Study: Analyzing Algorithms

- **Computing the prefix averages of a sequence of numbers.**

The ***i***-th prefix average of an array ***X*** is average of the first

(***i*** + 1) elements of ***X*****:**

$\boldsymbol{A}[\boldsymbol{i}] = (\boldsymbol{X}[0] + \boldsymbol{X}[1] + \ldots + \boldsymbol{X}[\boldsymbol{i}])/(i+1)$

- **Applications**
  - **Runtime analysis example:**

    Two algorithms for prefix averages

Case Study: Analyzing Algorithms

- The following algorithm computes prefix averages in quadratic time by applying the definition

Algorithm *prefixAverages1*(X, n)	
Input array X of n integers	
Output array A of prefix averages of X	#operations
$A \leftarrow$ new array of n integers	n
for $i \leftarrow 0$ **to** $n - 1$ **do**	n
$\quad s \leftarrow X[0]$	n
$\quad$ **for** $j \leftarrow 1$ **to** i **do**	$1 + 2 + \ldots + (n - 1)$
$\quad\quad s \leftarrow s + X[j]$	$1 + 2 + \ldots + (n - 1)$
$\quad A[i] \leftarrow s / (i + 1)$	n
return A	1

Algorithm ***prefixAverages1*** runs in $\boldsymbol{O}(n^2)$ time

Case Study: Analyzing Algorithms

- The following algorithm computes prefix averages in linear time by keeping a running sum

Algorithm ***prefixAverages2*(*X*, *n*)**
  Input array X of n integers
  Output array A of prefix averages of X

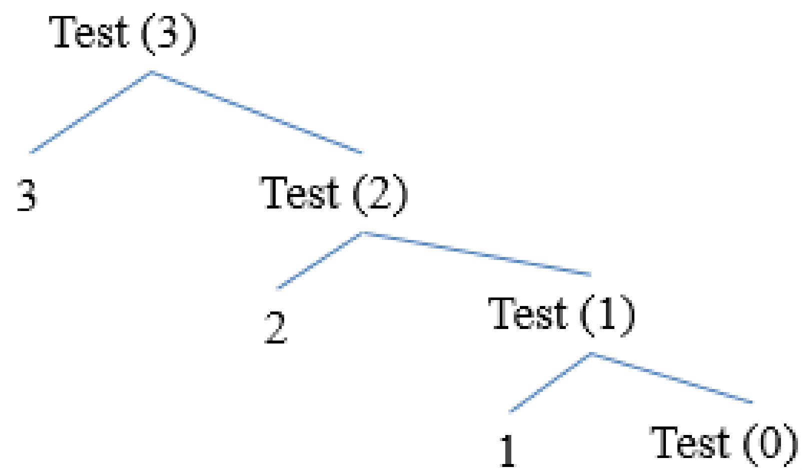
	#operations
$A \leftarrow$ new array of n integers	n
$s \leftarrow 0$	1
for $i \leftarrow 0$ **to** $n - 1$ **do**	n
$\quad s \leftarrow s + X[i]$	n
$\quad A[i] \leftarrow s / (i + 1)$	n
return A	1

Algorithm ***prefixAverages2*** runs in $O(n)$ time

Homework Problems

Solving recurrences : Iterative Method

```
void test(int n)
{
        if(n>0)
        {
        printf("%d",n);
        test(n-1);
        }
}
```



$$T(n) = \begin{cases} 0, & n = 0 \\ T(n-1) + 1 & n > 0 \end{cases}$$

Solving recurrences : Iterative Method

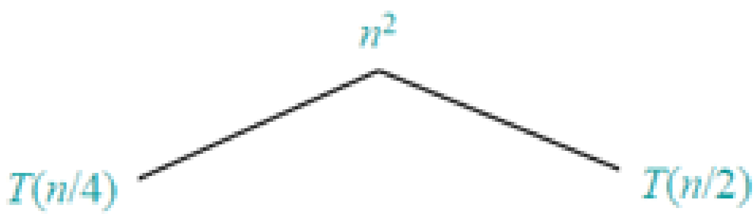
```
Void Test (int n)
{
        if(n>1)
        {
                for (i=1;i<n;i++)
                {
                        stmt;
                }
                Test(n/2);
                Test(n/2);
        }
}
```

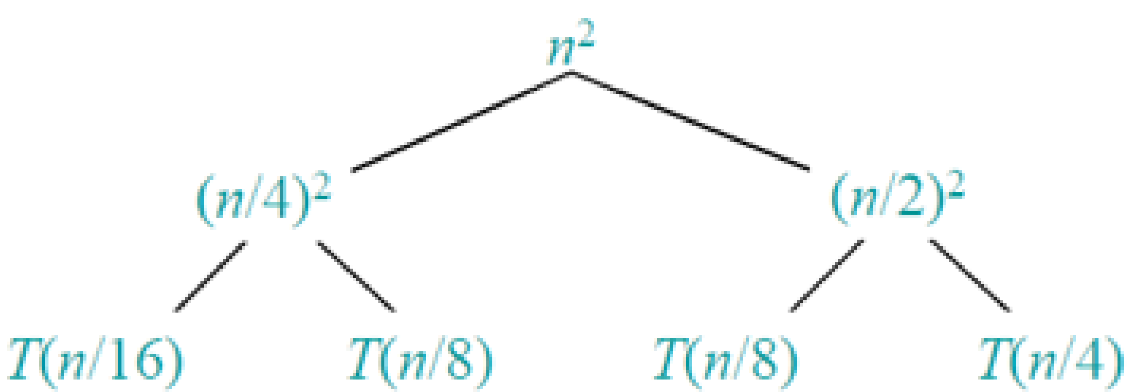
$$T(n) = \begin{cases} 0, & n=1 \\ 2T(n/2)+n & n>1 \end{cases}$$

Solving Recurrences-Recursion Tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

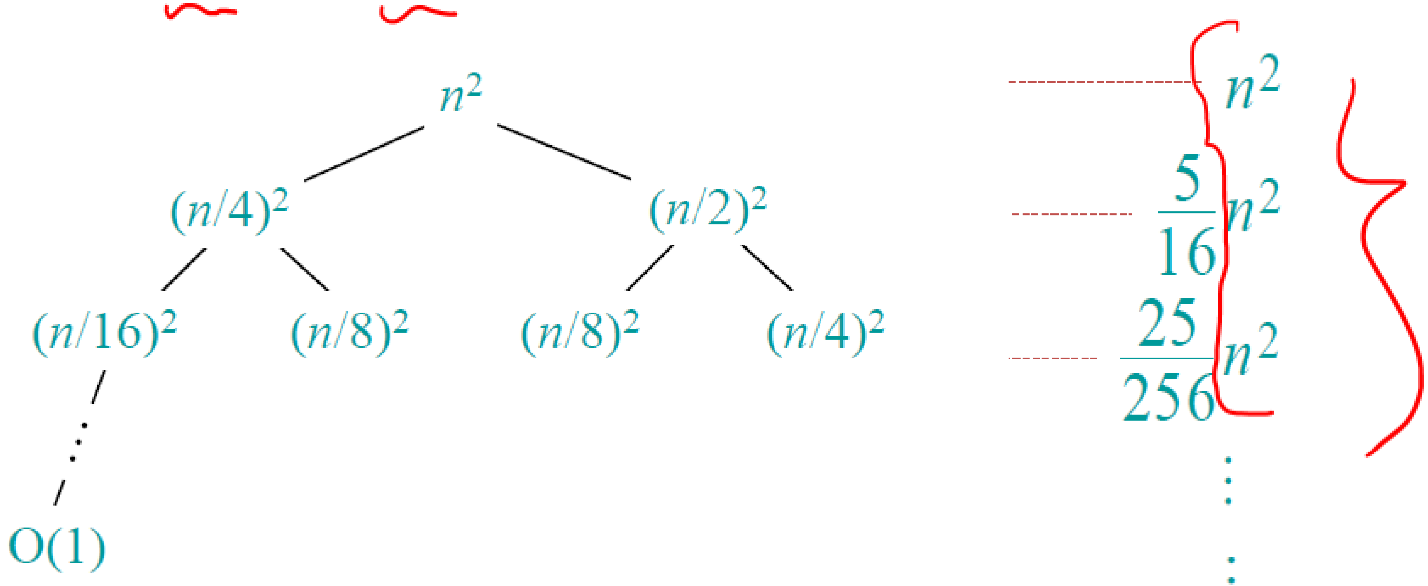
$T(n)$





Solving Recurrences-Recursion Tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



$$\text{Total} = n^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \cdots\right)$$

$$= O(n^2)$$ *geometric series*

Master method Problems

- $T(n)=9T(n/3)+n$
- $T(n)=T(2n/3)+1$
- $T(n)=3T(n/4)+n\log n$
- $T(n)=2T(n/2)+ n \lg n$
- $T(n)=8T(n/2)+\Theta(n^2)$

$a = 1$

$b = 3/2$

$f(n) = 1$

$n^{\log_b a} = n^{\log_{3/2} 1} = n^0 = 1$

Case 2 applies

$\Theta(n^{\log_b a} \log n)$

$= \Theta(\log n)$

$f(n) = n\log n$

$n^{\log_b a} = n^{\log_2 2}$

$= n$

$f(n)$ is $\Theta(n^{\log_b a} \log^k n)$

THANK YOU!

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