BITS Pilani
Hyderabad Campus

Data Structures and Algorithms Design

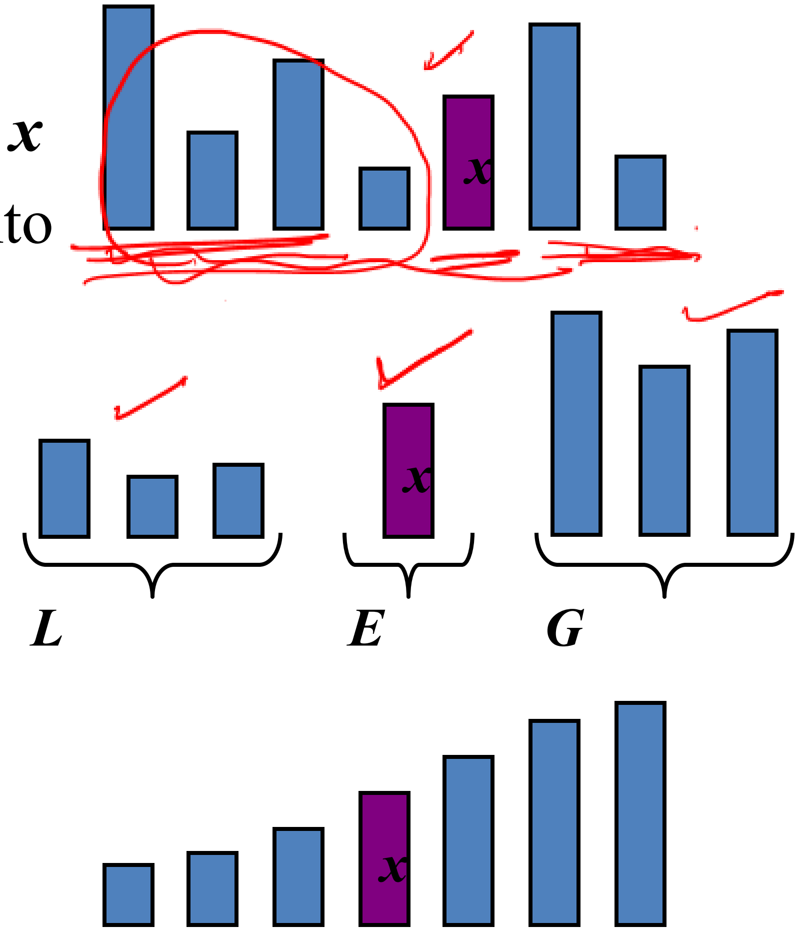
Febin.A.Vahab
2019-20

ONLINE SESSION 11 -PLAN

Sessions(#)	List of Topic Title	Text/Ref Book/external resource
12	Quick Sort Algorithm.	**T1: 5.2, 4.1, 4.3**

Quick-Sort

- Quick-sort is a randomized sorting algorithm based on the divide-and-conquer paradigm:
  - Divide: pick a random element x
  - (called pivot) and partition S into
    - L elements less than x
    - E elements equal x
    - G elements greater than x
  - Recur: sort L and G
  - Conquer: join L, E and G



Quicksort

innovate achieve lead

low, high, pivot

0	1	2	3	4	5	6	7	8
44	75	23	43	55	12	64	77	33
	i							j

Is low ≤ high ? Yes

i ← low + 1 ← 1

j ← high ← 8

pivot ← a[low] ← 44

a) a[i] ≤ pivot ? / pivot > a[i] ?

44 > 75 ? No

b) a[j] > pivot ?

33 > 44 ? No

c) Is i ≤ j ? 1 ≤ 8 ? Yes exchange a[i] and a[j]

75 ↔ 33

0	1	2	3	4	5	6	7	8
44	33	23	43	55	12	64	77	75
	~~i~~	~~i~~	~~i~~	i	j	~~j~~	~~j~~	~~j~~

a) pivot > a[i] ?

44 > 33 Yes

$i \leftarrow i{+}{+} \leftarrow 2$

pivot > a[i]

44 > 23 Yes

$i \leftarrow i+1 \leftarrow 3$

pivot > a[i]

44 > 43 Yes

$i \leftarrow i+1 \leftarrow 4$

pivot > a[i]

44 > 55 No

b) a[j] > pivot ?

75 > 44 ? Yes

$j{-}{-} \leftarrow 7$

a[j] > pivot ?

77 > 44 ? Yes

$j{-}{-} \leftarrow 6$

a[j] > pivot ?

64 > 44 ? Yes

$j{-}{-} \leftarrow 5$

a[j] > pivot ?

12 > 44 ? No

c) Is $i \leq j$? $4 \leq 5$? Yes

$55 \leftrightarrow 12$ //

innovate achieve lead

0	1	2	3	4	5	6	7	8
44	33	23	43	12	55	64	77	75
				i j	i j			

a) pivot > a[i] ?

44 > 12 ? Yes

$i \leftarrow i+1 = 5$

pivot > a[i] ?

44 > 55 ? No

b) a[j] > pivot ?

55 > 44 ? Yes

$j-- \leftarrow 4$

a[j] > pivot ?

12 > 44 ? No

c) Is $i \le j$? $5 \le 4$? No

Exchange a[j] and pivot

$12 \leftrightarrow \underline{44}$

$\underbrace{12 \quad 33 \quad 23 \quad 43}$ $\boxed{44}$ $\underbrace{55 \quad 64 \quad 77 \quad 75}$

77

Partition

- We partition an input sequence as follows:
  - We remove, in turn, each element y from S and
  - We insert y into L, E or G, depending on the result of the comparison with the pivot x
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes $O(1)$ time
- Thus, the partition step of quick-sort takes $O(n)$ time

Partition

Algorithm *partition*(*S*, *p*)
  Input sequence *S*, position *p* of pivot
  Output subsequences *L*, *E*, *G* of the
    elements of *S* less than, equal to,
    or greater than the pivot, resp.
  $L, E, G \leftarrow$ empty sequences
  $x \leftarrow S.remove(p)$
  while $\neg S.isEmpty()$
    $y \leftarrow S.remove(S.first())$
    if $y < x$
      $L.insertLast(y)$
    else if $y = x$
      $E.insertLast(y)$
    else { $y > x$ }
      $G.insertLast(y)$
  return *L*, *E*, *G*

Quick-Sort

innovate achieve lead

Algorithm inPlaceQuickSort(S, a, b):

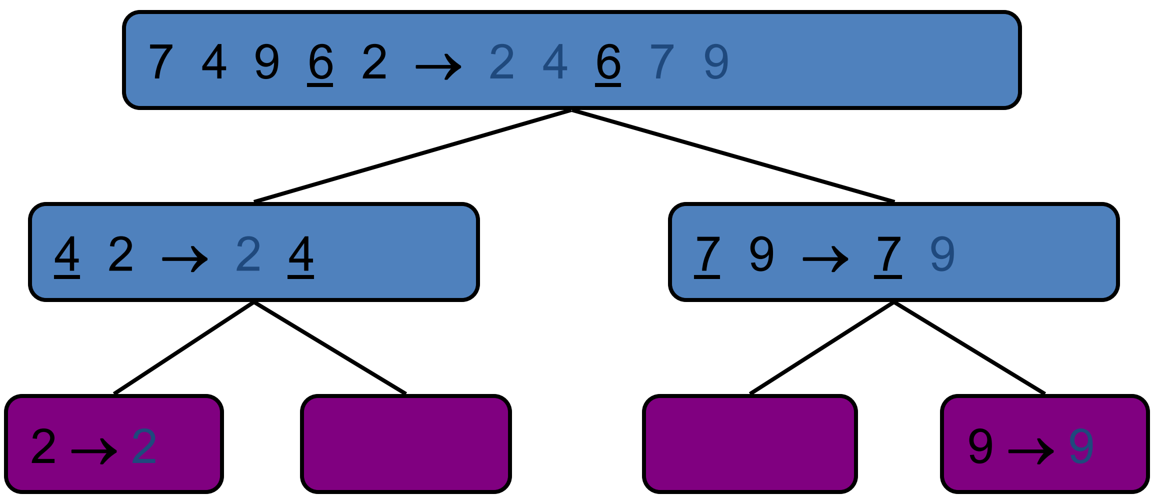
Input: Sequence S of distinct elements; integers a and b

Output: Sequence S with elements originally from ranks from a to b, inclusive, sorted in nondecreasing order from ranks a to b

```
if a ≥ b then return            {empty subrange}
p ← S.elemAtRank(b)             {pivot}
l ← a            {will scan rightward}
r ← b − 1            {will scan leftward}
while l ≤ r do
    {find an element larger than the pivot}
    while l ≤ r and S.elemAtRank(l) ≤ p do
        l ← l + 1
    {find an element smaller than the pivot}
    while r ≥ l and S.elemAtRank(r) ≥ p do
        r ← r − 1
    if l < r then
        S.swapElements(S.atRank(l), S.atRank(r))
{put the pivot into its final place}
S.swapElements(S.atRank(l), S.atRank(b))
{recursive calls}
inPlaceQuickSort(S, a, l − 1)
inPlaceQuickSort(S, l + 1, b)
```

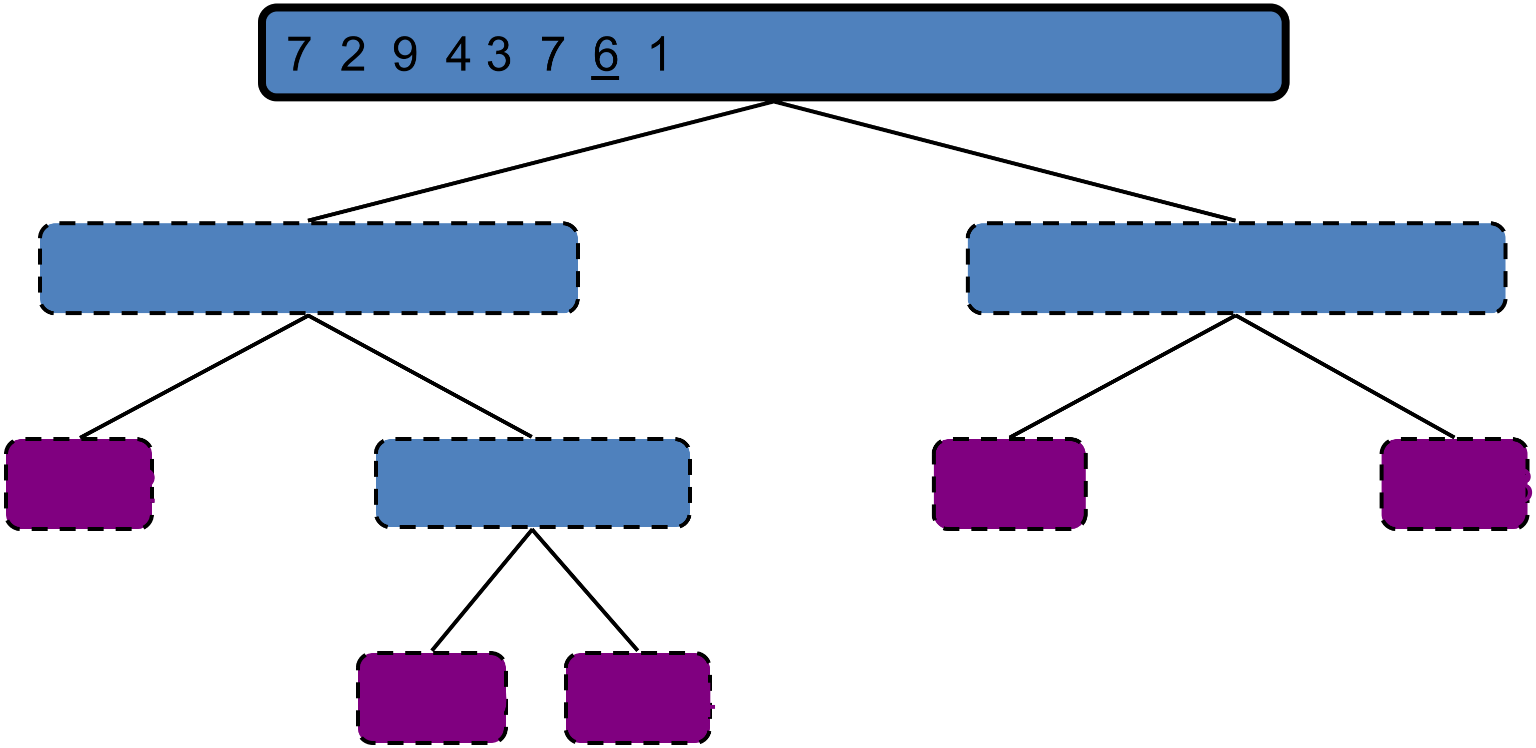
Quick-Sort Tree

- An execution of quick-sort is depicted by a binary tree
  - Each node represents a recursive call of quick-sort and stores
    - Unsorted sequence before the execution and its pivot
    - Sorted sequence at the end of the execution
  - The root is the initial call
  - The leaves are calls on subsequences of size 0 or 1



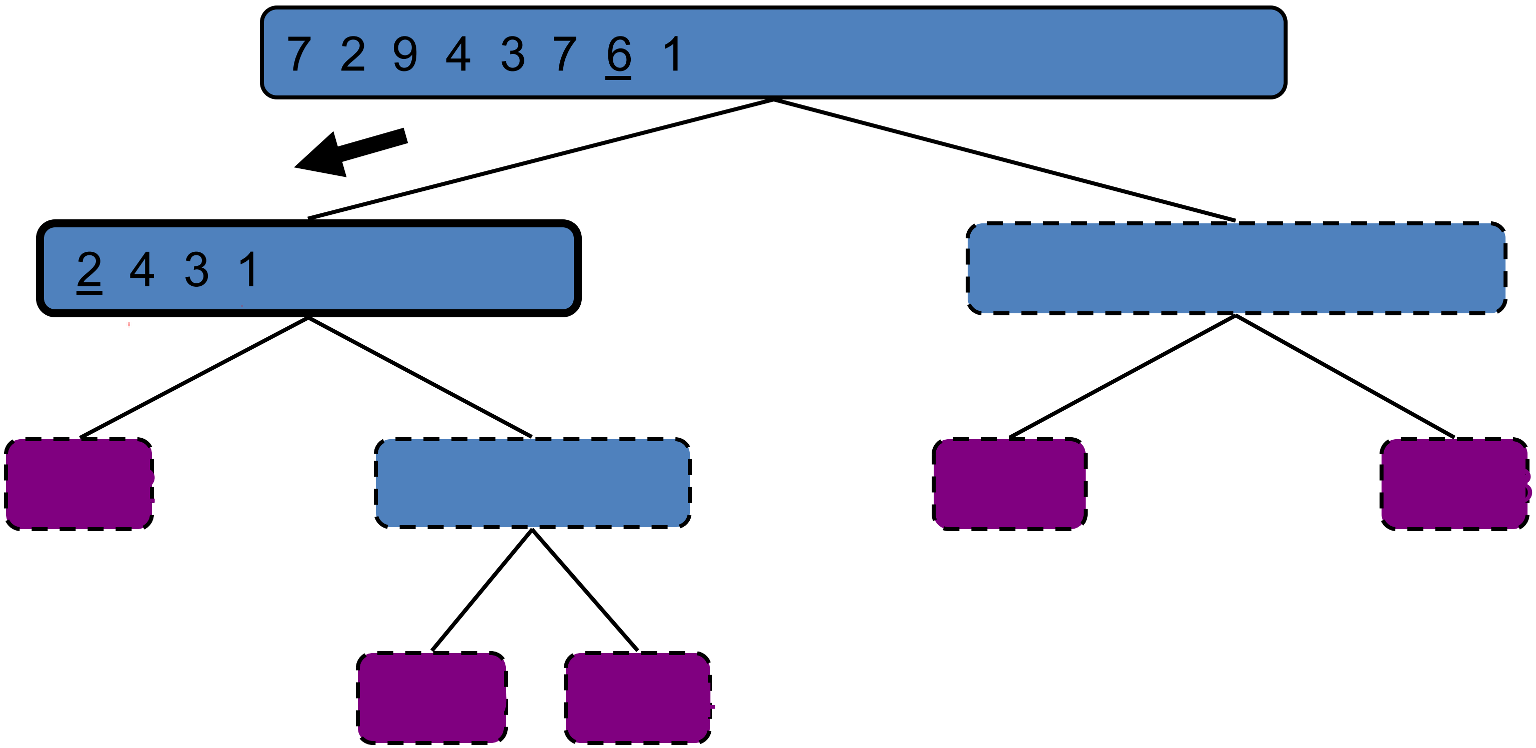
Execution Example

- Pivot selection



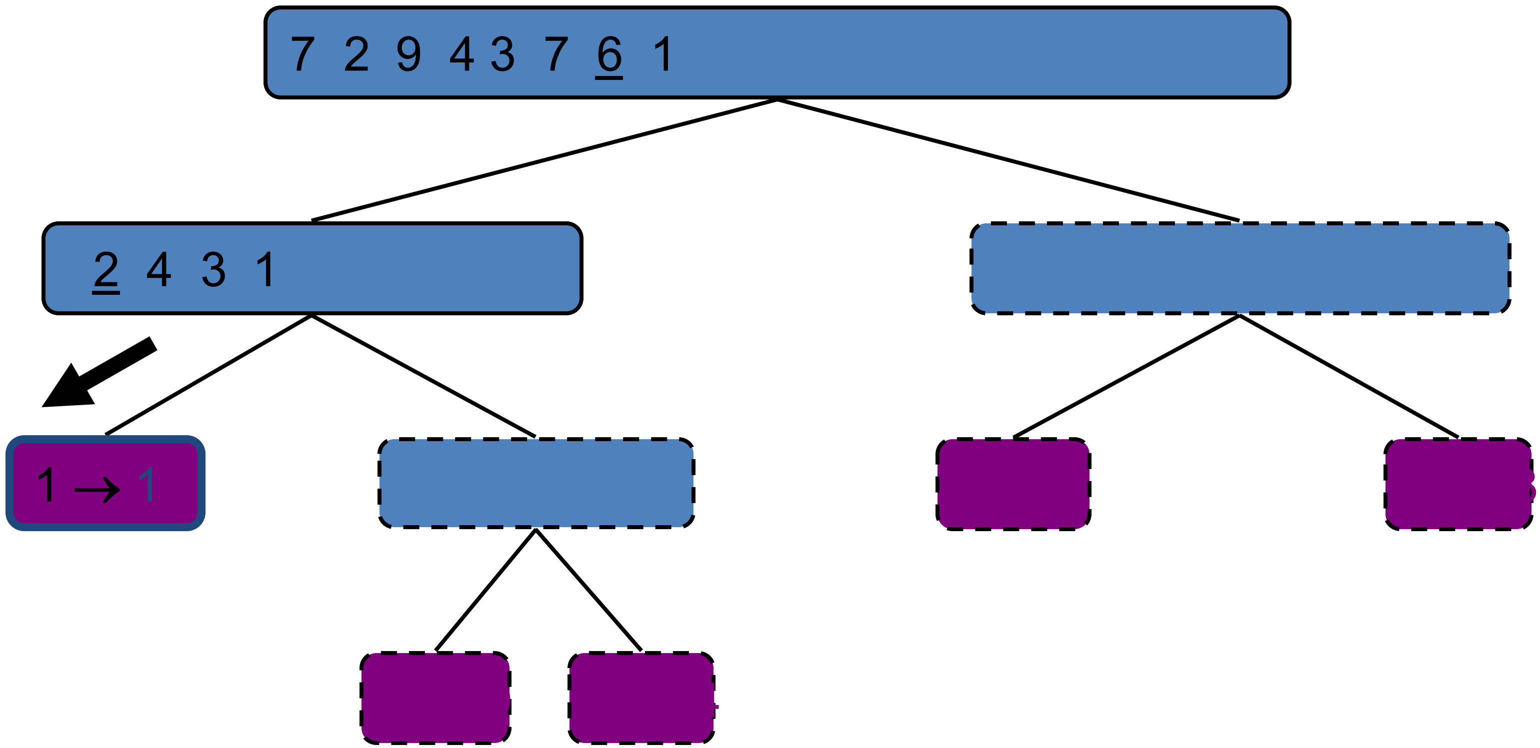
Execution Example (cont.)

- Partition, recursive call, pivot selection



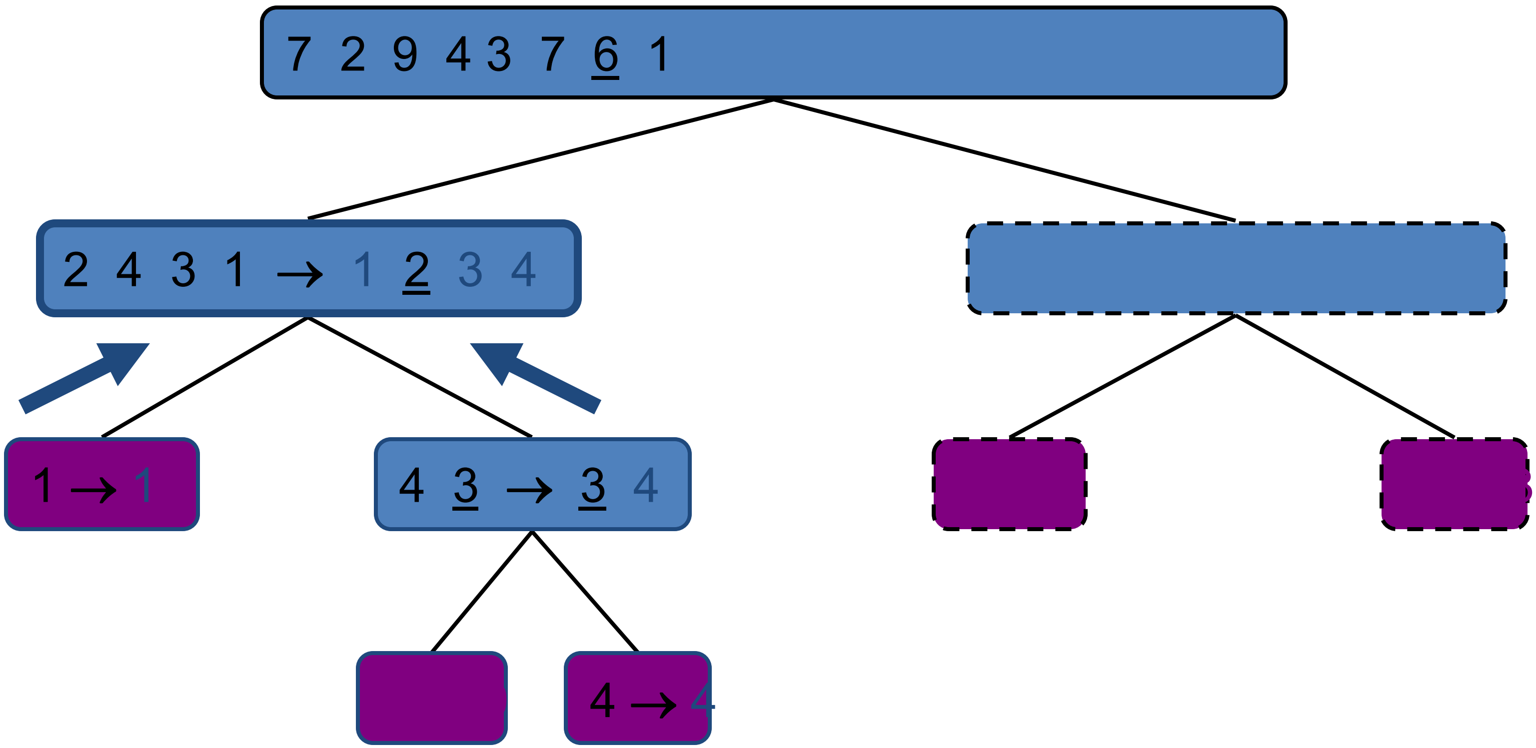
Execution Example (cont.)

- Partition, recursive call, base case



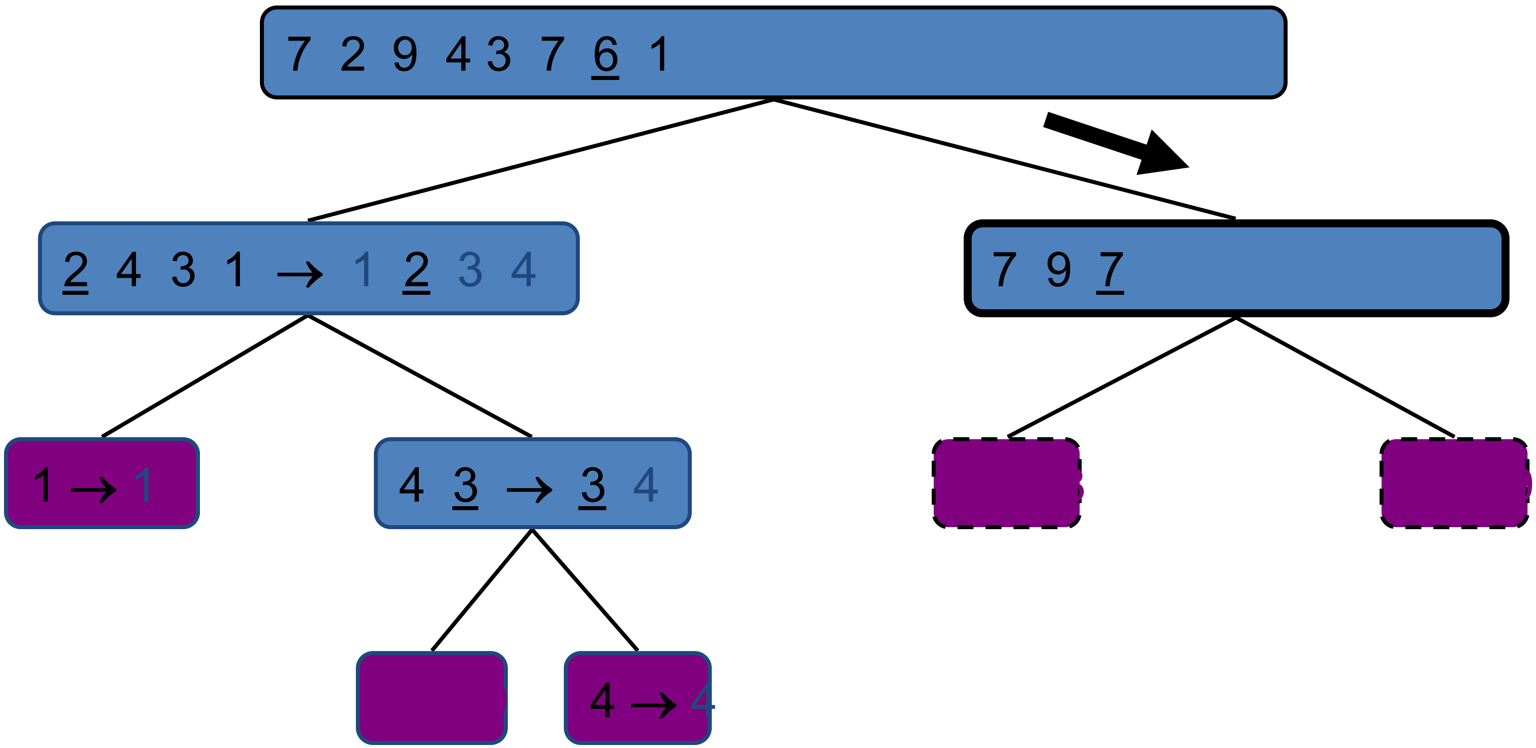
Execution Example (cont.)

- Recursive call, …, base case, join



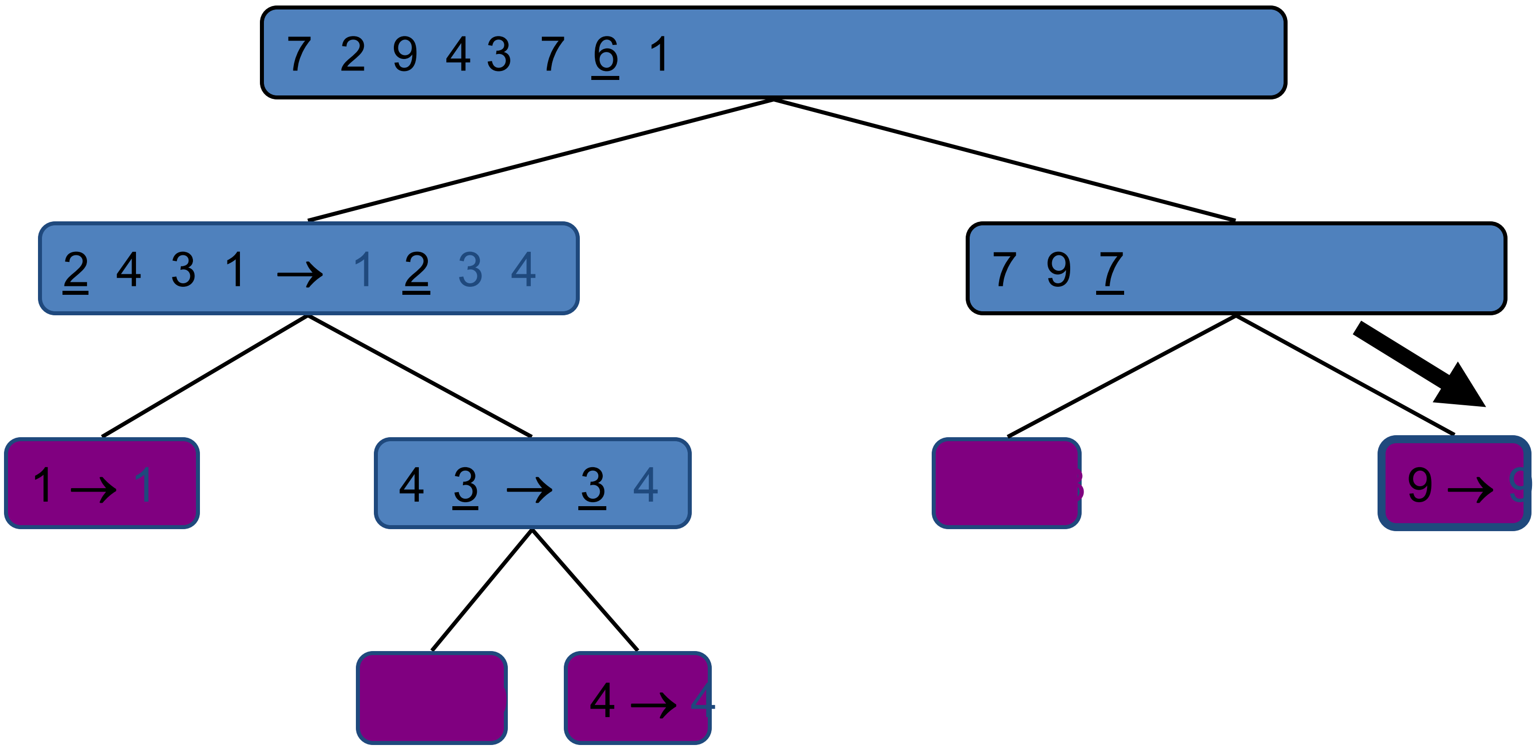
Execution Example (cont.)

- Recursive call, pivot selection



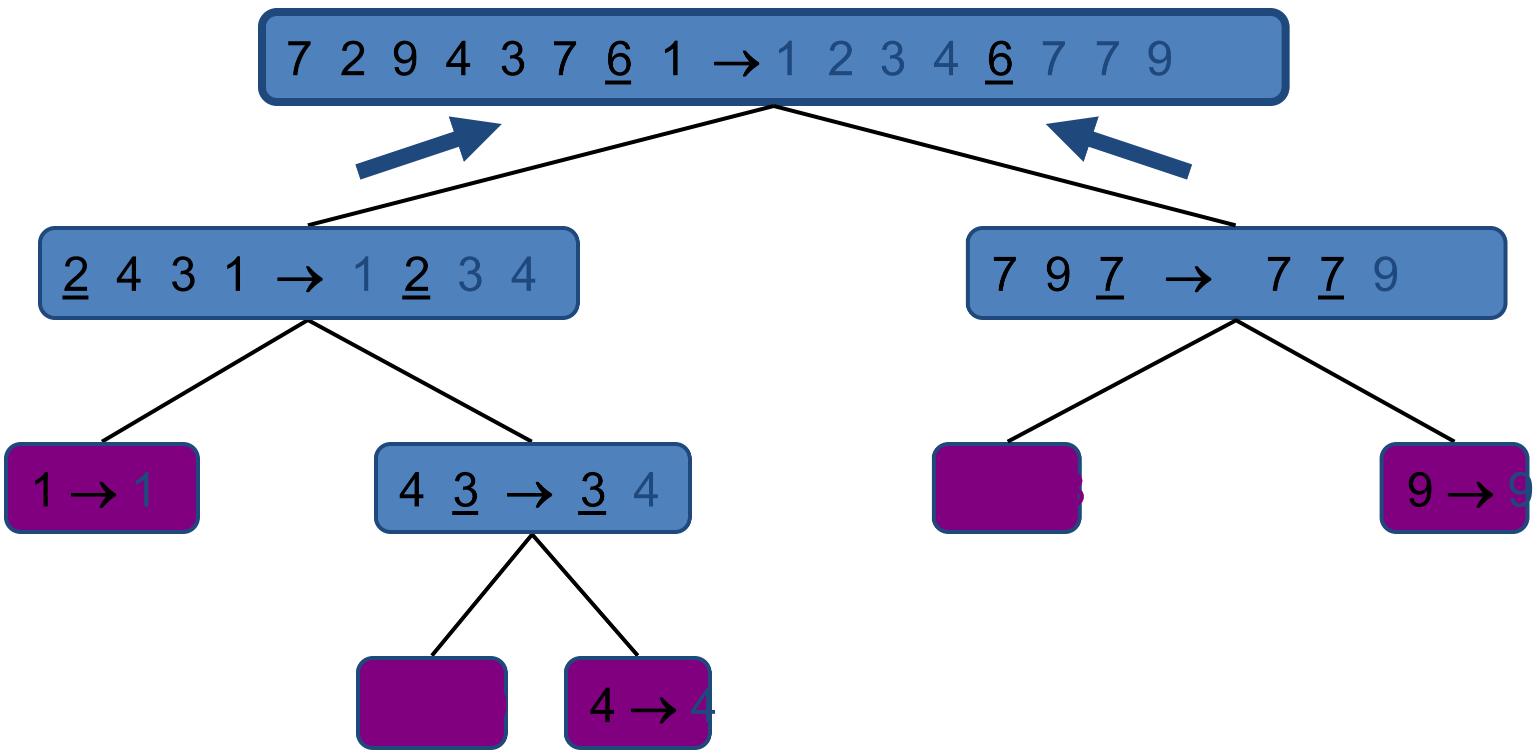
Execution Example (cont.)

- Partition, …, recursive call, base case



Execution Example (cont.)

- Join, join



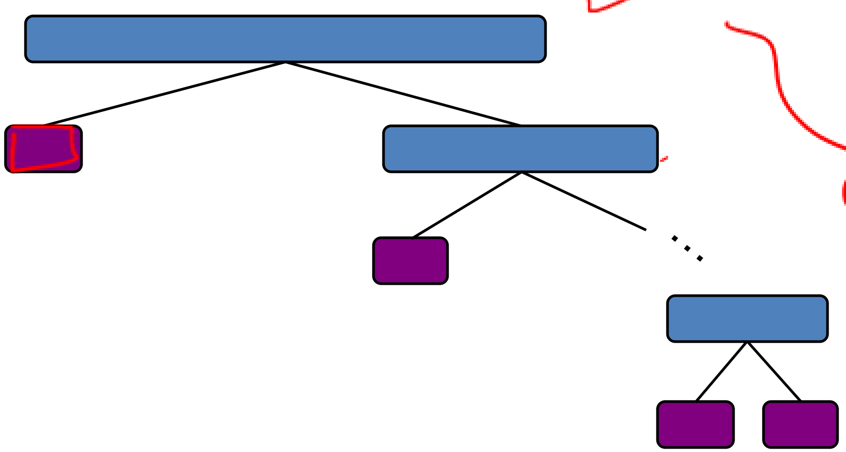
Worst-case Running Time

- The worst case for quick-sort occurs when the pivot is the unique minimum or maximum element
- One of $\boldsymbol{L}$ and $\boldsymbol{G}$ has size $\boldsymbol{n} - 1$ and the other has size 0
- The running time is proportional to the sum

$$\boldsymbol{n} + (\boldsymbol{n} - 1) + \ldots + 2 + 1$$

- Thus, the worst-case running time of quick-sort is $\boldsymbol{O}(\boldsymbol{n}^2)$

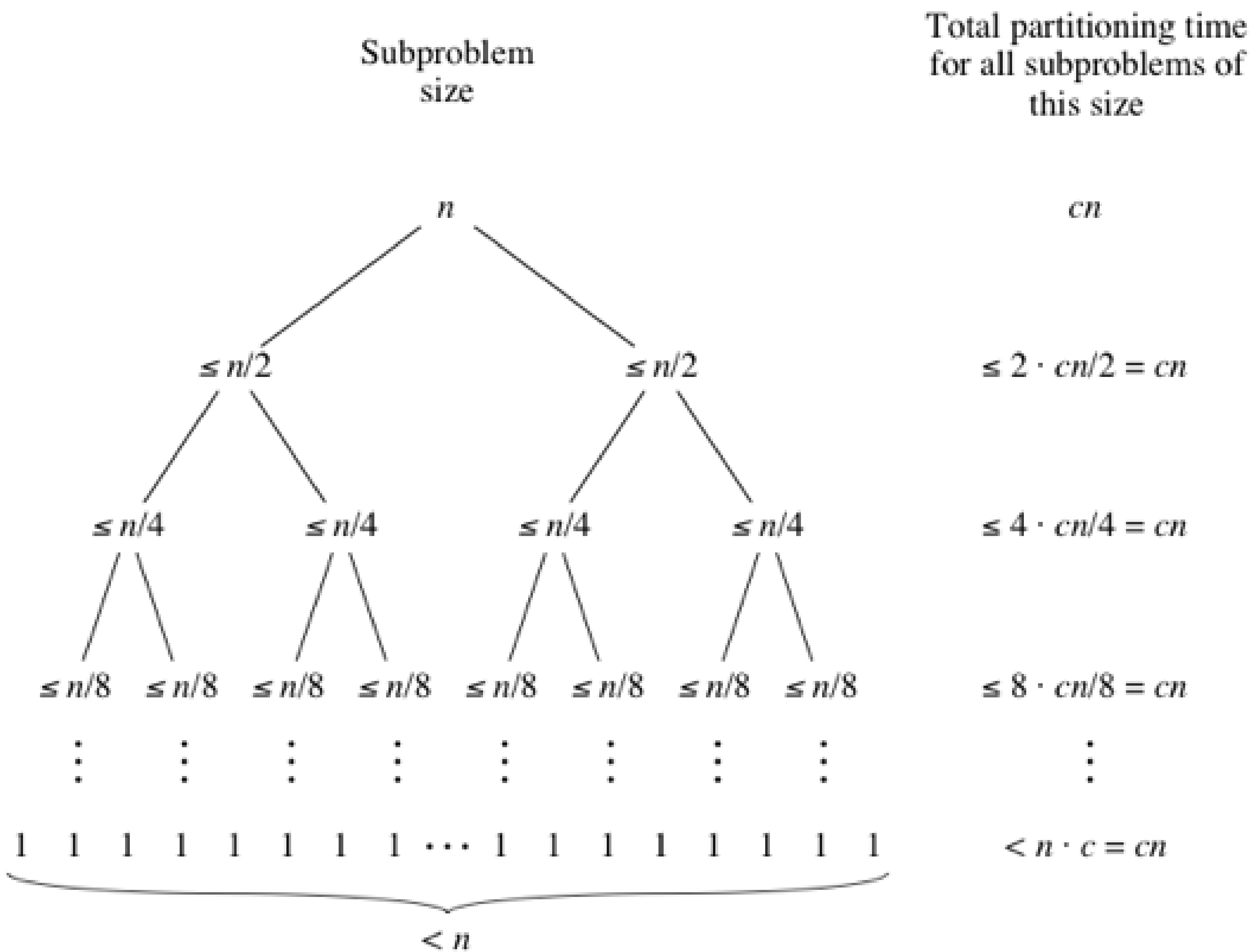
depth	time
0	$\boldsymbol{n}$
1	$\boldsymbol{n} - 1$
…	…
$\boldsymbol{n} - 1$	1



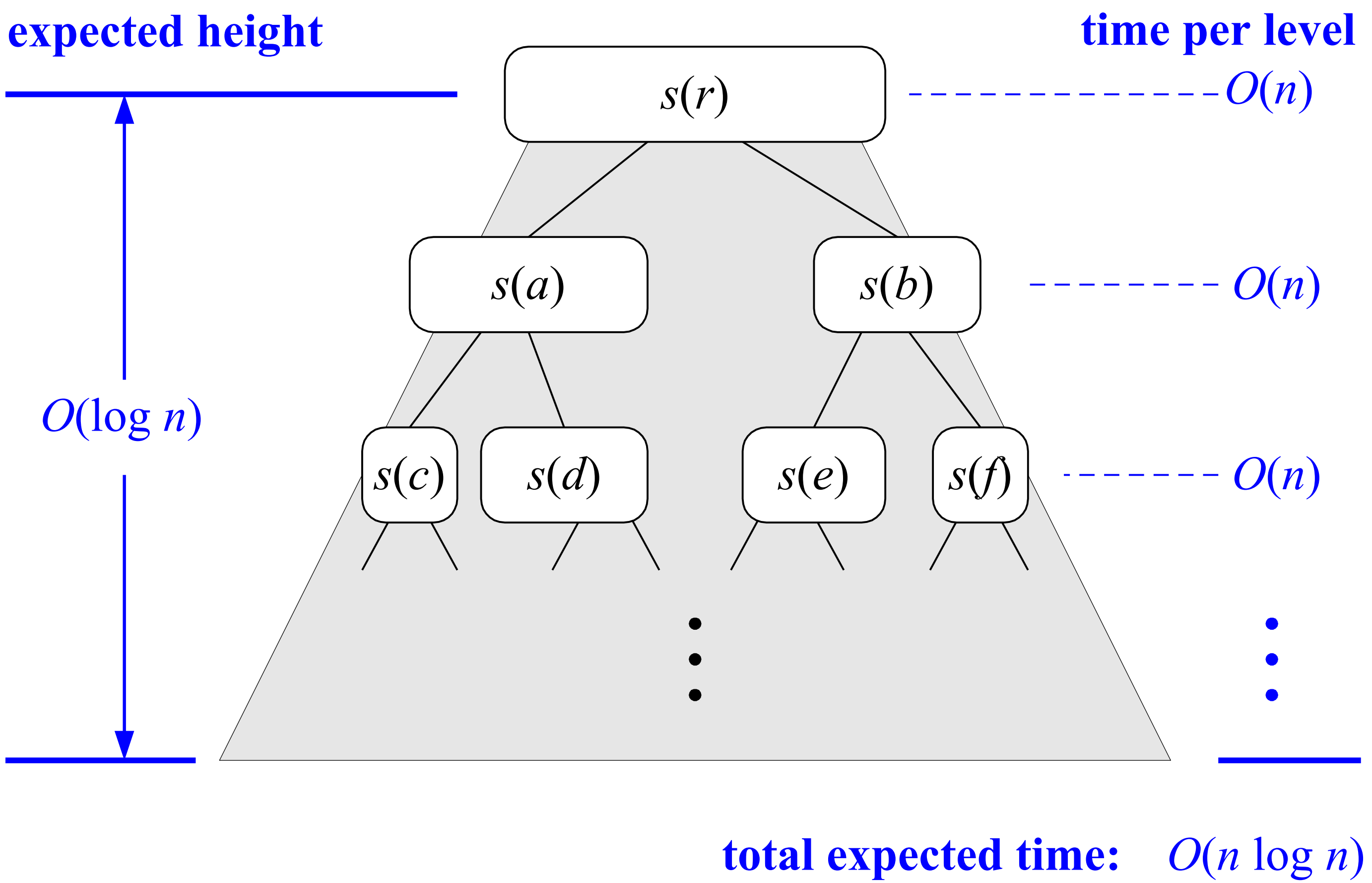
Best Case Analysis

- Best case occurs when the partitions are as evenly balanced as possible
- If the subarray has an odd number of elements and the pivot is right in the middle after partitioning, then each partition has (n-1)/2 elements.
- If the subarray has an even number n of elements and one partition has n/2 and other having n/2-1.
- In either of these cases, each partition has at most n/2 elements

Best Case Analysis



A visual time analysis of the quick-sort tree T

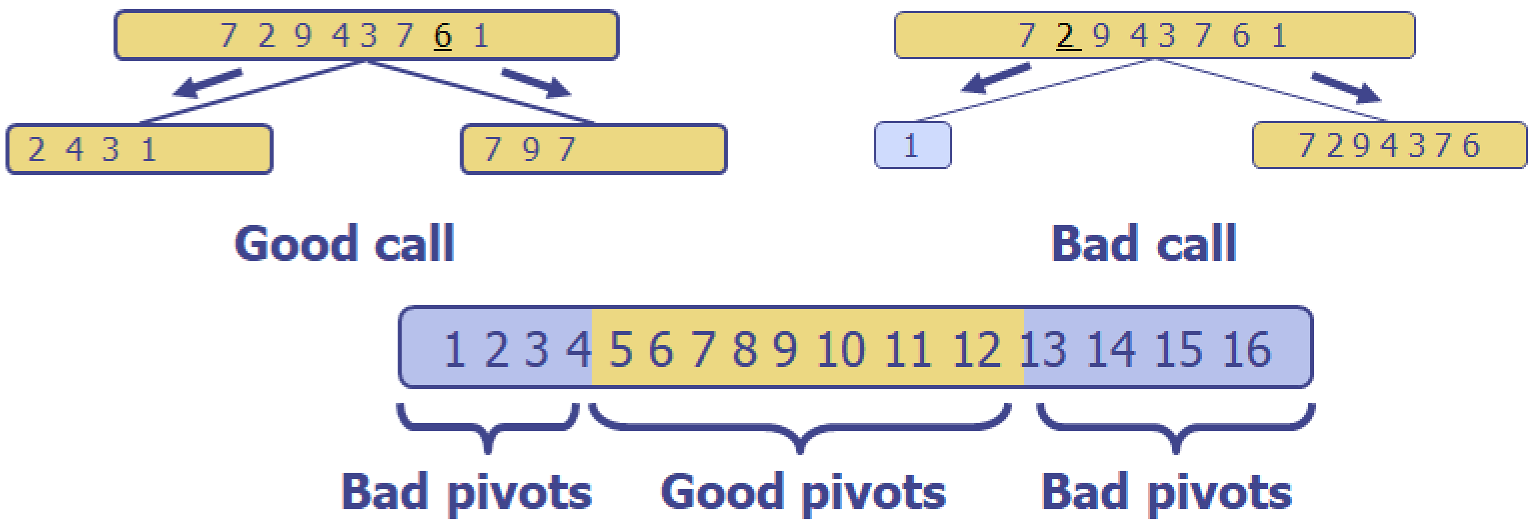


Best Case Running Time

- The amount or work done at the nodes of the same depth is O(n)
- The expected height of the quick-sort tree is O(log n)
- Thus, the expected running time of quick-sort is O(n log n)

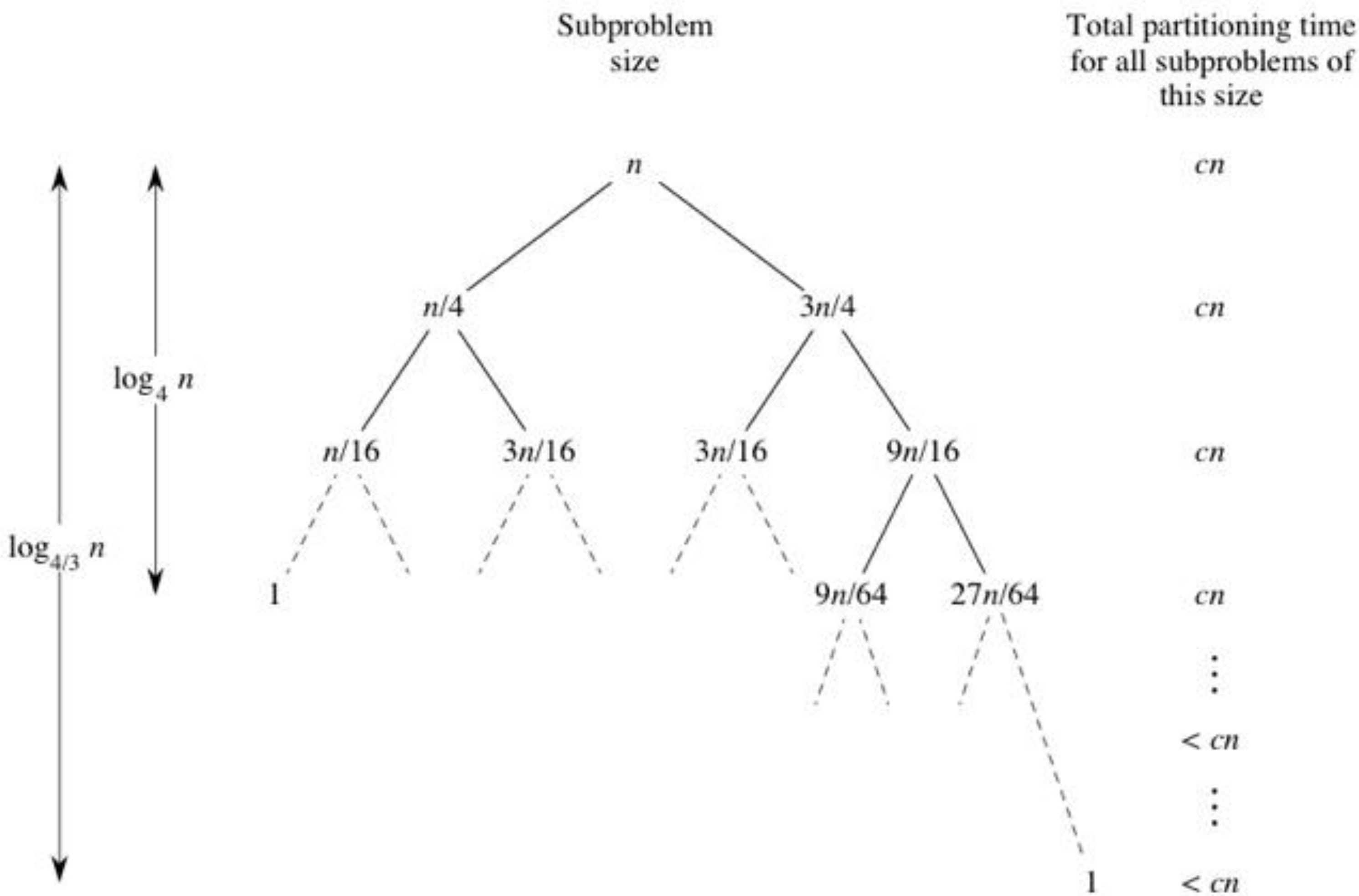
Expected Running Time

- Consider a recursive call of quick-sort on a sequence of size ***s***
  Good call: the sizes of ***L*** and ***G*** are each less than 3***s***/4
  Bad call: one of ***L*** and ***G*** has size greater than 3***s***/4



- A call is good with probability 1/2
  - 1/2 of the possible pivots cause good calls:

Expected Running Time



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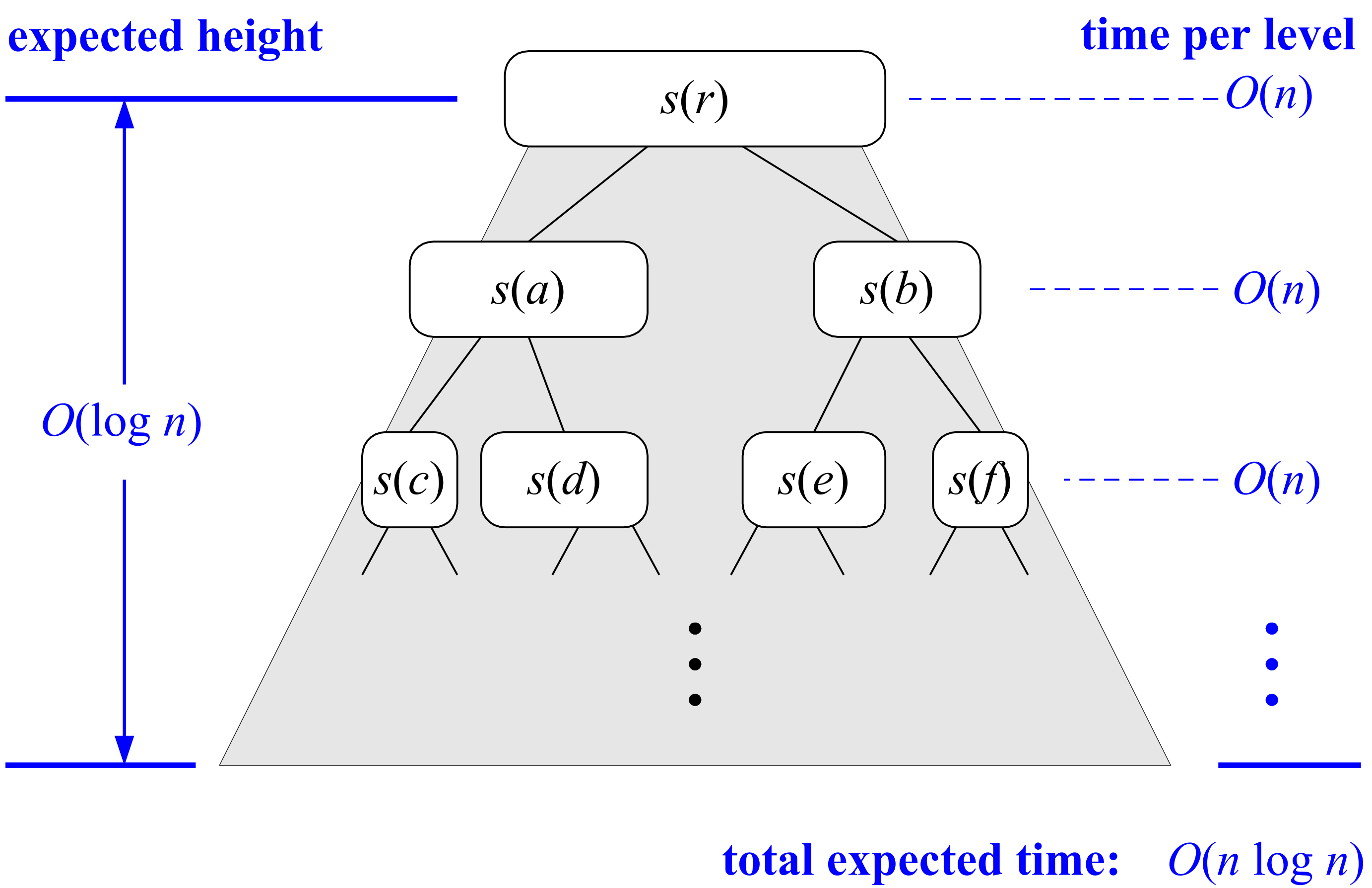


Expected Running Time

Intuitively,

- (In best case) For the tree to have height $\log_2 n$, with each step, the size of the sequence must shrink to at most (n/2).
  - That is, $\log_2 n$ invocations
- For each call to be good,
  - the input size should shrink to atmost (¾)n
  - If all calls are good calls, the expected height of tree is $\log_{(4/3)} n$
  - In other words, $\log_{(4/3)} n$ good calls are needed to get O(log n) height
  - If pivots are randomly chosen, we expect that, out of 2*log (4/3) n calls, log (4/3) n calls are good !
  - $\log_{(4/3)} n$ and log n differs by only a factor of $\log_2(4/3)$ which is a constant.
  - This implies that the height of tree is O(log n)

- Expected complexity of quick sort is O (n log n)

Expected Running Time

- The amount or work done at the nodes of the same depth is O(n)
- The expected height of the quick-sort tree is O(log n)
- Thus, the expected running time of quick-sort is O(n log n)

In-Place Quick-Sort

- A sorting algorithm is in-place if it uses only a constant amount of memory in addition to that needed for the objects being sorted themselves

In-Place Partitioning

- Perform the partition using two indices to split S into L and E U G (a similar method can split E U G into E and G).

j k

3	2	5	1	0	7	3	5	9	2	7	9	8	9	7	**6**	9

(pivot = 6)

- Repeat until j and k cross:
  - Scan j to the right until finding an element $\geq$ x.
  - Scan k to the left until finding an element < x.
  - Swap elements at indices j and k

$\longrightarrow$ j k $\longleftarrow$

3	2	5	1	0	7	3	5	9	2	7	9	8	9	7	**6**	9

In-Place Quick-Sort

- Quick-sort can be implemented to run in-place
- In the partition step, we use replace operations to rearrange the elements of the input sequence such that
  - the elements less than the pivot have rank less than h
  - the elements equal to the pivot have rank between h and k
  - the elements greater than the pivot have rank greater than k
- The recursive calls consider
  - elements with rank less than h
  - elements with rank greater than k

In-Place Quick-Sort

Algorithm inPlaceQuickSort(S, a, b):

Input: Sequence S of distinct elements; integers a and b

Output: Sequence S with elements originally from ranks from a to b, inclusive, sorted in nondecreasing order from ranks a to b

```
if a ≥ b then return            {empty subrange}
p ← S.elemAtRank(b)             {pivot}
l ← a                 {will scan rightward}
r ← b − 1                  {will scan leftward}
while l ≤ r do
    {find an element larger than the pivot}
    while l ≤ r and S.elemAtRank(l) ≤ p do
        l ← l + 1
    {find an element smaller than the pivot}
    while r ≥ l and S.elemAtRank(r) ≥ p do
        r ← r − 1
    if l < r then
        S.swapElements(S.atRank(l), S.atRank(r))
{put the pivot into its final place}
S.swapElements(S.atRank(l), S.atRank(b))
{recursive calls}
inPlaceQuickSort(S, a, l − 1)
inPlaceQuickSort(S, l + 1, b)
```

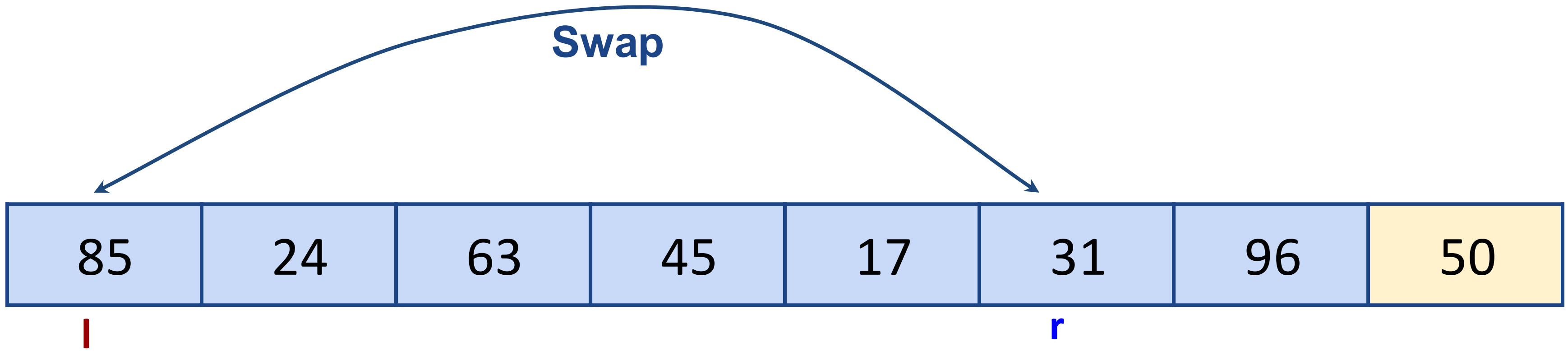
In-Place Quick-Sort

85	24	63	45	17	31	96	50
l						r	

In-Place Quick-Sort

85	24	63	45	17	31	96	50
l					r		

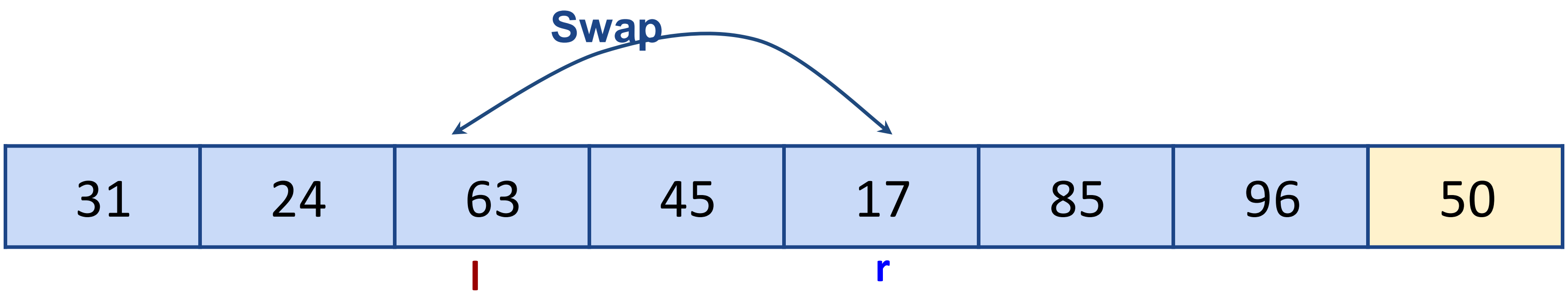
In-Place Quick-Sort



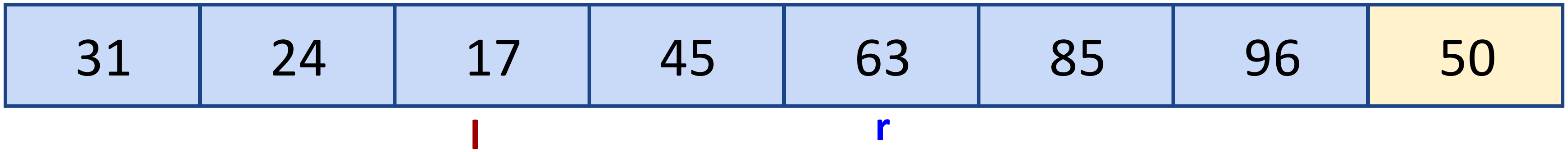
In-Place Quick-Sort

31	24	63	45	17	85	96	50
		l		r			

In-Place Quick-Sort



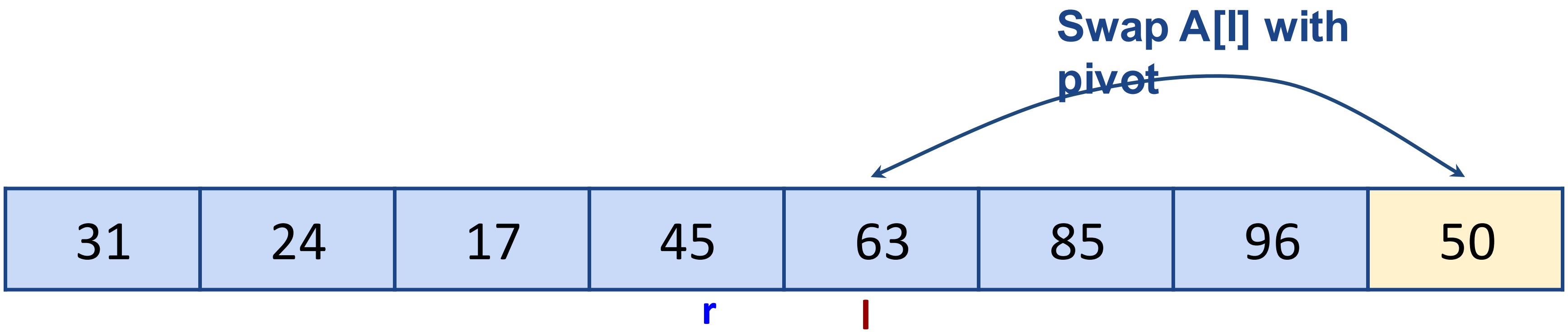
In-Place Quick-Sort



In-Place Quick-Sort

31	24	17	45	63	85	96	50
			r	l			

In-Place Quick-Sort



In-Place Quick-Sort

31	24	17	45	50	85	96	63
			r	l			

Next Recursive Calls:

inPlaceQuickSort(A, a, l – 1)
inPlaceQuickSort(A, l + 1, b)

Quick-Sort Properties

- **Not Adaptive**
- **Not Stable-Can be made stable.**
- **Not Incremental** : Incremental versions possible
- **Not online**
- **In Place**

QuickSort

- **Advantages of Quicksort**
- Its average-case time complexity to sort an array of n elements is O(nlogn).
- It requires no additional memory.
- **Disadvantages of Quicksort**
- Its worst-case running time, $O(n^2)$ to sort an array of n elements, happens when pivot is an extreme
- It is not stable.

QuickSort and Mergesort

- Quick Sort: traditionally built-in for many runtimes, hence used by programs that call the default. *Can't be used where worst-case behavior could be exploited or cause significant ramifications*, such as services that might receive denial-of-service attacks, or real-time systems.
- Java's systems programmers have chosen to use quicksort (with 3-way partitioning) to implement the primitive-type methods, and merge sort for reference-type methods. The primary practical implications of these choices are to trade speed and memory usage (for primitive types) for stability and guaranteed performance (for reference types).
- Merge Sort: used in *database scenarios*, because stable and external (results don't all fit in memory).

QuickSort and MergeSort

- In most practical situations, quicksort is the method of choice.
- If stability is important and space is available, merge sort might be best.
- In some performance-critical applications, the focus may be on just sorting numbers, so it is reasonable to avoid the costs of using references and sort primitive types instead.

THANK YOU!

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