3.3 A. We can use Gaussian Elimination when solving upper diagonal (**U**) matrix

$$\begin{bmatrix} 1 & 2 & 7 \\ 3 & 5 & -1 \\ 6 & 1 & 4 \end{bmatrix} ===> R_2-(\mathbf{3})R_1 => R_2, \begin{bmatrix} 1 & 2 & 7 \\ 0 & -1 & -22 \\ 6 & 1 & 4 \end{bmatrix} ===> R_3-(\mathbf{6})R_1 => R_3, \begin{bmatrix} 1 & 2 & 7 \\ 0 & -1 & -22 \\ 0 & -11 & -38 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 7 \\ 0 & -1 & -22 \\ 0 & -11 & -38 \end{bmatrix} ===> R_3-(\mathbf{11})R_2 => R_3, \begin{bmatrix} 1 & 2 & 7 \\ 0 & -1 & -22 \\ 0 & 0 & 204 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 1 & 2 & 7 \\ 0 & -1 & -22 \\ 0 & 0 & 204 \end{bmatrix}$$

To find lower diagonal (**L**) matrix, The linear operations can be answer while finding upper diagonal matrix.

In the linear operations, our **bold** numbers are elements of our lower diagonal matrix.

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ ? & 1 & 0 \\ ? & ? & 1 \end{bmatrix} ===> \begin{bmatrix} 1 & 0 & 0 \\ \mathbf{3} & 1 & 0 \\ \mathbf{6} & \mathbf{11} & 1 \end{bmatrix}$$

Let's we check whether our result is true or false.

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & 11 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 7 \\ 0 & -1 & -22 \\ 0 & 0 & 204 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 7 \\ 3 & 5 & -1 \\ 6 & 1 & 4 \end{bmatrix} = \mathbf{A}$$

3.13(a) A. If we check whether these matrices is equal to M or not, then we show that we can decompose M as the product of others.

$$M = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D-CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$M = \begin{bmatrix} A & 0 \\ \underbrace{CA^{-1}A}_{\text{C}} & D-CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$M = \begin{bmatrix} A & \overbrace{AA{-1}}^{\text{I}}B \\ C & \underbrace{CA^{-1}B - CA^{-1}B}_{0}+D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

3.13(b) A. Part (a) of this question, M = LDU. However, part (b) we will transform LDU to LU.

$$M = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

According to question, $A = L_1U_1$ and $D - CA^{-1}B = L_2U_2$

$$M = \begin{bmatrix} I & 0 \\ CU_1^{-1}L_1^{-1} & I \end{bmatrix} \begin{bmatrix} L_1U_1 & 0 \\ 0 & L_2U_2 \end{bmatrix} \begin{bmatrix} I & U_1^{-1}L_1^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$M = \begin{bmatrix} L_1U_1 & 0 \\ C\underbrace{U_1^{-1}L_1^{-1}L_1U_1}_{\text{I}} & L_2U_2 \end{bmatrix} \begin{bmatrix} I & U_1^{-1}L_1^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$M = \begin{bmatrix} L_1U_1 & 0 \\ C & L_2U_2 \end{bmatrix} \begin{bmatrix} I & U_1^{-1}L_1^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

After transforming $A = L_1U_1$ and $D - CA^{-1}B = L_2U_2$

$$M = \underbrace{\begin{bmatrix} A & 0 \\ C & D - CA^{-1}B \end{bmatrix}}_{\text{L}} \underbrace{\begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}}_{\text{U}} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Above that, M was written in terms of **LU method.** Let's check whether true or not.

$$M = \begin{bmatrix} A & 0 \\ C & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$M = \begin{bmatrix} A & \overbrace{AA^{-1}}^{\text{I}}B \\ C & CA^{-1}B + D - CA^{-1}B \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \checkmark$$

5.5 A. According to definition, **Involutory matrix** is a matrix that is **its own inverse.** It means that, $H_{\vec{v}} = H_{\vec{v}}{}^{-1}$

And also we can show that, $H_{\vec{v}}{}^2 = I_{nxn}$

$$(I_{nxn} - \frac{2\vec{v}\vec{v}^T}{\vec{v}^T\vec{v}})\ (I_{nxn} - \frac{2\vec{v}\vec{v}^T}{\vec{v}^T\vec{v}}) = I_{nxn} - \frac{2\vec{v}\vec{v}^T}{\vec{v}^T\vec{v}} - \frac{2\vec{v}\vec{v}^T}{\vec{v}^T\vec{v}} + \frac{4\vec{v}\,\overbrace{\vec{v}^T\vec{v}}\,\vec{v}^T}{\vec{v}^T\,\underbrace{\vec{v}\vec{v}^T}\,\vec{v}} = I_{nxn} - \frac{4\vec{v}\vec{v}^T}{\vec{v}^T\vec{v}} + \frac{4\vec{v}\vec{v}^T}{\vec{v}^T\vec{v}} = I_{nxn}$$

If a matrix is involutory, this mean is $A = A^{-1}$. We can find eigenvalues using this property.

$\mathrm{A}\vec{x} = \lambda\vec{x} ==> \underbrace{A^{-1}A}_{\mathrm{I}}\vec{x} = \mathrm{A}^{-1}\lambda\vec{x} ==> \vec{x} = \lambda^2\vec{x} ==> (\lambda^2 - 1)\vec{x} = 0 ==> ((\lambda - 1)(\vec{x} - 1) = 0$

$$\lambda = \mathbf{1} \quad \lambda = \mathbf{-1}$$

5.11 A. Firstly, we have two formulas with matrix multiplication to find sin and cos formulas;

$$R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad R_\theta\vec{x} = r\vec{e}_1$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

Above that solving the equation, we obtained two different equations.

(1) $\mathrm{x}_1 \cos\theta + x_2 \sin\theta = r$

(2) $-\mathrm{x}_1 \sin\theta + x_2 \cos\theta = 0$

By using equation **(2)**, we can obtain this equality;

$\mathrm{x}_1 = x_2 \dfrac{\cos\theta}{\sin\theta}$

And after that by using equation **(1)** with **above** equation, we can obtain third equation;

(3) $x_2 \dfrac{\cos^2\theta}{\sin\theta} + x_2 \sin\theta = r \quad ==> \quad x_2\cos^2\theta + x_2\sin^2\theta = r\sin\theta \quad ==> \quad x_2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} = r\sin\theta$

$x_2 = r\sin\theta \quad => \quad sin\theta = \dfrac{x_2}{r}$ (**formula for** $\sin\theta$ **that does not require trigonometric formula**)

$x_1 = x_2 \dfrac{\cos\theta}{\frac{x_2}{r}} \quad ==> \quad x_1 = r\cos\theta \quad ==> \quad \cos\theta = \dfrac{x_1}{r}$ (**formula for** $\cos\theta$ **that does not require trigonometric formula**)