**13.1** We can write degree-three polynomial interpolating using Lagrange Interpolation without monomial basis.

Our data points are (-2, 15), (0, -1), (1, 0), (3, -2).

$$\phi_1(x) = \frac{x(x-1)(x-3)}{(-2)(-3)(-5)} \;=\; -\frac{1}{30}(x^3 - 4x^2 + 3x)$$

$$\phi_2(x) = \frac{(x+2)(x-1)(x-3)}{(2)(-1)(-3)} \;=\; \frac{1}{6}(x^3 - 2x^2 - 5x + 6)$$

$$\phi_3(x) = \frac{(x+2)x(x-3)}{(3)(1)(-2)} \;=\; -\frac{1}{6}(x^3 - x^2 - 6x)$$

$$\phi_4(x) = \frac{(x+2)x(x-1)}{(5)(3)(2)} \;=\; \frac{1}{30}(x^3 + x^2 - 2x)$$

After finding lagrange basis for each x point, we can write general solution using formula.

$$P_3(x) = 15\phi_1(x) - \phi_2(x) + 0\phi_3(x) - 2\phi_4(x)$$

$$P_3(x) = \frac{-11x^3 + 34x^2 - 8x - 15}{15}$$

**13.2** When we interpolate firstly x1 and then x2 respectively, we have to get the same result with previous solution.

To find f($\frac{1}{4}$,$\frac{1}{2}$), we first interpolate $x_2$;

$$f(0, \frac{1}{2}) = \frac{1}{2}f(0,0) + \frac{1}{2}f(0,1) = \frac{1}{2} - \frac{3}{2} = -1$$

$$f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}\frac{1}{2}f(0,0) + \frac{1}{2}\frac{1}{2}f(0,1) + \frac{1}{2}\frac{1}{2}f(1,0) + \frac{1}{2}\frac{1}{2}f(1,1) = \frac{1}{4} - \frac{3}{4} - \frac{11}{4} - \frac{5}{4} = -\frac{8}{4} = -2$$

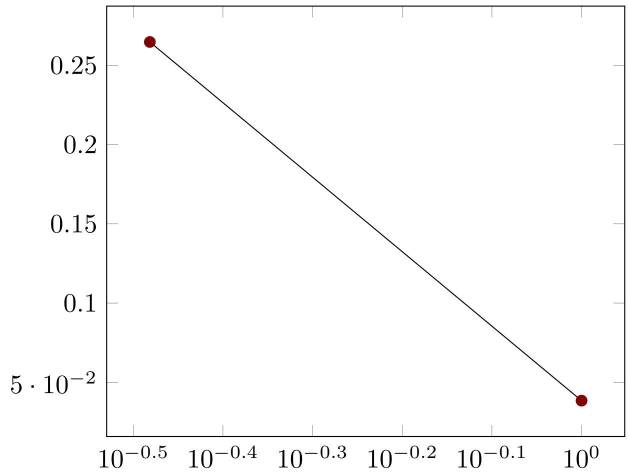
Now, we secondly interpolate x2 to find result.

$$f(\frac{1}{4}, \frac{1}{2}) = \frac{1}{2}f(0, \frac{1}{2}) + \frac{1}{2}f(\frac{1}{2}, \frac{1}{2}) = -\frac{1}{2} - \frac{2}{2} = -\frac{3}{2}$$

**13.3(a)** We can give three number to find quality of approximation for a few samples of k such as 3, 4, 5.

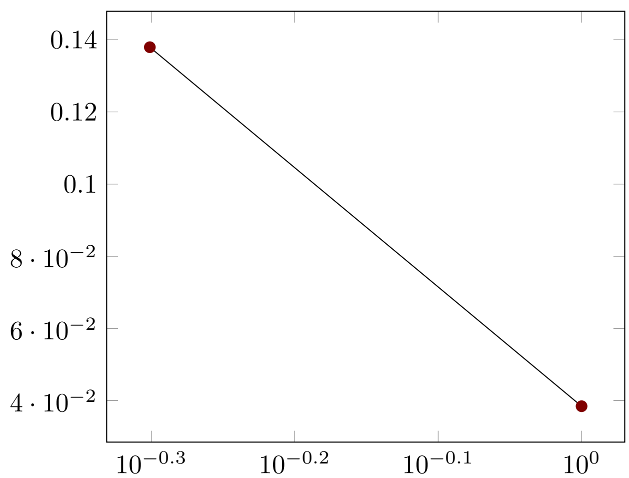
**For k = 3**,

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $x_k$ | -1 | -1/3 | 1/3 | 1 |
| $f(x_k)$ | 0.03846 | 0.26470 | 0.26470 | 0.03846 |



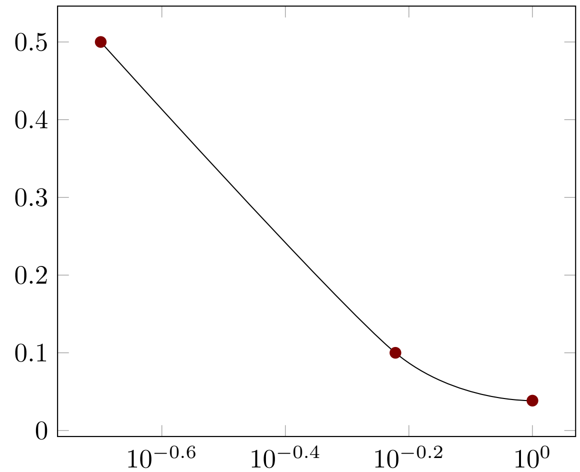

**For k = 4**,

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $x_k$ | -1 | -1/2 | 0 | 1/2 | 1 |
| $f(x_k)$ | 0.03846 | 0.13793 | 1 | 0.13793 | 0.03846 |




**For k = 5**,

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x_k$ | -1 | -3/5 | -1/5 | 1/5 | 3/5 | 1 |
| $f(x_k)$ | 0.03846 | 0.1 | 0.5 | 0.5 | 0.1 | 0.03846 |




**According to chapter 13.3.2 in main book, when number of k increases, interpolation error also increases.**

$$|f(x) - P_k(x)| \ \leq = \ \frac{1}{(n+1)!}\left[max \prod_{i=1} |x - x_k|\right]\left[max \prod_{i=1} |f^{n+1} - (x)|\right], \quad at\ any\ x \in [x_0, x_n]$$

**Above that, this formula shows interpolation error of k-degree polynomial. When increasing number of k, we can find that error increases using formula above. Thus, increasing k does not improve the quality of approximation.**

## 13.6(a)

Our function is P(t)=at$^3 + bt^2 + ct + d$

$P(0) = h_0 \qquad P'(0) = h_2$

$P(1) = h_1 \qquad P'(1) = h_3$

We can find functions and their derivatives that are showed above in terms of P function.

$P(t) = at^3 + bt^2 + ct + d \qquad P'(t) = 3at^2 + 2bt + c$

Here, we will use matrix to solve system of equations

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

$P(0) = h_0 = d \qquad P'(0) = h_2 = c$

$P(0) = h_1 = a + b + c + d \qquad P'(0) = h_3 = 3a + 2b + c$

We can also find a, b, c, d derived from h$_0, h_1, h_2, h_3$.

$a = h_3 + h_2 - 2h_1 + 2h_0$

$b = -h_3 - 2h_2 + 3h_1 - 3h_0$

$$c = h_2$$

$$d = h_0$$

**13.6(b)** P(t) = h$_0\phi_0(t) + h_1\phi_1(t) + h_2\phi_2(t) + h_3\phi_3(t)$

We can substitute h values with their equalities.

P(t) = d$\phi_0(t) + (a+b+c+d)\phi_1(t) + c\phi_2(t) + (3a+2b+c)\phi_3$

P(t) = a$(\phi_1(t) + 3\phi_3(t)) + b(\phi_1(t) + 2\phi_3(t)) + c(\phi_1(t) + \phi_2(t) + \phi_3(t)) + d(\phi_0(t) + \phi_1(t))$

**14.1** To find midpoint rule is exact for 1-degree polynomial, we can use method of undetermined coefficients .

$$\int_a^b f(x)dx = (b-a)f(\frac{a+b}{2}) \qquad \text{(Midpoint Rule Quadrature)}$$

$$\left.\begin{array}{r} \text{For f(x)=1,} \quad \int_a^b f(x)dx = x\Big|_a^b = (b-a) \\ \int_a^b f(x)dx = (b-a)\underbrace{f(\frac{a+b}{2})}_{1} = (b-a) \end{array}\right\} (b-a) = (b-a)\checkmark$$

$$\left.\begin{array}{r} \text{For f(x)=x,} \quad \int_a^b f(x)dx = \frac{x^2}{2}\Big|_a^b = \frac{b^2-a^2}{2} \\ \int_a^b f(x)dx = (b-a)\underbrace{f(\frac{a+b}{2})}_{\frac{a+b}{2}} = \frac{b^2-a^2}{2} \end{array}\right\} (\frac{b^2-a^2}{2}) = (\frac{b^2-a^2}{2})\checkmark$$

For $f(x) = x^2$, there is no equality between result of integral of f(x) and result of midpoint rule quadrature.

**So, according to result, midpoint rule is exact for the function f(x) = mx + c (1-degree polynomial).**

**14.2** We can solve this question using method of undetermined coefficients.

$\int_0^2 f(x)dx \approx \alpha f(0) + \beta f(x_1)$ (We must find $\alpha, \beta$, x$_1$)

For f(x) = 1, $\int_0^2 1dx = \ \alpha + \beta = 2$

For f(x) = x, $\int_0^2 xdx = \ 0\alpha + \beta x_1 = \beta x_1 = 2$

For f(x) = $x^2$, $\int_0^2 x^2dx = \ 0\alpha + \beta(x_1)^2 = \beta(x_1)^2 = \dfrac{8}{3}$

We can find $\alpha$, $\beta$ and $x_1$ using conditions above.

$\dfrac{\beta(x_1)^2}{\beta x_1} = \ x_1 = \ \dfrac{4}{3}$

$\beta x_1 = \ \dfrac{4}{3}\beta = \ 2 \ \ ==> \ \beta = \dfrac{3}{2}$

$\alpha + \beta = 2 \ \ ==> \ \alpha = \dfrac{1}{2}$

**14.4.a** We can derive infinite lengths ($\infty, -\infty$) into finite lengths using change of variables.

For $\int_{-\infty}^{\infty} f(x)dx =$

$$\lim_{x\to\infty} x = \lim_{t\to 1} \frac{t}{1-t^2} = \infty$$

$$\lim_{x\to-\infty} x = \lim_{t\to -1} \frac{t}{1-t^2} = -\infty$$

$$\frac{t}{1-t^2} \to x, \quad \frac{1+t^2}{(1-t^2)^2}dt \to dx$$

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-1}^{1} f(\frac{t}{1-t^2})\frac{1+t^2}{(1-t^2)^2}dt$$

For $\int_0^{\infty} f(x)dx =$

$$\lim_{x\to\infty} x = \lim_{t\to 0} -\ln t = \infty$$

$$\lim_{x\to 0} x = \lim_{t\to 1} -\ln t = 0$$

$$-\ln t \to x, \quad -\frac{1}{t}dt \to dx$$

$$\int_0^{\infty} f(x)dx = \int_1^0 -\frac{f(-\ln t)}{t}dt = \int_0^1 \frac{f(-\ln t)}{t}dt$$

For $\int_c^{\infty} f(x)dx =$

$$\lim_{x\to\infty} x = \lim_{t\to 1} c + \frac{t}{1-t} = \infty$$

$$\lim_{x\to c} x = \lim_{t\to 0} c + \frac{t}{1-t} = c$$

$$c + \frac{t}{1-t} \to x, \quad \frac{1}{(1-t)^2}dt \to dx$$

$\int_c^\infty f(x)dx = \int_0^1 f(c + \frac{t}{1-t})\frac{1}{(1-t)^2}dt$

**Actually, we do not change result when changing infinite lengths into finite lengths. This is just implementation for obtaining result easily. We limited infinite length using changing boundaries of integral, after that, we substituted new unknown variable with previous one.**

**After derivation infinite length integral, it gives determined approximation about our result because of evenly spacing t samples. For this reason, our approximation lose smoothness for our result.**

**14.5** First of all, understanding this question, we have to know what single-valued function is. A single-valued function is function that, for each point in the domain, has a unique value in the range. It is therefore one-to-one or many-to-one.

For example; $y = f(x) = x^4 + 10$ is single-valued function. Every x refers just y.

**When we use Jacobian with these single-valued function, we obtain just one-by-one matrix. And every derivation of x does not change the size of matrix.**

$\frac{\partial f}{\partial x} = 4x^3 ==> J = \begin{bmatrix} 4x^3 \end{bmatrix}$

$\frac{\partial^2 f}{\partial x^2} = 12x^2 ==> J = \begin{bmatrix} 12x^2 \end{bmatrix}$

$\frac{\partial^3 f}{\partial x^3} = 24x ==> J = \begin{bmatrix} 24x \end{bmatrix}$

$\frac{\partial^4 f}{\partial x^4} = 24 ==> J = \begin{bmatrix} 24 \end{bmatrix}$

**As it seems, At every derivation, jacobian f(x) has just one. After every derivation, n decrease by one because of x and m is constant because of y.**