Lecture 1

Wednesday, February 23, 2022 7:30 AM

# Dynamical Systems

Many laws of nature can be stated as dynamical systems, more precisely using either differential equations or difference equations.

I. $x : \mathbb{R} \to \mathbb{R}^n \quad t \mapsto x(t)$ $\quad$ $t$ is the time, a continuous time

$x(t)$ describes the state of a phenomenon that changes in time


A differential equation is a relation between $x(t)$ and its derivatives $x'(t)$, $x''(t)$, ... up to some order.

In mechanics $x(t)$ is the position of a material point, $x'(t)$ is its velocity, $x''(t)$ is the acceleration.

In chemistry $x(t)$ can be the quantity or the concentration of a chemical substance, $x'(t)$ is the rate of change.

In population dynamics $x(t)$ is the number or the density of individuals in a population (people, animals, fish ...) and $x'(t)$ is the rate of growth.

II $\quad x : \mathbb{N} \to \mathbb{R}^n \quad k \mapsto x(k) = x_k \quad (x_k)_{k \geq 0}$ is a sequence

$k$ is the time, a discrete time

A difference equation is a relation between $x_k, x_{k+1}, x_{k+2}, \ldots$ .

There are two main parts of dynamical systems theory

I Continuous dynamical systems

II Discrete dynamical systems

The aim of this theory is to make "predictions" for the evolution of a phenomenon when one knows the initial state and some "laws" of "nature".

Mathematically this means that either we intend to find the expression of $x(t)$ or $(x_k)_{k \geq 0}$ or, if this is not possible, to be able to establish its properties.

properties.

A general form of a differential equation is $F(t, x(t), x'(t), \dots, x^{(n)}(t)) = 0$

## 1. Linear differential equations of order $n$, $n \in \mathbb{N}^*$

$$(1) \quad x^{(n)} + a_1(t)x^{(n-1)} + \dots + a_{n-1}(t)x' + a_n(t)x = f(t)$$

where $a_1, \dots, a_n : I \to \mathbb{R}$ continuous (called the coefficients) are given (known)

$I \subset \mathbb{R}$ nonempty open interval

$f : I \to \mathbb{R}$ continuous (called the force or the nonhomogeneous part of (1)) is given (known)

The unknown is $x : I \to \mathbb{R}$ a scalar function.

**Definition** A solution of (1) is a function $\varphi : I \to \mathbb{R}$ which is $n$ times differentiable and $\varphi^{(n)} : I \to \mathbb{R}$ is continuous such that

$$\varphi^{(n)}(t) + a_1(t)\varphi^{(n-1)}(t) + \dots + a_n(t)\varphi(t) = f(t), \quad \forall t \in I.$$

**Notations** Let $I \subset \mathbb{R}$ interval

$C(I) = \{ f : I \to \mathbb{R} \text{ continuous} \}$

$C^n(I) = \{ f : I \to \mathbb{R} \text{ s.t. } \exists f, f', f'', \dots, f^{(n)} \text{ continuous on } I \}$

So $a_1, \dots, a_n, f \in C(I)$ and a solution $\varphi \in C^n(I)$.

We consider the addition between two functions

Let $f_1, f_2 \in C(I)$, $\quad f_1 + f_2 \in C(I) \quad (f_1 + f_2)(t) = f_1(t) + f_2(t), \forall t \in I$

$\alpha \in \mathbb{R}$ (scalar) $\quad \alpha \cdot f \in C(I) \quad (\alpha f)(t) = \alpha \cdot f(t) \quad \forall t \in I.$

$C(I)$ is a vector space $\quad C^n(I)$ is a vector space.

When (1) is called linear?

why (1) is called linear ?

For each $x \in C^n(I)$ we define the function $\mathcal{L}x : I \to \mathbb{R}$,

$$\mathcal{L}x(t) \overset{def}{=} x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \dots + a_n(t)x(t), \quad \forall t \in I.$$

Moreover, we define now $\mathcal{L} : C^n(I) \to C(I)$

$$x \longmapsto \mathcal{L}x$$

So, $\mathcal{L}$ is a map between two vector spaces.

<u>Proposition</u> $\mathcal{L}$ is a <u>linear map</u>, i.e.

$$\mathcal{L}(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \mathcal{L}(x_1) + \alpha_2 \mathcal{L}(x_2), \quad \forall x_1, x_2 \in C^n(I), \ \forall \alpha_1, \alpha_2 \in \mathbb{R}.$$

<u>Proof.</u> Let $t \in I$. $\mathcal{L}(\alpha_1 x_1 + \alpha_2 x_2)(t) =$

$$= (\alpha_1 x_1 + \alpha_2 x_2)^{(n)}(t) + a_1(t)(\alpha_1 x_1 + \alpha_2 x_2)^{(n-1)}(t) + \dots + a_n(t)(\alpha_1 x_1 + \alpha_2 x_2)(t) =$$

$$= \alpha_1 x_1^{(n)}(t) + \alpha_2 x_2^{(n)}(t) + a_1(t)\alpha_1 \cdot x_1^{(n-1)}(t) + a_1(t) \cdot \alpha_2 \cdot x_2^{(n-1)} + \dots + a_n(t)\alpha_1 x_1(t) + a_n(t)\alpha_2 x_2(t) =$$

$$= \alpha_1 \cdot \mathcal{L}x_1(t) + \alpha_2 \mathcal{L}x_2(t) \ \blacksquare$$

with this notation, eq (1) can be written as

(2) $\mathcal{L}(x) = f$ (equation with $x$ as unknown)

A particular case of (2) is when $f = 0$, i.e.

(3) $\mathcal{L}(x) = 0$ this is called <u>linear homogeneous</u> dif. eq.

The set of solutions of (3) is (called, from linear algebra course) the kernel of $\mathcal{L}$, denoted $\operatorname{Ker} \mathcal{L}$.

When $f \neq 0$ eq (2) is called <u>linear nonhomogeneous</u>.

Examples. 1) $x + 2x' = 0$ first order differential eq.

$x' + \frac{1}{2}x = 0$ linear homogeneous, with constant coeff.

$x = 0$ is a solution (in fact $x = 0$ is a solution of any lin. hom. dif. eq.)

$x = -2$ "$(-2)' + \frac{1}{2}\cdot(-2) = 0$" $\Leftrightarrow$ "$-1 = 0$" F

Thus $x = -2$ is not a solution.

$x = a$ $(a \in \mathbb{R})$ check "$\frac{a}{2} = 0$" True $\Leftrightarrow$ $a = 0$.

So, $x = 0$ is the only constant solution of $x' + \frac{1}{2}x = 0$

Try $x = at + b$ $x' = a$ check "$a + \frac{1}{2}(at + b) = 0 \Big|\cdot 2 \quad \forall t \in \mathbb{R}$" $\Leftrightarrow$

$\Leftrightarrow$ "$at + (2a + b) = 0 \quad \forall t \in \mathbb{R}$" $\Leftrightarrow \begin{cases} a = 0 \\ 2a + b = 0 \end{cases} \Leftrightarrow a = b = 0$.

$x = 0$

Try $x = e^{-\frac{t}{2}}$ $x' = -\frac{1}{2}e^{-\frac{t}{2}}$

Check "$-\frac{1}{2}e^{-\frac{t}{2}} + \frac{1}{2}e^{-\frac{t}{2}} = 0, \quad \forall t \in \mathbb{R}$" TRUE.

So, $x = e^{-\frac{t}{2}}$ is another solution. In addition, $x = c\, e^{-\frac{t}{2}}$, $c \in \mathbb{R}$ arbitrary is also a solution.

2) $x'' + t x' = 3$ second order linear nonhomogeneous diff. eq.

coeff. is not constant

nonhom. part

3) $x \cdot x' = 0$ first order nonlinear differential eq.

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Consequences of the linearity of $\mathcal{L}$ (The Linearity principle or The Superposition principle.)

(1) $\mathcal{L}(\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_k x_k) = \alpha_1 \mathcal{L}(x_1) + \alpha_2 \mathcal{L}(x_2) + \dots + \alpha_k \mathcal{L}(x_k)$

$\forall\, x_1, \dots x_k \in C^n(I)$, $\forall\, \alpha_1, \dots, \alpha_k \in \mathbb{R}$ $\quad \underline{k \geq 1}, k \in \mathbb{N}$.

Proof by induction.

(2) If $x_1, \dots, x_k$ are solutions of the LHDE $\mathcal{L}(x) = 0$

then, $\forall\, \alpha_1, \dots, \alpha_k \in \mathbb{R}$ we have that $\alpha_1 x_1 + \dots + \alpha_k x_k$ is another solution of $\mathcal{L}(x) = 0$.

(3) Let $f_1, \dots, f_k \in C(I)$, let $x_i$ be a sol. of $\mathcal{L}(x) = f_i$, $i = \overline{1,k}$

Let $\alpha_1, \dots, \alpha_k \in \mathbb{R}$. Then for the eq. $\mathcal{L}(x) = \alpha_1 f_1 + \dots + \alpha_k f_k$

we have the solution $\alpha_1 x_1 + \dots + \alpha_k x_k$.

For the proof of (2) and (3) we apply (1).

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$x' + \frac{1}{2} x = 0 \quad \Big| \cdot \boxed{e^{\frac{t}{2}}}$ we know $x = c \cdot e^{-\frac{t}{2}}$ is a solution $\forall c \in \mathbb{R}$

this is an <u>integrating factor</u> for <u>this</u> equation.

$$x' \cdot e^{\frac{t}{2}} + x \cdot \underbrace{\frac{1}{2} e^{\frac{t}{2}}}_{= \left(e^{\frac{t}{2}}\right)'} = 0 \quad \forall t \in \mathbb{R} \iff \left(x \cdot e^{\frac{t}{2}}\right)' = 0 \iff$$

$$x \cdot e^{\frac{t}{2}} = c \; \Big|\cdot e^{-t/2}, \quad c \in \mathbb{R} \iff$$

$$\boxed{x = c \cdot e^{-t/2}, \quad c \in \mathbb{R}}$$

Hence

There no other solutions.

$x = c \cdot e^{-t/2}$, $c \in \mathbb{R}$ is the <u>general solution</u> of our d.e.

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