Practice

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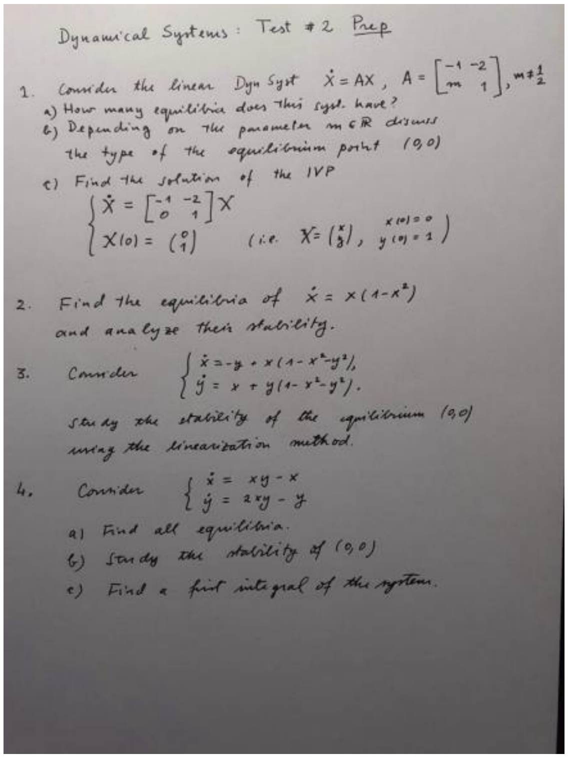

Dynamical Systems: Test #2 Prep

1. Consider the linear Dyn Syst $\dot{X} = AX$, $A = \begin{bmatrix} -1 & -2 \\ m & 1 \end{bmatrix}$, $m \neq \frac{1}{2}$
   a) How many equilibria does this syst. have?
   b) Depending on the parameter $m \in \mathbb{R}$ discuss the type of the equilibrium point $(0,0)$
   c) Find the solution of the IVP
   $$\begin{cases} \dot{X} = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} X \\ X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases} \quad \left(\text{i.e. } X = \begin{pmatrix} x \\ y \end{pmatrix},\ \begin{matrix} x(0) = 0 \\ y(0) = 1 \end{matrix}\right)$$

2. Find the equilibria of $\dot{x} = x(1-x^2)$ and analyze their stability.

3. Consider $\begin{cases} \dot{x} = -y + x(1-x^2-y^2), \\ \dot{y} = x + y(1-x^2-y^2). \end{cases}$
   Study the stability of the equilibrium $(0,0)$ using the linearization method.

4. Consider $\begin{cases} \dot{x} = xy - x \\ \dot{y} = 2xy - y \end{cases}$
   a) Find all equilibria.
   b) Study the stability of $(0,0)$
   c) Find a first integral of the system.

1) $\dot{X} = AX$, $A = \begin{bmatrix} -1 & -2 \\ m & 1 \end{bmatrix}$, $m \neq \frac{1}{2}$

a) equilibria??

x^* : $f(x^*) = 0$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ m & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{cases} \dot{x} = -x - 2y = 0 \\ \dot{y} = mx + y = 0 \end{cases} \Leftrightarrow \begin{cases} x = -2y \\ -2my + y = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \left.\begin{cases} x = -2y \\ (1-2m)y = 0 \quad (m \neq \frac{1}{2}) \end{cases}\right\} \Rightarrow y = 0;\ x = 0$$

$\Rightarrow (0,0)$ equilibrium point

b) $\det(A - \lambda I_2) = 0 \Leftrightarrow \begin{vmatrix} -1-\lambda & -2 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Leftrightarrow$

$\Leftrightarrow -1 + \lambda^2 = 0 \Leftrightarrow \lambda = \pm 1 \Rightarrow \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}$

Node: $(\lambda_1 \leq \lambda_2 < 0) \,||\, (0 < \lambda_1 \leq \lambda_2)$ (both pos or both neg)

Saddle: $\lambda_1 < 0 < \lambda_2$ (one pos, one neg)

Center: $\lambda_{1,2} = \pm i\beta$, $\beta \in \mathbb{R}^*$

Focus: $\lambda_{1,2} = \alpha \pm i\beta$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^{*}$

$\Rightarrow (0,0)$ saddle point

c) $\begin{cases} \dot{x} = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} x \\ x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$

$\dot{x}(0) = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Leftrightarrow \dot{x}(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\dot{x} = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix} x \Rightarrow \begin{cases} \dot{x} = -x - 2y \\ \dot{y} = y \end{cases} \Leftrightarrow \begin{cases} \dot{x} = -x - 2c_2 \cdot e^{t} \\ y = c_2 \cdot e^{t} \end{cases}$

$x(t) = x_H(t) + x_p(t)$

$x'(t) = x(t)$, $x(t) = ?$

$x(t) = Ce^{t}$

$x'(t) = -x(t)$, $x(t) = ?$

$x(t) = Ce^{-t}$

$x''(t) = -x(t)$, $x(t) = ?$

$(\sin t)'' = -\sin t$

$(\cos t)'' = -\cos t$

$x(t) = x_H(t) + x_p(t)$ $\qquad \dot{x} = -x - 2c_1 \cdot e^{t}$

$x_H(t) = c_1 e^{-t}$

$x_p(t) = me^{t}$

$$me^t = -(me^t) - 2C_2 e^t \iff me^t = -(m+2c_2)e \iff$$

$$\iff 2m - 2c_2 = 0 \iff m = C_2$$

$$\Rightarrow x(t) = C_1 e^{-t} + C_2 e^t$$

2) $\dot{x} = x(1-x^2)$

$f(x) = x(1-x^2) = 0 \iff x \in \{-1, 0, 1\}$

$f'(x) = x' - x^{3\prime} = 1 - 3x^2$

$f'(-1) = -2 < 0 \Rightarrow$ asymp. stable

$f'(0) = 1 > 0 \Rightarrow$ unstable

$f'(1) = -2 < 0 \Rightarrow$ asymp. stable

3) $$\begin{cases} \dot{x} = -y + x(1-x^2-y^2) = -y + x - x^3 - y^2 x \\ \dot{y} = x + y(1-x^2-y^2) = x + y - yx^2 - y^3 \end{cases}$$

Linearization Method

x^* equim. p. $\Rightarrow \begin{cases} f'(x^*) < 0 \Rightarrow x^* \text{ attractor} \\ f'(x^*) > 0 \Rightarrow x^* \text{ repeller} \\ f'(x^*) = 0 \Rightarrow ? \end{cases}$

$$J_{f(x)} = \begin{pmatrix} \frac{\partial f_1}{\partial x}(x,y) & \frac{\partial f_1}{\partial y}(x,y) \\ \frac{\partial f_2}{\partial x}(x,y) & \frac{\partial f_2}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} 1-3x^2-y^2 & -1-2xy \\ 1-2xy & 1-x^2-3y^2 \end{pmatrix}$$

$(0,0) \Rightarrow J_{f(0,0)} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\det(J_{f(0,0)} - \lambda I_2) = 0 \Leftrightarrow \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Leftrightarrow$

$\Leftrightarrow \lambda^2 - 2\lambda + 2 = 0$

$\Delta = 4 - 8 = -4 \Rightarrow \begin{cases} \lambda_1 = 1+i \\ \lambda_2 = 1-i \end{cases}$

$\Rightarrow$ instable

4) c) $\begin{cases} \dot{x} = xy - x \\ \dot{y} = 2xy - y \end{cases}$

$\frac{dy}{dx} = \frac{f_2(x,y)}{f_1(x,y)} \Leftrightarrow \frac{dy}{dx} = \frac{2xy - y}{xy - x} \Leftrightarrow$

$\frac{2x-1}{x - \frac{x}{y}} = \frac{2x-1}{x(1-\frac{1}{y})}$

$dy(1 - \frac{1}{y}) = \frac{2x-1}{x} dx$

$\int(1 - \frac{1}{y})dy = \int 2 - \frac{1}{x} dx \Leftrightarrow y - \ln|y| =$

$= 2x - \ln|x| + C, C \in \mathbb{R}$

$2x - \ln|x| + y - \ln|y| + C = 0 \Leftrightarrow 2x + y - \ln|xy| + C = 0$

$H(x,y)$ first int $\Leftrightarrow f_1(x,y)\frac{\partial H}{\partial x}(x,y)+f_2(x,y)\frac{\partial H}{\partial y}(x,y)=0$

$(xy-x)(2-\frac{1}{x})+(2xy-y)(1-\frac{1}{y})=0$ - true

$2xy-2x-y+1+2xy-y-2x+1=0$