Theory

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1) The characteristic equation method

- for linear differential and difference equations with constant coefficients;
- !! this can be applied only if all the coefficients are constant.

Steps:

I. $x^{(n)} + a_1 x^{(n-1)} + \dots + a_{n-1} x' + a_n x = 0$

write characteristic equation:

$r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_n = 0$

II. Find the roots $r_1, r_2, \dots, r_n$.

III. Find n functions $x_1(t), x_2(t), \dots, x_n(t)$ for every root.

IV. General solution:

$$x(t) = c_1 x_1(t) + c_2 x_2(t) + \dots + c_n x_n(t)$$

ex. • $x'' - x = 0$

$r^2 - 1 = 0 \iff (r-1)(r+1) = 0$

$\Rightarrow r_1 = -1, r_2 = 1$

$\boxed{r = c \Rightarrow x(t) = e^{rt}}$

$x_1(t) = e^{-t}$

$x_2(t) = e^{t}$

$\Rightarrow x(t) = c_1 e^{-t} + c_2 e^{t}, \quad c_1, c_2 \in \mathbb{R}$

• $x'' + x = 0$

$r^2 + 1 = 0 \iff r^2 = -1$

$\Rightarrow r_1 = -i, r_2 = i$

$$\boxed{r = a \pm bi \Rightarrow \begin{cases} x_1(t) = e^{at} \sin(bt) \\ x_2(t) = e^{at} \cos(bt) \end{cases}}$$

$x_1(t) = \cos t$

$x_2(t) = \sin t$

$x(t) = c_1 \cos t + c_2 \sin t$

- $x'' - 2x' + x = 0$

$r^2 - 2r + 1 = 0 \Leftrightarrow (r-1)^2 = 0$

$\Rightarrow r_1 = 1, r_2 = 1$

$x_1(t) = e^t$

$x_2(t) = te^t$

$x(t) = c_1 e^t + tc_2 e^t$

For multiplicity:

$r_1 = c, r_2 = c, \ldots, r_n = c$

$\mathbb{R}$: $\Downarrow$

$x_1(t) = e^t$

$x_2(t) = te^t$

$\vdots$

$x_n(t) = t^{n-1} e^t$

$\mathbb{C}$:

2^n functions (for each one: sin/cos

2) The reduction method

- for a linear planar system with constant coefficients;
- !! this can be applied only if all the coefficients are constant;
- we need to find 2^{nd} order LDE and relation between x, x', x''.

$$(1) \begin{cases} x' = a_{11}x + a_{12}y \\ y' = a_{21}x + a_{22}y \end{cases} \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_n(\mathbb{R})$$

$$x' = a_{11}x + a_{12}y \Rightarrow y = \frac{1}{a_{12}}(x' - a_{11}x) \quad (2)$$

$$x' = a_{11}x + a_{12}y \Rightarrow x'' = a_{11}x' + a_{12}y' \xrightarrow{y' = a_{21}x + a_{22}y}$$

$$\Rightarrow x'' = a_{11}x' + a_{12}a_{21}x + a_{12}a_{22}y \overset{(2)}{\Longrightarrow} x'' = a_{11}x' + a_{12}a_{21}x + a_{22}x' - a_{22}a_{11}x(t)$$

$$\Leftrightarrow x'' - (a_{11} + a_{22})x' + (a_{11}a_{22} - a_{12}a_{21})x = 0 \quad (3)$$

Now we use characteristic equation method.

$$l(r) = r^2 - (\underbrace{a_{11} + a_{22}}_{\operatorname{tr} A})r + (\underbrace{a_{11}a_{22} - a_{12}a_{21}}_{\det A}) = 0$$

$$\ell(r) = r^2 - (\underbrace{a_{11}+a_{22}}_{\operatorname{tr} A})\, r + (\underbrace{a_{11}a_{22} - a_{12}a_{21}}_{\det A})$$

roots of $\ell(r)$ = eigenvalues of A

$$\boxed{\det(A - \lambda I_2) = 0}$$

ex. $e^{t\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}}$

$$A = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

$$\begin{cases} x' = y \\ y' = -x \end{cases} \Rightarrow x'' = y' \Rightarrow x'' = -x \Rightarrow \begin{array}{l} x'' + x = 0 \\ y = x' \end{array}$$

$$r^2 + 1 = 0 \Rightarrow r = \pm i \Rightarrow \begin{cases} x_1(t) = \cos t \\ x_2(t) = \sin t \end{cases}$$

$$\Rightarrow x(t) = c_1 \cos t + c_2 \sin t$$

$$\Rightarrow \begin{cases} x = c_1 \cos t + c_2 \sin t \\ y = -c_1 \sin t + c_2 \cos t \end{cases}$$

$$\text{ivp} = \begin{cases} x' = y \\ y' = -x \\ x(0) = 1 \\ y(0) = 0 \end{cases}$$

$x(0) = c_1$

$x(0) = c_2$

$\Rightarrow c_1 = 1$

$c_2 = 0$

$$\Rightarrow \varphi(t) = \begin{pmatrix}\cos t \\ -\sin t\end{pmatrix}$$

or

$$\begin{pmatrix}\sin t \\ \cos t\end{pmatrix}$$

$t\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ $\begin{pmatrix}\cos t & \sin t \\ \dots \end{pmatrix}$

$$e^{t\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}} = \begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}$$

3) <u>Phase portrait of a dynamical system</u>

- the DS is associated to a scalar differential equation,

<u>Steps</u>

I. Find equilibra ($f(x)=0$).

II. Find sign of f on the intervals determined by the equilibria.

III. Represent orbits on $\mathbb{R}$ and draw arrows.

$$\begin{cases} \text{if } f > 0 \text{ on orbit} \Rightarrow > \\ \text{if } f < 0 \text{ on orbit} \Rightarrow < \end{cases}$$

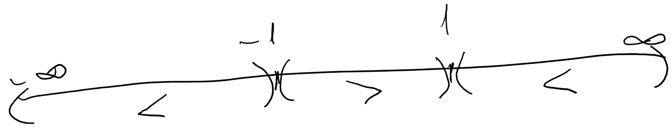
<u>Orbits</u> are intervals containing the equilibria.

ex. $f(x) = 1 - x^2$

$f(x) = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x_{1,2} = \pm 1$

x	$-\infty$		-1		1		∞
$f(x)$	$-$	$-$	0	$+ \ +$	0	$-$	$-$

orbits: $(-\infty, -1)$, $\{-1\}$, $(-1, 1)$, $\{1\}$, $(1, \infty)$



4) <u>Attractor for continuous/discrete dynamical systems</u>

For Continuous DSs:

If x^* is equilibrium point ($f(x)=0$) $\Rightarrow \begin{cases} f'(x^*) < 0 \Rightarrow \text{attractor} \\ f'(x^*) > 0 \Rightarrow \text{repellor} \\ f'(x^*) = 0 \Rightarrow \text{method fails} \end{cases}$

$(f(x)=0)!$ | $f'(x^*) = 0 \Rightarrow$ method fails

For Discrete DSs:

If x^* is fixed point $\Rightarrow$ if $|f'(x^*)| < 1$, then x^* attractor
$(f(x) = x)$

Continuous: system, equation
Discrete: function, map

5) <u>Co-web diagram for scalar maps</u>

$$g(x) = x - \frac{f(x)}{f'(x)}$$

- plot and intersect with $y = x$ (basin of attraction)