Fundamental notions of quantum computing

Basic Principles; Qubits; Bloch Sphere

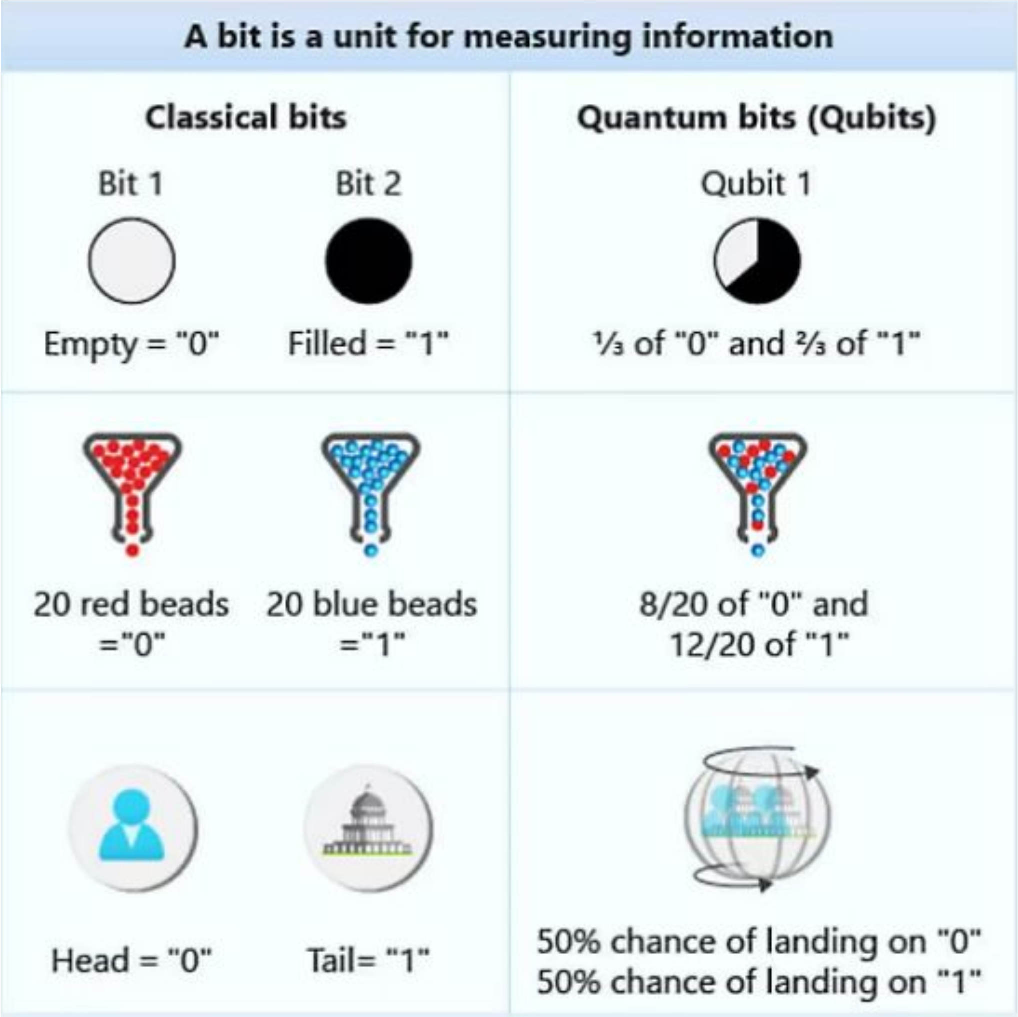
T.D. Mihoc 2024

Outline

- Qubits and Qudits
- Bloch Sphere
- Basic Principles

Qubits and Qudits

- What is a qubit
- Computational basis
- Qudis
- Multiple qubits



Qubits

Physically, **a quantum bit**, or **qubit**, is a closed quantum system that has two energy levels: *the ground state* (lower energy level) $|0\rangle$ and *the first excited state* $|1\rangle$.

Qubits

Mathematically, a qubit $|\psi\rangle \in \mathbb{C}^2$ (a two-unit vector on the complex number set $|\psi\rangle \in \mathbb{C}^2$) is

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

such that $|\alpha|^2 + |\beta|^2 = 1$ where $\alpha, \beta \in \mathbb{C}$, and $|\cdot|$ denotes the modulus $|\alpha| = \alpha \cdot \alpha$.

Qubits

Computationally, a qubit is the fundamental unit of information on a quantum computer.

Computational basis

For qubits, the vector space (more specifically, the Hilbert space) is $\mathbb{C}^2$.

The **computational basis**, known as the standard or natural basis, consists of the vectors $|0\rangle,\ |1\rangle \in \mathbb{C}^2$.

For example, the spin state of a particle can be applied as a model, with the z-spin up state $|z,+\rangle$ representing 0 and the z-spin down state $|z,-\rangle$ representing the value 1.

Qubit as a Linear Combination of Basis Vectors

Any qubit

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

can be written as a linear combination of computational basis vectors, that is,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

The Hadamard basis

is an orthonormal basis $\mathcal{H} = \{|+\rangle, |-\rangle\}$ that consists of the two basis elements

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

In order to see that $|+\rangle$ and $|-\rangle$ are indeed a basis for $\mathbb{C}^2$ we first verify that $|+\rangle$ and $|-\rangle$ are normalized.

The Hadamard basis

$$\langle +|+\rangle = \langle +||+\rangle = \frac{1}{\sqrt{2}}(1\ 1)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\cdot 2 = 1$$

$$\langle -|-\rangle = \langle -||-\rangle = \frac{1}{\sqrt{2}}(1\ \ -1)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}\cdot 2 = 1$$

The inner product of the two vectors is:

$$\langle +|-\rangle = \langle +||-\rangle = \frac{1}{2}(1\ 1)\begin{pmatrix}1\\-1\end{pmatrix} = 0.$$

Therefore, the two elements are orthogonal to each other.

Qudits

A **qudit**, or a **d-dimensional quantum system**, can be represented as a d-dimensional ket vector $|\sigma\rangle \in \mathbb{C}^d$,

$$|\sigma\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle, \text{ where } \forall i = \overline{0, d-1} \text{ we have } \alpha_i \in \mathbb{C}, \text{ and } \sum_{i=0}^{d-1} |\alpha_i|^2 = 1.$$

Multiple qubits

To write more than one qubit, we can associate each of the two classical bits $x_1, x_2 \in \{0,1\}^2$ with a vector. We have four orthonormal vectors:

$$00 \longrightarrow |00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad 01 \longrightarrow |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$10 \longrightarrow |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad 11 \longrightarrow |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Multiple qubits – example

Let us consider a state $|\Psi_{AB}\rangle$ that is a superposition of all these states:

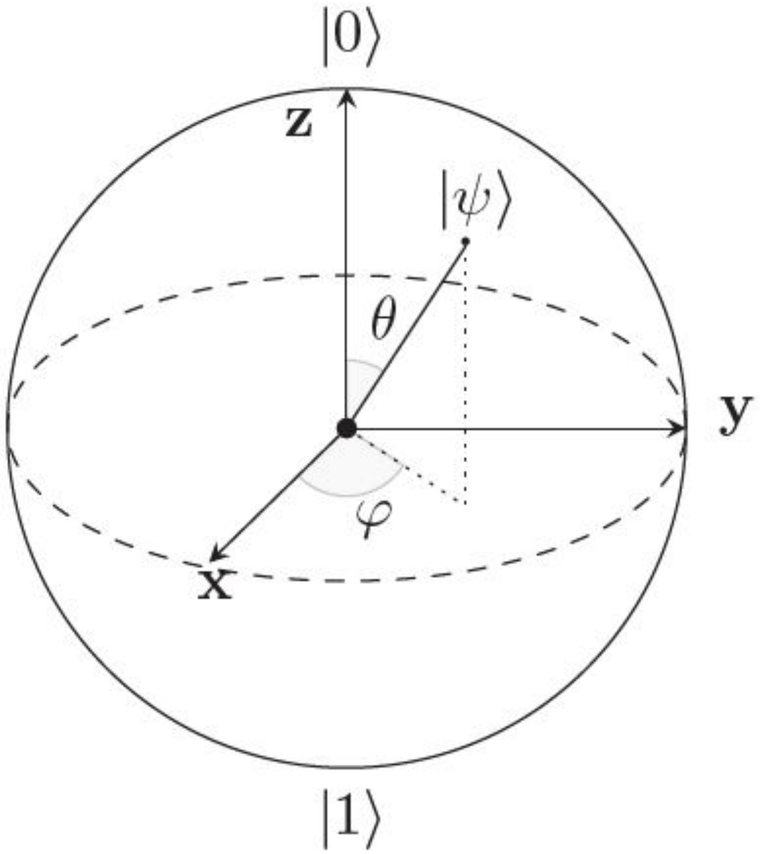
$$|\Psi\rangle_{AB} = \tfrac{1}{2}|00\rangle_{AB} + \tfrac{1}{2}|01\rangle_{AB} + \tfrac{1}{2}|10\rangle_{AB} + \tfrac{1}{2}|11\rangle_{AB}$$

$$= \tfrac{1}{2}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \tfrac{1}{2}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \tfrac{1}{2}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \tfrac{1}{2}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$= \tfrac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

Bloch Sphere

- What is a bloch sphere
- Examples
- Tools for creating such representations



What is it?

A very intuitive way to visualize one single qubit!

In quantum mechanics and computing, the Bloch sphere is a **geometrical representation** of the pure state space of a two-level quantum mechanical system (qubit), named after the physicist Felix Bloch.

Bloch Sphere

is a three-dimensional sphere with a radius R = 1, in which each point on its surface represents a unique pure state of a qubit.

- north pole represents the basis state $|0\rangle$
- south pole represents the basis state $|1\rangle$

Any other point on the sphere's surface is a linear combination (superposition) of these basis states.

Bloch Sphere

According to Euler's formula, we can write the complex number z_c in trigonometric form as:

$$z_c = r_c e^{i\theta_c}$$

In equation $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, we express the complex numbers α and β in this form:

$$|\psi\rangle = r_\alpha e^{i\theta_\alpha}|0\rangle + r_\beta e^{i\theta_\beta}|1\rangle$$

Quantum global phase

if two quantum states in polar form (as described above) differ only by a factor of some $e^{i\theta}$, they are considered indistinguishable. These two states, $|\psi\rangle$and $|\psi'\rangle$, can be treated as mathematically identical.

$$|\psi'\rangle = e^{-i\theta_\alpha}|\psi\rangle$$

This concept will allow us to eliminate a variable.

$$|\psi\rangle = e^{-i\theta_\alpha}\left(r_\alpha e^{i\theta_\alpha}|0\rangle + r_\beta e^{i\theta_\beta}\right)|1\rangle = r_\alpha|0\rangle + r_\beta e^{i(\theta_\alpha - \theta_\beta)}|1\rangle$$

Qubits in polar coordinates

In polar coordinates, a qubit can be written as:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

where $0 \leq \theta \leq \pi$ and $0 \leq \varphi < 2\pi$

Remark: the norm of this qubit is indeed 1:

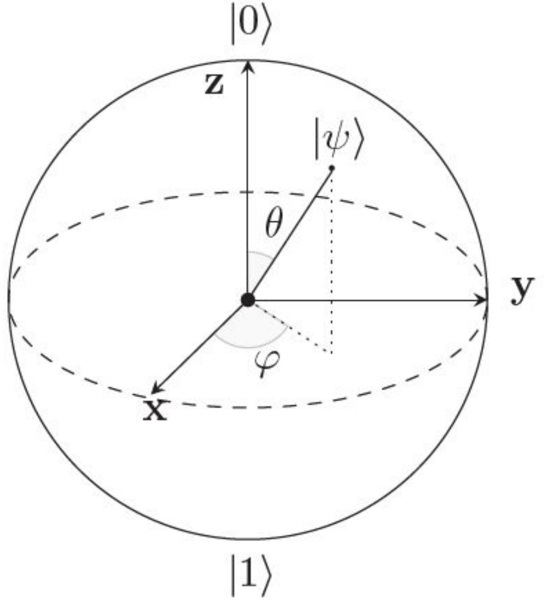
$$\langle\psi|\psi\rangle = cos^2\left(\frac{\theta}{2}\right) + e^{i\varphi}e^{-i\varphi}sin^2\left(\frac{\theta}{2}\right) = cos^2\left(\frac{\theta}{2}\right) + sin^2\left(\frac{\theta}{2}\right) = 1$$

The qubit state function in the Bloch Sphere

In spherical coordinates, a point is defined by R (radius) and two angles θ and φ.

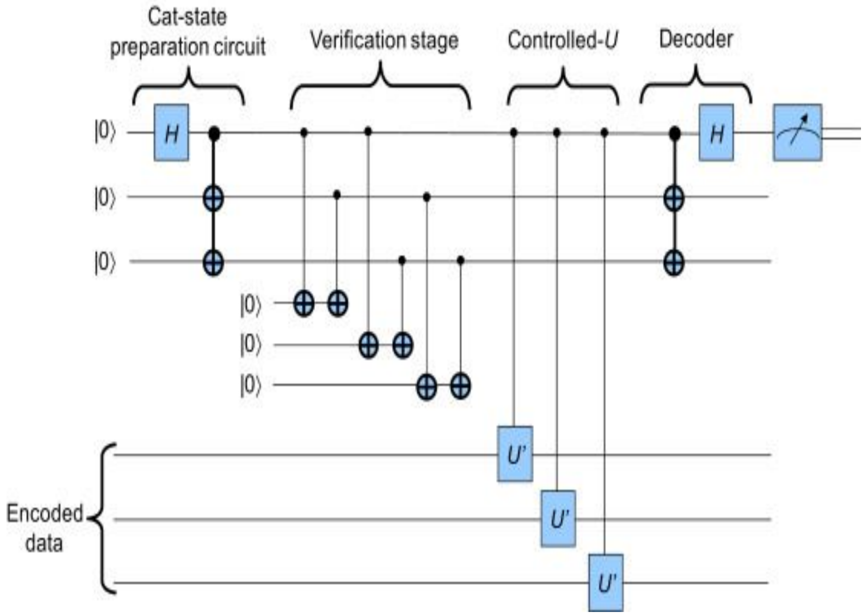
Since R is fixed to 1, only φ and θ are variable.

$$|\psi\rangle = cos^2\left(\frac{\theta}{2}\right)|0\rangle + (cos(\varphi) + i\ sin(\varphi)) \cdot sin^2\left(\frac{\theta}{2}\right)|1\rangle$$



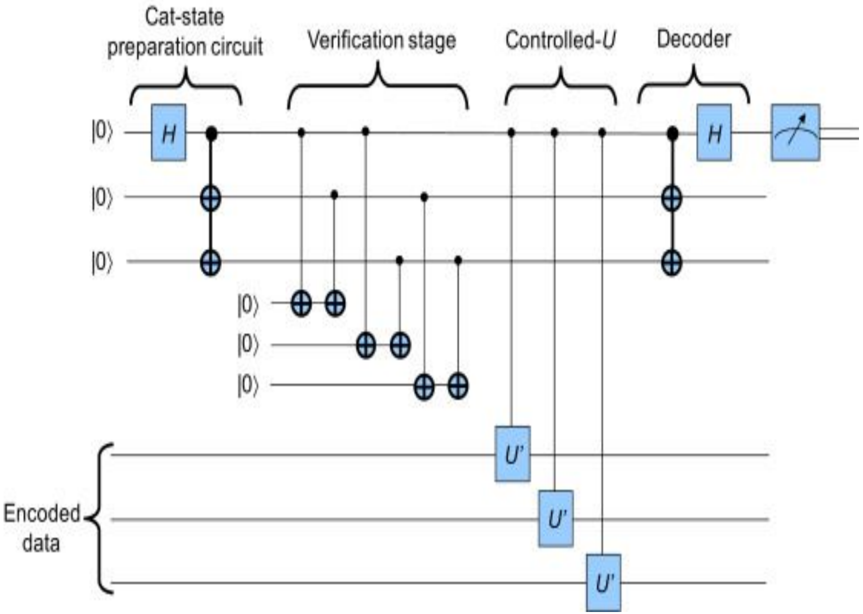
Basic Principles

- Circuits
- Measurement
- Superposition
- Randomness
- Entanglement



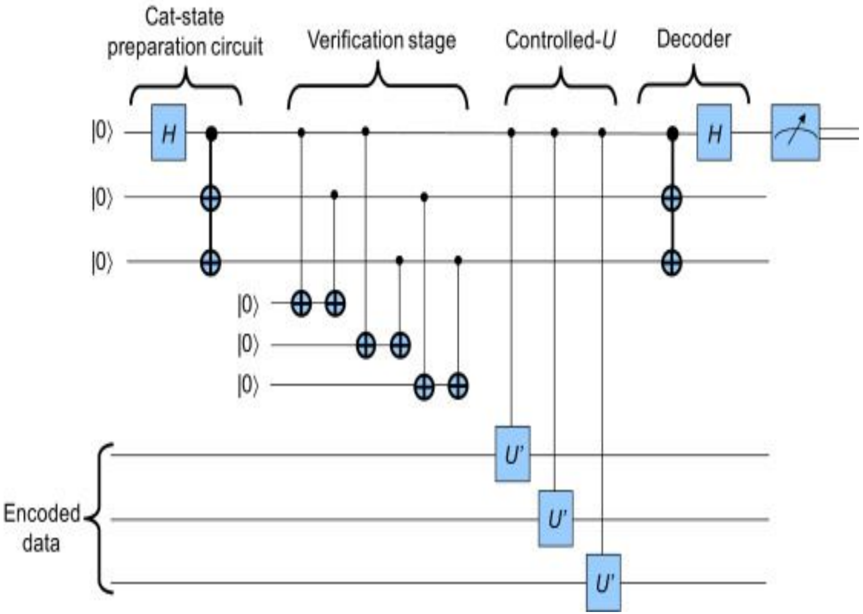
Quantum circuits

- a model for quantum computation,
- similar to classical circuits,
- a computation is a sequence of:
    - quantum gates,
    - measurements,
    - initializations of qubits to known values,
    - other actions.



Quantum circuits

- The horizontal axis is time, starting at the left hand side and ending at the right;
- Horizontal lines are qubits;
- Doubled lines represent classical bits;
- The items that are connected by these lines are operations performed on the qubits (measurements or gates);
- Lines define the sequence of events (not physical cables).



Vector states

As we know each qubit can be represented by a vertical vector:

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

This vector is a representation of the quantum state, and is called a **vector state**.

State-changing

Each instruction i may be viewed as a matrix (i.e. linear) transformation from one basis-state vector to another.

matrix transformation $\longrightarrow$ an $n \times n$ matrix A_i

If $|v_i\rangle$ is a state vector, then $A_i|v_i\rangle$ is another state vector that encodes the configuration that results when executing i on the configuration encoded by $|v_i\rangle$.

Linear transformations

Points of interest:

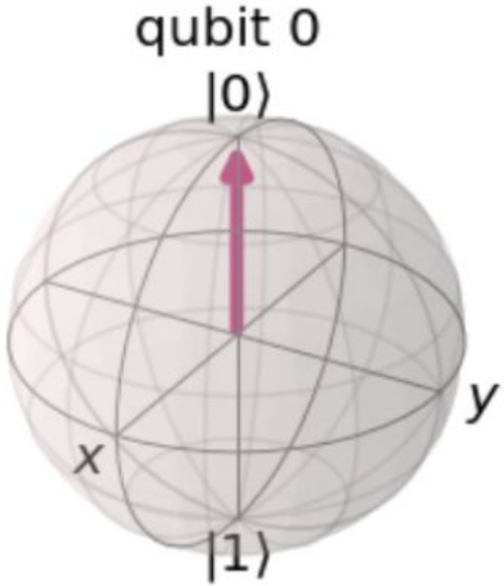
- Eigenvalues and eigenvectors
- Diagonalizable matrices
- Normal and Hermitian matrices
- Unitary matrices

Linear transformations

Advantage:

We can express a sequence of linear transformations as matrix multiplications!

Visualise a state vector on a Bloch Sphere




```
from qiskit.visualization import plot_bloch_multivector

qc = QuantumCircuit(1)

# execute the quantum circuit

backend = Aer.get_backend('statevector_simulator')

result = execute(qc, backend).result()

stateVectorResult = result.get_statevector(qc)

#Display the Bloch sphere

plot_bloch_statevector(stateVectorResult)
```

Registers

A **Quantum register** is a collection of n qubits.

A **Classic register** is a collection of m bits.

Quantum logic gates

Similar to their classical counterparts: they are used to perform operations by manipulating the qubits in such a way that the results serve to provide a solution.

Different in part because **they perform linear transformations** on the qubit in a complex vector space.

Quantum logic gates

Unique property:

REVERSIBILITY

you can reverse the operation of the qubit gate to obtain the previous state

Measure

Considering a qubit:

$$|\varphi\rangle = a|0\rangle + b|1\rangle$$

Measuring a qubit make it collapse in the state $|0\rangle$with the probability a^2 or in $|1\rangle$ with the probability b^2.

As a direct consequence is the **Holevo's theorem**:

n qubits can not hold more information than n bits.

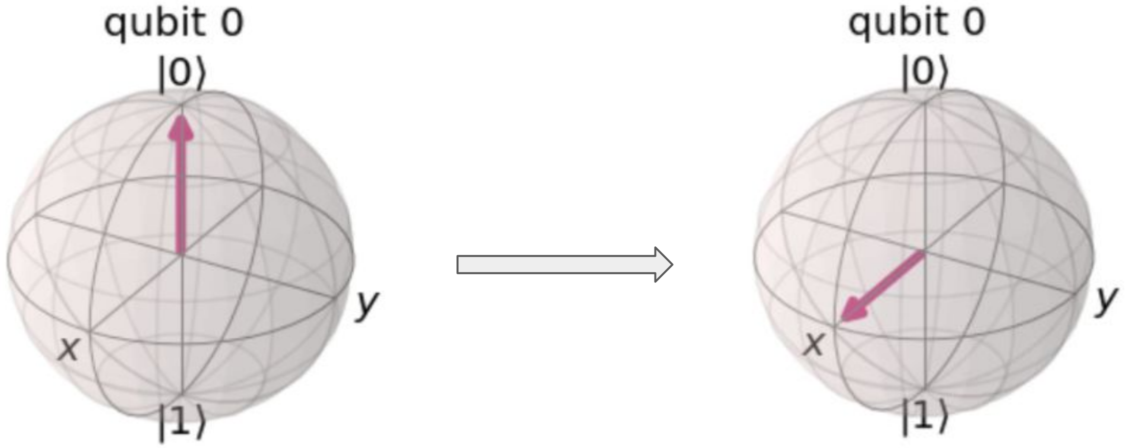
Superposition

We use a **Hadamard gate**.

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

an operational gate that places the qubit in a superposition state – a complex linear combination of the basis states, which means that when we measure the qubit, it will have an equal probability result of measuring a 0 or 1.

H gate on a Bloch Sphere



Superposition of a qubit after 90° rotation around the X and Z axes

The H gate placed the vector on the positive X axis from the |0⟩basis state.

Randomness

Observe that **measuring** a qubit put in superposition with a Hadamard gate from the state $|0\rangle$or $|1\rangle$will make it collapse with equal probability in a pure state thus allowing to create **a true random bit**.

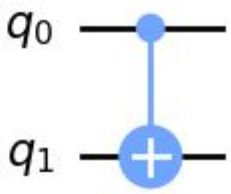
Entanglement

Is defined as a quantum mechanical phenomenon that occurs when two or more particles have **correlated states**.

If you have two qubits that are entangled when we measure one qubit, we can determine the result of the other qubit based on the measurement of the first qubit.

Entanglement

We use a multi-qubit gate called a **Control-NOT (CNOT)**. It has two connecting points – one called **Control** and another called **Target**.



Before CNOT		After CNOT	
Control	Target	Control	Target
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$

Thank you for your attention!