1.4 Birkhoff Interpolation

Consider the following situation: We have a moving object and the times $t_0, t_1, \ldots, t_m$. For some of these nodes, we know the *distances* traveled $d_i = d(t_i), i \in I \subset \{0, 1, \ldots, m\}$, for others, the *velocities* $v_j = d'(t_j), j \in \tilde{I} \subset \{0, 1, \ldots, m\}$ and for others, the *accelerations* $a_k = d''(t_k), k \in I^* \subset \{0, 1, \ldots, m\}$. Having all these data, can we find a polynomial approximation of the distance function $d = d(t)$ on the entire interval containing the points $t_0, \ldots, t_m$?

Obviously, this is *not* a Lagrange interpolation problem, because we do not have the values of the function at all the nodes. We *cannot* find a Hermite polynomial, either, because at some nodes, only the value of the derivative (or the second derivative) is given (without the values of the function). This is a **Birkhoff interpolation** problem, also known as *lacunary Hermite interpolation* (because not *all* the functional or derivative values for all points are provided) and it is more general than Hermite interpolation.

1.4.1 Birkhoff interpolation polynomial

Birkhoff interpolation problem. Let $x_k \in [a, b], k = \overline{0, m}$, be $m + 1$ distinct nodes, $r_k \in \mathbb{N}$ and $I_k \subseteq \{0, , \ldots, r_k\},\ k = 0, \ldots, m$. Consider the function $f : [a, b] \to \mathbb{R}$ whose derivatives $f^{(j)}(x_k), k = 0, \ldots, m, j \in I_k$ exist. Find a polynomial $P(x)$ of minimum degree, satisfying the interpolation conditions

$$P^{(j)}(x_k) \quad = \quad f^{(j)}(x_k),\ k = \overline{0, m},\ j \in I_k. \tag{1.1}$$

Denote by $n + 1 = |I_0| + \cdots + |I_m|$, where $|I_k|$ is the cardinality (number of elements) of I_k. There are $n + 1$ interpolation conditions in (1.1), so we seek a polynomial of degree at most n,

$$P_n(x) \quad = \quad a_0 + a_1 x + \cdots + a_n x^n,$$

whose coefficients are found from the linear system generated by the interpolation conditions (1.1). If the determinant of this system is not equal to zero, then the Birkhoff interpolation problem has a unique solution.

Remark 1.1. If $I_k = \{0, 1, \ldots, r_k\}$, for every $k = 0, \ldots, m$, then the Birkhoff interpolation problem is reduced to Hermite interpolation (which, in turn, is reduced to Lagrange interpolation when $r_k = 0, k = 0, \ldots, m$). Hence, Birkhoff interpolation is more general.

Unlike Lagrange and Hermite interpolation, the Birkhoff interpolation problem (1.1) *does not* always have a solution. When such a polynomial, denoted by $B_n f$, exists, it has the form

$$B_n f(x) = \sum_{k=0}^{m} \sum_{j \in I_k} b_{kj}(x) f^{(j)}(x_k). \tag{1.2}$$

The terms $b_{kj}(x)$ are called **Birkhoff fundamental polynomials** and they satisfy the relations:

$$\begin{aligned} b_{\boldsymbol{k}j}^{(p)}(x_{\boldsymbol{\nu}}) &= 0,\ \boldsymbol{\nu} \neq \boldsymbol{k},\ p \in I_\nu, \\ b_{\boldsymbol{kj}}^{(\boldsymbol{p})}(x_{\boldsymbol{k}}) &= \delta_{\boldsymbol{jp}},\ p \in I_k,\ \text{for } j \in I_k \text{ and } \nu, k = 0, 1, \ldots, m, \end{aligned} \tag{1.3}$$

where

$$\delta_{jp} = \begin{cases} 0, & j \neq p \\ 1, & j = p \end{cases}$$

is Kronecker's symbol.

Remark 1.2. Because some of the functional (or derivative) values are missing, finding mathematical expressions for the Birkhoff fundamental polynomials $b_{kj}, k = 0, \ldots, m; j \in I_k$, is, in general, difficult. They can be determined (when possible) directly from din conditions (1.3).

Example 1.3. Let $f \in C^2[0,1]$ and consider the nodes $x_0 = 0,\ x_1 = 1$, for which the values $f(0) = 1$ and $f'(1) = 2$ are given. Find the Birkhoff polynomial that interpolates these data.

Solution.
We have $m = 1$, two nodes, with $I_0 = \{0\}, I_1 = \{1\}$, so $n = 1 + 1 - 1 = 1$. We want a polynomial of degree 1,

$$P(x) = a_0 + a_1 x,$$

satisfying the conditions

$$\begin{aligned} P(0) &= f(0), \\ P'(1) &= f'(1). \end{aligned}$$

From here, we have the linear system

$$\begin{cases} a_0 = f(0) \\ a_1 = f'(1) \end{cases}.$$

The determinant of this system is

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0,$$

so this problem has a unique solution

$$\begin{cases} a_0 &= f(0) \\ a_1 &= f'(1) \end{cases},$$

i.e. the polynomial we seek is

$$P(x) = f(0) + f'(1)x = 1 + 2x.$$

On the other hand, by (1.2), the Birkhoff polynomial is of the form

$$B_1 f(x) = b_{00}(x) f(0) + b_{11}(x) f'(1).$$

Let us find the fundamental polynomials $b_{00}(x)$ şi $b_{11}(x)$. Both have degree 1, hence,

$$\begin{aligned} b_{00}(x) &= ax + b, \\ b_{11}(x) &= cx + d. \end{aligned}$$

By conditions (1.3), for $b_{00,}$ we have

$$\begin{cases} b_{00}(x_0) &= 1 \\ b'_{00}(x_1) &= 0 \end{cases} \iff \begin{cases} b_{00}(0) &= 1 \\ b'_{00}(1) &= 0 \end{cases} \iff \begin{cases} b &= 1 \\ a &= 0 \end{cases},$$

thus,

$$b_{00}(x) = 1.$$

Similarly, for b_{11}, we have

$$\begin{cases} b_{11}(x_0) &= 0 \\ b'_{11}(x_1) &= 1 \end{cases} \iff \begin{cases} b_{11}(0) &= 0 \\ b'_{11}(1) &= 1 \end{cases} \iff \begin{cases} d &= 0 \\ c &= 1 \end{cases},$$

so we get

$$b_{11}(x) = x.$$

Thus,

$$B_1 f(x) = f(0) + xf'(1) = 1 + 2x.$$

■

Example 1.4. Find a polynomial of smallest degree (if it exists) satisfying the conditions

$$P(-1) = P(1) = 0, \; P'(0) = 1.$$

Solution.
Here, we have 3 nodes, $x_0 = -1, x_1 = 0, x_2 = 1$, $m = 2$, for which $I_0 = \{0\}, I_1 = \{1\}, I_2 = \{0\}$. Hence, we seek a polynomial of degree $n = 1 + 1 + 1 - 1 = 2$. This is of the form

$$P(x) = a_0 + a_1 x + a_2 x^2$$

and must satisfy the relations

$$\begin{aligned} P(-1) &= 0, \\ P'(0) &= 1, \\ P(1) &= 0. \end{aligned}$$

We obtain the linear system

$$\left\{ \begin{array}{ccccccc} a_0 & - & a_1 & + & a_2 & = & 0 \\ & & a_1 & & & = & 1 \\ a_0 & + & a_1 & + & a_2 & = & 0 \end{array} \right. .$$

Subtracting the first equation from the third, we get $a_1 = 0$, which contradicts the second equation. The system is incompatible and, thus, this interpolation problem *does not have* a solution.

■

Example 1.5. [Abel-Goncearov interpolation] Let $f \in C^{n+1}[0, nh]$, with $h > 0, \; n \in \mathbb{N}$. Find a polynomial of smallest degree satisfying the relations

$$\begin{aligned} P(0) &= f(0), \\ P'(h) &= f'(h), \\ \dots \\ P^{(n)}(nh) &= f^{(n)}(nh). \end{aligned}$$

This problem has a unique solution for every $h > 0,\ n \in \mathbb{N}$. Notice that the problem in Example 1.3 was of this type, with $n = 1$ and $h = 1$.

1.4.2 Peano's theorem and the error for Birkhoff interpolation

To find an error formula for Birkhoff interpolation (when the Birkhoff polynomial exists), we need an important result from linear operator theory.

Let us recall some notions and properties:

- Let $n \in \mathbb{N}^*$. We define the space

$$H^n[a,b] = \{f : [a,b] \to \mathbb{R} \mid f \in C^{n-1}[a,b], f^{(n-1)} \text{ absolutely continuous on } [a,b]\}.$$

- A function $f : [a,b] \to \mathbb{R}$ is *absolutely continuous* on $[a,b]$, if, for instance, it has a derivative f' almost everywhere, the derivative is Lebesgue integrable, and

$$f(x) = f(a) + \int_a^x f'(t)dt,\ \forall x \in [a,b].$$

- $H^n[a,b]$ is linear space.
- Any function $f \in H^n[a,b]$ has a Taylor-type representation, with the remainder in integral form

$$f(x) = \sum_{k=0}^{n-1} \frac{(x-a)^k}{k!} f^{(k)}(a) + \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)\, dt.$$

- The function

$$z_+ = \begin{cases} z, & z \geq 0 \\ 0, & z < 0 \end{cases}$$

is called the *positive part* of z, and $z_+^n = (z_+)^n$ is called a *truncated power*.

- For $m \in \mathbb{N}$, $\mathbb{P}_m$ denotes the space of all polynomials of degree at most m. Obviously, $\mathbb{P}_m \subset C^\infty[a,b], \forall m \geq 0$.
- The *kernel* of a linear map $L : V \to W$ between two vector spaces V and W, is the set of all vectors in V that are mapped to zero:

$$\ker L \quad = \quad \{v \in V \mid L(v) = \mathbf{0}_{\boldsymbol{W}}\},$$

where $\mathbf{0}_{\boldsymbol{W}}$ is null vector in W.

The next theorem is **paramount** in Numerical Analysis. It gives a representation of real linear functionals defined on a space $H^m[a,b]$. This result provides means for expressing the errors in many approximating procedures.

Theorem 1.6. [Peano]
Let $L : H^{n+1}[a,b] \to \mathbb{R}$ *be a linear functional that commutes with the definite integral operator. If* $\ker L = \mathbb{P}_n$, *then*

$$Lf(x) \quad = \quad \int_a^b K_n(x,t) f^{(n+1)}(t) dt, \tag{1.4}$$

where

$$K_n(x,t) \quad = \quad \frac{1}{n!} L\Big((x-t)_+^n\Big) \tag{1.5}$$

is called the ***Peano kernel****.*

So, this is saying that if L maps *all* polynomials of degree at most n to the *function identically equal to* 0, i.e. $Le_k \equiv 0, e_k(x) = x^k, k = 0, 1, \ldots, n, Le_{n+1} \not\equiv 0$, then Lf can be expressed as in (1.4).

Corollary 1.7. If the kernel K has constant sign on $[a,b]$ and $f^{(n+1)}$ is continuous on $[a,b]$, then there exists $\xi \in (a,b)$ such that

$$Lf \quad = \quad \frac{1}{(n+1)!} f^{(n+1)}(\xi) Le_{n+1}, \tag{1.6}$$

where $e_k(x) = x^k, \; k \in \mathbb{N}$.

Proof. If the kernel K has constant sign on $[a, b]$, we can apply the mean value theorem in (1.4):

$$Lf = f^{(n+1)}(\xi) \int_a^b K_n(x,t)\, dt, \ \xi \in (a,b). \tag{1.7}$$

Notice that the kernel K *does not* depend on f and the relation above is true *regardless* of the function f. Then, taking $f = e_{n+1}$, we get

$$\begin{aligned} Le_{n+1} &= e_{n+1}^{(n+1)}(\xi) \int_a^b K_n(x,t)\, dt \\ &= (n+1)! \int_a^b K_n(x,t)\, dt, \end{aligned}$$

from which we get

$$\int_a^b K_n(x,t)\, dt = \frac{1}{(n+1)!} Le_{n+1}.$$

Using this in (1.7), we obtain (1.6).

□

Remark 1.8. This corollary is the one that is mostly used in applications, to assess the approximation error. We apply Theorem 1.6 or Corollary 1.7 to the *remainder functional*. We derive an approximation formula

$$f(x) = B_n f(x) + R_n f(x).$$

Since B_n is a polynomial of degree n, the formula above is *exact* for all polynomials of degree n, i.e.

$$R_n e_k = (f - B_n) e_k = \mathbf{0}.$$

In this case, we say that the approximation formula has *degree of precision* (or *degree of exactness*)

$d = n$. Then,

$$
\begin{aligned}
K_n(x,t) &= \frac{1}{n!} R_n\Big((x-t)_+^n\Big) \\
&= \frac{1}{n!}\Big[(x-t)_+^n - B_n\big((x-t)_+^n\big)\Big]. \qquad (1.8)
\end{aligned}
$$

Now, it is easy to check (from the definition) that the function $F(x) = (x-t)_+^n$ has the *derivative*

$$
F'(x) = \frac{\partial\big[(x-t)_+^n\big]}{\partial x} = n(x-t)_+^{n-1}
$$

and the *integral* (this will only be needed later on, in Chapter 4)

$$
\int_a^b F(x)dx = \frac{1}{n+1}(x-t)_+^{n+1}\Big|_{x=a}^{x=b} = \frac{1}{n+1}\left[(b-t)_+^{n+1} - (a-t)_+^{n+1}\right]
$$

If $K_n(x,t)$ above has constant sign on $[a,b]$ (the smallest interval containing the interpolation nodes), then we have an expression for the error of the approximation, as

$$
R_n f = \frac{1}{(n+1)!} f^{(n+1)}(\xi) R_n e_{n+1},\ \xi \in (a,b). \qquad (1.9)
$$

Example 1.9. Let us find a formula for the rest of the Birkhoff polynomial in Example 1.3.

Solution. We found the Birkhoff polynomial

$$
B_1 f(x) = f(0) + f'(1)x,\ x \in [0,1],
$$

so, we have

$$
f(x) = B_1 f(x) + R_1 f(x).
$$

We apply Peano's theorem to the remainder operator,

$$
\begin{aligned}
Lf &= R_1 f = f - B_1 f, \\
Lf(x) &= (f - B_1 f)(x) = f(x) - \big(f(0) + f'(1)x\big).
\end{aligned}
$$

We have

$$
\begin{aligned}
R_1e_0(x) &= e_0(x) - B_1e_0(x) = e_0(x) - \big(e_0(0) + e_0'(1)x\big) = 1 - (1+0) \equiv 0,\\
R_1e_1(x) &= e_1(x) - B_1e_1(x) = e_1(x) - \big(e_1(0) + e_1'(1)x\big) = x - (0 + 1\cdot x) \equiv 0,\\
R_1e_2(x) &= e_2(x) - B_1e_2(x) = e_2(x) - \big(e_2(0) + e_2'(1)x\big) = x^2 - 2x \not\equiv 0,
\end{aligned}
$$

(the first two were obvious, since B_1 is a polynomial of degree 1, but it is a good computational exercise). Thus,

$$
R_1f(x) = \int_0^1 K_1(x,t)f''(t)\,dt,
$$

with

$$
\begin{aligned}
K_1(x,t) &= \frac{1}{1!}R_1\left((x-t)_+^1\right)\\
&= \underbrace{(x-t)_+}_{f(x)} - \Big(\underbrace{(0-t)_+}_{f(0)} + \underbrace{1\cdot x}_{f'(1)x}\Big)\\
&= (x-t)_+ - (-t)_+ - x.
\end{aligned}
$$

Since $x,t\in[0,1]$, we have $(-t)_+ = 0$. Fix an arbitrary $x\in[0,1]$.
If $0\le t\le x$, then $(x-t)_+ = x-t$ and

$$
K_1(x,t) = x - t - x = -t \le 0.
$$

If $x\le t\le 1$, then $(x-t)_+ = 0$ and

$$
K_1(x,t) = 0 - x = -x \le 0.
$$

So, either way, K_1 has constant sign on $[0,1]$. By Corollary 1.7, it follows that

$$
\begin{aligned}
R_1f(x) &= \frac{1}{2!}f''(\xi)R_1e_2(x)\\
&= \frac{x^2-2x}{2!}f''(\xi),\ \xi\in(0,1).
\end{aligned}
$$

Now, $x(x-2)$ is a parabola with the minimum value at the midpoint value of the interval of the roots. Thus,

$$|x(x-2)| \le 1,$$

for $x \in [0,1]$ and we have the following estimate for the interpolation error:

$$|R_1 f(x)| \le \frac{1}{2}||f''||_\infty,\ \forall x \in [0,1].$$

■

Example 1.10. Let $f : [0,2] \to \mathbb{R}$ be a function in $C^3[0,2]$.

a) Find a polynomial of minimum degree that interpolates the data $f'(0), f(1)$ and $f'(2)$;

b) If $f'(0) = 1, f(1) = 2$ and $f'(2) = 1$, use the approximation found in part **a)** to approximate $f(1/2)$ and estimate the error.

Solution.
a) Since, for instance, for $x_0 = 0$, *only* the derivative is given, without the function value, this is Birkhoff interpolation. We have $m+1 = 3$ nodes, with $I_0 = \{1\}, I_1 = \{0\}, I_2 = \{1\}$, so $n = 1+1+1-1 = 2$. The Birkhoff polynomial of degree (at most) 2,

$$\begin{aligned} B_2(x) &= ax^2 + bx + c, \\ B_2'(x) &= 2ax + b, \end{aligned}$$

must satisfy the relations

$$\begin{aligned} B_2'(0) &= b = f'(0), \\ B_2(1) &= a + b + c = f(1) \\ B_2'(2) &= 4a + b = f'(2). \end{aligned}$$

The determinant of the corresponding linear system is

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} = 4 \ne 0,$$

hence, there exists a unique Birkhoff interpolation polynomial, of the form

$$B_2 f(x) = b_{01}(x) f'(0) + b_{10}(x) f(1) + b_{21}(x) f'(2).$$

The fundamental polynomials (of degree at most 2) must satisfy the conditions

$$\left\{\begin{array}{rcl} b'_{01}(0) & = & 1 \\ b_{01}(1) & = & 0 \\ b'_{01}(2) & = & 0 \end{array}\right., \quad \left\{\begin{array}{rcl} b'_{10}(0) & = & 0 \\ b_{10}(1) & = & 1 \\ b'_{10}(2) & = & 0 \end{array}\right. \text{ and } \left\{\begin{array}{rcl} b'_{21}(0) & = & 0 \\ b_{21}(1) & = & 0 \\ b'_{21}(2) & = & 1 \end{array}\right..$$

Each of them is of the form $ax^2 + bx + c$.
For b_{01}:

$$\left\{\begin{array}{cccccccc} & & b & & & = & 1 \\ a & + & b & + & c & = & 0 \\ 4a & + & b & & & = & 0 \end{array}\right. \iff \left\{\begin{array}{rcr} a & = & -1/4 \\ b & = & 1 \\ c & = & -3/4 \end{array}\right.,$$

so

$$b_{01}(x) = -\frac{x^2 - 4x + 3}{4} = -\frac{(x-1)(x-3)}{4}.$$

For b_{10}, we have:

$$\left\{\begin{array}{cccccccc} & & b & & & = & 0 \\ a & + & b & + & c & = & 1 \\ 4a & + & b & & & = & 0 \end{array}\right. \iff \left\{\begin{array}{rcr} a & = & 0 \\ b & = & 0 \\ c & = & 1 \end{array}\right.,$$

hence,

$$b_{10}(x) = 1.$$

Finally, for b_{21}:

$$\left\{\begin{array}{cccccccc} & & b & & & = & 0 \\ a & + & b & + & c & = & 0 \\ 4a & + & b & & & = & 1 \end{array}\right. \iff \left\{\begin{array}{rcr} a & = & 1/4 \\ b & = & 0 \\ c & = & -1/4 \end{array}\right.$$

and, thus,

$$b_{21}(x) = \frac{x^2 - 1}{4}.$$

From these, we find the interpolation polynomial

$$B_2 f(x) = \frac{1}{4}(x-1)(3-x)f'(0) + f(1) + \frac{1}{4}(x^2-1)f'(2),$$

with derivative

$$(B_2 f)'(x) \;=\; \frac{1}{2}(2-x)f'(0) + \frac{1}{2}x f'(2).$$

Check that $B_2 f$ satisfies the interpolation conditions.

The remainder is computed as

$$R_2 f(x) = f(x) - B_2 f(x) \;=\; f(x) - \left[\frac{1}{4}(x-1)(3-x)f'(0) + f(1) + \frac{1}{4}(x^2-1)f'(2)\right].$$

We have

$$\begin{aligned}
R_2 e_0(x) &= 1 - [0+1+0] \;\equiv\; 0, \\
R_2 e_1(x) &= x - \left[\frac{-x^2+4x-3}{4}\cdot 1 + 1 + \frac{1}{4}(x^2-1)\cdot 1\right] \;\equiv\; x - \left[x - \frac{3}{4} + 1 - \frac{1}{4}\right] \;=\; 0, \\
R_2 e_2(x) &= x^2 - \left[0 + 1 + \frac{1}{4}(x^2-1)\cdot 4\right] \;=\; x^2 - 1 - (x^2-1) \;\equiv\; 0, \\
R_2 e_3(x) &= x^3 - \left[0 + 1 + \frac{1}{4}(x^2-1)\cdot 12\right] \;=\; (x-1)(x^2-2x-2) \;\not\equiv\; 0.
\end{aligned}$$

(again, the first three relations *needed not* be checked). So the approximation formula

$$f(x) \;\approx\; B_2 f(x)$$

has degree of precision $d = 2$. Then the remainder is given by

$$R_2 f(x) \;=\; \int_0^2 K_2(x,t) f'''(t)\, dt,$$

where

$$\begin{aligned}
K_2(x,t) &= R_2\left(\frac{(x-t)_+^2}{2!}\right) \\
&= \frac{1}{2}\left[(x-t)_+^2 - \left(\frac{1}{4}(x-1)(3-x)\cdot 2(0-t)_+ + (1-t)_+^2 + \frac{1}{4}(x^2-1)\cdot 2(2-t)_+\right)\right] \\
&= \frac{1}{2}\left[(x-t)_+^2 - \left((1-t)_+^2 + \frac{1}{2}(x^2-1)(2-t)\right)\right],
\end{aligned}$$

because for $t \in [0,2]$, $(-t)_+ = 0$ and $(2-t)_+ = 2-t$.

b) For the numerical values $f'(0) = 1$, $f(1) = 2$ and $f'(2) = 1$, we have

$$B_2 f(x) \;=\; \frac{1}{4}(x-1)(3-x) + 2 + \frac{1}{4}(x^2-1) \;=\; x+1.$$

Note that, for these particular values, the degree is 1. However, the interpolation formula *still* has degree of exactness $d = 2$ and everything done in part **a)** still holds.

Then the approximation is

$$f(1/2) \;\approx\; B_2 f(1/2) \;=\; 3/2.$$

We compute the remainder:

$$\begin{aligned} K_2(1/2, t) &= \frac{1}{2}\left[(1/2 - t)_+^2 - \left((1-t)_+^2 + \frac{1}{2}((1/2)^2 - 1)(2-t)\right)\right] \\ &= \frac{1}{2}\left(\frac{1}{2} - t\right)_+^2 - \frac{1}{2}(1-t)_+^2 + \frac{3}{16}(2-t). \end{aligned}$$

We have the following cases:

1. $0 \le t \le \frac{1}{2}$, when $\left(\frac{1}{2} - t\right)_+^2 = \left(\frac{1}{2} - t\right)^2$ and $(1-t)_+^2 = (1-t)^2$, so

$$\begin{aligned} K_2(1/2, t) &= \frac{1}{8}(1-2t)^2 - \frac{1}{2}(1-t)^2 + \frac{3}{16}(2-t) \\ &= \frac{1}{16}\Big(2 - 8t + 8t^2 - (8 - 16t + 8t^2) + 3(2-t)\Big) \;=\; \frac{5}{16}t \;\ge\; 0. \end{aligned}$$

2. $\frac{1}{2} \le t \le 1$, when $\left(\frac{1}{2} - t\right)_+^2 = 0,\; (1-t)_+^2 = (1-t)^2$ and we have

$$\begin{aligned} K_2(1/2, t) &= -\frac{1}{2}(1-t)^2 + \frac{3}{16}(2-t) \\ &= \frac{1}{16}\Big(-(8 - 16t + 8t^2) + 3(2-t)\Big) \;=\; -\frac{1}{16}(8t^2 - 13t + 2) \;\ge\; 0, \end{aligned}$$

because the roots of the quadratic polynomial above, $\dfrac{13 \pm \sqrt{105}}{16} = \{0.1721, 1.4529\}$, lie *outside* the interval $\left[\frac{1}{2}, 1\right]$, so for $t \in \left[\frac{1}{2}, 1\right] \subset [0.1721, 1.4529]$, the quadratic polynomial above has positive sign (opposite to the sign of the leading coefficient).

3. $1 \le t \le 2$, when $\left(\frac{1}{2} - t\right)_+^2 \;=\; (1-t)_+^2 \;=\; 0$ and

$$K_2(1/2, t) \;=\; \frac{3}{16}(2-t) \;\geq\; 0.$$

So, in all three cases, K_2 has constant sign on $[0, 2]$ and, thus,

$$R_2 f(1/2) \;=\; \frac{1}{3!} f'''(\xi) R_2 e_3(1/2) \;=\; \frac{1}{6} \cdot \frac{11}{8} f'''(\xi) \;=\; \frac{11}{48} f'''(\xi),\; \xi \in (0, 2).$$

In the end, we have the approximation

$$f(1/2) \;\approx\; B_2 f(1/2) \;=\; 3/2,$$

with the error

$$|R_2 f(1/2)| \;\leq\; \frac{11}{48} ||f'''||_\infty.$$

■

2 Spline Interpolation

Polynomial interpolation has a major setback: the difference between the values of the function f and the values of the interpolation polynomial *outside* the nodes' interval can be quite large. Choosing more nodes and finding a higher degree polynomial does not solve this problem, but increases the computational cost. So, even though polynomials are smooth and easy to work with functions, they are not always the best choice for approximating functions.

From these considerations came the idea of changing polynomials to *piecewise polynomials* that satisfy some continuity conditions (of the interpolation function and some of its derivatives). Such functions are called *splines*.

Historically, spline functions can be traced all the way back to ancient mathematics. The term "spline" was first used by I. J. Schoenberg in 1946, but a thorough spline function theory started developing in 1964, as their good approximating properties became more evident. They can be used in a large variety of ways in approximation theory, computer graphics, data fitting, numerical integration and differentiation, and the numerical solution of integral, differential, and partial differential equations.

Over time, there have been several world renowned research groups in spline theory, scattered all over the world. One such group, with remarkable contributions, was a Romanian research group (based especially in Cluj).

The basic idea of approximating a function on an interval $[a, b]$ with spline functions, is to use different polynomials (of lower degree) on different parts of the interval. The reason for this is the fact that on a sufficiently small interval, functions can be approximated arbitrarily well by polynomials of low degree, even degree 1, or 0.

Definition 2.1. *Let Δ be a grid of the interval $[a, b]$,*

$$\Delta \; : \; a = x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The set

$$\mathbb{S}_m^k(\Delta) \;\; = \;\; \{s \mid s \in C^k[a, b],\; s|_{[x_i, x_i+1]} \in \mathbb{P}_m,\; i = 1, 2, \ldots, n-1\} \qquad (2.10)$$

is called the ***the space of polynomial spline functions*** *of degree m and class k on Δ.*

These are piecewise polynomial functions, of degree $\leq m$, continuous at $x_1, \ldots, x_{n-1}$, together with all their derivatives of order up to k. In general, we assume $0 \leq k < m$. For $k = m$,

$$S_m^m \; = \; \mathbb{P}_m.$$

If $k = -1$, we allow discontinuities at the grid points.

2.1 Linear Splines

For $m = 1$ and $k = 0$, we have *linear spline functions*. We determine a function $s_1 \in \mathbb{S}_1^0(\Delta)$ such that

$$s_1(x_i) \; = \; f(x_i) \; = \; f_i,\; i = 1, 2, \ldots, n.$$

That means that on the interval $[x_i, x_{i+1}]$, the function s_1 is the interpolation polynomial of degree 1

$$s_1(f; x) \;\; = \;\; f_i + f[x_i, x_{i+1}](x - x_i) \; = \; f_i + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i). \qquad (2.11)$$

The graph of this function is shown in Figure 1.
The error is given by

$$|f(x) - s_1(f;x)| \;\le\; \frac{|(x-x_i)(x-x_{i+1})|}{2!} \max_{x\in[x_i,x_{i+1}]} |f''(x)| \le \frac{h_i^2}{8} \max_{x\in[x_i,x_{i+1}]} |f''(x)|, \quad (2.12)$$

where we denoted by $h_i = x_{i+1} - x_i$.
Hence, if $|\Delta|$ denotes

$$|\Delta| \;=\; \max_{i=\overline{1,n-1}} h_i,$$

we have

$$||f(\cdot) - s_1(f,\cdot)||_\infty \;\le\; \frac{|\Delta|^2}{8} ||f''||_\infty. \quad (2.13)$$

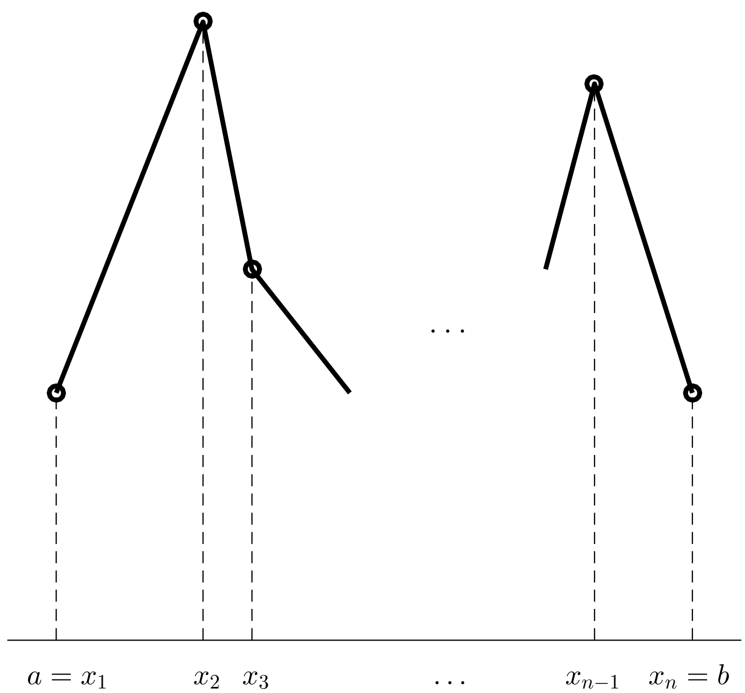

Fig. 1: Linear splines

Obviously, $\mathbb{S}_1^0(\Delta)$ is a vector space. To find its dimension, we count the number of degrees of freedom and the number of constraints. There are $n-1$ subintervals and 2 coefficients to be determined (i.e. 2 degrees of freedom) on each, for a total of $2(n-1)$. We have continuity conditions at each interior node, so $n-2$ constraints. Thus, in the end we have

$$\dim \mathbb{S}_1^0(\Delta) \;=\; 2(n-1)-(n-2) \;=\; n.$$

A basis for this space is given by the so-called ***B-spline functions***. Taking $x_0 = x_1 = a,\ x_{n+1} = x_n = b$, for $i = \overline{1,n}$, we define

$$B_i(x) \;=\; \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & \text{If } x_{i-1} \le x \le x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & \text{If } x_i \le x \le x_{i+1} \\ 0, & \text{in rest.} \end{cases} \tag{2.14}$$

Note that the first equation for $i = 1$, and the second equation for $i = n$, are to be ignored. The functions B_i are sometimes referred to as "hat functions" (Chinese hats), but note that the first and the last hat are cut in half. Their graphs are depicted in Figure 2. They are linearly independent and have the property

$$B_i(x_j) \;=\; \delta_{ij}.$$

Any function $s \in \mathbb{S}_1^0(\Delta)$ can be written uniquely as

$$s(x) \;=\; \sum_{i=1}^{n} c_i B_i(x).$$

B-spline functions play the same role as fundamental Lagrange polynomials l_i.

A linear spline agrees with the data, but it has the disadvantage of not having a smooth graph. Most data will represent a smooth curved graph, one without the corners of a linear spline. Consequently, we usually want to construct a smooth curve that interpolates the given data points, but one that follows the shape of the linear spline.

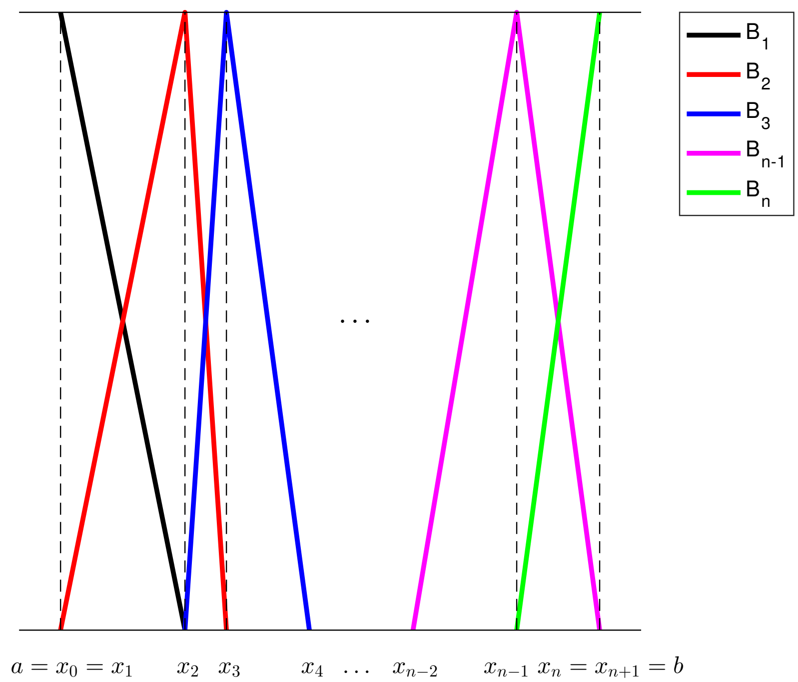

Fig. 2: Linear B-splines