Lecture 2

Wednesday, March 2, 2022 7:29 AM

Linear differential equations of arbitrary order (continuation)

<u>Given:</u> $I \subset \mathbb{R}$ open interval, $a_1, a_2, \dots, a_n \in C(I)$, $f \in C(I)$, $n \in \mathbb{N}^*$

<u>Find</u> $x : I \to \mathbb{R}$ s.t.

$$(1) \quad x^{(n)} + a_1(t)x^{(n-1)} + \dots + a_{n-1}(t)x' + a_n(t)x = f(t), \quad t \in \mathbb{R}$$

<u>Remark.</u> n is the order of eq. (1).

<u>L1</u>: $\mathcal{L} : C^n(I) \to C(I)$ $\qquad \mathcal{L}(x)(t) \overset{\text{not}}{=} x^{(n)} + a_1(t)x^{(n-1)} + \dots + a_n(t)x, \ \forall t \in I, \ \forall x \in C^n(I)$

$x \mapsto \mathcal{L}(x)$

$\boxed{(1) \iff \mathcal{L}(x) = f}$

$f \neq 0$ $\quad \mathcal{L}(x) = f$ is said to be a linear non-hom. d.e. (LnonHDE)

$f = 0$ $\quad \mathcal{L}(x) = 0$ — " — " linear hom. d.e. (LHDE)

we proved that $\mathcal{L}$ is a <u>linear map</u> between the vector spaces $C^n(I)$ and $C(I)$

1. Fundamental theorems for LDE's
2. LHDE's with constant coefficients

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1. Fundamental theorems for LDE's

<u>Def.</u> Given $\eta = (\eta_1, \eta_2, \dots, \eta_n) \in \mathbb{R}^n$ and $t_0 \in I$. The following problem is called Initial Value Problem (IVP) or Cauchy Problem (CP):

$$(2) \quad \begin{cases} \mathcal{L}(x) = f & (n^{th} \text{ order LDE}) \\ x(t_0) = \eta_1 \\ x'(t_0) = \eta_2 \\ \dots \\ x^{(n-1)}(t_0) = \eta_n \end{cases} \quad \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} n \text{ initial conditions}$$

$t_0 \sim$ <u>the initial time</u>

<u>Example.</u> <u>$n=1$</u> $\quad \begin{cases} x' + a_1(t)x = f(t) \\ x(t_0) = \eta_1 \end{cases}$ $\qquad$ <u>$n=2$</u> $\quad \begin{cases} x'' + a_1(t)x' + a_2(t)x = f(t) \\ x(t_0) = \eta_1 \\ x'(t_0) = \eta_2 \end{cases}$

<u>Theorem 1</u> (The existence and uniqueness theorem for the sol. of an IVP)

In the above hypotheses the IVP (2) has a unique solution $\varphi \in C^n(I)$.

<u>Example.</u> Solve the IVP:

Example. Solve the IVP :

(i) $\begin{cases} x''+t^2x = 0 & \text{(second order LHDE)} \\ x(0)=0 \\ x'(0)=0 \end{cases}$ } 2 initial cond.

First we notice that $x=0$ is a solution. Then we apply the $\exists!$ th. and deduce that there are no other solutions. Hence, $x=0$ is the unique sol.

Theorem 2 (The fundamental theorem for LHDE's)

$\operatorname{Ker} \mathcal{L}$ is a vector space of dimension n.

Hence, if $x_1, x_2, \dots, x_n$ are linearly independent solutions of $\mathcal{L}(x)=0$, then the general solution of $\mathcal{L}(x)=0$ is

$$x = c_1x_1 + c_2x_2 + \dots + c_nx_n, \quad c_1, \dots, c_n \in \mathbb{R} \text{ arbitrary}.$$

Examples (i) Prove that $x = ce^{-\frac{t}{2}}$, $c \in \mathbb{R}$ is the general sol. of $x' + \frac{1}{2}x = 0$

(Recall that in Lecture 1 we already proved this by integrating this DE).

Note that $x_1 = e^{-\frac{t}{2}}$ satisfies $x_1 \neq 0$ and x_1 is a sol of $x' + \frac{1}{2}x = 0$. (Lecture 1)

$\xRightarrow{\text{Th2}}$ the gen sol. is $x = c \cdot x_1$, $c \in \mathbb{R}$, i.e. $x = ce^{-\frac{t}{2}}$, $c \in \mathbb{R}$.

(ii) Prove that $x = c_1e^t + c_2e^{-t}$, $c_1, c_2 \in \mathbb{R}$ is the general sol. of $x'' - x = 0$

denote $x_1 = e^t$ and $x_2 = e^{-t}$.

we immediately check that x_1 and x_2 are solutions of $x'' - x = 0$;

and that they are lin. independent : take $\alpha_1, \alpha_2 \in \mathbb{R}$ s.t. $\alpha_1e^t + \alpha_2e^{-t} = 0$ $\forall t \in \mathbb{R}$

HW: prove that $\alpha_1 = \alpha_2 = 0$.

Then, by Th2 we get the conclusion

(iii) Prove that $x = c_1 \operatorname{ch} t + c_2 \operatorname{sh} t$, $c_1, c_2 \in \mathbb{R}$ is the general sol. of $x'' - x = 0$.

Notation: $\begin{cases} \operatorname{ch} t = \frac{1}{2}(e^t + e^{-t}) \quad \text{and} \quad \operatorname{sh} t = \frac{1}{2}(e^t - e^{-t}) \\ \operatorname{ch} 0 = 1, \ \operatorname{sh} 0 = 0 \qquad (\operatorname{ch} t)' = \operatorname{sh} t, \ (\operatorname{sh} t)' = \operatorname{ch} t \\ \operatorname{ch}^2 t - \operatorname{sh}^2 t = 1 \end{cases}$

$\frac{\cos}{\operatorname{ch}}$

HW: represent their graph.

$\begin{cases} x = \operatorname{ch} t \\ y = \operatorname{sh} t \end{cases}$, $t \in \mathbb{R}$ $\quad x^2 - y^2 = 1$ hyperbola | $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$, $t \in \mathbb{R}$ $\quad \cos^2 t + \sin^2 t = 1$

$\begin{cases} x = \operatorname{ch} t \\ y = \operatorname{sh} t \end{cases}, t \in \mathbb{R}$ $x^2 - y^2 = 1$ hyperbola (represent this hyperbola) | $\begin{cases} x = \cos t \\ y = \sin t, t \in \mathbb{R} \end{cases}$ $\cos^2 t + \sin^2 t = 1$, $x^2 + y^2 = 1$ circle

There are two methods: Method 1 Using that $x = c_1 e^t + c_2 e^{-t}$, $c_1, c_2 \in \mathbb{R}$ is the general sol. of $x'' - x = 0$.
(HW)

Method 2. Prove first that $x_1 = \operatorname{ch} t$ and $x_2 = \operatorname{sh} t$ are linearly indep. solutions of $x'' - x = 0$.
The conclusion comes from Th 2. (HW)

Theorem 3 (The fundamental theorem for L non-HDE's)

The set of solutions of $\mathcal{L}(x) = f$ is $\operatorname{Ker} \mathcal{L} + \{x_p\}$, where x_p is a particular solution of $\mathcal{L}(x) = f$.

Hence, the general sol of $\mathcal{L}(x) = f$ can be written as $x = x_h + x_p$ where x_h is the general sol. of $\mathcal{L}(x) = 0$ / called the LHDE associated to $\mathcal{L}(x) = f$) and x_p is a particular sol of $\mathcal{L}(x) = f$.

Example. Find the general sol of $x'' - x = -3$.

Step 1 we write the LHDE associated $x'' - x = 0$. and we find its general sol.
we know from a previous example that $x_h = c_1 e^t + c_2 e^{-t}$, $c_1, c_2 \in \mathbb{R}$

Step 2. we look for a partic. sol. of $x'' - x = -3$.
Note that $x_p = 3$.

Step 3 the gen sol is $x = c_1 e^t + c_2 e^{-t} + 3$, $c_1, c_2 \in \mathbb{R}$.

2. The characteristic equation method for LHDE's with CC (constant coefficients)

(3) $x^{(n)} + a_1 \cdot x^{(n-1)} + \ldots + a_{n-1} x' + a_n \cdot x = 0$ where $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

Idea: Look for solutions of the form $x = e^{rt}$.

$x' = r e^{rt}$, $x'' = r^2 e^{rt}$, ..., $x^{(k)} = r^k e^{rt}$

we replace in (3) and obtain: $r^n e^{rt} + a_1 r^{n-1} e^{rt} + \ldots + a_{n-1} \cdot r \cdot e^{rt} + a_n \cdot e^{rt} = 0, \forall t \in \mathbb{R}$

(4) $r^n + a_1 r^{n-1} + \ldots + a_{n-1} r + a_n = 0$ this is a polynomial eq. of degree n with the same coeff.

as (3)

(4) is called the <u>characteristic equation</u> of the DE (3).

Recall that: (4) has n roots in $\mathbb{C}$, counting with the corresp. <u>multiplicities</u>, denote the roots by $r_1, r_2, \dots, r_m \in \mathbb{C}$ (some of them are repeated)

e^{rt} when $r \in \mathbb{R}$ of course e^{rt} is a solution of (3) ✓

when $r \in \mathbb{C}$ what e^{rt} means ? ?

if $r=-2$ has multiplicity $m=3$ we have only one sol. e^{-2t}. From where we take other 2 solutions of (3) ? ?

There is a particular case in which this idea is sufficient to find the general sol. of (3): when the roots of the ch. eq (4) <u>are real and distinct</u>

<u>The characteristic equation method to find the general sol. of (3) in any situation</u>

<u>Step 1</u> we write the ch. eq. (4).

<u>Step 2.</u> we find the roots $r_1, r_2, \dots, r_n \in \mathbb{C}$ and their multiplicities (of (4))

<u>Step 3.</u> we associate n functions using the rules:

if $r \in \mathbb{R}$ is a root of (4) of multiplicity $m \geq 1$ $\longmapsto$

$$e^{rt}, \; te^{rt}, \; t^2e^{rt}, \; \dots, \; t^{m-1}e^{rt}$$

if $r = \alpha \pm i\beta$ with $\alpha, \beta \in \mathbb{R}$, $\beta > 0$ are roots of (4) of multiplicity $m \geq 1$ $\mapsto$

$$e^{\alpha t} \cdot \cos\beta t, \; e^{\alpha t} \cdot \sin\beta t, \; te^{\alpha t}\cos\beta t, \; te^{\alpha t}\sin\beta t, \; \dots, \; t^{m-1} e^{\alpha t}\sin\beta t$$

<u>Step 4.</u> We use the following proposition

<u>Proposition</u> The n functions found at Step 3 are n linearly independent solutions of eq (3). Denote them by $x_1, x_2, \dots, x_n$.

and Th 2 to deduce that the general solution of (3) is

$x = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$, $c_1, c_2, \ldots c_n \in \mathbb{R}$ arbitrary.

<u>For the next lecture (optional)</u>

- present something about $c h t$, $s h t$
- Proof of Th 2 (using Th 1) : this is a nice application of linear algebra
- Proof of Proposition above in the case that the roots are real and distinct
- other questions formulated by you.

Euler's formula $e^{i\beta} = ?$ e^{z}, $z \in \mathbb{C}$?