1. For each of the following, find the matrix $M \in SO(2)$ which diagonalizes the given symmetric matrix:

1. $\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$
2. $\begin{bmatrix} 5 & -13 \\ -15 & 5 \end{bmatrix}$
3. $\begin{bmatrix} 7 & -2 \\ -2 & 5/3 \end{bmatrix}$

2. For each of the symmetric matrices A in the previous exercise write down a quadratic equation with associated matrix A.

3. For each of the following equations write down the associated matrix.

1. $-x^2 + xy - y^2 = 0,$
2. $6xy + x - y = 0.$

4. Bring the equations from the previous exercise in canonical form.

5. Decide if the following equations describe an ellipse, a hyperbola or a parabola.

1. $x^2 - 4xy + 2y^2 = 1,$
2. $x^2 + 4xy + 2y^2 = 2,$
3. $x^2 + 4xy + 4y^2 = 3.$

6. Using the classification of quadrics, decide what surfaces are described by the following equations.

1. $x^2 + 2y^2 + z^2 + xy + yz + zx = 1,$
2. $xy + yz + zx = 1,$
3. $x^2 + xy + yz + zx = 1,$
4. $xy + yz + zx = 0.$

7. Consider the rotation R_{90° of $\mathbb{E}^2$ around the origin and the translation $T_{\mathbf{v}}$ of $\mathbb{E}^2$ with vector $\mathbf{v}(1,0)$.

1. Give the algebraic form of the isometries R_{90°, $T_{\mathbf{v}}$ and $T_{\mathbf{v}} \circ R_{90^\circ}$.
2. Determine the equations of the hyperbola $\mathcal{H} : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the parabola $\mathcal{P} : y^2 - 8x = 0$ after transforming them with R_{90° and with $T_{\mathbf{v}} \circ R_{90^\circ}$ respectively.

8. Let $\boldsymbol{e}$ and $\boldsymbol{f}$ be two orthonormal bases of a $\mathbb{V}^n$. Show that $\mathrm{M}_{\boldsymbol{e},\boldsymbol{f}}$ is orthogonal, i.e. that $\mathrm{M}_{\boldsymbol{e},\boldsymbol{f}} \in O(n)$.

9. Let $\boldsymbol{e} = (\mathbf{e}_1, \dots, \mathbf{e}_n)$ be an orthonormal basis of $\mathbb{V}^n$. If π is a permutation of $\{1, \dots, n\}$ let $\pi(\boldsymbol{e}) = (\mathbf{e}_{\pi(1)}, \dots, \mathbf{e}_{\pi(n)})$. Show that $\mathrm{M}_{\boldsymbol{e},\pi(\boldsymbol{e})} \in O(n)$.

1. For each of the following, find the matrix $M \in SO(2)$ which diagonalizes the given symmetric matrix:

1. $\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} = A$

the matrix M is the base change matrix $M_{e',e}$ where e is the basis with respect to which the matrix A is given and e' is an orthonormal basis of eigenvectors

$$\det(\lambda I_2 - A) = \begin{vmatrix} \lambda-6 & -2 \\ -2 & \lambda-9 \end{vmatrix} = (\lambda-6)(\lambda-9) - 4 = \lambda^2 - 15\lambda + 54 - 4 = 0$$

$$\lambda^2 - 15\lambda + 50 = 0 \iff (\lambda-10)(\lambda-5) = 0$$

so, the eigenvalues are 5 and 10

$\lambda = 5$ $\quad A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix} \iff (5 \cdot I - A)\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$$\iff \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix} \quad \text{so } -x - 2y = 0$$

$$\Rightarrow x = -2y$$

the eigenspace for the eigenvalue $\lambda = 5$ is

$$\left\{ \begin{bmatrix} -2t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\langle \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\rangle$$

$\lambda = 10$ $\quad (10 I_2 - A)\begin{bmatrix} x \\ y \end{bmatrix} = 0 \iff \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{so } -2x + y = 0 \Rightarrow y = 2x$$

the eigenspace for the eigenvalue $\lambda = 10$ is

$$\left\{ \begin{bmatrix} t \\ 2t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\rangle$$

$\Rightarrow$ an orthonormal basis of eigenvectors is $\left\{ \frac{1}{\sqrt5}\begin{bmatrix}-2\\1\end{bmatrix}, \frac{1}{\sqrt5}\begin{bmatrix}1\\2\end{bmatrix} \right\}$

so M could be $M_{e,e'} = \frac{1}{\sqrt5}\begin{bmatrix}-2 & 1\\ 1 & 2\end{bmatrix}$ $\quad M\cdot M^t (= M^2) = \frac{1}{5}\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix} = I_2$

$\det M = \frac{1}{5}\cdot(-5) = -1$

$\Downarrow$

we can replace $\frac{1}{\sqrt5}\begin{bmatrix}-2\\1\end{bmatrix}$ by $\frac{1}{\sqrt5}\begin{bmatrix}2\\-1\end{bmatrix}$

then $M_{e,e'} = \frac{1}{\sqrt5}\begin{bmatrix}2 & 1\\ -1 & 2\end{bmatrix}$ has $M\cdot M^t = \frac{1}{5}\begin{bmatrix}2 & 1\\ -1 & 2\end{bmatrix}\cdot\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix} = I_2$

and $\det M = \frac{1}{5}5 = 1$

$\Rightarrow M \in SO(2)$

Moreover $\quad M^{-1}AM = M^tAM = \frac{1}{\sqrt5}\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}\cdot\begin{bmatrix}6 & 2\\ 2 & 9\end{bmatrix}\cdot\frac{1}{\sqrt5}\begin{bmatrix}2 & 1\\ -1 & 2\end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}\begin{bmatrix}10 & 10\\ -5 & 20\end{bmatrix} = \frac{1}{5}\begin{bmatrix}25 & 0\\ 0 & 50\end{bmatrix} = \begin{bmatrix}5 & 0\\ 0 & 10\end{bmatrix}$

2. For each of the symmetric matrices A in the previous exercise write down a quadratic equation with associated matrix A.

1. $\begin{bmatrix}6 & 2\\ 2 & 9\end{bmatrix}$ $\qquad 6x^2 + 9y^2 + 4xy = 0$

3. For each of the following equations write down the associated matrix.

1. $-x^2 + xy - y^2 = 0,$
2. $6xy + x - y = 0.$

1. $A = \begin{bmatrix}-1 & 1/2\\ 1/2 & -1\end{bmatrix}$

4. Bring the equations from the previous exercise in canonical form.

as in exer 1, see also lecture notes

5. Decide if the following equations describe an ellipse, a hyperbola or a parabola.

1. $x^2 - 4xy + 2y^2 = 1$,
2. $x^2 + 4xy + 2y^2 = 2$,
3. $x^2 + 4xy + 4y^2 = 3$.

1. the associated matrix is $\begin{pmatrix} 1 & -2 \\ -2 & 2 \end{pmatrix}$

the characteristic polynomial is $\begin{vmatrix} 1-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 4$

$$= \lambda^2 - 3\lambda - 2$$

$$\Delta = 9 + 8$$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{17}}{2}$$

$\Rightarrow$ signature is $(1,1)$

in this case we have a hyperbola

6. Using the classification of quadrics, decide what surfaces are described by the following equations.

1. $x^2 + 2y^2 + z^2 + xy + yz + zx = 1$,

$\hookrightarrow$ has matrix $\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 2 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$ $\rightarrow$ has char. poly $-x^3 + 4x^2 - \frac{17}{4}x + \frac{5}{4}$

$$= -\frac{1}{4}(x-1)(2x-5)(2x-1)$$

$\Rightarrow$ has signature $(3,0)$

in this case the equation describes an ellipsoid (possibly imaginary)

7. Consider the rotation R_{90° of $\mathbb{E}^2$ around the origin and the translation $T_{\mathbf{v}}$ of $\mathbb{E}^2$ with vector $\mathbf{v}(1,0)$.

1. Give the algebraic form of the isometries R_{90°, $T_{\mathbf{v}}$ and $T_{\mathbf{v}} \circ R_{90^\circ}$.
2. Determine the equations of the hyperbola $\mathcal{H} : \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the parabola $\mathcal{P} : y^2 - 8x = 0$ after transforming them with R_{90° and with $T_{\mathbf{v}} \circ R_{90^\circ}$ respectively.

$$R_{90^\circ} = \begin{bmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$T_v \circ R_{90^\circ} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathcal{P}: \; y^2 - 8x = 0 \iff [x \;\; y] \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + [-8 \;\; 0] \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

replace this with $\begin{bmatrix} x' \\ y' \end{bmatrix} = Rot_{90^\circ} \begin{bmatrix} x \\ y \end{bmatrix}$

and $\begin{bmatrix} x' \\ y' \end{bmatrix} = T_v \circ Rot \begin{bmatrix} x \\ y \end{bmatrix}$

in order to get the equations.

8. Let $\boldsymbol{e}$ and $\boldsymbol{f}$ be two orthonormal bases of a $\mathbb{V}^n$. Show that $\mathrm{M}_{e,f}$ is orthogonal, i.e. that $\mathrm{M}_{e,f} \in O(n)$.

$$M_{e,f} = \begin{bmatrix} \cdots & f_i & \cdots \end{bmatrix}$$

$\hookrightarrow$ components of f_i w.r.t e

$$M_{e,f}^t M_{e,f} = \begin{bmatrix} \vdots \\ - f_i - \\ \vdots \end{bmatrix} \begin{bmatrix} \cdots & f_j & \cdots \end{bmatrix} = \begin{bmatrix} \cdots & f_i \cdot f_j & \cdots \end{bmatrix}$$

scalar product $f_i \cdot f_j$

$$\Rightarrow M_{e,f}^t M_{e,f} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$