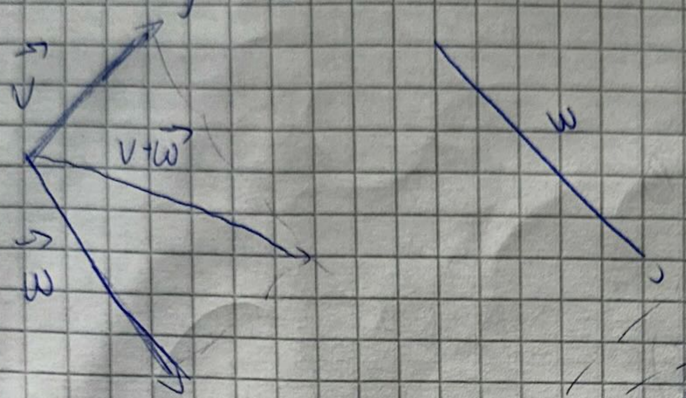
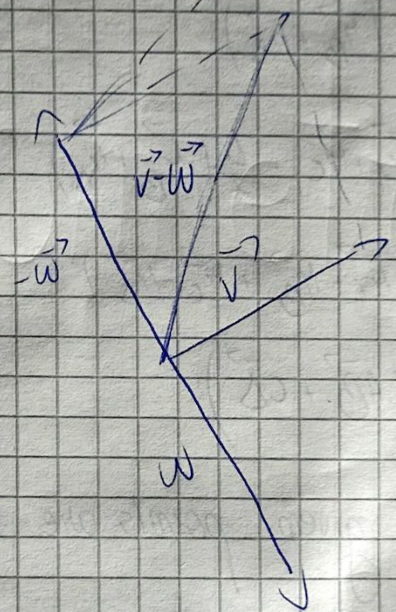
Seminar 1

$\mathbb{E}^n$ n-dimensional Euclidean space (usually $n=2$ or $n=3$),
"space of points"

$\mathbb{V}^n$ - space of vectors in $\mathbb{E}^n$



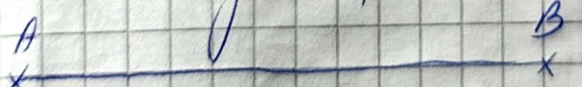


$A_1, A_2, \dots, A_k \in \mathbb{E}^n$

$\overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} + \dots + \overrightarrow{A_{k-1}A_k} + \overrightarrow{A_kA_1} = \vec{0}$

Fix $O \in \mathbb{E}^n$

$\forall A \in \mathbb{E}^n$ we define $\vec{r_A} = \overrightarrow{OA}$ = the position vector of A (w.r.t. O)



$A, B, M \in \mathbb{E}^n$

M = midpoint of $(AB) \Leftrightarrow \vec{r_M} = \dfrac{\vec{r_A} + \vec{r_B}}{2}$

$\vec{BC} = \vec{i} + \vec{j} + 2\vec{k}$

$\vec{AB} = 2\vec{BC} \Rightarrow A, B, C$ - col

Seminar 2

$K = (O, B)$

Ref. system for E^n

$O \in E^n$

B basis of V^n

$\forall P \in E^n$: $[P]_K := [\vec{OP}]_B$

Notations: V, V' vector spaces

B, B' bases of V, V'

$[M]_{B,B'}(\varphi) = [\varphi]_{B,B'} = ([\varphi(v_1)]_{B'}, \ldots, [\varphi(v_n)]_{B'})$

where $B = (v_1, \ldots, v_n)$

$\varphi: V \to V'$ linear map

V v.s.

B, B' bases of V

The base change matrix from B to B' is:

$$M_{B',B} := M_{B'B}(id) = [id]_{B,B'}$$

$$= \left([v_1]_{B'}, [v_2]_{B'}, \dots [v_m]_{B'} \right)$$

We use this in the following way:

$$\forall v \in V: \quad [v]_{B'} = M_{B',B} \cdot [v]_B$$

$$[\varphi \circ \psi] = [\varphi][\psi]$$

$$M_{B_1,B_3}(\varphi \circ \psi) = M_{B_1,B_2}(\varphi) \cdot M_{B_2,B_3}(\psi)$$

$K = (O, B)$

$K' = (O', B')$

$$[P]_{K'} = [\overrightarrow{O'P}]_{B'} = M_{B',B} \cdot [\overrightarrow{O'P}]_B = M_{B',B}[\overrightarrow{OP} - \overrightarrow{OO'}]_B$$

$$= M_{B'B}\left([\overrightarrow{OP}]_B - [\overrightarrow{OO'}]_B\right) = M_{B',B}\left([P]_K - [O']_K\right)$$

$$= M_{B'B}([P]_K - M_{B',B}[\overrightarrow{OO'}]_B = M_{B'B}[P]_K - [\overrightarrow{OO'}] =$$

$$= M_{B'B}[P]_K + [\overrightarrow{O'O}]_{B'} = M_{B'B}[P]_K + [O]_{K'}$$

$$M^{-1}_{B',B} = M_{B,B'}$$

From now on $M_{B',B} = M_{K',K}$

Seminar 3

Affine variety

$A = a + U = \{ a + \vec{v} \mid \vec{v} \in U \}$

$a \in E^n$, $U \leq_{\mathbb{R}} V^n$

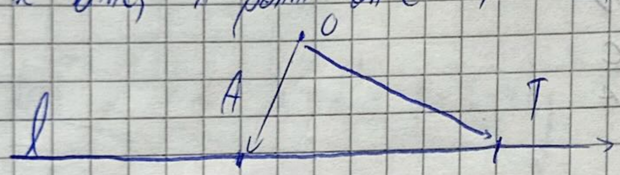
$$\boxed{\dim A = \dim_{\mathbb{R}} U}$$

$U = \Delta(A)$ = direction subspace of A

If $\dim A = 1 \Rightarrow A$ = line

If $\dim A = 2 \Rightarrow A$ = plane

l line, A point $\in l$, $\vec{v} \in \Delta(l)$ (i.e. $\vec{v} \parallel l$)



Fix an origin O

$\vec{r_T} = \vec{r_A} + \vec{AT}$

$\vec{AT} \in \Delta(l) \Rightarrow \exists\, \lambda \in \mathbb{R} : \vec{AT} = \lambda \vec{v}$

$$\boxed{\vec{r_T} = \vec{r_A} + \lambda \vec{v}}$$ - vector equation of l

Fix a reference system

$K = (O, B)$

Suppose $n = 2$

$[T]_K = \begin{pmatrix} x \\ y \end{pmatrix}$, $[A] = \begin{pmatrix} x_A \\ y_A \end{pmatrix}$, $[\vec{v}]_K = \begin{pmatrix} x_{\vec{v}} \\ y_{\vec{v}} \end{pmatrix}$

$$\boxed{l : \begin{cases} x = x_A + \lambda x_{\vec{v}} \\ y = y_A + \lambda y_{\vec{v}} \end{cases}}$$

(parameter equation of l)

If $x_{\vec{v}}, y_{\vec{v}} \neq 0 \Rightarrow \frac{x - x_A}{x_{\vec{v}}} = \frac{y - y_A}{y_{\vec{v}}}$

If $x_{\vec{v}} = 0 \Rightarrow l : x = x_A$

If $y_{\vec{v}} = 0 \Rightarrow l : y = y_A$

Implicit form: $\boxed{y_{\vec{v}}(x - x_A) - x_{\vec{v}}(y - y_A) = 0}$

Explicit form:

If $AB \neq 0 \Rightarrow y = -\frac{A}{B}x - \frac{C}{B}$

If $A = 0,\ B \neq 0 \Rightarrow y = -\frac{C}{B}$

If $A = 0,\ B = 0 \Rightarrow x = -\frac{C}{A}$

$\Delta(l) = \langle \vec{v} \rangle$

2 1 1, 2 2 Det. the parameter and Cartesian equat

Seminar 4

De făcut: 2.1, 2.2, 2.5, 2.10, 2.11,

Dot product (scalar product)

$\vec{v}, \vec{w} \in V$

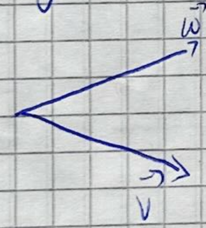
$\vec{v} \cdot \vec{w} = 0$ if $\vec{v} = 0$ or $\vec{w} = 0$

$\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\vec{v}, \vec{w})$ otherwise

Properties

- Bilinearity: $\forall \vec{v_1}, \vec{v_2}, \vec{w} \in V$ $\forall \alpha, \beta \in \mathbb{R}$: $(\alpha \vec{v_1} + \beta \vec{v_2}) \cdot \vec{w} =$
$= \alpha \vec{v_1} \cdot \vec{w} + \beta \vec{v_2} \cdot \vec{w}$

- Symmetry: $\forall \vec{v}, \vec{w} \in V$, $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$



- Positive definedness
$\forall \vec{v} \in V$, $\vec{v} \cdot \vec{v} \in \mathbb{R}_{\geq 0}$ if $\vec{v} \neq \vec{0}$ then $\vec{v} \cdot \vec{v} > 0$

Consequence

$\vec{v} \cdot \vec{v} = |\vec{v}|^2$

$\vec{v}^2 - \vec{w}^2 = (\vec{v} - \vec{w})(\vec{v} + \vec{w})$

If we have an orthogonal basis $B = (\vec{v_1}, \dots, \vec{v_n})$

$\forall i \neq j$: $\vec{v_i} \cdot \vec{v_j} = 0$

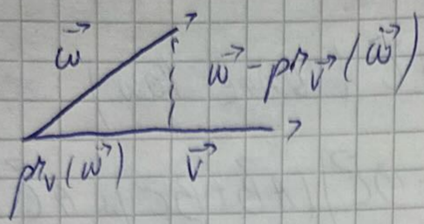
$\forall i$: $|\vec{v_i}| = 1$

Let $\vec{w}, \vec{w'} \in V$

$[\vec{w}]_B = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ $\Rightarrow$ $\vec{w} \cdot \vec{w'} = x_1 x_1' + \dots + x_n x_n'$

$|\vec{w}| = \sqrt{x_1^2 + \dots x_n^2}$

$[\vec{w'}]_B = \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}$



~~$pr_{\vec v}(\vec w)$~~

$$|pr_{\vec v}(\vec w)| = |\vec w| \cos(\vec v, \vec w)$$

$$pr_{\vec v}(\vec w) = \vec v \cdot \frac{|pr_{\vec v}(\vec w)|}{|\vec v|}$$

$$= \frac{|\vec w| \cdot \cos(\vec v, \vec w)}{|\vec v|}$$

$$\boxed{= \frac{\vec v \cdot \vec w}{\vec v \cdot \vec v} \cdot \vec v}$$ - formula for projection

Gram Schmidt

$B = (v_1, v_2, \dots v_m)$ basis for V

I Orthogonalization

1. $\vec{v_1'} = v_1$

2. $\vec{v_2'} = \vec{v_2} - pr_{\vec{v_1}}(\vec{v_2})$

3. $\vec{v_3'} = \vec{v_3} - pr_{Span(\vec{v_1'}, \vec{v_2'})}(\vec{v_3})$

$$= \vec{v_3} - pr_{\vec{v_1'}}(\vec{v_3}) - pr_{\vec{v_2'}}(\vec{v_3})$$

$$= \vec{v_3} - \frac{\vec{v_3} \cdot \vec{v_1'}}{\vec{v_1'} \cdot \vec{v_1'}} \cdot \vec{v_1'} - \frac{\vec{v_3} \cdot \vec{v_2'}}{\vec{v_2'} \cdot \vec{v_2'}} \cdot \vec{v_2'}$$

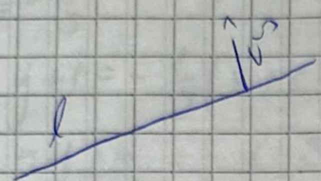
Repeat this until we get the orthogonal basis $B(\vec{v_1'}, \vec{v_2'}, \dots \vec{v_4'})$

Seminar 5

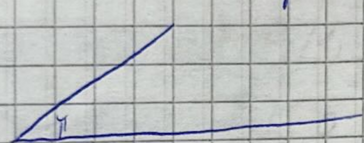
We will work wrt an orthogonal reference sys.

$l: ax + by + c = 0$ (in 2D)

$\vec{v}(a, b)$ normal vector for l



$\pi: Ax + By + Cz + D = 0$ plane in 3D $\Rightarrow \vec{v}(A, B, C)$ normal vector for π

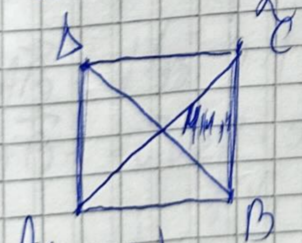


$\pi: Ax + By + Cz + D = 0$

$P(x_0, y_0, z_0)$

$$dist(P, \pi) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

3.26. The point $A(3, -2)$ is the vertex of a square $ABCD$ and $M(1, 1)$ is the intersection point of the diagonals. Det. Cartesian equations for the sides of the square



$x_M = \frac{x_A + x_C}{2} \Rightarrow 1 = \frac{3 + x_C}{2} \Rightarrow x_C = -1$

$y_M = \frac{y_A + y_C}{2} \Rightarrow 1 = \frac{-2 + y_C}{2} \Rightarrow y_C = 4$

$AM = \sqrt{(1-3)^2 + (1+2)^2} = \sqrt{4+9} = \sqrt{13}$

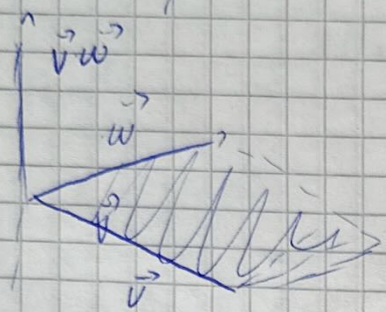
$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{4+2}{-4} = -\frac{3}{2} \Rightarrow m_{BD} = \frac{2}{3}$

$BD: y - y_M = m_{BD}(x - x_M)$

$y - 1 = \frac{2}{3}(x - 1) \Leftrightarrow 2x - 3y + 1 = 0$

Seminar 6

Cross product
(vector product)



$|\vec{v} \times \vec{w}| = |\vec{v}| \cdot |\vec{w}| \sin(\widehat{\vec{v}, \vec{w}})$

$\vec{v} \times \vec{w} \perp \vec{v}, \vec{w}$ the orientation of $\vec{v} \times \vec{w}$ is given by the right hand rule

Properties

- bilinearity : $\forall \alpha_1, \alpha_2 \in \mathbb{R}, \forall \vec{v_1}, \vec{v_2}, \vec{w} \in V^3$
  $(\alpha_1 \vec{v_1} + \alpha_2 \vec{v_2}) \times \vec{w} = \alpha_1 (\vec{v_1} \times \vec{w}) + \alpha_2 (\vec{v_2} \times \vec{w})$
  (same for $\vec{w}$)
- skew - symmetry : $\forall \vec{v}, \vec{w} : \vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$
- If $\vec{v}$ and $\vec{w}$ are lin. dependent then $\vec{v} \times \vec{w} = \vec{0}$

- We work over a right oriented orthogonal system

$\vec{v_1}(x_1, y_1, z_1)$
$\vec{v_2}(x_2, y_2, z_2)$

$$\vec{v_1} \times \vec{v_2} = \boxed{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}}$$

4.13. $A(1, 2, -1)$

$B(0, 1, 5)$

$C(-1, 2, 1)$

$\Delta(2, 1, 3)$

Are the points coplanar?

$\boxed{A, B, C, \Delta \text{ coplanar} \Leftrightarrow [\vec{AB}; \vec{AC}; \vec{A\Delta}] = 0}$

$$[\vec{AB}, \vec{AC}, \vec{A\Delta}] = \begin{vmatrix} -1 & -1 & 6 \\ -2 & 0 & 2 \\ 1 & -1 & 4 \end{vmatrix} = 0 - 2 + 12 - 0 - 8 - 2 = 0$$

Seminar 4