Seminar 7

Ex: 3.17, 3.30, 3.33, 3.34, 3.36, 3.40, 3.42
4.11, 4.13, 4.16, 4.17

3.34. Solve ex. 2.16. using normal vectors

2.16. Show that the pairwise intersection of the planes

$\pi_1: 3x+y+z-5=0$

$\pi_2: 2x+y+3z+2=0$

$\pi_3: 5x+2y+4z+1=0$

are parallel lines

$\Delta(l) = \langle \vec{n_{\pi_1}} \times \vec{n_{\pi_2}} \rangle$

~~$\pi_1 \times \pi_2 = (3x+y+z-5)\times(2x+y+3z+2)$~~ n_{π_k}

~~$= 6 \ldots 10 \ldots 18$~~

$\vec{n_{\pi_1}}(3,1,1)$

$\vec{n_{\pi_2}}(2,1,3)$

$n_{\pi_3}(5,2,4)$

$n_{\pi_1} \times n_{\pi_2} =$ ~~$6 \ldots 3$~~

$$\begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$

$= 2i - k - 7j$

$n_{\pi_1} \times n_{\pi_3} = 2i + k - 7j$ ✓

$n_{\pi_2} \times n_{\pi_3}$

3.36. Determine the angles between the planes

$\pi_1: x - \sqrt{2}y + z - 1 = 0$

$\pi_2: x + \sqrt{2}y - z + 3 = 0$

$\vec{n}_{\pi_1}(1, -\sqrt{2}, 1)$

$\vec{n}_{\pi_2}(1, \sqrt{2}, -1)$

$\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}||\vec{n_2}| \cos(\vec{n_1}, \vec{n_2})$

$|n_1| = \sqrt{1+2+1} = \sqrt{6}$

$|n_2| = \sqrt{6}$

$n_1 \cdot n_2 = 1 - 2 - 1 = -2$

$-2 = \sqrt{6} \cdot \sqrt{6} \cdot \cos(n_1, n_2)$

$-2 = 6 \cos(n_1, n_2)$

$\cos(n_1, n_2) = -\frac{1}{2}$

$\sphericalangle(n_1, n_2) = \arccos\left(-\frac{1}{2}\right)$

$\pi - \frac{\pi}{3} = \boxed{\frac{2\pi}{3}}$

$\boxed{\frac{\pi}{3}}$

3.40 $A(1,3,5)$ line l

$l: 2x+y+z-1=0 \wedge 3x+y+2z-3=0$

Determine the orthogonal projection and reflection of A in l

π plane containing A that is perpendicular to l

Then $\{p'\} = \pi \cap l$

$$l \begin{cases} 2x+y+z-1=0 \\ 3x+y+2z-3=0 \end{cases}$$

$z = -2x-y+1$

$3x+y+2(-2x-y+1)-3=0$

$-x-y-1=0 \Leftrightarrow x+y+1=0$

$y=\lambda \in \mathbb{R}$

$\Rightarrow x=-1-\lambda$

$z=-2(-1-\lambda)-\lambda+1=2+\lambda+1=\lambda+3$

$$l: \begin{cases} x=1-\lambda \\ y=\lambda \\ z=\lambda+3 \end{cases} \Rightarrow \Delta(l) = \langle(-1,1,1)\rangle$$

$\pi: -1(x-1)+1(y-3)+1(z-5)=0$

$-x+1+y-3+z-5=0$

$\pi: x-y-z+7=0$

$-1-\lambda-\lambda-\lambda-3+7=0$

$-3\lambda + 3 = 0$

$\Rightarrow \lambda = 1$

$A(-2, 1, 4)$

A' - midpoint of $[AA'']$

$\Rightarrow x_{A'} = \dfrac{x_A + x_{A''}}{2}$

$y_{A'} = \dfrac{y_A + y_{A''}}{2}$

$z_{A'} = \dfrac{z_A + z_{A''}}{2}$

$-2 = \dfrac{1 + x_{A''}}{2} \Rightarrow x_{A''} = -5$

$1 = \dfrac{3 + y_{A''}}{2} \Rightarrow y_{A''} = -1$

$4 = \dfrac{5 + z_{A''}}{2} \Rightarrow z_{A''} = 3$

5.18. ABCD tetrahedron

$A(2, -1, 1),\ B(5, 5, 4),\ C(3, 2, -1),\ D(4, 1, 3)$

Find the common perpendicular of AB and CD

$A_1 \in l_1,\ A_2 \in l_2$

$v_1 \in D(l_1),\ \vec{v_2} \in D(l_2)$

We can build π_1, π_2:

π_1 given by $A_1, \vec{v_1}, \vec{v_2}$

π_2 given by $A_2, \vec{v_1}, \vec{v_2}$

$\vec{v_1} \times \vec{v_2}$ normal vector for both π_1 and π_2

ρ_1 = plane given by $A_1, \vec{v_1}, \vec{v_1} \times \vec{v_2}$

ρ_2 = plane given by $A_2, \vec{v_2}, \vec{v_1} \times \vec{v_2}$

$\rho_1 \nparallel \rho_2$ because:

$\vec{n}_{\rho_1} = \vec{v_1} \times (\vec{v_1} \times \vec{v_2})$

$\vec{n}_{\rho_2} = \vec{v_2} \times (\vec{v_1} \times \vec{v_2})$

not parallel $\Rightarrow \exists\, l = \rho_1 \cap \rho_2$ (this is the common perpend.)

$\vec{AB} : (3, 6, 3)$

$\vec{CD} : (1, -1, 4)$

$$\vec{AB} \times \vec{CD} = \begin{vmatrix} i & j & k \\ 3 & 6 & 3 \\ 1 & -1 & 4 \end{vmatrix} = 24i - 9 + 24 - 6k + 9 - 24$$

$= \cancel{24i - 6k}\ 27\vec{i} - 9\vec{j} - 9\vec{k}$

$= $

$$\rho_1 : \begin{vmatrix} x-2 & y+1 & z-1 \\ 3 & 6 & 3 \\ 27 & -9 & -9 \end{vmatrix} = 27(-2\ x+4 \cdot z + 1 + 3y + 3 - 6z + x - 2 + y + 1) = 0$$

$$\pi_2: \begin{vmatrix} x-3 & y-2 & z+1 \\ 1 & -1 & 4 \\ 27 & -9 & -9 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-3 & y-2 & z+1 \\ 1 & -1 & 4 \\ 3 & -1 & -1 \end{vmatrix} = 0$$

$$5x+13y+2z-36=0$$

$$\pi_1 \cap \pi_2: \begin{cases} -x+4y-7z+9=0 \\ 5x+13y+2z-36=0 \end{cases}$$

$$-x=+4y-7z+9$$

$$y=\lambda \in \mathbb{R}$$

$$x=4\lambda-7z+y$$

$$x=-\frac{13}{5}\lambda-\frac{2}{5}z+\frac{36}{5}$$

Seminar 8

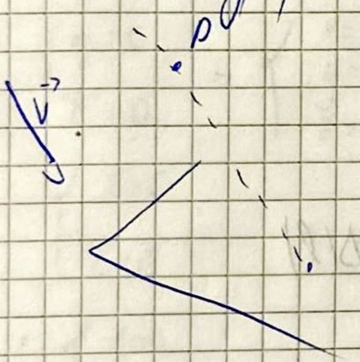
1, 2, 4, 5, 11, 12, 13, 14, 16

$$\boxed{\begin{array}{l} f: \mathbb{A}^n \to \mathbb{A}^m \text{ affine morphism} \\ f(\vec{AB}) = \overrightarrow{f(A)\,f(B)} \end{array}}$$

$f(P) = A \cdot P + b$

$A \in M_{m,n}(K)$, $b \in M_m(\mathbb{R})$

Projection on a hyperplane H parallel to a vector v



$P' = Pr_{H,v}(P)$

$H: a_1 x_1 + a_2 x_2 + \dots + a_n x_n + a_{n+1} = 0$

$Pr_{H,v}(P) = \left(I_n - \dfrac{v \otimes a}{\langle v, a \rangle} \right) \cdot P - \dfrac{a_{n+1}}{\langle v, a \rangle} \cdot a$, where $a = (a_1, \dots a_n)$

$v(x_1, \dots x_n)$, $w(y_1, \dots, y_n)$

$v \otimes w = v \cdot w^T$

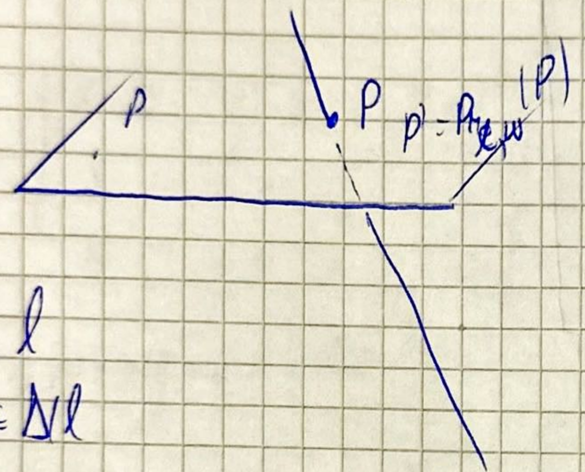
$v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$w = (y_1, \dots y_n)$

$\Rightarrow v \cdot w = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} (y_1, \dots, y_n) = \begin{pmatrix} x_1 y_1 & \dots & x_1 y_n \\ \vdots & & \vdots \\ x_n y_1 & \dots & x_n y_n \end{pmatrix}$

$$\langle v, a\rangle = v^T \cdot a = (x_1 \dots x_m)\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = x_1 y_1 + \dots + x_m y_m$$

Projection on a line ℓ parallel to a hyperplane ω



$Q \in \ell$

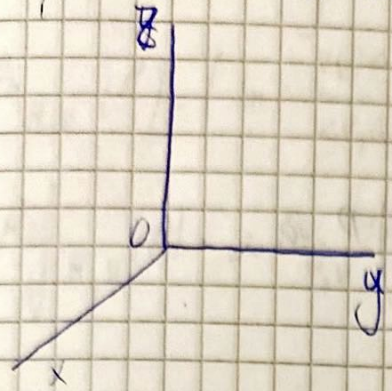
$v \in \Delta \ell$

$$Pr_{\ell,\omega}(P) = \frac{v \otimes a}{\langle v, a\rangle} \cdot P + \left(I - \frac{v \otimes a}{\langle v, a\rangle}\right) \cdot Q$$

$$Pr_H^{\perp} = Pr_{H,a}$$

$Pr_{\ell}^{\perp} = Pr_{\ell, W_\ell}$ where ~~W_ℓ is~~ $\vec{n}_{W_\ell} \in \Delta(\ell)$

5.1 Write the formulas for the orthogonal projections on the coordinate ~~axis~~ planes and coordinate axes in E^3



$(xOy): z=0$
$(yOz): x=0$
$(xOz): y=0$

$Ox: \begin{cases} y=0 \\ z=0 \end{cases} \Rightarrow \begin{cases} x=0+1\cdot\lambda \\ y=0+0\lambda \\ z=0+0\lambda \end{cases}$

$Oy: \begin{cases} x=0 \\ z=0 \end{cases}$

$Oz: \begin{cases} x=0 \\ y=0 \end{cases}$

$$P_{H\,xOy,\,Oz} = \left(I_n - \frac{v \otimes a}{\langle v, a\rangle}\right) P - \frac{a_{n+1}}{\langle v, a\rangle}$$

~~OOa~~ $H=(xOy) \Rightarrow a = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$a \otimes a = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}(0\ 0\ 1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \langle a, a\rangle = 1$$

$$P_{H,V} | P_H = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot P - 0 \cdot a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot P$$

$H_2 = (yOz) \Rightarrow a = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = v$

$a \otimes a = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1\ 0\ 0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$a_{n+1} = 0 \quad , \langle a, a \rangle = 1$

$P_{H_2, v}(P) = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) P - 0$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P$

$l_1 = (Ox)$

$a = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = v$

$Q(0,0,0)$

$v \otimes a = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1\ 0\ 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow P_{Ox, yOz} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P$

$l_2 = Oy$

$a = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = v$

$Q(0,0,0)$

$v \otimes a = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (0\ 1\ 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \langle v, a \rangle = 1$

$P_{Oy, xOz} = P_{Ox}^{\perp} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} P$

$$f_1(P) = A_1 P + b_1$$
$$f_2(P) = A_2 P + b_2$$

$$(f_2 \circ f_1)(P) = f_2(A_1 P + b_1) = A_2(A_1 P + b_1) + b_2$$

$$P = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \bar{P} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$f(P) = AP + b$$

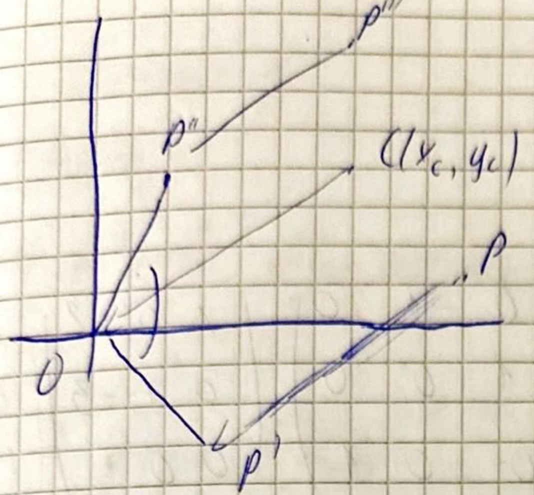
$$\bar{f}(P) = \left(\begin{array}{c|c} A & b \\ \hline 0 & 1 \end{array}\right) \bar{P} =$$

$$\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n \\ \hline 0 & \cdots & 0 & 1 \end{array}\right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_1 x_1 + a_{12} x_2 + \cdots + a_{1n} x_n + b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n + b_2 \\ \vdots \\ a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} + b_n \end{pmatrix}$$

$$Rot_\theta(P) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot P$$

$$\bar{Rot}_\theta(\bar{P}) = \left(\begin{array}{cc|c} \cos\theta & -\sin & 0 \\ \sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$T_{\vec{v}}(P) = P + \vec{v} = I_n \cdot P + \vec{v}$$



$$P' = T_{(-x_c, -y_c)}(P)$$

$$P = Rot_{\theta}(P'')$$

$$P''' = T_{(x_c, y_c)}(P'')$$

$$Rot_{C\theta}(P) = \left(T_{(x_c, y_c)} \circ Rot_{\theta} \circ T_{(-x_c, y_c)}\right)(\bar{P})$$

$$= \left(\begin{array}{cc|c} 1 & 0 & x_c \\ 0 & 1 & y_c \\ \hline 0 & 0 & 1 \end{array}\right)\left(\begin{array}{cc|c} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ \hline 0 & 0 & 1 \end{array}\right)\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

14. $A(1,1)$, $B(4,1)$, $C(2,3)$

$Ref^{1}_{AB} \circ Rot_{C,\frac{\pi}{2}}(A)$

$$\overline{Rot}_{C,\frac{\pi}{2}}(\bar{A}) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \Rightarrow Rot_{C,\frac{\pi}{4}}(A) = (4,2)$$

$$Ref^{1}_{AB}(P) = \left(I_n \, 2 \cdot \frac{\vec{AB} \times \vec{AB}}{\langle AB \times AB \rangle} - I_n \right) \cdot P - \left(I_n - \frac{\vec{AB} \otimes \vec{AB}}{\langle \vec{AB}, \vec{AB} \rangle} \right) \cdot A$$

$$\vec{AB} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\vec{AB} \times \vec{AB} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} (3 \;\; 0) = \begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}$$

$$\langle \vec{AB}, \vec{AB} \rangle = 9$$

$$Ref^{1}_{AB}(P) = \left(\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot P - \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\overline{Ref}^{1}_{AB}(\bar{A}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \bar{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$

Seminar 9

Chapter 5

$\varphi : \mathbb{E}^4 \to \mathbb{E}^4$ affine morphism

$\begin{cases} \forall P \in \mathbb{E}^4 \\ \varphi(P) = A \cdot P + b \\ A \in M_4(\mathbb{R}),\ b \in \mathbb{R}^4 \end{cases}$

φ isometry $\Leftrightarrow \forall\ P, Q \in \mathbb{E}^4\ dist(\varphi(P), \varphi(Q)) = dist(P, Q)$

$\Leftrightarrow A \in O(n) = \left\{ M \in M_n(\mathbb{R}) \;\middle|\; M^{-1} = M^T \right\} \Leftrightarrow A \cdot A^T = I_n$

$\det A = 1 \Rightarrow$ direct isometry

$\hookrightarrow A \in SO(4)$

$\det A = -1 \Rightarrow$ indirect isometry

$n = 2$: $\varphi : \mathbb{E}^2 \to \mathbb{E}^2$ isometry

- direct
  - $\to$ identity
  - $\to$ translation by $\vec{v} \in V^2$
    $T_{\vec{v}}$
  - $\to$ rotation around a point Q by an angle θ
    $Rot_{Q,\theta}$
- indirect
  - $\to$ reflexion wrt a line l
    Ref_l
  - $\to$ glide-reflection
    $T_{\vec{v}} \circ Ref_l$

$\vec{v} \in \Delta(l)$

If φ is a rotation then $\mathrm{Tr}A = 2\cos\theta$

$\mathrm{Fix}(\varphi) = \{P \in E^3 \mid \varphi(P) = P\}$

5.16. F isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a translation with vector $(-2, 5)$

Determine the inverse transformation F^{-1}

$$F = Rot_{-\frac{\pi}{3}} \circ T_{(-2,5)}$$

$$F^{-1} = T^{-1}_{(-2,5)} \circ Rot^{-1}_{-\frac{\pi}{3}} \qquad T^{-1}_{(-2,5)} = T_{(2,-5)}$$

$$Rot^{-1}_{-\frac{\pi}{3}} = Rot_{\frac{\pi}{3}}$$

$$= T_{(2,-5)} \circ Rot_{\frac{\pi}{3}}$$

$$\hat{F}^{-1}(\hat{P}) = \hat{T}(2,-5) \circ \hat{Rot}_{\frac{\pi}{3}}(\hat{P})$$

$$= \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 2 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = F^{-1}(P) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} P + \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

18. $f: \mathbb{E}^2 \to \mathbb{E}^2$ affine morphism

$$f(P) = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ +4 & 5 \end{pmatrix} P + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Show that f is a rotation, find its axis and angle

$$f(P) = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ +\frac{4}{5} & \frac{3}{5} \end{pmatrix} P + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

1. Check isometry

$A \cdot A^t = I_2$

$$\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ +\frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix} = \begin{pmatrix} \frac{9}{25}+\frac{16}{25} & \frac{12}{25}-\frac{12}{25} \\ \frac{12}{25}-\frac{12}{25} & \frac{16}{25}+\frac{9}{25} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow A \in O(2) \Rightarrow f$ isometry

2. Check if direct ($\det A = 1$)

$$\begin{vmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{vmatrix} = \cancel{1 - 0 = 1 \Rightarrow A \in SO(2)}$$

$$\frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1 \Rightarrow A \in SO(2)$$

III Show it is a rotation

$Fix(f) = \{ P \in \mathbb{E}^n \mid f(P) = P \}$

$$f(P) = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} P + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = P \Leftrightarrow \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} =$$

$$= \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\left(\frac{3}{5}x - \frac{4}{5}y, \frac{4}{5}x + \frac{3}{5}y\right) = (x-1, y+2)$

$$\begin{cases} \frac{3}{5}x - \frac{4}{5}y = x-1 \quad | \cdot 5 \\ \frac{4}{5}x + \frac{3}{5}y = y+2 \quad | \cdot 5 \end{cases}$$

$$\begin{cases} -2x - 4y = -5 \quad | \cdot 2 \\ 4x - 2y = 10 \end{cases}$$

$-10y = 0$

$y = 0$

$-2x = -5 \Rightarrow x = \frac{5}{2}$

$Fix(f) = \{(\frac{5}{2}, 0)\} \neq \emptyset \Rightarrow f$ is a rotation

$Tr(A) = \frac{6}{5} = 2\cos\theta \Rightarrow \cos\theta = \frac{3}{5}$

$n = 3$

- ~~direction~~
- direct isometry
  - identity
  - translation $T_{\vec{v}}$
  - rotation around an axis by angle θ Rot_{θ}
  - glide rotation

    $T_{\vec{v}} \circ Rot_{\ell, \theta}$

    $\vec{v} \in \Delta(\ell)$
- indirect isometry
  - reflection wrt a plane π — rotation

    Ref_{π}
  - glide reflection

    $T_{\vec{v}} \circ Rot_{\pi}$, $\vec{v} \in \Delta(\pi)$

If φ is a rotation: $Tr A = 2\cos\theta + 1$

$$Rot_{z,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

19. Verify that the matrix

$$A = \frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix} \in SO(3)$$

Determine the axis and angle of rotation

$$A \cdot A^T = \frac{1}{9}\begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & -2 & -2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = I_3 \Rightarrow A \in O(3) \Rightarrow \text{is isometry}$$

$$\det A = \frac{1}{27}\begin{vmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{vmatrix} = \frac{1}{27}(4+4+4+8+8-1)$$

$$= \frac{1}{27} \cdot 27 = 1 \Rightarrow \text{direct isometry}$$

Check if it's rotation

$A \cdot P + b$

Let $P = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} 4x + 2y - 2z = 0 \\ -2x - 5y \cdot z = 0 \\ -2x + y - 5z = 0 \end{cases}$$

Euler - Rodrigues

$$Rot_{l,\theta}(P) = \cos\theta \cdot P + \sin\theta(\vec{v} \times p) + (1-\cos\theta) \langle \vec{v}, P \rangle \cdot \vec{v}$$

$\vec{v} \in \Delta(l)$, $|\vec{u}| = 1$

22. Write down the matrix form of a rotation around the axis $\mathbb{R}\vec{v}$, where $\vec{v} = (1,1,0)$ and use it to give a parametrization of a cylinder with axis $\mathbb{R}\vec{v}$ and diameter $\sqrt{2}$

$$\vec{w} = \frac{1}{|\vec{u}|} \cdot \vec{v} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$Rot_{l,\theta}(P) = \cos\theta \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \sin\theta \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ x & y & z \end{vmatrix} +$$

$$+ (1-\cos\theta)\left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \ldots = M \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$Rot_{s,\theta}(P) = \cos\theta \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \sin\theta \cdot \begin{vmatrix} i & j & k \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ x & y & z \end{vmatrix} + (1-\cos\theta)\cdot\left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \dots = M \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

30.IV.2024

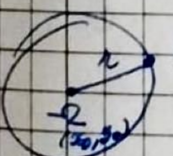
Seminar 10
Chapter 6

1, 3, 4, 5, 6, 7, 8, 9, 10

Quadratic curves (elipses)

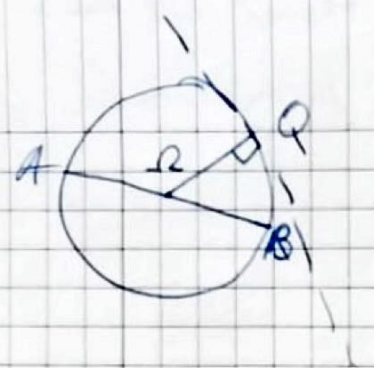
$Q: a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{01}y + a_{00}$

Circles = locus of points in the plane whose distance to a fixed point $\Omega(x_0, y_0)$ is $R>0$ (the radius)



$\mathcal{C}(\Omega, R): (x-x_0)^2 + (y-y_0)^2 = R^2$
(implicit, cartesian form)

$\mathcal{C}(\Omega, R): \begin{cases} x = x_0 + r\cos t \\ y = y_0 + r\sin t \end{cases} \quad t\in[0, 2\pi)$
(parametric equation)



6.1. Find the equation of:

a) the circle of diameter (AB) with $A(1,2)$, $B(-3,-1)$,

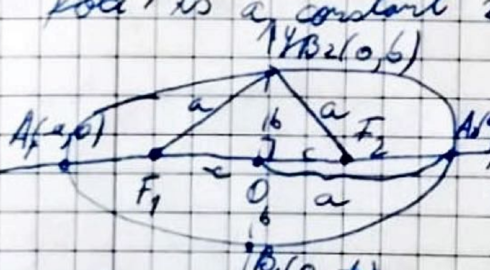
b) passing through $A(3,1)$, and $B(-1,3)$ and having the centre on the line $l: 3x - y - 2 = 0$

~~$AB = \sqrt{16+9} = 5$~~ $\quad AB = 5 \Rightarrow R = \frac{5}{2}$

~~AB~~: $x_\Omega = \frac{1-3}{2} = -1$, $y_\Omega = \frac{2-1}{2} = \frac{1}{2}$ $\Rightarrow \Omega\left(-1, \frac{1}{2}\right)$

$\Rightarrow (x+1)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{25}{4}$

Def. Ellipse: locus of points in the plane whose sum of distances to two distinct fixed points F_1 and F_2 (called the focal points or foci) is a constant $2a$.



If we fix the reference system $\mathcal{R} = (O, \vec{i}, \vec{j})$ where O is the midpoint of $[F_1, F_2]$

$\vec{i} \in D(F_1, F_2)$

then the equation is of the form $\boxed{\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$, $a, b > 0$, $a \geq b$

$c = \sqrt{a^2 - b^2}$, $F_1(-c, 0)$, $F_2(c, 0)$

$e = \frac{c}{a}$ = eccentricity $\in (0,1)$

6.3. Determine the foci of the ellipse $\mathcal{E}: 9x^2+25y^2-225=0$

$9x^2+25y^2=225 \,\big|:225 \Rightarrow \frac{9x^2}{225}+\frac{25y^2}{225}=1$

$\frac{x^2}{25}+\frac{y^2}{9}=1 \Rightarrow \frac{x^2}{5^2}+\frac{y^2}{3^2}=1 \Rightarrow \begin{array}{l} a=5 \\ b=3 \end{array}$

$c=\sqrt{16}=4 \Rightarrow F_1(-4,0),\ F_2(4,0)$

6.4. Determine the intersection of the line $l: x+2y-7=0$ and the ellipse $\mathcal{E}: x^2+3y^2-25=0$

$\cancel{x^2+3y^2-25}$
$$\begin{cases} x+2y-7=0 \Rightarrow x=7-2y \\ x^2+3y^2-25=0 \quad \Rightarrow (7-2y)^2+3y^2-25=0 \end{cases}$$

$49-28y+4y^2+3y^2-25=0 \Rightarrow 7y^2-28y+24=0$

$\Delta = 28^2-4\cdot 24\cdot 7$

$=28^2-28\cdot 24$

$=28(28-24)=28\cdot 4$

$y_{1,2}=\frac{28\pm 4\sqrt{7}}{14}=\frac{14\pm 2\sqrt{7}}{7}$

$\Rightarrow x_{1,2}=7-2\cdot\frac{14\pm 2\sqrt{7}}{7}$

$=\frac{49-28\mp 4\sqrt{7}}{7}=\frac{21\mp 4\sqrt{7}}{7} \Rightarrow \left(\frac{14+2\sqrt{7}}{7}, \frac{21-4\sqrt{7}}{7}\right)$

$\Rightarrow \left(\frac{14-2\sqrt{7}}{7}, \frac{21+4\sqrt{7}}{7}\right)$

$l: y=kx+m$

$\mathcal{E}: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$l \cap \mathcal{E}: \begin{cases} y=kx+m \\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \end{cases}$

$\frac{x^2}{a^2}+\frac{(kx+m)^2}{b^2}=1 \Rightarrow x^2\left(\frac{1}{a^2}+\frac{k^2}{b^2}\right)+\frac{2km}{b^2}x+\frac{m^2-b^2}{b^2}=0$

$$\Delta = \frac{4\,\cancel{k^2 n^2}}{a^2 b^4}\left(\cancel{a^2 k^2 n^2} - b^2 n^2 - \cancel{k^2 a^2 n^2} + b^4 + k^2 a^2 b^2\right)$$

$$= \frac{4}{a^2 b^2}\left(b^2 + k^2 a^2 - n^2\right)$$

$b^2 + k^2 a^2 - n^2$	intersection
<0	$\emptyset$
$=0$	one point (l tangent)
>0	two point (l secant)

So ~~if~~ l is tangent then $n = \pm\sqrt{b^2 + k^2 a^2}$

So: $l : y = kx \pm \sqrt{a^2 k^2 + b^2}$

---

6. 6. Determine an equation of a line which is orthogonal to l $2x - 2y - 13 = 0$ and tangent to the ellipse $\mathcal{E}: x^2 + 4y^2 - 20 =$

Tangent to $\mathcal{E}$ in (x_0, y_0):

$$T_{(x_0,y_0)}\mathcal{E} : \frac{x x_0}{a^2} + \frac{y y_0}{b^2} = 1$$

$$T_{(x_0,y_0)}\mathcal{E} : \frac{\partial f}{\partial x}(x_0, y_0)\cdot(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)\cdot(y - y_0) = 0$$

$$,\ f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$$

$$T\mathcal{E} : \frac{x_0}{a^2}(x - x_0) + \frac{y_0}{b^2}(y - y_0) = 0$$

~~T_1 $m_l = -\frac{a}{b} = 1 \Rightarrow$~~

$l : y = x - \frac{13}{2} \Rightarrow m_l = 1$

$\underbrace{m_t}_{=k} \cdot m_l = -1 \Rightarrow m_t = -1 \Rightarrow t: y = kx + n$

$t : y = -x + n$

$t \cap \mathcal{E} = \{$

$$\begin{cases} y = -x + n \\ \cancel{y = \frac{x}{2} - \frac{13}{2}} \\ x^2 + 4y^2 - 20 = 0 \end{cases} \Rightarrow x^2 + 4(-x+n)^2 - 20 = 0$$

$$x^2 + 4(x^2 - 2nx + n^2) - 20 = 0$$

$$x^2 + 4x^2 - 8nx + 4n^2 - 20 = 0$$

$$5x^2 - 8nx + 4n^2 - 20$$

$$\Delta = 0$$

$$\Delta = 64n^2 - 4 \cdot 5(4n^2 - 20) = 0$$

$$64n^2 - 16n^2 + 80 = 0$$

$$\cancel{16n^2}$$

$$48n^2 + 80 = 0 \quad |:4$$

$$12n^2 + 20 = 0 \quad |:2$$

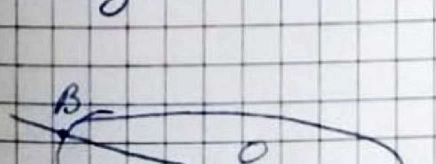
$$6n^2 + 10 = 0 \quad |:2$$

$$3n^2 + 5 = 0$$

$$\Rightarrow 3n^2 = -5 \Rightarrow n^2 = \frac{25}{\cancel{3}} \Rightarrow \underline{n = \pm\sqrt{25}} \quad \underline{5}$$ (Calcul greșit uitat 5 la Δ)

$$\Rightarrow t: y = -x \pm 5$$

6.7. diameter for an ellipse line segment obtained by ~~intersecting~~ intersecting a line through the center with the ellipse



Show that the tangent lines to an ellipse ~~are~~ at the endpoints of a diameter are parallel

$l: y = kx$

$l \cap \mathcal{E} = \{A, B\}$

$\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$l \cap \mathcal{E} = \begin{cases} y = kx \\ \frac{x^2}{a^2} + \frac{k^2x^2}{b^2} = 1 \end{cases} \Rightarrow x^2\left(\frac{b^2 + a^2k^2}{a^2b^2}\right) = 1$$

$$x_{1,2} = \pm \frac{ab}{\sqrt{b^2+a^2k^2}}$$

$$y_{1,2} = \pm \frac{abk}{\sqrt{b^2+a^2k^2}}$$

Let $A(x_1, y_1)$, $B(x_2, y_2)$

$\Rightarrow x_B = -x_A$, $y_B = -y_A$

$$T_A\mathcal{E}: \frac{x_A \cdot x}{a^2} + \frac{y_A \cdot y}{b^2} = 1$$

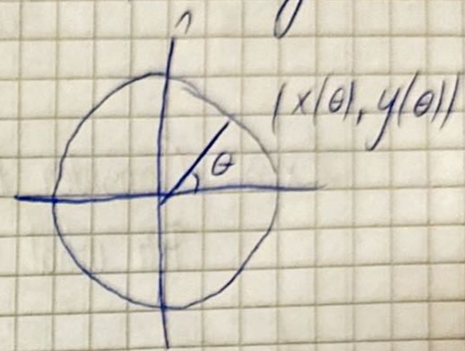
$$m_{T_A\mathcal{E}} = -\frac{x_A}{a^2} \cdot \frac{b^2}{y_A}$$

$$m_{T_B\mathcal{E}} = -\frac{b^2}{a^2} \cdot \frac{x_B}{y_B} = -\frac{b^2}{a^2} \cdot \frac{-x_A}{-y_A} = -\frac{b^2}{a^2} \cdot \frac{x_A}{y_A} = m_{T_A\mathcal{E}} \Rightarrow T_A\mathcal{E} \parallel T_B\mathcal{E}$$

Seminar 11

Quadratic curves

Exercise 7: For a circle $\mathcal{C}$ of radius R, use the param. $\theta \mapsto (R\cos\theta, R\sin\theta)$ to deduce the param. of tangent lines to $\mathcal{E}$



$\vec{r} \perp \vec{u} \Rightarrow \vec{r} \cdot \vec{u} = 0$

$(x(\theta), y(\theta)) \cdot (x_u, y_u) = 0$

$x(\theta) \cdot x_u + y(\theta) \cdot y_u = 0$

$$x_u = -\frac{y(\theta) \cdot y_u}{x(\theta)}$$

let $y_u = x(\theta)$

$$\Rightarrow x_u = -\frac{y(\theta) \cdot \cancel{x(\theta)}}{\cancel{x(\theta)}} = -y(\theta)$$

$\vec{u}(-y(\theta), x(\theta))$

$$\begin{cases} x = x(\theta) + t(-y(\theta)) \\ y = y(\theta) + t\,x(\theta) \end{cases}, t \in \mathbb{R}$$

Exercise 8: Consider the family of ellipses $\mathcal{E}_a : \frac{x^2}{a^2} + \frac{y^2}{16} = 1$. For what value $a \in \mathbb{R}$ is $\mathcal{E}_a$ tangent to the line $l: x - y + 5 = 0$?

$$l \cap \mathcal{E}_a : \begin{cases} y = x + 5 \\ \frac{x^2}{a^2} + \frac{y^2}{16} = 1 \end{cases}$$

$$\frac{x^2}{a^2} + \frac{(x+5)^2}{16} = 1$$

$$\frac{x^2}{a^2} + \frac{x^2 + 10x + 25}{16} = 1 \quad | \, a^2 16$$

$16x^2 + a^2x^2 + 10a^2x + 25a^2 = a^2 16$

$x^2(16 + a^2) + x10a^2 + 9a^2$

$\Delta = a^2(136a^2 - 576)$

I $a = 0$ invalid

II $136a^2 - 576 = 0$

$a^2 = \frac{576}{164} \Rightarrow a = \pm\frac{24}{8} \Rightarrow$

$a = 3$

$$\boxed{\begin{array}{l} \mathcal{E}: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \\ l: y=kx+m \\ m^2=a^2k^2+b^2 \end{array}}$$

- if this is satisfied, the line and the ellipse are tangent

Ex 9: Consider the family of lines $l_c: \sqrt{5}x-y+c=0$
For what values $c\in\mathbb{R}$ is l_c tangent to $\mathcal{E}: x^2+\frac{y^2}{4}=1$?

$l_c: \sqrt{5}x-y+c=0 \Rightarrow l_c: y=\underbrace{\sqrt{5}}_{k}x+\underbrace{c}_{m}=0$

$m^2=a^2k^2+b^2$

$c^2=1^2\cdot\sqrt{5}^2+4=9 \Rightarrow c=\pm 3$

E10: Determine the common tangents to the ellipses
$\frac{x^2}{45}+\frac{y^2}{9}=1$, $\frac{x^2}{9}+\frac{y^2}{18}=1$

$$\boxed{\begin{array}{l} \mathcal{E}: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \\ \text{tangents: } kx\pm\sqrt{a^2k^2+b^2} \end{array}}$$

$\frac{x^2}{45}+\frac{y^2}{9}=1$

$l_h: hx\pm\sqrt{45h^2+9}=y$ $\quad l_k: y=kx\pm\sqrt{9h^2+18}$

$\Rightarrow hx\pm\sqrt{45h^2+9}=kx\pm\sqrt{9h^2+18} \quad |\ (\,)^2$

$\Rightarrow 36k^2=9 \Rightarrow k=\pm\frac{1}{2}$

$9\cdot\frac{1}{4}+18=\frac{9+72}{4}=\frac{81}{4}$

The tangents are: $y=\pm\frac{1}{2}x\pm\frac{9}{2}$

Ex18: Determine the tangents to the hyperbola

$H: x^2 - y^2 = 16$ which contain $M(-1, 7)$

$l_{k_1}: y = kx \pm \sqrt{16k^2 - 16}$

$H: \frac{x^2}{16} - \frac{y^2}{16} = 1$

$l_{k_1}: 7 = -k \pm \sqrt{16k^2 - 16}$

$k^2 + 14k + 49 = 16k^2 - 16$

$15k^2 - 14k - 65 = 0$

$\Delta = 14^2 - 4 \cdot 16(-65) = 4096$

$k = \frac{14 \pm 64}{30} \Rightarrow k_1 = -\frac{5}{3}$

$k_2 = \frac{13}{5}$

$k = \frac{5}{3} \Rightarrow y = \frac{5}{3}x$

$\pm 4 \cdot \frac{4}{3} = \frac{16}{3}$

$y = \frac{13}{5}x \pm \sqrt{16\frac{169}{25} - 16}$

$y = \frac{13}{5}x + \frac{48}{5}$

$M(-1, 7)$

$y = \frac{-5}{3}x + \frac{16}{3}$

$7 = \frac{5}{3} + \frac{16}{3}$

$y = \frac{-5}{3}x - \frac{16}{3}$

$7 = +\frac{5}{3} - \frac{16}{3}$ not ok

$y = \frac{13}{5}x + \frac{48}{5}$

$7 = -\frac{13}{5} + \frac{48}{5}$ ok

$y = \frac{13}{5}x - \frac{48}{5}$ not ok

Exercise 20: Find the area of the triangle determined by the asymptotes of

$H: \frac{x^2}{4} - \frac{y^2}{9} = 1$ and the line $l: 9x + 2y - 24 = 0$

$$\boxed{\begin{array}{l} H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \\ \text{Asymptotes: } y = \pm \frac{b}{a} x \end{array}}$$

$y = \pm \frac{3}{2} x$

$O(0,0)$

$\begin{cases} y = \frac{3}{2} x \\ 9x + 2y - 24 = 0 \end{cases} \Rightarrow \begin{cases} y = \frac{3}{2} x \\ \cancel{9x - \frac{4}{3}x - 24 = 0 \Rightarrow 24x = 72 \Rightarrow x = 3 \Rightarrow y = 2} \\ 9x - 2x - 24 = 0 \Rightarrow 6x = 24 \Rightarrow x = 4 \\ y = -6 \end{cases}$

$\cancel{\Rightarrow A(3,2)}$

$\Rightarrow A(4, -6)$

$\begin{cases} y = \frac{3}{2} x \\ 9x + 2y - 24 = 0 \end{cases} \Rightarrow \begin{cases} y = \frac{3}{2} x \\ 9x + 3x - 24 = 0 \Rightarrow x = 2 \Rightarrow y = 3 \end{cases}$

Exercise 26:

$P: y^2 = 4x$

$l: y = k + 2$

$k = ?$ s.t. l tangent to P

$\frac{2}{2k} = 2 \Rightarrow \frac{1}{k} = 2 \Rightarrow k = \frac{1}{2}$

Seminar 12 - Ch 7

1, 2, 4, 6, 7, 8

$Q: a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{01}y + a_{00} = 0$

$Q \xrightarrow{\text{isometry}} \lambda_1 x^2 + \lambda_2 y^2 = k$

(rotation and translation)

or

$\lambda_1 x^2 + \lambda_2 y = k$

$\downarrow$ rescaling

$k, \lambda_1, \lambda_2 \in \{-1, 0, 1\}$

$M_Q = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$Q: (x \ \ y) \cdot M_Q \begin{pmatrix} x \\ y \end{pmatrix} + (x \ \ y)(2a_{10} \ \ 2a_{01}) + a_{00} = 0$

$r = \operatorname{rank} M_Q$	$(p, r-p)$	equation	name
2	$(2,0)$ or $(0,2)$	$x^2 + y^2 + 1 = 0$	immaginary ellipse
2	$(2,0)$ or $(0,2)$	$x^2 + y^2 - 1 = 0$	ellipse
2	$(1,1)$	$x^2 - y^2 - 1 = 0$	hyperbola
2	$(2,0)$ or $(0,2)$	$x^2 + y^2 = 0$	2 complex lines
2	$(1,1)$	$x^2 - y^2 = 0$	2 real lines
1	$(0,1)$ or $(1,0)$	$x^2 + 1 = 0$	2 complex lines
1	$(1,0)$	$x^2 - 1 = 0$	2 real lines
1	$(1,0)$	$x^2 = 0$	a double line
1	$(1,0)$	$x^2 - y = 0$	parabola

7.2. Write down a quadratic equation with associated matrix A and find the matrix $M \in SO(2)$ which diagonalizes A

$$A = \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}$$

$$P_A(x) = \det(A - xI_n) = \begin{vmatrix} 6-x & 2 \\ 2 & 9-x \end{vmatrix} = x^2 - 15x + 50$$

$$\Rightarrow \lambda_{1,2} = \frac{15 \pm \sqrt{225-200}}{2} = \frac{15 \pm \sqrt{25}}{2} \begin{cases} 10 \\ 5 \end{cases}$$

$$S(\lambda) = \{(x,y) \mid A\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}\} = \{(x,y) \mid (A - \lambda I_2)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

$$S(\lambda_1) = \{(x,y) \mid \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

$$= \{(x,y) \mid \begin{cases} -4x + 2y = 0 \\ 2x - y = 0 \end{cases}\}$$

$$= \{(x,y) \mid y = 2x\} = \{(x, 2x) \mid x \in \mathbb{R}\} = \langle (1,2) \rangle$$

Choose $v_1 = \frac{1}{\sqrt{5}} \cdot (1, 2)$

$$S(\lambda_2) = \{(x,y) \mid \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$$

$$= \{(x,y) \mid \begin{cases} x + 2y = 0 \\ 2x + 4y = 0 \end{cases}\}$$

$$= \{(x,y) \mid x = -2y\}$$

$$= \{(-2y, y) \mid y \in \mathbb{R}\}$$

$$= \langle (-2, 1) \rangle$$

Choose $v_2 = \frac{1}{\sqrt{5}}(2, -1)$

$$M = M_{B', B} = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix}$$

$\det M = -1$ and we want $M \in SO(2)$, so we choose instead

$v_2 = \frac{1}{\sqrt{5}}(-2, 1) \Rightarrow M = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \in SO(2)$

$\Delta = M^{-1} A M =$

$= M^T$ because $M \in O(2)$

$\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}$

$= \frac{1}{5}\begin{pmatrix} 10 & 20 \\ -10 & 5 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

$= \frac{1}{5}\begin{pmatrix} 50 & 0 \\ 0 & 25 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$

Let

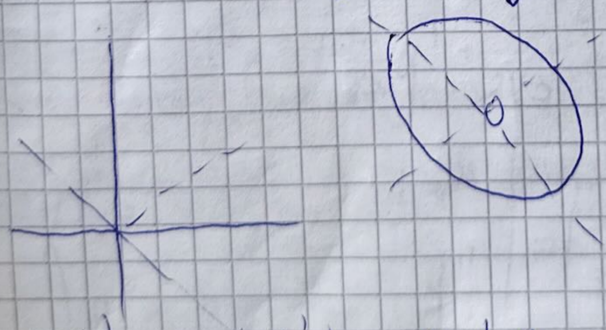
$Q: (x\ y)\ A \begin{pmatrix} x \\ y \end{pmatrix} + (x\ y)\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1 = 0$

$Q: (x, y)\begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + (x\ y)\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1 = 0$

$Q: (6x + 2y, 7x - 9y)\begin{pmatrix} x \\ y \end{pmatrix} + x + 2y - 1 = 0$

$Q: 6x^2 + 2yx + 2xy + 9y^2 + 2x + 2y - 1 = 0$

$Q: 6x^2 + 4xy + 9y^2 + x + 2y - 1 = 0$



$\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} \Rightarrow \begin{cases} x = \frac{1}{\sqrt{5}}x' - \frac{2}{\sqrt{5}}y' \\ y = \frac{2}{\sqrt{5}}x' + \frac{1}{\sqrt{5}}y' \end{cases}$

$$(x\ y) = \begin{pmatrix} x \\ y \end{pmatrix}^T = \left(M \begin{pmatrix} x' \\ y' \end{pmatrix} \right)^T = (x'\ y') \cdot M^T$$

$$Q: (x'y')\ M^T \cdot A \cdot M \begin{pmatrix} x' \\ y' \end{pmatrix} + (x'y')\ M^T \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1 = 0$$

$$Q: (x'y') \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + (x'y') \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1 = 0$$

$$Q: 10x'^2 + 5y'^2 + \sqrt{5}x' - 1 = 0$$

$$Q: (10x'^2 + \sqrt{5}x') + 5y'^2 - 1 = 0$$

$$Q: \left(10x'^2 + 2 \cdot \frac{\sqrt{5}}{20} x' + \frac{5}{400}\right) + 5y'^2 - \frac{1}{8} = 0$$

$$Q: 10\left(x' + \frac{\sqrt{5}}{20}\right)^2 + 5y'^2 - \frac{9}{8} = 0$$

$$\begin{cases} x'' = x' + \frac{\sqrt{5}}{20} \\ y'' = y' \end{cases}$$

$$Q: 10x''^2 + 5y''^2 = \frac{9}{8}$$

$$Q: \frac{x''^2}{\frac{9}{80}} + \frac{y''^2}{\frac{9}{40}} = 1$$

$$x''' = \frac{x''}{\frac{3}{\sqrt{80}}}$$

$$y''' = \frac{y''}{\frac{3}{\sqrt{40}}}$$

$$\Rightarrow Q: x'''^2 + y'''^2 = 1$$

Second approach (Lagrange)

$Q: (6x^2+4xy)+9y^2+x+2y-1=0$

$6x^2+2\cdot\sqrt{6}\cdot x\cdot\frac{2}{\sqrt{6}}+\frac{2}{3}y^2)+9y^2-\frac{2}{3}y^2+x+2y-1=0$

$(\underbrace{\sqrt{6}x+\frac{2}{\sqrt{6}}y}_{x'})^2+\frac{25}{3}y^2+x+2y-1=0$

$$\begin{cases} x'=\sqrt{6}x+\frac{2}{\sqrt{6}}y \\ y'=y \end{cases}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix}=\begin{pmatrix} \sqrt{6} & \frac{2}{\sqrt{6}} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$x=\frac{x'-\frac{2}{\sqrt{6}}y'}{\sqrt{6}}$$

$Q: x'^2+\frac{25}{3}y'^2+\frac{1}{\sqrt{6}}x'-\frac{1}{3}y'+2y-1=0$

$Q: \left(x'^2+\frac{1}{\sqrt{6}}x'\right)+\left(\frac{25}{3}y'^2+\frac{5}{3}y'\right)-1=0$

$Q: \left(x'^2+2\cdot x'\cdot\frac{1}{2\sqrt{6}}+\frac{1}{24}\right)+\left(\frac{25}{3}y'^2+2\cdot\frac{5}{\sqrt{3}}y'\cdot\frac{1}{2\sqrt{3}}+\frac{1}{12}\right)-1-\frac{1}{24}-\frac{1}{12}=0$

$Q: \left(x'+\frac{1}{2\sqrt{6}}\right)^2+\left(\frac{5}{\sqrt{3}}y'+\frac{1}{2\sqrt{3}}\right)^2=\frac{27}{24}=\frac{9}{8}$

~~Q~~ $\begin{cases} x''=x'+\frac{1}{2\sqrt{6}} \\ y''=\frac{5}{\sqrt{3}}y'+\frac{1}{2\sqrt{3}} \end{cases}$ $\begin{pmatrix} x'' \\ y'' \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & \frac{5}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}+\begin{pmatrix} \frac{1}{2\sqrt{6}} \\ \frac{1}{2\sqrt{3}} \end{pmatrix}$

$\alpha: 6x^2+4xy+9y^2+x+2y-1=0$

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$$Q: a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{01}y + a_{00} = 0$$

$$M_Q = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

$$\hat{M}_Q = \left(\begin{array}{cc|c} a_{11} & a_{12} & \\ a_{12} & a_{22} & \\ \hline a_{10} & a_{01} & a_{00} \end{array}\right)$$

$$Q: (x\ y\ 1) \cdot \hat{M}_Q \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

The isometric invariants:

$\hat{\Delta} = \det \hat{M}_Q$

$\Delta = \det M_Q$

$T = \mathrm{Tr}\, M_Q$

$\hat{\Delta}$	Δ	T	Curve Q
$\hat{\Delta} = 0$	$\Delta > 0$		a point
	$\Delta = 0$		two lines on the empty set
	$\Delta < 0$		two lines
$\hat{\Delta} \neq 0$	$\Delta > 0$	$T < 0$	ellipse
	$\Delta > 0$	$T > 0$	empty set
	$\Delta = 0$		parabola
	$\Delta < 0$		hyperbola

7.9. Discuss the type of the curve $x^2+\lambda xy+y^2-6x-16=0$ in terms of $\lambda \in \mathbb{R}$

$$A_Q = \begin{pmatrix} 1 & \frac{\lambda}{2} \\ \frac{\lambda}{2} & 1 \end{pmatrix}$$

$$\hat{A}_Q = \begin{pmatrix} 1 & \frac{\lambda}{2} & -3 \\ \frac{\lambda}{2} & 1 & 0 \\ -3 & 0 & -16 \end{pmatrix}$$

$$\hat{\Delta} = \begin{vmatrix} 1 & \frac{\lambda}{2} & -3 \\ \frac{\lambda}{2} & 1 & 0 \\ -3 & 0 & -16 \end{vmatrix} = 4\lambda^2-25$$

$$\Delta = \begin{vmatrix} 1 & \frac{\lambda}{2} \\ \frac{\lambda}{2} & 1 \end{vmatrix} = 1-\frac{\lambda^2}{4}$$

$T = 2$

If $\hat{\Delta}=0 \Rightarrow 4\lambda^2-25=0$

$4\lambda^2=25$

$\lambda^2=\frac{25}{4}$

$\lambda=\pm\frac{5}{2}$

For $\lambda=\pm\frac{5}{2} \Rightarrow \Delta = 1-\frac{25}{16} = -\frac{9}{16} < 0 \Rightarrow$

Q - two lines

$\hat{\Delta} \neq 0 \Rightarrow \lambda \in \mathbb{R} \setminus \{\pm\frac{5}{2}\}$

If $\Delta > 0 \Rightarrow 1-\frac{\lambda^2}{4} > 0 \Rightarrow \frac{\lambda^2}{4} < 1 \Rightarrow \lambda^2 < 4$

$\Rightarrow \lambda \in (-2,2) \Rightarrow$ Q empty set

$\Delta < 0 \Rightarrow \lambda \in (-\infty,-2) \cup (2,\infty) \Rightarrow$ Q hyperbola

$\Delta = 0 \Rightarrow \lambda = \pm 2 \Rightarrow$ Q - parabola

7.10. Decide what surfaces are described by the following equations:

a). $x^2+2y^2+z^2+xy+yz+zx=1$

b). $xy+yz+zx=1$

(use the Lagrange method)

$(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ac$

a). $\left(x^2+xy+xz+\frac{y^2}{4}+\frac{z^2}{4}+\frac{yz}{2}\right)-\frac{y^2}{4}-\frac{z^2}{4}-\frac{yz}{2}+2y^2+z^2+yz=1$

$\left(x+\frac{y}{2}+\frac{z}{2}\right)^2+\frac{7y^2}{4}+\frac{3z^2}{4}+\frac{yz}{2}=1$

$x'^2+\frac{7y^2}{4}+\frac{3z^2}{4}+\frac{yz}{2}=1$

$x'^2+\frac{7y^2}{4}+\left(\frac{3z^2}{4}+\frac{yz}{2}\right)=1$

$x'^2+\frac{7y}{4}+\left(\frac{3z^2}{4}+2\cdot\frac{\sqrt{3}z}{2}\cdot\frac{y}{2\sqrt{3}}+\frac{y^2}{12}\right)-\frac{y^2}{12}=1$

$x'^2+\frac{20y^2}{12}+\left(\underbrace{\frac{\sqrt{3}}{2}z+\frac{y}{2\sqrt{5}}}_{z'}\right)=1$

$x'^2+\frac{20y^2}{12}+z'^2=1$

b). $xy + yz + zx = 1$

Put $x = y + x'$

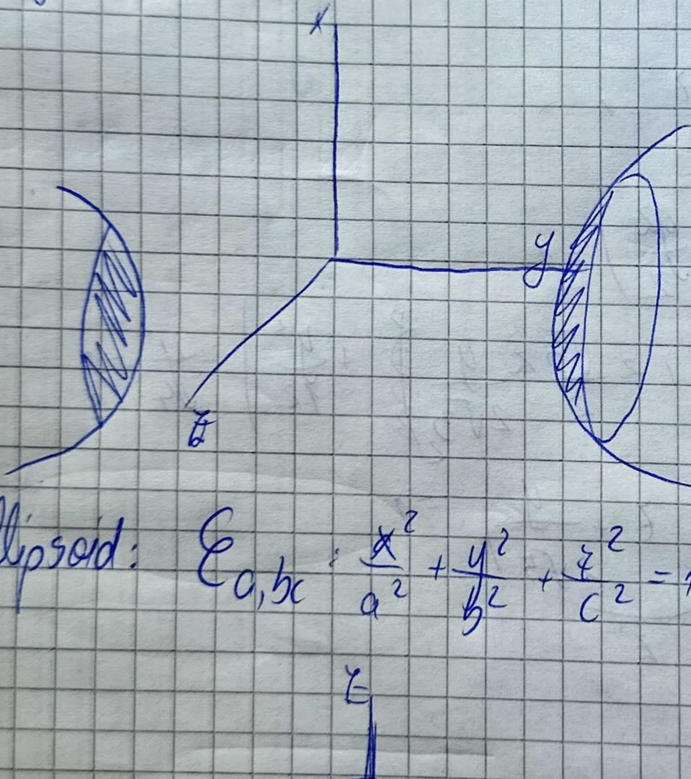
$(y + x')y + yz + z(y + \cancel{z}x') = 1$

$y^2 + yx' + 2yz + zx' = 1$

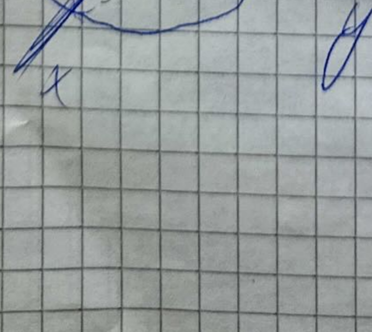
$y^2 + 2 \cdot y \cdot \frac{x'}{2} + 2yz + \left(\frac{x'}{2}\right)^2 + z^2 + 2 \cdot \frac{x'}{2} z - \left(\frac{x'}{2}\right)^2 - z^2 = 1$

$\left(y + \frac{x'}{2} + z\right)^2 - \left(\frac{x'}{2}\right) - z^2 = 1$

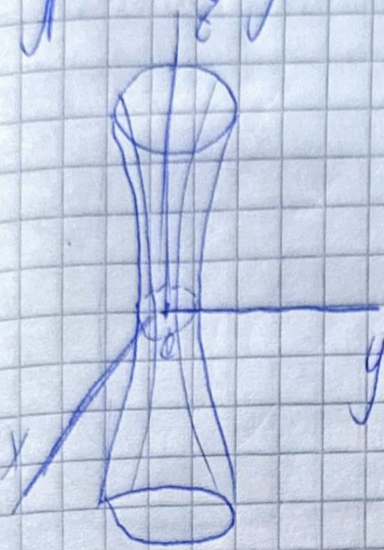
$y'^2 - \left(\frac{x'}{2}\right)^2 - z^2 = 1$



- ellipsoid: $\mathcal{E}_{a,b,c} : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

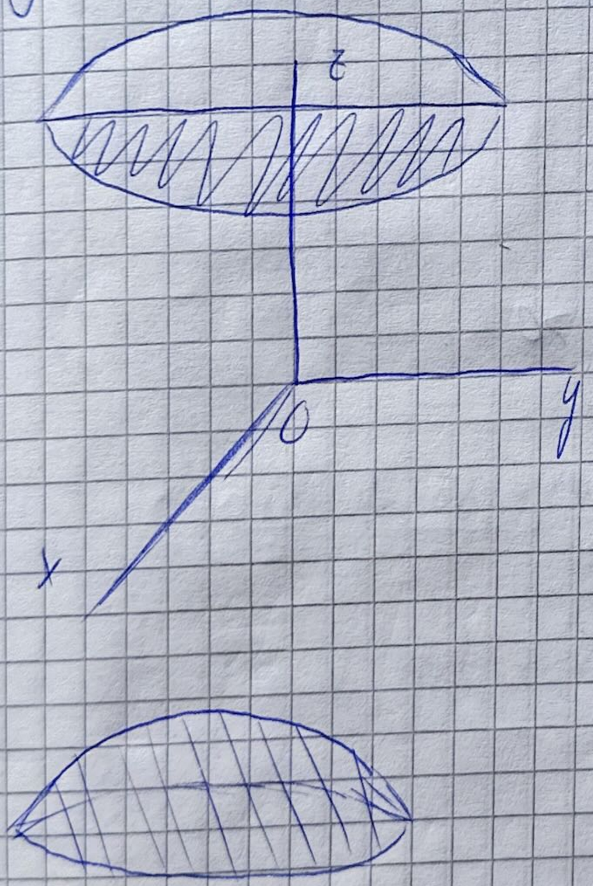


- hyperboloid of one sheet



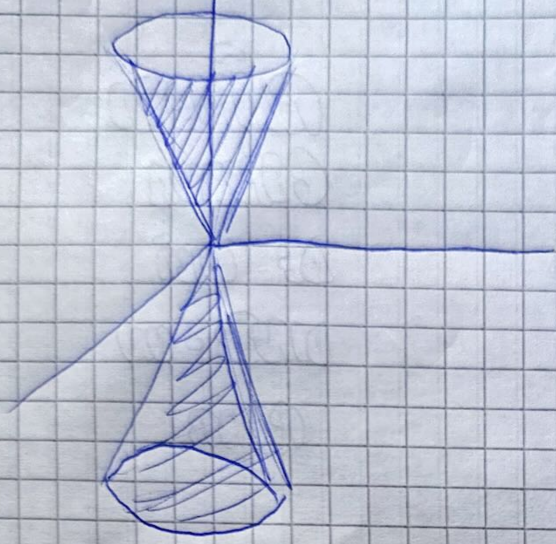
$$H_{a,b,c}: \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

- hyperboloid of two sheets



$$H^2_{a,b,c}: -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

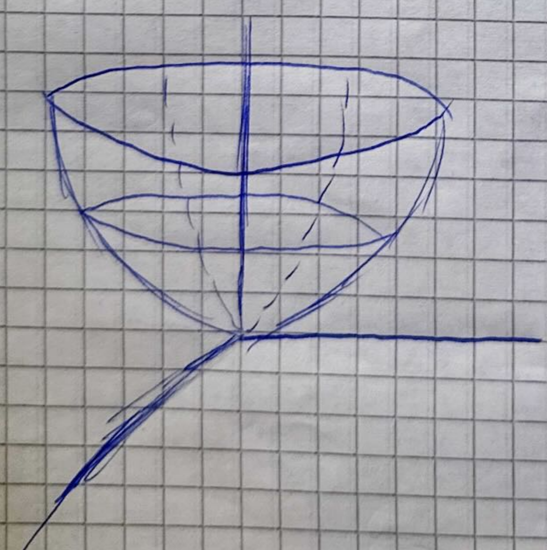
Ellipse cone



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

Ellipse parabolloid

$$P^{e}_{p,q}: \frac{x^2}{p}+\frac{y^2}{q}=2z \quad , p,q>0$$

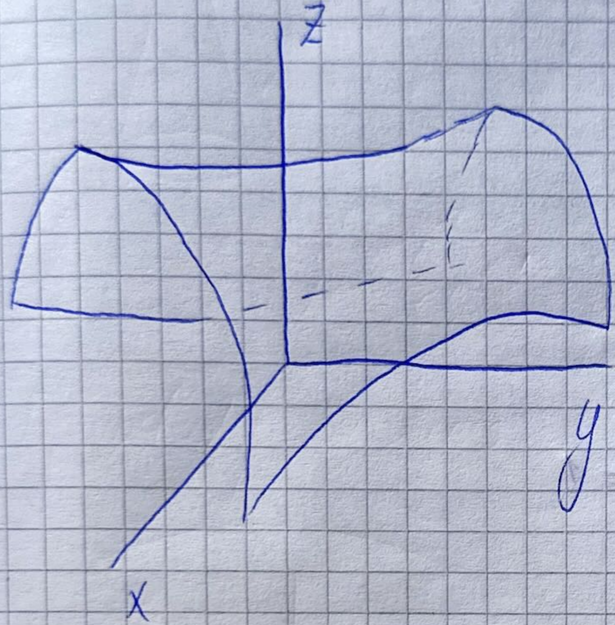


- hyperbolic paraboloid

$\mathcal{P}_h$

$\mathcal{P}_{p,q} : \frac{x^2}{p} - \frac{y^2}{q} = 2z$

Number of sheets = no. of negatives
(nu știu ce înseamnă asta domnu Micu)



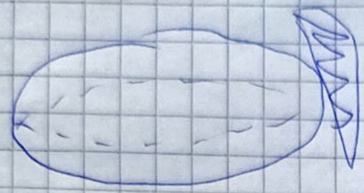
$\mathcal{Y} : f(x, y, z) = 0$
↳ surface

$T_{(x_0, y_0, z_0)}\mathcal{Y} : \frac{\partial f}{\partial x}(x_0, y_0, z_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0) \cdot (y - y_0) +$

$+ \frac{\partial f}{\partial z}(x_0, y_0, y_0)(z - z_0) = 0$

8.4. Determine the tangent planes to the ellipsoid:

$\mathcal{E}_{2,3,2\sqrt{2}}: \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{8} = 1$ which are parallel to the plane $\pi: 3x - 2y + 5z + 1 = 0$



$$T_{x_0,y_0,z_0}\mathcal{E}: \frac{xx_0}{4} + \frac{yy_0}{9} + \frac{zz_0}{8} = 1$$

$$\vec{n}_{T\mathcal{E}} = \left(\frac{x_0}{4}, \frac{y_0}{9}, \frac{z_0}{8}\right)$$

$$\vec{n}_{\pi} = (3, -2, 5)$$

$$T_{\mathcal{E}} \parallel \pi \Leftrightarrow \vec{n}_{T\mathcal{E}} \parallel \vec{n}_{\pi} \Leftrightarrow \frac{x_0/4}{3} = \frac{y_0/9}{-2} = \frac{z_0/8}{5}$$

$$\Leftrightarrow \frac{x_0}{12} = \frac{y_0}{-18} = \frac{z_0}{40} = t_0$$

$$x_0 = 12t_0$$
$$y_0 = -18t_0$$
$$z_0 = 40t_0$$

$$P(x_0, y_0, z_0) \in \mathcal{E} \Rightarrow \frac{(12t_0)^2}{4} + \frac{(18t_0)^2}{9} + \frac{(40t_0)^2}{8} = 1 \Rightarrow$$

$$\Rightarrow 36t_0^2 + 36t_0^2 + 200t_0^2 = 1 \Rightarrow t_0^2 = \frac{1}{272} \Rightarrow t_0 = \pm\frac{1}{4\sqrt{17}}$$

$$\Rightarrow P_1\left(\frac{12}{4\sqrt{17}}, \frac{-18}{4\sqrt{17}}, \frac{40}{4\sqrt{17}}\right), P_2\left(\frac{-12}{4\sqrt{17}}, \frac{18}{4\sqrt{17}}, \frac{-40}{4\sqrt{17}}\right) \Rightarrow T_{P_1}\mathcal{E}, T_{P_2}\mathcal{E}$$