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# OPTICAL PROPERTIES OF WIGGLERS WITH HIGH FIELD-TO-ENERGY RATIO *

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*Abstract*

One of the options to bring electron cooling to high energies is to employ an electron storage ring, which utilizes damping wigglers to counteract emittance growth of electron bunches used to cool hadrons. An example of such a cooler is the Ring Electron Cooler (REC) that can find potential future applications in Electron Ion Collider (EIC). The REC is designed to operate at 150 MeV and requires wigglers with peak field of 2.4 T. This unique combination a strong field wiggler operated at a relatively low energy results in unusual optical properties. In this paper we derive analytic formulas for focusing and chromaticity of different wiggler options and compare analytic and beam tracking results. While our analysis was used to optimize chromaticity in the REC, the derived formulas have a general applicability to wigglers with a high field-to-energy ratio.


## INTRODUCTION

The Ring Electron Cooler [1] is designed to provide cooling of EIC protons at top energy ($\gamma = 273$).

The REC is a non-magnetized RF-based 150 MeV electron cooler, in which electrons are kept in the storage ring and re-utilized for multiple turns. The REC is equipped with 18 damping wigglers, which keep electrons' emittance constant by counteracting both electrons' IBS and a proton-electron beam-beam scattering. These wigglers operate at a relatively low energy but must have a rather high peak field $B_0$ =2.4 T to provide the required radiation cooling of electrons. As a result of this unique range of parameters, the wigglers' optics exhibits interesting features [2].

The REC design requires that throughout the wiggler horizontal $\beta$-function is $\beta_x = 29.5$ cm. Thus, a wiggler must produce a strong focusing in the wiggling direction.

Below we show that to limit contribution of the wigglers to the REC chromaticity while producing necessary horizontal focusing, a special field profile is required.

We consider wigglers with period $\lambda = 23.28$ cm, 2 cm gap and number of periods $N_p = 18$.

## EQUATIONS OF MOTION

We consider motion in the wiggler's reference frame, with $x$-axis directed in the wiggling direction, $y$-axis orthogonal to the wiggling plane and $z$ being in the direction along the wiggler's axis. The equations of motion in approximation of $x' \ll 1$ and $y' \ll 1$ are:

$$x'' = -\frac{B_y}{B\rho} + y'\frac{B_z}{B\rho};\; y'' = \frac{B_x}{B\rho} - x'\frac{B_z}{B\rho} \qquad (1)$$

### Quadratic Focusing

Consider the following representation of wiggler field:

$$\begin{aligned} B_x &= \frac{k_x}{k_y} B_0 \sinh(k_x x)\sinh(k_y y)\sin(kz) \\ B_y &= B_0 \cosh(k_x x)\cosh(k_y y)\sin(kz) \\ B_z &= \frac{k}{k_y} B_0 \cosh(k_x x)\sinh(k_y y)\cos(kz) \end{aligned} \qquad (2)$$

where $k = \frac{2\pi}{\lambda} \approx 27$ m$^{-1}$, $k_x$ and $k_y$ are geometric parameters such that $k_x^2 + k_y^2 = k^2$.

In this wiggler the poles are shaped to produce a quadratic dependence of $B_y$ on $x$ near the axis and a focusing in $x$-direction is happening due to particles going with an offset through a sextupole-like field.

Substituting Eq. (2) into Eq. (1) we get:

$$\begin{aligned} x'' &= -b\cosh(k_x x)\cosh(k_y y)\sin(kz) + \\ &\quad + b\frac{k}{k_y} y' \cosh(k_x x)\sinh(k_y y)\cos(kz) \\ y'' &= b\frac{k_x}{k_y}\sinh(k_x x)\sinh(k_y y)\sin(kz) - \\ &\quad - b\frac{k}{k_y} x' \cosh(k_x x)\sinh(k_y y)\cos(kz) \end{aligned} \qquad (3)$$

where $b = \frac{B_0}{B\rho}$.

An approximate solution of Eq. (3) is given by [2]:

$$\begin{aligned} x &= \frac{b}{k^2}\sin(kz) + x_0\cos\left(\frac{bk_x}{\sqrt{2}k}z\right) + \frac{\sqrt{2}k}{bk_x}\left(x_0' - \frac{b}{k}\right)\sin\left(\frac{bk_x}{\sqrt{2}k}z\right) \\ y &= y_0\cos\left(\frac{b\sqrt{k^2-k_x^2}}{\sqrt{2}k}z\right) + y_0'\frac{\sqrt{2}k}{b\sqrt{k^2-k_x^2}}\sin\left(\frac{b\sqrt{k^2-k_x^2}}{\sqrt{2}k}z\right) \end{aligned} \qquad (4)$$

There is a good agreement between Eq. (4) and numerical integration of Eq. (3) (see Fig. 1).

### Linear Focusing

An example of linear focusing is a wiggler immersed in a continuous quadrupole field (the assumption of continuous field simplifies equations, a periodic quadrupole field gives similar final results):

$$\begin{aligned} B_x &= B_0\cos(k_q x)\sinh(k_q y) \\ B_y &= B_0\cosh(ky)\sin(kz) + B_0\sin(k_q x)\cosh(k_q y) \\ B_z &= B_0\sinh(ky)\cos(kz) \end{aligned} \qquad (5)$$

where $k_q$ is a quadrupole parameter.

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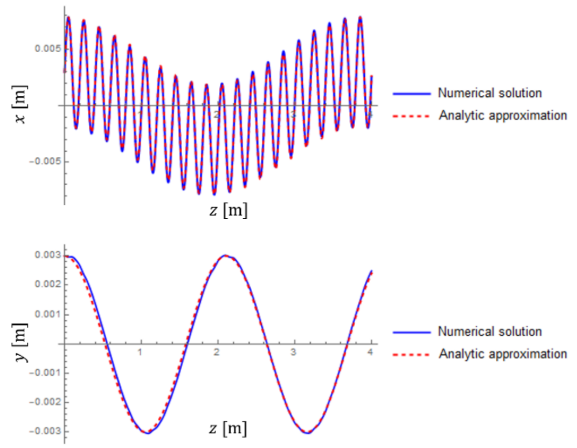


Figure 1: Numerical solution of Eq. (3) and approximate analytic trajectory given by Eq. (4). The results are shown for $k_x = 15$ m$^{-1}$ and $x_0' = b/k$.

Substituting Eq. (5) into Eq. (1) we get:

$$\begin{aligned} x'' &= -b\left(\cosh(ky)\sin(kz) + \sin(k_q x)\cosh(k_q y)\right) + \\ &\quad + by'\sinh(ky)\cos(kz) \\ y'' &= b\cos(k_q x)\sinh(k_q y) - bx'\sinh(ky)\cos(kz) \end{aligned} \tag{6}$$

An approximate solution of Eq. (6) is given by [2]:

$$\begin{aligned} x &= \frac{b}{k^2}\sin(kz) + x_0\cos\left(\sqrt{bk_q}z\right) + \left(x_0' - \frac{b}{k}\right)\frac{\sin\left(\sqrt{bk_q}z\right)}{\sqrt{bk_q}} \\ y &= y_0\cos\left(\sqrt{\frac{b^2}{2} - bk_q}z\right) + \frac{y_0'}{\sqrt{\frac{b^2}{2} - bk_q}}\sin\left(\sqrt{\frac{b^2}{2} - bk_q}z\right) \end{aligned} \tag{7}$$

Numerical solution of Eq. (6) and analytic results (7) are plotted in Fig. 2.

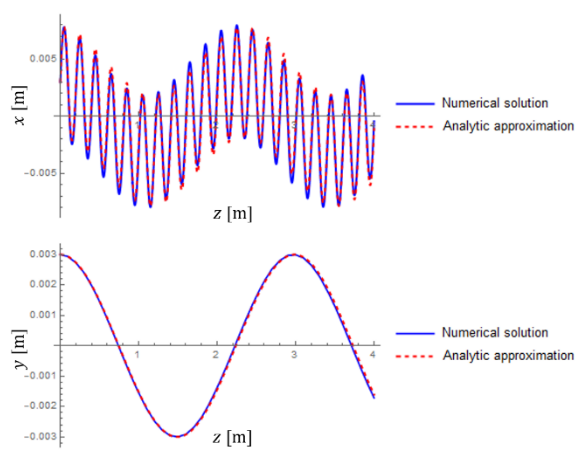


Figure 2: Numerical solution of Eq. (6) and approximate analytic trajectory given by Eq. (7). The results are shown for $k_q = 1.5$ m$^{-1}$ and $x_0' = b/k$.

## OPTICAL FUNCTIONS

### *Wiggler with Quadratic Focusing*

Starting with Eq. (4) one can show [2] that the wiggler's transfer matrix in a wiggling plane is given by:

$$M_x = \begin{pmatrix} \cos\left(\frac{bk_x}{\sqrt{2}k}L\right) & \frac{\sqrt{2}k}{bk_x}\sin\left(\frac{bk_x}{\sqrt{2}k}L\right) \\ -\frac{bk_x}{\sqrt{2}k}\sin\left(\frac{bk_x}{\sqrt{2}k}L\right) & \cos\left(\frac{bk_x}{\sqrt{2}k}L\right) \end{pmatrix} \tag{8}$$

where $L = N_p\lambda$.

For the orthogonal plane:

$$M_y = \begin{pmatrix} \cos\left(\frac{b\sqrt{k^2-k_x^2}}{\sqrt{2}k}L\right) & \frac{\sqrt{2}k\sin\left(\frac{b\sqrt{k^2-k_x^2}}{\sqrt{2}k}L\right)}{b\sqrt{k^2-k_x^2}} \\ -\frac{b\sqrt{k^2-k_x^2}\sin\left(\frac{b\sqrt{k^2-k_x^2}}{\sqrt{2}k}L\right)}{\sqrt{2}k} & \cos\left(\frac{b\sqrt{k^2-k_x^2}}{\sqrt{2}k}L\right) \end{pmatrix} \tag{9}$$

These are equations of a thick lens and the wiggler's matched $\beta$-functions are given by:

$$\beta_x = \frac{\sqrt{2}k}{bk_x}; \quad \beta_y = \frac{\sqrt{2}k}{b\sqrt{k^2-k_x^2}} \tag{10}$$

The respective phase advances are:

$$\phi_x = \frac{bk_x}{\sqrt{2}k}L; \quad \phi_y = \frac{b\sqrt{k^2-k_x^2}}{\sqrt{2}k}L \tag{11}$$

According to Eq. (11), a number of betatron oscillations per wiggler depend on electron's relative momentum $\delta$ as:

$$Q_x = \frac{1}{2\pi}\frac{bk_xL}{\sqrt{2}k}(1-\delta); \quad Q_y = \frac{1}{2\pi}\frac{b\sqrt{k^2-k_x^2}L}{\sqrt{2}k}(1-\delta) \tag{12}$$

The resulting chromaticities ($\eta_{x,y} = \frac{\partial Q_{x,y}}{\partial\delta}$) are given by:

$$\eta_x = -\frac{1}{2\sqrt{2}\pi}\frac{LB_0}{B\rho}\frac{k_x}{k}; \quad \eta_y = -\frac{1}{2\sqrt{2}\pi}\frac{LB_0}{B\rho}\frac{\sqrt{k^2-k_x^2}}{k} \tag{13}$$

More information about wiggler's optical functions can be found in [2].

### *Wiggler with linear focusing*

For a wiggler with linear focusing the optical functions are found to be [2]:

$$\beta_{x1} = \frac{1}{\sqrt{bk_q}}; \quad \beta_{y1} = \frac{1}{\sqrt{\frac{b^2}{2} - bk_q}} \tag{14}$$

$$\phi_{x1} = \sqrt{bk_q}L; \quad \phi_{y1} = \sqrt{\frac{b^2}{2} - bk_q}L \tag{15}$$

$$\eta_{x1} = -\frac{L\sqrt{B_0k_q}}{4\pi\sqrt{B\rho}}; \quad \eta_{y1} = -\frac{LB_0}{2\sqrt{2}\pi B\rho}\frac{1-k_qB\rho/B_0}{\sqrt{1-2k_qB\rho/B_0}} \tag{16}$$

## CHROMATICITY IN CASE OF STRONG FOCUSING

For a wiggler with quadratic focusing, a strong horizontal focusing producing the required small $\beta_x$ corresponds to $k_x \approx 26.97$ m$^{-1}$. That is, the required focusing is achieved for $k_x \to k$.

Comparing equations (10) and (14) one can see that for the wiggler with linear focusing the same focusing is achieved when $k_q \to \frac{b}{2}$.

Notice, that in both cases the motion in an orthogonal plane is approaching the motion in the drift. It is easy to see from Eqs. (4) and (7).

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The horizontal chromaticity in the quadratic focusing case becomes $\eta_x \approx -\frac{1}{2\sqrt{2}\pi}\frac{LB_0}{B\rho}$. In the linear focusing case $\eta_{x1} \approx -\frac{1}{4\sqrt{2}\pi}\frac{LB_0}{B\rho}$. Yet, while the vertical chromaticity for the quadratic focusing case is approaching zero $\eta_y \to 0$, for the linear focusing it becomes infinitely large $\eta_{y1} \to \infty$.

Let us explore the reasons for this drastic difference in behaviour of vertical chromaticity.

The equation of motion (1) for $y$ contains two terms, which have different physical meaning.

The term $\frac{x' B_z}{B\rho}$ represents a fringe focusing from wiggler's poles. Since fringes have a non-zero $B_z$ field, a particle with a non-zero $x'$ experiences a force in $y$ direction. In the wiggler with sinusoidal $B_y$ field, $B_z$ field must be sinusoidal as well. One can say that the wiggler introduces a continuous edge focusing.

Assuming $x_0 = 0$ and $x_0' = \frac{b}{k}$, both Eqs. (4) and (7) give $x = \frac{b}{k^2}\sin(kz)$ for the beam trajectory. On the other hand, from Eqs. (2), (5), for both types of wigglers: $B_z \approx y \cdot kB_0\cos(kz)$. Therefore, the ``edge-focusing term" averaged over the wiggler's period gives:

$$\left\langle \frac{x' B_z}{B\rho} \right\rangle \approx y\langle b^2\cos^2(kz)\rangle = y\frac{b^2}{2} \tag{17}$$

The term $\frac{B_x}{B\rho}$ represents an additional focusing. In our case it is a defocusing term in $y$, because we need focusing in $x$ direction.

For the wiggler with a quadratic focusing $B_x \approx yxk_x^2 B_0 \sin(kx)$. Hence, the additional term is:

$$\left\langle \frac{B_x}{B\rho} \right\rangle \approx y\langle xk_x^2 b\sin(kz)\ \rangle = y\left\langle \frac{k_x^2 b^2}{k^2}\sin^2(kz) \right\rangle = y\frac{b^2}{2}\frac{k_x^2}{k^2} \tag{18}$$

Equations (17) and (18) show that for a wiggler with quadratic focusing both the edge-focusing and the additional focusing have the same dependence on $b$, and thus the same dependence on a relative momentum offset $\delta$ for an off-momentum particle. The respective equation of motion in $y$ plane is:

$$y'' + \kappa_y^2 y = 0; \quad \kappa_y^2 = \frac{b^2(k^2-k_x^2)}{2k^2} \tag{19}$$

As $\kappa_y$ approaches zero, the chromaticity $\eta_y$ approaches zero as well, since $y$-phase advance in such a wiggler $\kappa_y L \propto \frac{1}{1+\delta}$.

For the wiggler with a linear focusing the situation is different. Equation (5) gives $B_x \approx yk_q B_0$. Therefore, the additional focusing term for such a wiggler is $\frac{B_x}{B\rho} = yk_q b$. This means that the additional focusing term has a $\delta$ dependence $\propto \frac{1}{1+\delta}$, while the wiggler's edge-focusing term depends on $\delta$ as $\frac{1}{1+\delta^2}$. The respective equation of motion in $y$-direction is:

$$y'' + \kappa_{y1}^2 y = 0; \quad \kappa_y^2 = \frac{b^2}{2} - bk_q \tag{20}$$

As focusing $\kappa_{y1}$ approaches zero, the wiggler's chromaticity becomes infinitely large because two parts of $\kappa_{y1}$ have different dependence on $\delta$.

# DISSCUSSION OF RESULTS

The wigglers with a linear focusing are not suitable for the REC, since their contribution to vertical chromaticity becomes uncontrollably large for the required REC settings.

The wigglers with the quadratic dependence of $B_y$ on $x$ near the axis have an almost zero vertical chromaticity for the REC parameters. A preliminary design for the wigglers with the required field distribution was created [3]. Such wigglers were included in the REC lattice.

The wigglers with the field described by Eq. (2) were utilized in the REC lattice optimizations [4], including optimization of momentum and dynamic apertures, misalignment studies, emittance optimization etc.

It is worth noting that while wigglers with the linear focusing are not compatible with REC design, the high field low energy wigglers of that type can be used to introduce a very large negative chromaticity in a lattice.

Another curious observation is that such wigglers can be used as devices producing the same vertical focusing for two beams with different energies.

Consider two beams with rigidities $B\rho_1$ and $B\rho_2$. It follows from Eq. (20) that the same average vertical focusing will be achieved for:

$$k_q = \frac{B_0}{2}\left(\frac{1}{B\rho_1} + \frac{1}{B\rho_2}\right) \tag{21}$$

# CONCLUSION

The EIC Ring Electron Cooler utilizes eighteen wigglers with high field at a relatively low beam energy ($\frac{B_0}{B\rho} = 4.8$ m$^{-1}$).

We showed that such a wiggler works as a thick lens in both the wiggling and the orthogonal planes. The phase advance through these wigglers is substantial. As a result, one must pay special attention to the choice of additional focusing inside the wiggler, so that the wiggler's contribution to chromaticity is kept small.

The explicit analytic formulas for chromaticities of the wigglers with two possible field configurations were derived.

For the wiggler with a quadratic dependence of vertical field on horizontal displacement of the beam, an additional focusing is provided by a beam moving with an offset in each pole. The contributions to chromaticities in both planes for such a wiggler are small.

For the wiggler where additional focusing in horizontal direction is provided by a quadrupole field the chromaticity in the plane orthogonal to the wiggling plane becomes infinitely large as the focusing in that plane is approaching zero.

The Ring Electron Cooler lattice requires wigglers with a strong focusing in the wiggling plane and preferably zero focusing in the orthogonal plane. Therefore, it was decided to make the wiggler's field near the axis as close as possible to the field given by Eq. (2).

The design of wigglers with the required field profile was created. These wigglers were included into the REC lattice and used in various lattice optimizations.

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