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ANALYTICAL MODEL FOR THE TRANSITION TO SUPERRADIANCE IN SEEDED FREE-ELECTRON LASERS

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Abstract

Free-electron lasers (FEL) seeded by short radiation pulses can exhibit superradiant behavior. In the superradiant regime, the pulse simultaneously compresses and amplifies as it propagates through the FEL, making superradiance very promising for pushing the performance limits of attosecond x-ray FELs. To date, this regime has been studied in asymptotic limits, but there is no model for how the initially linear dynamics of the seeded FEL transition into the nonlinear superradiant behavior. We derive an analytical model for the 1D FEL seeded by a short pulse which accurately captures the linear dynamics, the nonlinear superradiant evolution, and the smooth transition between them. Our model fills a critical gap in our understanding of FEL superradiance and nonlinear time-dependent FEL physics more broadly, and may provide a bridge to the corresponding problem in three-dimensions, and analogous problems in other fields exhibiting soliton behavior.

INTRODUCTION

The dynamics of the one-dimensional (1D) FEL are accurately described by a set of equations of motion for the particles and a wave equation for the electric field [1–3]. The particles are described by a ponderomotive phase $\theta_j = (k_r + k_u)\bar{z}_j(t) - \omega_r t$ and a scaled relative energy deviation $\hat{\eta}_j = (\gamma_j - \bar{\gamma})/\rho\bar{\gamma}$, where k_r and k_u are the radiation and undulator wavenumbers, respectively, $\bar{\gamma}$ is the average Lorentz factor of the bunch, and ρ is the Pierce parameter. These coordinates satisfy the equations of motion

$$\frac{d\theta_j}{d\hat{z}} = \hat{\eta}_j, \quad (1)$$

$$\frac{d\hat{\eta}_j}{d\hat{z}} = -(ae^{-i\theta_j} + \text{c.c.}), \quad (2)$$

where $\hat{z} = 2\rho k_u z$, with z the distance along the undulator, and a is the scaled electric field strength, which satisfies

$$\frac{\partial a}{\partial \hat{z}} + \frac{\partial a}{\partial \zeta} = \langle e^{i\theta_j} \rangle, \quad (3)$$

where $\zeta = 2\rho\theta$ describes the scaled position within the electron bunch, and the average $\langle e^{i\theta_j} \rangle$ is taken over a thin slice centered at ζ. Figure 1 shows the typical behavior of the numerical solution of these equations seeded by a short Gaussian field, in this case with amplitude 10^{-4} and width $\sigma_\zeta = 0.5$. Panel (a) shows the evolution of the temporal power profile along the undulator length. The short Gaussian seed initially undergoes exponential amplification and lengthens as a result of its slippage across the electron bunch. This is called the linear regime, as the electron dynamics in this region are only a small perturbation to their initial conditions. Once the field amplitude reaches the order unity, the electron dynamics become non-perturbative, leading to saturation of the field amplitude (reflected in panel (b)). After that saturation point, the pulse enters the solitonlike superradiant regime, in which the highly nonlinear dynamics lead to rapid pulse shortening and quadratic amplification of the peak power [4–7].

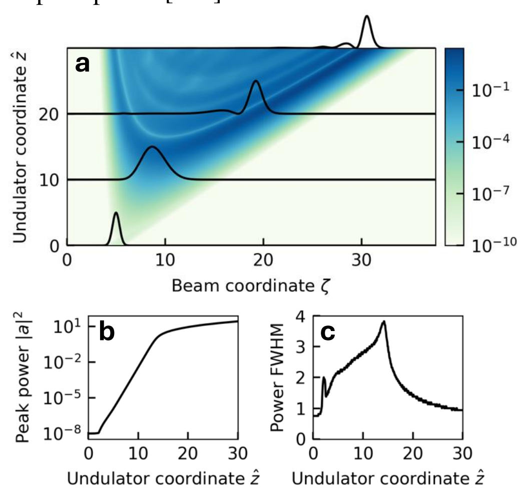

Figure 1: Typical dynamics of an FEL seeded by a short, low-power pulse. (a) The evolution of the field power versus beam coordinate and undulator position. (b) The peak power of the pulse versus undulator length. (c) The full-width at half-maximum (FWHM) of the power profile versus undulator length.

The dynamics in the linear ($\hat{z} \lesssim 10$ in Fig. 1) and nonlinear ($\hat{z} \gtrsim 15$) regimes have been studied extensively analytically. The linear regime has an exact Green's function solution [8–10]. The nonlinear solitonlike regime has been studied in isolation using scaled coordinates, yielding, among other results, an approximate hyperbolic secant form for the pulse shape [11, 12]. However, studies of the nonlinear regime to date have ignored the initial value problem, providing asymptotic scaling laws but failing to connect the nonlinear evolution to the particular initial conditions seeding the FEL instability.

In the remainder of this paper we develop an analytical solution of the full initial value problem of the solitonlike superradiant FEL in one dimension. We focus on the case

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of seeding by a short, weak field. This work fills a critical gap in time-dependent FEL theory, which has to date been understood only in the linear and nonlinear regimes in isolation, completely missing the transition between them. Our approach may find use in other fields where analogous soliton dynamics exist, such as laser optics, plasma physics, or hydrodynamics. Furthermore, the 1D solution provides a bridge to the more challenging 3D problem [12].

THEORETICAL APPROACH

The key to patching the linear and nonlinear regimes together is the realization that the leading edge of the pulse, propagating at the speed of light along the bunch, always remains in the linear regime. As a result, the leading edge of the pulse is always well-described by the Green's function solution from the linear regime even while the rest of the pulse is undergoing the highly nonlinear superradiant dynamics. Equating the Green's function solution, which contains information about the precise initial conditions that seeded the FEL, with the analytical superradiant solution in the appropriate limits allows us to transfer information about the initial value problem into the scaled nonlinear solution.

In the linear Green's function formalism, the scaled field amplitude a can be written as

$$a(\zeta,\hat{z}) = \int_{-\infty}^{\infty} a_0(\zeta')g(\zeta-\zeta',\hat{z})d\zeta', \tag{4}$$

where $a_0(\zeta') \equiv a(\zeta',0)$, and in the exponential gain regime the Green's function has the approximate form [10]

$$g(\zeta,\hat{z}) \simeq \frac{1}{2}\sqrt{\frac{\zeta}{3\pi}}\left(\frac{2}{y}\right)^{2/3} e^{3\frac{\sqrt{3}-i}{2}\left(\frac{y}{2}\right)^{2/3}-\frac{i\pi}{12}}H(\zeta), \tag{5}$$

where $H(\zeta)$ is the Heaviside function defined as 1 for $\zeta > 0$ and 0 otherwise. The variable $y = \sqrt{\zeta}(\hat{z}-\zeta)$ is maximized when $\zeta = \hat{z}/3$, reflecting the well-known group velocity of the FEL in the exponential gain regime. Close to that maximum, the magnitude of the Green's function behaves like a Gaussian:

$$|g(\zeta,\hat{z})| \propto e^{\frac{\sqrt{3}}{2}\hat{z}-\frac{9\sqrt{3}}{8\hat{z}}\left(\zeta-\frac{\hat{z}}{3}\right)^2}. \tag{6}$$

This exhibits the exponential gain behavior expected in the linear regime, and its width in time grows as $\sigma_{\zeta,g} = \frac{2}{3}\sqrt{\frac{\hat{z}}{\sqrt{3}}}$.

Here we focus on the case that the seed field is short compared to the FEL cooperation length – the slippage in one gain length. In that limit, the seed field acts like a delta function in the convolution with the Green's function, and contributes only through its integrated strength:

$$a(\zeta,\hat{z}) \simeq g(\zeta,\hat{z})\int_{-\infty}^{\infty} a_0(\zeta')d\zeta' \equiv A_0 g(\zeta,\hat{z}). \tag{7}$$

We have assumed that $a_0(\zeta)$ is centered on $\zeta = 0$. In other words, for a short seed field, the Green's function accurately describes the temporal shape of the field and the integral of the seed field A_0 determines the complex amplitude. Furthermore, we argue that this form for the field will always apply at the leading edge of the pulse assuming the seed field integral is small compared to one. Figure 2 shows a comparison of the field calculated according to Eq. (7) with that from the full time-dependent simulation at $\hat{z} = 30$. We see that the linear solution accurately describes the initial growth of the field, but diverges near the point that the field amplitude saturates and the beam dynamics become nonlinear.

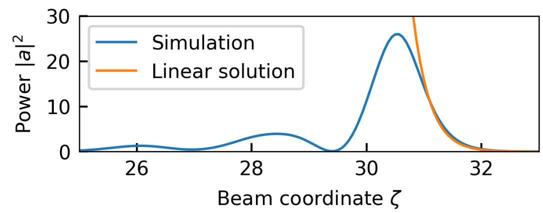

Figure 2: Comparison of the FEL power predicted by a full simulation and the linear Green's function solution.

Those nonlinear dynamics can be studied independently in appropriately scaled coordinates. In the solitonlike superradiant regime, the field is expected to propagate at the speed of light with a duration shrinking like $1/\sqrt{\hat{z}}$ and amplitude growing like $\hat{z}$. We note that in the leading edge of the pulse to which the superradiant pulse is confined, ζ is very close to $\hat{z}$, so these scalings can also be written as $1/\sqrt{\zeta}$ and ζ, respectively. This suggests moving from variables $(\zeta,\hat{z})$ to (ζ,y) where $y = \sqrt{\zeta}(\hat{z}-\zeta)$, and scaling the field to $A = a/\zeta$. In these coordinates, the FEL equations become

$$\frac{d\theta_j}{dy} = p_j, \tag{8}$$

$$\frac{dp_j}{dy} = -(Ae^{-i\theta_j} + \text{c.c.}), \tag{9}$$

$$\frac{y}{2}\frac{dA}{dy} + A = b, \tag{10}$$

where $p_j = \hat{\eta}_j/\sqrt{\zeta}$ and A is assumed to be a function only of y. The numerical solution of these equations yields a self-similar pulse shape which agrees qualitatively with the shape of the field found from time-dependent simulations after saturation. Furthermore, the scaled equations have a known approximate solution in terms of a hyperbolic secant function which describes the first peak in the superradiant pulse where most of the pulse energy is contained. The result is, using techniques similar to Refs. [12, 13],

$$A(y) = \sqrt{\frac{3}{2}}\left(\frac{2}{y}\right)^{2/3} \operatorname{sech}\left[\frac{3\sqrt{3}}{2}\left(\frac{y}{2}\right)^{2/3} + \psi\right] e^{-\frac{i}{2}3\left(\frac{y}{2}\right)^{2/3}-i\phi_0}, \tag{11}$$

where ψ is an offset determined by the initial conditions for the equations and ϕ_0 is a phase offset. In the limit of small y, this solution behaves exponentially as $\operatorname{sech}(x) \simeq 2e^x$, reproducing the same exponential behavior as the Green's function. By matching the properly scaled Green's function

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solution to this nonlinear solution, we find $\phi_0 = \frac{\pi}{12} - \arg(A_0)$ and

$$\psi = \ln\left(\frac{|A_0|}{6\sqrt{2\pi\zeta}}\right), \tag{12}$$

giving the ultimate form for the nonlinear solution, after scaling back to the original variables,

$$a(\zeta,\hat{z}) \simeq \zeta\sqrt{\frac{3}{2}}\left(\frac{2}{y}\right)^{2/3} e^{-\frac{3i}{2}\left(\frac{y}{2}\right)^{2/3} - \frac{i\pi}{12} + i\arg(A_0)}$$
$$\times \mathrm{sech}\left[\frac{3\sqrt{3}}{2}\left(\frac{y}{2}\right)^{2/3} + \ln\left(\frac{|A_0|}{6\sqrt{2\pi\zeta}}\right)\right]. \tag{13}$$

NUMERICAL BENCHMARKS

Figure 3 illustrates the behavior of this nonlinear solution as compared to the time-dependent simulation and the linear Green's function solution. Panel (a) demonstrates similar dynamics as in Fig. 1, where an exponential gain regime accompanied by pulse lengthening transitions around $\hat{z} \simeq 15$ into a post-saturation regime accompanied by pulse shortening. Panels (b)-(e) show several metrics of the field - the peak power, power FWHM, position of the power peak ζ_0, and velocity of the power peak $d\zeta_0/d\hat{z}$ - as calculated from the simulation, the linear solution, and the nonlinear solution. Inaccuracies in the nonlinear solution early in the undulator derive from the fact that we include only the contributions from the exponentially growing root of the FEL dispersion relation. The nonlinear solution accurately describes the initial exponential growth and pulse lengthening, as well as the peak velocity of $1/3$ of the slippage rate. It also captures fine details of the transition from the linear into the nonlinear regime, and the resulting slowing of the amplification, shrinking of the pulse duration, and acceleration of the peak velocity. The nonlinear model even reproduces the previously reported result that the velocity of the peak is superluminal [14].

Using Eq. (13) we can derive estimates for some of the metrics plotted in Fig. 3. For example, we can approximate the location of the peak of the pulse by asking that the argument of the sech vanishes. In doing that, we neglect that the amplitude of the pulse also depends on ζ, but we will find the error to be small in the superradiant regime. Since the superradiant pulse is localized near $\zeta = \hat{z}$, we can approximate y using a Taylor series $\sqrt{\zeta}(\hat{z} - \zeta) \simeq (\hat{z}^2 - \zeta^2)/2\sqrt{\hat{z}}$, and furthermore replace ζ by $\hat{z}$ inside of the logarithm, after which we can set the sech argument to zero to find

$$\zeta_0 \simeq \sqrt{\hat{z}^2 - 4\sqrt{\hat{z}}\left(\frac{2}{3\sqrt{3}}\ln\left(\frac{6\sqrt{2\pi\hat{z}}}{|A_0|}\right)\right)^{3/2}}. \tag{14}$$

Using this, we can evaluate other properties of interest such as the velocity of the peak of the pulse $d\zeta_0/d\hat{z}$. We can also use this to roughly estimate the saturation point, which can be approximated by the point where this solution matches the peak position in the linear regime, $\zeta_0 \simeq \hat{z}/3$. That point

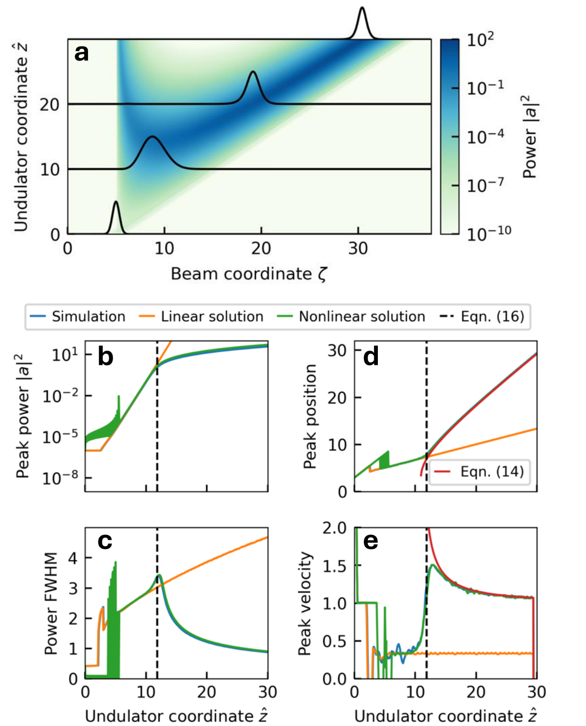

Figure 3: Summary of the model behavior for the same input parameters as Fig. 1. (a) Power evolution along the undulator. (b-e) The peak power, power FWHM, peak position, and peak velocity, respectively, versus undulator length. The black dashed line is the saturation length estimate from Eq. (16). The red lines are based on Eq. (14).

solves the equation

$$\left(\frac{\sqrt{3}}{2}\right)^{1/3}\hat{z} = \ln\left(\frac{6\sqrt{2\pi\hat{z}}}{|A_0|}\right). \tag{15}$$

The solution to this is

$$\hat{z} = -\frac{1}{2^{2/3}\times 3^{1/6}} W_{-1}\left(-\frac{|A_0|^2}{12\pi\times 2^{1/3}\times 3^{5/6}}\right), \tag{16}$$

where $W_{-1}(z)$ is the $k = -1$ branch cut of the Lambert W function, defined as the solution to $z = We^W$. The estimated peak position, peak velocity, and saturation point are all shown in Fig. 3, agreeing excellently with the simulated results and the full nonlinear solution.

CONCLUSIONS

In summary, we have presented an analytical model for free-electron lasers seeded by weak, short pulses. Our model accurately captures the physics of the linear and nonlinear superradiant regimes and the smooth transition between them. As a result, our model reveals how the initial conditions seeding an FEL translate to the particular properties of the eventual superradiant pulse. This same approach that we have taken towards the 1D problem may eventually be applied to the more complicated, but more physically relevant, 3D problem including diffraction.

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