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ASPECTS OF STROBOSCOPIC AVERAGING FOR THE INVARIANT SPIN FIELD

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Abstract

A new method is formulated for calculating the invariant spin field (ISF) at a phase space point by leveraging the property that spins which are distributed along the ISF achieve maximum time-averaged polarization. To quantify this, we construct the time-average of spin rotation matrices beginning at a certain phase space point. It is recognized that the ISF vector at that point achieves the matrix-norm, meaning that the ISF corresponds to the first right-singular vector of that matrix. We show the relation of this method with traditional stroboscopic averaging, such that these methods are two sides of the same coin. This approach offers a new perspective in invariant spin field calculations.

INTRODUCTION

The semi-classical spin dynamics used in accelerator physics is described by the Thomas-Bargmann-Michel-Telegdi (T-BMT) equation [1–3]. For purely magnetic fields, it is

$$\frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} = -\frac{q}{m\gamma}\left[(1+G\gamma)\boldsymbol{B}_\perp + (1+G)\boldsymbol{B}_\parallel\right] \times \boldsymbol{S}, \qquad (1)$$

where G is the anomalous gyromagnetic ratio, γ is the Lorentz factor, $\boldsymbol{B}_\parallel$ is the component of the magnetic field parallel to the particle velocity, and $\boldsymbol{B}_\perp = \boldsymbol{B} - \boldsymbol{B}_\parallel$. In a circular accelerator, the static magnetic fields along the co-moving coordinate system are periodic in space with the ring's circumference C:

$$\boldsymbol{B}(x, y, s + C) = \boldsymbol{B}(x, y, s). \qquad (2)$$

However, in general, the fields along a particle's trajectory are not periodic with the circumference because the trajectory itself is not periodic. Therefore, except for a particle traveling on the closed orbit, the spin motion is not periodic. However, if the T-BMT equation is expressed in action-angle coordinates, it becomes evident that the spin motion is quasiperiodic in the orbital phases and in the azimuth θ around the ring [4]. As in Floquet's theorem, the solution can then be written as

$$\boldsymbol{S}(\theta) = U_{\boldsymbol{J}}(\theta, \boldsymbol{\phi}(\theta))\ \exp(\mathcal{J}\nu_{\boldsymbol{J}}\theta)\ U_{\boldsymbol{J}}(0, \boldsymbol{\phi}(0))^{\mathrm{T}}\, \boldsymbol{S}(0) \qquad (3)$$

where $\mathcal{J}$ is the generator of rotations around the third axis, $\nu_{\boldsymbol{J}}$ is an *amplitude-dependent spin tune* (ADST), and $U_{\boldsymbol{J}} = [\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{n}]_{\boldsymbol{J}}$ is an *invariant frame field* (IFF). There are situations where an ADST and/or an IFF do not exist, such as when the beam is sitting on an orbital resonance. We restrict further discussion to situations where an ADST and an IFF both exist. The inverse matrix of the IFF in the above equation transports the initial spin coordinates into a system in which the spin rotates around the third axis with uniform frequency ν. Then, the IFF transports the spin back into the accelerator's coordinate system. For spin dynamics, the IFF is therefore analogous to a normal-form transformation of the orbital motion.

The axis around which the spin rotates in the IFF is $\boldsymbol{n}$, known as the *invariant spin field* (ISF). The projection of a particle's spin vector onto the ISF, $J_S \equiv \boldsymbol{S} \cdot \boldsymbol{n}$, has been shown to be an adiabatic invariant of spin motion [5]. It is the non-adiabatic change of this quantity that is responsible for beam depolarization. The computation of the ISF in accelerator simulations is therefore of great interest, and that subject is the topic of this paper.

STROBOSCOPIC AVERAGING

The first non-perturbative algorithm available for efficiently computing the ISF with three-dimensional orbital motion was *stroboscopic averaging* [6]. To begin, we define a spin field as a function $\boldsymbol{f} : \mathbb{R}^6 \times \mathbb{R} \to S^2$ satisfying the T-BMT equation along any orbital trajectory, where S^2 is the unit sphere in $\mathbb{R}^3$. The ISF is a spin field which is 2π-periodic in the azimuth, i.e., $\boldsymbol{n}(\boldsymbol{z}, \theta + 2\pi) = \boldsymbol{n}(\boldsymbol{z}, \theta)$ with $\boldsymbol{z} = (\boldsymbol{J}, \boldsymbol{\phi})$. If the orbital motion is integrable and an IFF exists, any sufficiently-smooth spin field $\boldsymbol{f}$ satisfies

$$\begin{aligned}&\lim_{N\to\infty} \frac{1}{N+1} \sum_{j=0}^{N} \boldsymbol{f}(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0 + 2\pi j) \\ &= \langle \boldsymbol{f}(\boldsymbol{J}_0, \boldsymbol{\phi}, \theta_0) \cdot \boldsymbol{n}(\boldsymbol{J}_0, \boldsymbol{\phi}, \theta_0)\rangle_{\boldsymbol{\phi}}\, \boldsymbol{n}(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0).\end{aligned} \qquad (4)$$

As long as the expectation value is not zero, this property can be used to recover an approximate ISF by choosing a large N.

The intuitive derivation of this method is that shifting θ_0 by an integer multiple of 2π does not affect the average in the limit $N \to \infty$, so the result must be 2π-periodic in the azimuth. Additionally, by the linearity of the T-BMT equation, the average in Eq. (4) is also a solution of the T-BMT equation. Hence, the result is either zero or parallel to an ISF.

For simplicity, the spin field $\boldsymbol{f}$ is usually chosen to be constant across phase space at the starting azimuth, i.e.,

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$\boldsymbol{f}(\boldsymbol{J}, \boldsymbol{\phi}, \theta_0) \equiv \boldsymbol{S}_0$. In simulations, the most common choice is $\boldsymbol{S}_0 = \boldsymbol{n}_0$, the ISF on the closed orbit. The spin field evaluated at the required points can then be found using

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0 + 2\pi j) = & \mathsf{R}_j(\boldsymbol{J}_0, \boldsymbol{\phi}_0 - 2\pi j\boldsymbol{Q}, \theta_0) \\ & \times \boldsymbol{f}(\boldsymbol{J}_0, \boldsymbol{\phi}_0 - 2\pi j\boldsymbol{Q}, \theta_0), \end{aligned} \tag{5}$$

where $\boldsymbol{Q}$ contains the orbital tunes and $\boldsymbol{R}_j(\boldsymbol{J}, \boldsymbol{\phi}, \theta)$ is the j-turn spin-transfer matrix beginning at phase-space position $(\boldsymbol{J}, \boldsymbol{\phi})$ and azimuth θ. In practice, this often cannot be used because winding back the angle requires inverting the orbital-transfer map, which is only feasible in the linear case. However, note that the periodicity condition of the ISF is

$$\boldsymbol{n}(\boldsymbol{J}_0, \boldsymbol{\phi}_0 + 2\pi\boldsymbol{Q}, \theta_0) = \boldsymbol{R}(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0)\boldsymbol{n}(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0), \tag{6}$$

where R is the one-turn spin-transfer matrix. This leads to

$$\boldsymbol{n}(\boldsymbol{J}_0, \boldsymbol{\phi}_0 - 2\pi\boldsymbol{Q}, \theta_0) = \boldsymbol{R}(\boldsymbol{J}_0, \boldsymbol{\phi}_0 - 2\pi\boldsymbol{Q}, \theta_0)^{\mathrm{T}}\boldsymbol{n}(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0). \tag{7}$$

Therefore, the ISF of the forward motion is the same as the ISF of the backward motion. Hence, we can instead use

$$\begin{aligned} \boldsymbol{f}(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0 - 2\pi j) = & \mathsf{R}_j(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0)^{\mathrm{T}} \\ & \times \boldsymbol{f}(\boldsymbol{J}_0, \boldsymbol{\phi}_0 + 2\pi j\boldsymbol{Q}, \theta_0). \end{aligned} \tag{8}$$

This leads to

$$\begin{aligned} & \lim_{N\to\infty} \frac{1}{N+1} \sum_{j=0}^{N} \mathsf{R}(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0)^{\mathrm{T}}\boldsymbol{S}_0 \\ & = \left[\boldsymbol{S}_0 \cdot \langle \boldsymbol{n}(\boldsymbol{J}_0, \boldsymbol{\phi}, \theta_0)\rangle_{\boldsymbol{\phi}}\right] \boldsymbol{n}(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0). \end{aligned} \tag{9}$$

A VARIATIONAL APPROACH

We now formulate an alternative approach to finding the ISF based on a variational principle. Consider a particle with an initial spin vector $\boldsymbol{S}_0$ at a phase-space point $(\boldsymbol{J}_0, \boldsymbol{\phi}_0)$ and azimuth θ_0. The time-averaged spin vector after N turns is given by the sum

$$\langle \boldsymbol{S}\rangle_N = \frac{1}{N+1} \sum_{j=0}^{N} \mathsf{R}_j(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0)\boldsymbol{S}_0. \tag{10}$$

This motivates the definition of the *time-averaged spin transport matrix* $\langle \mathsf{R}\rangle_N$:

$$\langle \mathsf{R}\rangle_N(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0) = \frac{1}{N+1} \sum_{j=0}^{N} \mathsf{R}_j(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0). \tag{11}$$

Under assumptions similar to those of stroboscopic averaging, one can show that [7]

$$\begin{aligned} & \lim_{N\to\infty} \langle \mathsf{R}\rangle_N(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0)\boldsymbol{S}_0 \\ & = \left[\boldsymbol{S}_0 \cdot \boldsymbol{n}(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0)\right] \langle \boldsymbol{n}(\boldsymbol{J}_0, \boldsymbol{\phi}, \theta_0)\rangle_{\boldsymbol{\phi}}. \end{aligned} \tag{12}$$

Note that this average is qualitatively different from that of stroboscopic averaging. The averaging does not give us a vector which is parallel to the ISF, but the result does depend on the angle between the ISF and the initial spin. Hence, it is simple to see that

$$\boldsymbol{n}(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0) = \underset{\|\boldsymbol{S}_0\|=1}{\operatorname{argmax}} \left\| \lim_{N\to\infty} \langle \mathsf{R}\rangle_N(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0)\boldsymbol{S}_0 \right\|. \tag{13}$$

Additionally, the norm of the average ISF on a torus, called the *maximum time-averaged polarization* P_{lim}, is given by [6]

$$P_{\mathrm{lim}}(\boldsymbol{J}_0, \theta_0) = \max_{\|\boldsymbol{S}_0\|=1} \left\| \lim_{N\to\infty} \langle \mathsf{R}\rangle_N(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0)\boldsymbol{S}_0 \right\|. \tag{14}$$

By definition, P_{lim} is thus given by the matrix 2-norm of the time-averaged spin-transport matrix, and the ISF is given by the corresponding maximizer. Both of these quantities are provided by a singular-value decomposition (SVD). The matrix 2-norm is the first singular value of the matrix, and the maximizer is the corresponding right-singular vector. Therefore, the ISF can be approximated by constructing $\langle \mathsf{R}\rangle_N$ for large N and finding its SVD. Additionally, unlike stroboscopic averaging, this procedure gives P_{lim} for free.

CONNECTION TO STROBOSCOPIC AVERAGING

It turns out that stroboscopic averaging and the variational method are closely connected through $\langle \mathsf{R}\rangle_N$. To begin, note that Eq. (9) can be written as

$$\begin{aligned} & \lim_{N\to\infty} \langle \mathsf{R}\rangle_N(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0)^{\mathrm{T}}\boldsymbol{S}_0 \\ & = \left[\boldsymbol{S}_0 \cdot \langle \boldsymbol{n}(\boldsymbol{J}_0, \boldsymbol{\phi}, \theta_0)\rangle_{\boldsymbol{\phi}}\right] \boldsymbol{n}(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0). \end{aligned} \tag{15}$$

That is to say, the transpose of the time-averaged spin-transport matrix is rank-one because its image is the line along the ISF. Any rank-one matrix can be written as the outer product of two vectors, and it is fairly simple to deduce that in this case the result is

$$\lim_{N\to\infty} \langle \mathsf{R}\rangle_N(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0)^{\mathrm{T}} = \boldsymbol{n}(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0)\langle \boldsymbol{n}(\boldsymbol{J}_0, \boldsymbol{\phi}, \theta_0)\rangle_{\boldsymbol{\phi}}^{\mathrm{T}}. \tag{16}$$

Then, we have

$$\lim_{N\to\infty} \langle \mathsf{R}\rangle_N(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0) = \langle \boldsymbol{n}(\boldsymbol{J}_0, \boldsymbol{\phi}, \theta_0)\rangle_{\boldsymbol{\phi}}\, \boldsymbol{n}(\boldsymbol{J}_0, \boldsymbol{\phi}_0, \theta_0)^{\mathrm{T}}. \tag{17}$$

Applying this to a vector $\boldsymbol{S}_0$ gives Eq. (12). Therefore, stroboscopic averaging and the variational method are connected through the time-averaged spin-transport matrix.

The time-averaged spin-transport matrix in the form of Eq. (17) also gives some insight into the SVD mentioned in the previous section. It can be seen from Eq. (17) that there is at most one nonzero singular value, namely P_{lim}, and the corresponding right-singular vector is the ISF.

The previous arguments demonstrate that, in the limit $N \to \infty$, the variational method gives the same result as stroboscopic averaging, which is to be expected for a system with a unique (up to a sign) ISF. However, no statements have been made about the behavior of these methods when

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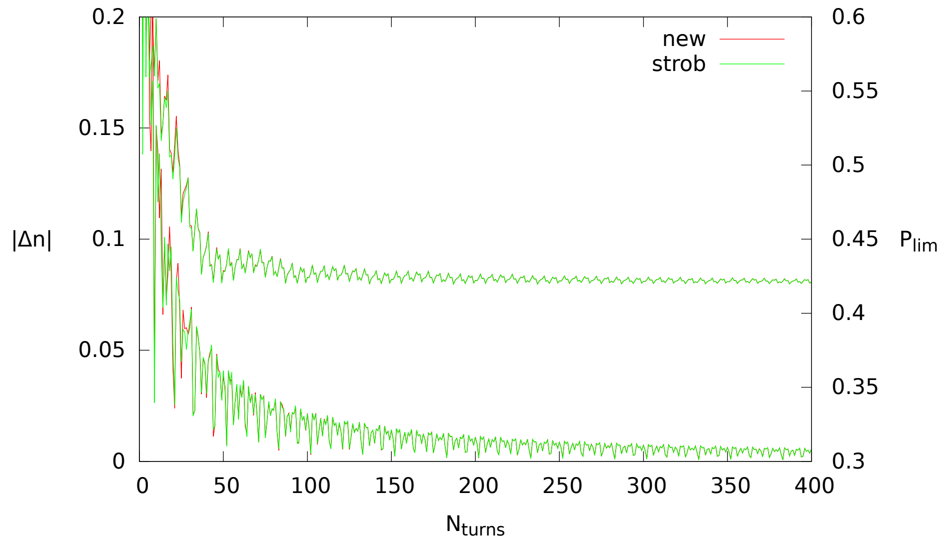

Figure 1: Comparison of the (new) variational method and (strob)oscopic averaging for simulations of the HSR. The horizontal axis is the number of turns used in computing $\langle \mathbf{R} \rangle_N$. The top curve is the maximum time-averaged polarization P_{lim} and the bottom curve is the difference (measured by the 2-norm) between the N-turn ISF and an ISF calculated with 10,000 turns of stroboscopic averaging.

approximating the ISF using finite N, as must be done in numerical calculations. In simulations, we have observed that the ISF from the two methods is very similar for all values of N, even before convergence is reached. This is exemplified by Fig. 1, which compares the two methods of computing P_{lim} and the ISF for simulations of the Hadron Storage Ring (HSR) of the Electron-Ion Collider. However, no rigorous statements about the relationship between the two methods for finite N have been made thus far.

CONCLUSION

This work has introduced a new formulation for the calculation of the invariant spin field based on a variational principle: the ISF is the initial spin direction that maximizes the norm of the time-averaged spin vector. We have shown that this principle leads to a direct computational method where the ISF is identified as the principal right-singular vector of a time-averaged spin-transport matrix. This "forward-propagation" approach was then contextualized by establishing its mathematical duality with the traditional "backward-propagation" method of stroboscopic averaging.

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