

# DESIGN OF FLAT-TO-ROUND (VORTEX) BEAM ADAPTER WITH STRONG SPACE CHARGE\*

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## Abstract

We describe the design of a symmetrical skew-quadrupole triplet and associated four-quadrupole matching section for a flat-to-round (vortex) (FTR) beam transformation in a low-energy, high-current electron experiment at the University of Maryland. We review the basic principles involved, from the Courant-Snyder parameters, beam (sigma) matrix, conservation of canonical angular momentum and emittances, to the evolution of the beam envelopes, with emphasis in practical aspects of the design. The initial optimization involves the use of TRACE 3-D, a classic sigma-matrix code that can include direct space-charge effects. Particle-in-cell (PIC) computer simulations and preliminary results from experiments are presented in an accompanying paper.

## INTRODUCTION

The manipulation of phase space of charged-particle beams is an active area of research in the science and technology of particle accelerators and related devices. An important example is the transformation of angular-momentum-dominated beams (“vortex” beams) into flat beams and vice versa. These beam transformers find applications in electron cooling of hadron beams, linear colliders, and space-charge compensation, among others. Typically, in a round-to-flat transformer (RTF), an electron beam is generated in a photocathode that is immersed in a strong axial field (0.5 – 200 kG), accelerated to 10 – 500 MeV and then passed through a triplet of skew quadrupoles to remove the coupling between space and velocity coordinates. The flat beam that results normally has a high emittance ratio of the order of 100:1 (horizontal to vertical).

Interestingly, the optics of high-energy, high  $B$ -field, beam transformers can be scaled to low-energy, low  $B$ -field systems, which facilitate parametric studies, while possibly introducing significant space-charge effects. At the University of Maryland Long-Solenoid Experiment (LSE), we are investigating the detailed beam dynamics in a FTR transformer with the parameters summarized in Table 1.

The key parameter for the FTR design is the Larmor  $\beta$  function, or  $\beta_L$  (Table 1); it corresponds to the inverse of the focusing constant of a matched solenoid following the FTR triplet, for zero beam current. Furthermore,  $\beta_L$  sets a length scale for the FTR, 0.3 m in our case [1].

The tabulated effective emittances for the flat beam are 4RMS, unnormalized values. These emittances are obtained

Table 1: Basic Parameters for Beam Transformer

Parameter	LSE FTR
Energy	5 keV
Magnetic Rigidity, $B\rho$	$2.39 \times 10^{-4}$ Tm
Magnetic Field, $B_S$	15 G, at solenoid
$\beta_L = 2(B\rho)/B_S$	0.319 m
Charge	0.049 nC
Peak Current (Slit 1)	0.49 mA
Beam Perveance, $K$	$2.1 \times 10^{-5}$
2RMS Beam Radius @ Solenoid	2.3 mm
Flat Beam Eff. Emitt., $(\epsilon_X, \epsilon_Y)$	(23.2, 2.23) $\mu\text{m}$
Norm. Avg. Angular Momentum	$\epsilon_X - \epsilon_Y = 21 \mu\text{m}$
Flat Beam Laminarity Par., $S_X, S_Y$	0.44, 1.4

with a 10:1 mm (horizontal:vertical) slit housed in the chamber that follows the two short solenoids, as shown in Fig. 1. The slit intercepts a 19 mA round beam that is focused to near a beam waist by the solenoids. The average angular momentum is normalized with the linear momentum  $p_z$ . The 2RMS beam radius at the solenoid includes the effect of direct space charge, obtained from smooth-approximation calculations discussed below. The laminarity parameters measure the ratio of space charge to emittance terms in the  $X - Y$  coupled Kapchinskij–Vladimirskij (K-V) envelope equations [2].

## FLAT AND VORTEX BEAM MATRICES

The matrix representing a *flat* beam with  $\alpha_X = 0 = \alpha_Y$ , that is, at a beam waist, is given by

$$\Sigma_{Flat} = \begin{bmatrix} \beta_L \epsilon_X & 0 & 0 & 0 \\ 0 & \epsilon_X/\beta_L & 0 & 0 \\ 0 & 0 & \beta_L \epsilon_Y & 0 \\ 0 & 0 & 0 & \epsilon_Y/\beta_L \end{bmatrix}. \quad (1)$$

This matrix is transformed by the transport matrix of a symmetrical triplet of skew *thin* quadrupoles into the matrix representing a round *and vortex* beam, Eq. (2).

From Eq. (2), the 2RMS beam radius of the vortex beam at the output of the FTR is  $\sqrt{2\beta_L(\epsilon_X + \epsilon_Y)}$ , while the normalized average angular momentum at the output is  $(\epsilon_X - \epsilon_Y)$ . We emphasize that these results are valid for a zero-current beam. Space-charge effects will be discussed later.

## LATTICE DESIGN

There are four main components in the LSE experiment: the two-solenoid and slit aperture plate, the skew quadrupole

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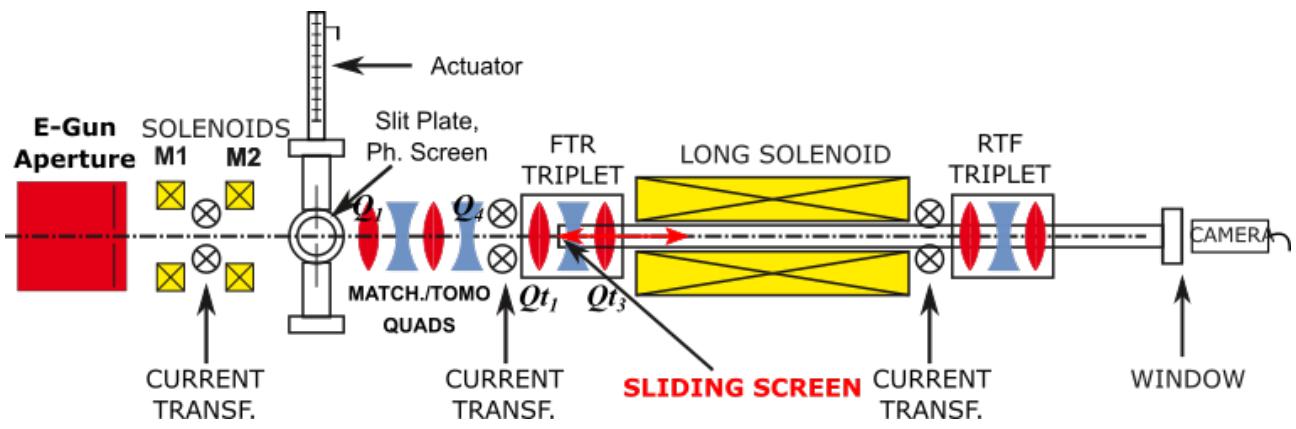


Figure 1: Schematics (not to scale) of the Long Solenoid Experiment (LSE) at the University of Maryland. FTR: Flat-to-Round, and RTF: Round-to-Flat transformers.

$$\Sigma_R = \begin{bmatrix} \frac{1}{2}\beta_L(\varepsilon_X + \varepsilon_Y) & 0 & 0 & \frac{1}{2}(\varepsilon_X - \varepsilon_Y) \\ 0 & \frac{1}{2\beta_L}(\varepsilon_X + \varepsilon_Y) & -\frac{1}{2}(\varepsilon_X - \varepsilon_Y) & 0 \\ 0 & -\frac{1}{2}(\varepsilon_X - \varepsilon_Y) & \frac{1}{2}\beta_L(\varepsilon_X + \varepsilon_Y) & 0 \\ \frac{1}{2}(\varepsilon_X - \varepsilon_Y) & 0 & 0 & \frac{1}{2\beta_L}(\varepsilon_X + \varepsilon_Y) \end{bmatrix}. \quad (2)$$

triplet, the four-quadrupole matching section, and the long solenoid. The function of the two short solenoids and aperture plate are described elsewhere [3].

### Symmetrical Skew Quadrupole Triplet

The design starts with evaluations of the *inverse* focal lengths of the FTR triplet quadrupoles [4]:

$$d = \frac{\beta_L}{2\sqrt{1+\sqrt{2}}}, \quad q_1 = q_3 = -\frac{\sqrt{2}+1}{\beta_L}, \quad q_2 = \frac{2\sqrt{2}}{\beta_L}. \quad (3)$$

In this geometry,  $d$  is the distance between the outer *thin* quadrupoles  $Q_{t1}$ ,  $Q_{t3}$  and the central one  $Q_{t2}$  (Fig. 1). Furthermore, the flat beam is incident directly on the first skew quadrupole.

Under these conditions, the phase difference between betatron oscillations in the two transverse planes is  $90^\circ$ , resulting in a beam that is not only round but also obeys the “vortex” condition  $xx' + yy' = 0$ . In terms of the Courant-Snyder parameters [5], we need  $\beta_1 = \beta_L = \beta_2$ ,  $\alpha_1 = 0 = \alpha_2$ , where the subscripts refer to the initial (1) and final values (2).

Since Eqs. (3) are valid only for thin lenses and without input/output drifts, the first calculation involves optimization with *thick* quadrupoles and *non-zero* input/output drifts. The air-core magnetic quadrupoles used are based on flexible printed-circuits and have an *effective length* equal to 5.164 cm; the input/output drifts in the triplet are 28.4 mm. The optimization uses  $d$ , and hard top gradients  $g_{\text{outer}}$ , and  $g_{\text{inner}}$  as matching variables to fit six elements of the sigma matrix in the TRACE 3-D code (match type MT = 11) [6–8].

The six elements of the target sigma matrix are the 2RMS semi-axis beam dimensions in the two transverse planes, *after space-charge correction*, the two elements related to

$\alpha_{2x,y} = 0$ , and the two elements connected to the angular momentum. Specifically,  $X_{\max} = Y_{\max} = R = 2.3$  mm,  $\mathbf{r}_{12} = \mathbf{0} = \mathbf{r}_{34}$ , and  $\mathbf{r}_{14} = -0.824$ ,  $\mathbf{r}_{23} = 0.824$ . The elements  $\mathbf{r}_{ij}$  refer to the reduced sigma matrix in TRACE 3-D, defined as  $\mathbf{r}_{ij} = \Sigma_{ij}/\sqrt{\Sigma_{ii}\Sigma_{jj}}$ .

Equally important for optimizing the triplet, the initial conditions include the emittances (Table 1) and the  $\beta$  parameters. The latter are corrected for space charge by using the smooth approximation. In this approximation [2], the 2RMS radius in the matched solenoid is

$$R = \left[ \frac{K}{2\kappa_0} + \sqrt{\left( \frac{K}{2\kappa_0} \right)^2 + \frac{\varepsilon^2}{\kappa_0}} \right]^{1/2}, \quad (4)$$

where  $\kappa_0^{-\frac{1}{2}} = \beta_L = 0.319$  m is the zero current  $\beta$ , and  $\varepsilon = (\varepsilon_X + \varepsilon_Y)/2$ . The space-charge corrected  $\beta$  is  $\kappa^{-\frac{1}{2}}$ , with

$$\kappa = \kappa_0 - \frac{K}{R^2}, \quad (5)$$

so  $\kappa^{-\frac{1}{2}} = 0.413$  m. This value applies to  $\beta_1$  and  $\beta_2$ , the initial and final betas in the symmetrical triplet.

The results for  $d$  and the quadrupole gradients are summarized in Table 2. The spacing between quadrupole triplets is larger with space charge, reflecting the corrected value of  $\beta$  from Eq. (5). Notice also that for zero current, the phase advance difference between  $X$  and  $Y$  planes is  $90.0^\circ$ , but  $89.1^\circ$  for 0.49 mA. Thus, with the approach followed to include space charge, the vortex condition is not perfectly satisfied. Improvement may be possible if the rotation angle of the quadrupoles and their location are included in optimization [9]. However, the experiment currently does not have the ability to vary the angles of the triplet quadrupoles.

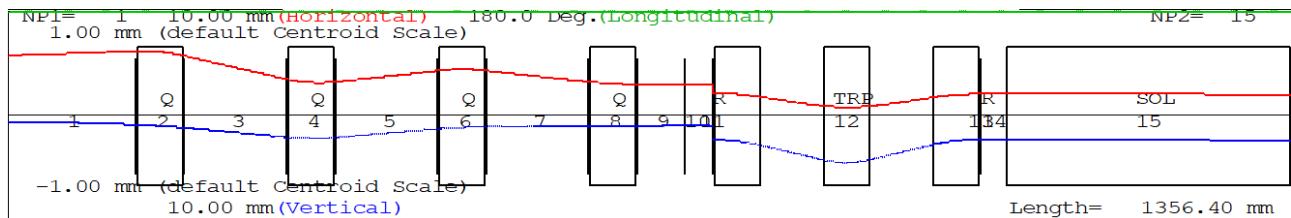


Figure 2: TRACE 3-D results (0.491 mA electron beam) for the four-quadrupole matching section followed by the skew triplet and the matched solenoid: horizontal (red) and vertical (blue) envelopes. See also text.

Table 2: Results of Sigma-Matrix Optimization for Symmetrical Triplet

Parameter	Zero Current	0.49 mA
Larmor Beta, $\beta_L$	0.319 m	0.413 m
Triplet spacing, $d$	45.3 mm	64.5 mm
$Qt_{1,3}$ Grad., $g_{\text{outer}}$	-0.0430 T/m	-0.0355 T/m
$Qt_2$ Grad., $g_{\text{inner}}$	+0.0545 T/m	+0.0463 T/m
X/Y Phase Adv. Diff.	90.0°	89.1°

Table 3: Summary of Courant-Snyder Parameters

Lattice	Initial	Final
Quadrupole Triplet	$\alpha_{1x}, \alpha_{1y} = 0$ $\beta_{1x}, \beta_{1y} = \beta_L$	$\alpha_{2x}, \alpha_{2y} = 0$ $\beta_{2x}, \beta_{2y} = \beta_L$
Matching Section	$\alpha_{1x} = -0.658$ $\alpha_{1y} = -0.097$ $\beta_{1x} = 1.495$ m $\beta_{1y} = 0.149$ m	$\alpha_{2x} = 0$ $\alpha_{2y} = 0$ $\beta_{2x} = \beta_L$ $\beta_{2y} = \beta_L$

### Matching Section

The four-quadrupole matching section transforms a sheet beam from a 10:1 mm horizontal slit into the required flat beam. In this case, the optimization uses standard Courant-Snyder parameters (match type MT = 8 in TRACE 3-D) instead of sigma matrix elements directly.

The *initial* emittances, and  $\alpha$ 's and  $\beta$ 's are derived from second moments obtained with the PIC code WARP (see the accompanying paper "Evolution of Realistic Beam Distributions in Space-Charge Dominated Electron Beams", by S. Wang *et al.*) The *target* Courant-Snyder parameters, on the other hand, are  $\alpha_2 = 0$ ,  $\beta_2 = \beta_L$ , as in Table 2, for zero current and 0.49 mA. Table 3 summarizes the Courant-Snyder initial and final values. The quadrupole spacing, center to center, is 16 cm, and the distance from the slit to the center of the first matching quadrupole  $Q_1$  is also 16 cm.

Figure 2 shows the TRACE 3-D envelopes, including a 15-G long solenoid. The discontinuous change at the entrance to the skew triplet (including the initial drift) reflect a coordinate transformation to a rotated coordinate system.

## DISCUSSION

We used a straightforward approach to the design of a symmetrical FTR based on the standard Courant-Snyder parametrization of the optics. The design is aided by matching optimization with the TRACE 3-D matrix code, including direct space-charge effects. Because of the smaller emittance in the vertical  $Y$  direction, the flat beam is space-charge dominated only in that direction, as characterized by a lamination parameter  $S_Y > 1$  (Table 1).

However, questions remain about the effects of space charge on the vortex condition. The target sigma matrix, which contains terms related to angular-momentum in the off-axis elements, is based on single-particle theory. But the (normalized) angular-momentum term in the sigma matrix for the vortex beam [ $\Sigma_R$ , Eq. (3)] is just the difference in the emittances,  $\varepsilon_X - \varepsilon_Y$ , which is independent of space charge. Naturally, emittance growth from non-linear effects could affect the vortex condition. For example, it is feasible that the vertical emittance is more impacted by these effects. Details of the initial particle phase-space distribution, on the other hand, can be important, as shown in PIC simulations presented in an accompanying paper.

Unlike the angular momentum, the beam size at the output of the FTR (and entrance to the solenoid) is clearly modified by direct space charge: a 14% increase relative to the zero-current case. The effect can be easily calculated using the smooth approximation, Eq. (4), [2], which also leads to a correction of the target Larmor  $\beta$ , ( $\beta_L$ ), as shown in Table 2. This correction is *ad hoc*, but appears to work, at least within the K-V framework of the space-charge model implicit in TRACE 3-D [6].

Another related issue is the matching of the vortex beam into the solenoid, which we have studied before under different optics parameters and space charge [9]. TRACE 3-D uses a hard-edge solenoid in the laboratory frame, but the actual matching is complicated by the solenoid fringe fields in addition to effects from a non K-V particle distribution.

Important future work includes devising a method to calculate and measure the angular momentum of the beam at the output of the FTR triplet. One possible way is to use an internal feature of the beam and compare the beams with and without quadrupole rotations. Naturally, this could be done first in PIC simulations. The effects of space charge could be explored in this way as well.

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