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DERIVATION OF CONDITIONS UNDER WHICH BOUSSARD'S CRITERION FOR THE MICROWAVE INSTABILITY MAY APPLY*

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Abstract

The microwave instability is typically driven by perturbations whose characteristic wavelength is much shorter than the bunch. In this case, Boussard argued that the microwave instability threshold can be found using the predictions of an infinite (coasting) beam, with the average current replaced by the peak current. We revisit this problem, and theoretically show that Boussard's hypothesis holds provided 1) the longitudinal ring impedance is dominated by frequencies much shorter that the inverse bunch length; 2) the single-particle wakefield is much shorter than the bunch length, or, equivalently, the impedance is slowly varying over frequencies longer than the inverse bunch length; 3) the resulting instability has a sudden onset with growth rate of the order of the synchrotron frequency. The first two conditions imply that perturbations are localized within distances much less than the bunch length, while the last condition means that the instability experiences significant growth before the particles can make one synchrotron oscillation. While these conditions may be "obvious" in retrospect, we believe that the last two have not been clearly stated or widely appreciated.

INTRODUCTION

The microwave instability is a classic single bunch instability in which collective longitudinal wakefields drive small-scale density perturbations. Its signature is a sudden increase in the energy spread when the beam current is above a threshold value. The microwave instability is naturally self-limiting; in some cases the final state is a seemingly stable beam with inflated energy spread [1–4], while in other cases one observes "saw-tooth" or "bursting" behavior in which the energy spread periodically grows to a large value followed by a slow damping towards equilibrium [5–8].

The standard theoretical treatment of the microwave instability applies to a coasting beam with constant current. We suppose our ring has a circumference C_R, slip factor η_s, and longitudinal impedance $Z_\parallel(k)$, and we wish to consider the linear stability of perturbations whose wavelength $\lambda = 2\pi/k$. For a coasting beam with average current I, energy γmc^2, and Gaussian energy distribution with rms width σ_δ, the complex frequency Ω describing the stability of such perturbations satisfies the dispersion relation [9, 10]

$$1 = \frac{eI}{\eta_s\sigma_\delta^2\gamma mc^2}\frac{Z_\parallel(k)}{ikC_R} \times\left[1+\hat{\Omega}\frac{e^{-\hat{\Omega}^2/2}}{\sqrt{2/\pi}}\left(i\,\mathrm{sgn}(\eta_s)-\mathrm{erfi}(\hat{\Omega}/\sqrt{2})\right)\right]. \tag{1}$$

Here, the scaled frequency $\hat{\Omega} = \Omega/(ck\eta_s\sigma_\delta)$ with $\Im(\Omega) > 0$ implying instability, $\mathrm{sgn}(x)$ picks the sign of x, while the imaginary error function $\mathrm{erfi}(x) = 2/\sqrt{\pi}\int_0^x dt\; e^{t^2}$.

Since the microwave instability is typically driven by fine-scale perturbations, Boussard suggested that the longitudinal stability of bunched beams may also be governed by a dispersion relation analogous to Eq. (1), only with the average current being replaced by the peak current [11]; for a Gaussian current profile of rms width σ_z this means that the microwave instability is given by Eq. (1) but with $I \to (C_R/\sqrt{2\pi}\sigma_z)I$. The goal of this paper is to determine in detail under what conditions Boussard's hypothesis applies.

THEORY

Our analysis considers the Vlasov equation of a bunched beam under the assumption that the rf potential is quadratic in z. In this case the zero-current motion is simple harmonic with frequency $\omega_{s0} = |\eta_s|\,\sigma_\delta c/\sigma_z$, and we therefore transform longitudinal coordinates to the harmonic oscillator action-angle variables $(\Phi, \mathcal{I})$ by replacing the position $z = \sigma_z\sqrt{2\mathcal{I}/\sigma_z\sigma_\delta}\cos\Phi$, and the energy deviation with $(\gamma/\gamma_0 - 1) = \mathrm{sgn}(\eta_s)\sigma_\delta\sqrt{2\mathcal{I}/\sigma_z\sigma_\delta}\sin\Phi$. To the unperturbed, simple harmonic dynamics we add the longitudinal collective forces derived from the wakefield potential

$$V_\mathrm{w}(z) = i\chi\sigma_\delta^2\,|\eta_s|\int d\hat{\mathcal{I}}\,d\hat{\Phi}\;F(\hat{\mathcal{I}},\hat{\Phi})\int d\kappa\;\frac{Z_\parallel(\kappa)}{2\pi\kappa}e^{i\kappa(z-\hat{z})},$$

where $\chi = eI/(\eta_s\sigma_\delta^2\gamma mc^2)$ and $F(\Phi, \mathcal{I}; s)$ denotes the single particle distribution function. The latter then satisfies the following Vlasov equation

$$0 = \frac{\partial F}{\partial s} + \frac{\omega_{s0}}{c}\frac{\partial F}{\partial\Phi} + \frac{|\eta_s|}{\eta_s}\left[\frac{\partial V_\mathrm{w}}{\partial\mathcal{I}}\frac{\partial F}{\partial\Phi} - \frac{\partial V_\mathrm{w}}{\partial\Phi}\frac{\partial F}{\partial\mathcal{I}}\right]. \tag{2}$$

We consider linear stability by dividing the distribution function into its equilibrium and perturbation,

$$F(\Phi, \mathcal{I}; s) = F_0(\mathcal{I}) + f(\Phi, \mathcal{I}; \Omega)e^{-i\Omega s/c}, \tag{3}$$

and assume that $|F_0| \gg |f|$. The static contribution $(\partial V_\mathrm{w}/\partial\Phi)(\partial F_0/\partial\mathcal{I})$ where the V_w is driven by F_0 accounts for the "potential well distortion," in which the impedance perturbs the harmonic potential and alters the equilibrium current profile from a Gaussian. We will approximate the perturbed potential as a quadratic function in z, and partially account for potential well distortion [12] by setting $\omega_{s0} \to \omega_s = |\eta_s|\,\sigma_\delta c/\sigma_z$, with σ_z the self-consistent (Haissinski) rms bunch length. In other words, we allow the impedance to stretch (or shorten) the bunch, but model

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the current profile as a Gaussian and assume that the equilibrium particle dynamics is approximately simple harmonic with $F_0(\mathcal{I}) = e^{-\mathcal{I}/\sigma_z\sigma_\delta}/(2\pi\sigma_z\sigma_\delta)$.

Under these assumptions and simplifications the linearized Vlasov equation can be written as

$$\begin{aligned}\left[\frac{\Omega}{ic} + \frac{\omega_s}{c}\frac{\partial}{\partial\Phi}\right] f &= \frac{\omega_s}{c} e^{i\Omega\Phi/\omega_s}\frac{\partial}{\partial\Phi}\left[e^{-i\Omega\Phi/\omega_s} f\right] \\ &= -\frac{\chi\omega_s\sigma_\delta}{2\pi\sigma_z/c}\frac{\partial F_0}{\partial\mathcal{I}}\frac{\partial}{\partial\Phi}\int d\kappa\,\frac{Z_\parallel}{i\kappa}e^{i\kappa z(\Phi,\mathcal{I})}\mathscr{B}(\kappa;\Omega),\end{aligned} \tag{4}$$

where the bunching $\mathscr{B} = \int d\mathcal{I}\,d\Phi\; e^{-ikz}f$. As indicated by Eq. (4), we can "solve" for the perturbation f by multiplying by $e^{-i\Omega\Phi/\omega_s}$ and integrating over angle from Φ to $\Phi + 2\pi$; this technique and related calculations were first applied to similar problems by Refs. [13–15]. Since the perturbation f is 2π-periodic in the angle Φ and the Gaussian equilibrium has $\partial F_0/\partial\mathcal{I} = -F_0/\sigma_\delta\sigma_z$, we derive

$$\begin{aligned}f = \frac{\chi F_0(\mathcal{I})}{2\pi}\int d\kappa\,\frac{Z_\parallel(\kappa)}{i\kappa}\mathscr{B}(\kappa;\Omega)\Bigg[e^{i\kappa z(\Phi,\mathcal{I})} \\ -\int_\Phi^{\Phi+2\pi} d\Phi'\,\frac{e^{i\Omega(\Phi-\Phi')/\omega_s}e^{i\kappa z(\Phi',\mathcal{I})}}{i(\omega_s/\Omega)(e^{-2\pi i\Omega/\omega_s}-1)}\Bigg],\end{aligned} \tag{5}$$

where we integrated Φ' by parts. We reduce the left-hand-side to the bunching $\mathcal{B}(k)$ by multiplying both sides by e^{-ikz} and then integrating over the phase space $(\Phi, \mathcal{I})$. The first term in brackets from Eq. (5) gives

$$\begin{aligned}&\int d\Phi d\mathcal{I}\;F_0(\mathcal{I})e^{i(\kappa-k)z} = \int d\Phi dr\,\frac{e^{-r}}{2\pi}e^{i(\kappa-k)\sigma_z\sqrt{2r}\cos\Phi} \\ &= \int_0^\infty dr\int_0^{2\pi} d\Phi\,\frac{e^{-r}}{2\pi}\sum_{\ell=-\infty}^{\infty} i^\ell J_\ell[(\kappa-k)\sigma_z\sqrt{2r}]e^{i\ell\Phi} \\ &= \int_0^\infty dr\;e^{-r}J_0[(\kappa-k)\sigma_z\sqrt{2r}] = e^{-\sigma_z^2(k-\kappa)^2/2},\end{aligned} \tag{6}$$

where the first line uses $r = \mathcal{I}/\sigma_z\sigma_\delta$, the second applies the Jacobi-Anger identity, while the third evaluates the integrals.

Meanwhile, we simplify the integrals over the second term in brackets from Eq. (5) by changing variables to Φ and $\theta = \Phi - \Phi' + 2\pi$ and then simplifying as

$$\begin{aligned}&\int_0^{2\pi} d\Phi\int_\Phi^{\Phi+2\pi} d\Phi'\;e^{i\Omega(\Phi-\Phi')/\omega_s}e^{i\sigma_z\sqrt{2r}(\kappa\cos\Phi'-k\cos\Phi)} \\ &= \int_0^{2\pi} d\Phi d\theta\;e^{i\Omega(\theta-2\pi)/\omega_s}e^{i\sigma_z\sqrt{2r}[\kappa\cos(\Phi-\theta)-k\cos\Phi]} \\ &= \int_0^{2\pi} d\theta\;e^{i\Omega(\theta-2\pi)/\omega_s}\int_0^{2\pi} d\Phi\;e^{i\sigma_z K\cos(\Phi+\psi)}.\end{aligned} \tag{7}$$

The second line combines the harmonic functions using

$$\kappa\cos(\Phi-\theta) - k\cos\Phi = K\cos(\Phi+\psi), \tag{8}$$

with $K = \sqrt{(\kappa\cos\theta - k)^2 + (\kappa\sin\theta)^2}$ and ψ a phase that we will not need to know. The integral over Φ in Eq. (7) and the subsequent one over $\mathcal{I}$ can be evaluated as we did to arrive at Eq. (6); the bunching therefore obeys

$$\begin{aligned}\mathscr{B}(k) = \frac{\chi}{\sqrt{2\pi}}\int d\kappa\;\mathscr{B}(\kappa)\frac{Z_\parallel(\kappa)}{i\kappa\sigma_z}\frac{e^{-\sigma_z^2(\kappa-k)^2/2}}{\sqrt{2\pi}/\sigma_z} \\ \times\Bigg[1 + \frac{i\Omega}{\omega_s}\int_0^\pi d\theta\;\frac{e^{-\kappa k\sigma_z^2(1-\cos\theta)}}{1-e^{2\pi i\Omega/\omega_s}} \\ \times\left(e^{i\Omega\theta/\omega_s} + e^{2\pi i\Omega/\omega_s}e^{-i\Omega\theta/\omega_s}\right)\Bigg].\end{aligned} \tag{9}$$

Equation (9) is an integral equation for the bunching $\mathscr{B}$ whose kernel depends upon the impedance, the equilibrium parameters, and the complex frequency Ω. Analysis of a different but equivalent formula was presented in [16].

Here, we wish to investigate Eq. (9) in the limit that the bunch length is much longer than the characteristic wavelength of the instability, such that the instability "doesn't know" about the finite bunch length and the coasting beam dispersion relation Eq. (1) applies with the average current I being replaced by the peak current $(C_R/\sqrt{2\pi}\sigma_z)I$. This is essentially Boussard's hypothesis [11], and our goal is to see under what circumstances Eq. (9) can be converted into a Boussard-Keil-Schnell-type criterion for the microwave instability in bunched beams. Detailed analysis that had some similar goals and comparable results was previously done in [15]. However, by restricting our approach to a harmonic potential we will be able to find specific conditions under which the the coasting beam dispersion applies. Hence, we sacrifice some generality in exchange for what we think is a cleaner, more compact derivation in which the underlying assumptions are simpler to understand.

To begin, we will assume that the integrand is dominated by contributions such that $|\kappa\sigma_z| \gg 1$, in which case the right-hand-side of Eq. (9) is small unless $\kappa \approx \pm k$. If $\kappa \approx k \gg 1/\sigma_z$, then the integral over θ is dominated by the contribution when $\theta \ll 1$, and we write

$$\begin{aligned}&\int_0^\pi d\theta\;e^{-\kappa k\sigma_z^2(1-\cos\theta)}e^{\pm i\Omega\theta/\omega_s} \\ &\approx \int_0^\infty d\theta\;e^{-\kappa k\sigma_z^2\theta^2/2}e^{\pm i\Omega\theta/\omega_s}\end{aligned} \tag{10}$$

$$= \frac{\exp\left(-\frac{\Omega^2/\omega_s^2}{2\kappa k\sigma_z^2}\right)}{\sigma_z\sqrt{2\kappa k/\pi}}\left[1 \pm i\,\mathrm{erfi}\left(\frac{\Omega/\omega_s}{\sigma_z\sqrt{2\kappa k}}\right)\right]. \tag{11}$$

When $k \approx -\kappa$ a similar expansion applies about $\theta \approx \pi$, with comparable results. We then find that the bunching satisfies the integral equation

$$\begin{aligned}\mathscr{B}(k) \approx \frac{\chi}{\sqrt{2\pi}}\int_{\kappa\approx+k} d\kappa\;\frac{e^{-\sigma_z^2(\kappa-k)^2/2}}{\sqrt{2\pi}/\sigma_z}\mathcal{M}_+ \\ -\frac{\chi}{\sqrt{2\pi}}\int_{\kappa\approx-k} d\kappa\;\frac{e^{-\sigma_z^2(\kappa+k)^2/2}}{\sqrt{2\pi}/\sigma_z}\mathcal{M}_-,\end{aligned} \tag{12}$$

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where

$$\mathcal{M}_{+} = \mathscr{B}(\kappa)\frac{Z_\parallel(\kappa)}{i\kappa\sigma_z}\left\{1 - \frac{\Omega\exp\left(-\frac{\Omega^2/\omega_s^2}{2\kappa k\sigma_z^2}\right)}{\omega_s\sigma_z\sqrt{2\kappa k/\pi}}\right. \times\left.\left[\frac{1}{\tan(\pi\Omega/\omega_s)} + \mathrm{erfi}\left(\frac{\Omega/\omega_s}{\sigma_z\sqrt{2\kappa k}}\right)\right]\right\} \tag{13}$$

$$\mathcal{M}_{-} = \mathscr{B}(\kappa)\frac{Z_\parallel(\kappa)}{i\kappa\sigma_z}\frac{(\Omega/\omega_s)\exp\left(\frac{\Omega^2/\omega_s^2}{2\kappa k\sigma_z^2}\right)}{\sigma_z\sqrt{-2\kappa k/\pi}\sin(\pi\Omega/\omega_s)}. \tag{14}$$

We're not quite at the dispersion relation Eq. (1), and to get closer we next assume that $\mathcal{M}_\pm$ vary slowly when $\kappa \approx \pm k \gg 1/\sigma_z$. In particular, this requires that any peaks in the impedance must be much broader than $1/\sigma_z$. For example, if $Z_\parallel(\kappa)$ is described by a resonator impedance with resonant frequency ck_r and quality factor Q, then we require both $k_r\sigma_z \gg 1$ and $k_r\sigma_z/Q \gg 1$. The former condition leads to a perturbation whose characteristic wavelength is much smaller than the bunch length, and is needed to get to Eq. (12). The latter condition implies that the wakefield driven by any particle is confined to distances that are also much smaller than the bunch length. This additional assumption allows us to approximate $(\sigma_z/\sqrt{2\pi})e^{-(k\pm\kappa)^2/2\sigma_z^2} \approx \delta(k\pm\kappa)$ in the integrand, which in turn leads to

$$1 \approx \frac{\chi}{\sqrt{2\pi}}\frac{Z_\parallel(k)}{ik\sigma_z}\left[1 + \frac{\mathscr{B}(-k)/\mathscr{B}(k)}{\sin(\pi\Omega/\omega_s)}\right. - \left.\frac{\hat{\Omega}e^{-\hat{\Omega}^2/2}}{\sqrt{2/\pi}}\left(\frac{\mathrm{sgn}(\eta_s)}{\tan(\pi\Omega/\omega_s)} + \mathrm{erfi}(\hat{\Omega}/\sqrt{2})\right)\right], \tag{15}$$

where $\hat{\Omega} = \Omega/ck\eta_s\sigma_\delta = \mathrm{sgn}(\eta_s)(\Omega/\omega_s)/(k\sigma_z)$ as in Eq. (1).

The dispersion relation Eq. (15) is almost what we want but for the term $\propto \mathscr{B}(-k)/\mathscr{B}(k)$ and the presence of the tangent. The former term leaves Eq. (15) with no closed-form solutions, and furthermore implies that the microwave instability typically involves coupling between different k vectors that is not present in a simple, Keil-Schnell-like dispersion relation. Getting the bunched beam result Eq. (15) to mirror that of a coasting beam requires one last assumption, namely, that the instability growth rate is much larger than the synchrotron frequency. In fact, taking the formal $\omega_s \to \infty$ limit of Eq. (15) leads to a dispersion relation that is discontinuous at $\Im(\Omega) = 0$ in the exact same way as is the usual dispersion integral. A suitably continuous dispersion relation is found by analytically continuing the result from $\Im(\Omega) > 0$ to $\Im(\Omega) \le 0$ as prescribed by Landau [17].

For finite ω_s, the bunched beam dispersion relation Eq. (15) and its coasting beam counterpart Eq. (1) agree to within a few percent when the instability growth rate $\Im(\Omega) \gtrsim \omega_s$ and the second term of Eq. (15) $\lesssim 1/\sinh(\pi) \ll 1$ is neglected. Hence, if the onset of the instability is sudden with a large growth rate, as is often observed, then we can conclude that a reasonable approximation to Eq. (15) matches the coasting beam result but with the average current replaced with the peak current:

$$1 = \frac{eI}{\eta_s\sigma_\delta^2\gamma mc^2}\frac{C_R}{\sqrt{2\pi}\sigma_z}\frac{Z_\parallel(k)}{ikC_R} \times\left[1 + \hat{\Omega}\frac{e^{-\hat{\Omega}^2/2}}{\sqrt{2/\pi}}\left(i\,\mathrm{sgn}(\eta_s) - \mathrm{erfi}(\hat{\Omega}/\sqrt{2})\right)\right]. \tag{16}$$

The dispersion relation Eq. (16) manifests Boussard's conjecture that the microwave instability threshold for short bunches can be related to that of a coasting beam. Furthermore, our derivation indicates that the coasting beam dispersion relation will predict the microwave instability threshold with the average current replaced by the peak current when

1. The longitudinal impedance is dominated by wavevectors that satisfy $k\sigma_z \gg 1$ so that $Z_\parallel(k)$ drives fine-scale perturbations much shorter than the bunch length;
2. The single-particle wakefield is much shorter than the bunch length, so that in reciprocal space we require $Z_\parallel(k)$ to be slowly varying over distances $\Delta k \lesssim 1/\sigma_z$;
3. The resulting microwave instability has a sudden onset with growth rate $\Im(\Omega) \gtrsim \omega_s$.

The first two conditions imply that perturbations are localized within distances much less than the bunch length, while the last condition means that the instability experiences significant growth before the particles can make one synchrotron oscillation. In retrospect these conditions may be somewhat obvious, but we think that the last two have not been clearly stated or widely appreciated.

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