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EXPLICIT SYMPLECTIC REPRESENTATIONS OF NONLINEAR DIPOLE FRINGE FIELD MAPS

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Abstract

The representation of beam transport through magnetic dipole fringe fields using effective "thin" maps has a long history. More recent work has extended the second-order Taylor model of Brown by providing Lie generators for symplectic maps that capture effects at higher order, expressed in terms of field integrals. These maps can be cumbersome to evaluate in an explicit symplectic form appropriate for particle tracking codes, and existing approaches often require additional approximations. We show how such maps can be recovered in a simple, explicit form as rational symplectic functions using a mixed-variable generating function approach.

INTRODUCTION

Second-order models for "thin" maps describing the dynamical effects of dipole fringe fields are well-known [1–3]. Recent works [4, 5] extend these models to capture additional effects, such as closed orbit deviation, octupole-like focusing effects, and quadrupole or sextupole content in the magnet body. The result is a symplectic map $\mathcal{M}_F$ defined by:

$$\mathcal{M}_F = e^{:\Omega_M:}, \tag{1}$$

where Ω_M is approximated (through third or fourth-order terms) in the phase space variables and the magnetic gap. Approaches to express this map in an explicit form include: i) extracting the corresponding Taylor map, sacrificing symplecticity, or ii) splitting Ω_M into exactly-solvable terms, which requires ingenuity, approximates Eq. (1) in the size of Ω_M, and results in complicated expressions.

We describe how to bring Eq. (1) into an explicit form by constructing a mixed-variable generating function directly from the generator Ω_M. Maps obtained from mixed-variable generating functions are always symplectic, but they are typically implicit. However, in the cases of interest the Lie generator is (at most) linear in the momenta p_x and p_y, and this feature allows for a complete treatment [6, 7]. The result is an explicit, rational symplectic map that agrees with the underlying model to the same order of approximation.

HAMILTONIANS AT MOST LINEAR IN THE MOMENTA

Consider a Hamiltonian of the following general form:

$$H(q,p,\bar{q},\bar{p}) = p\cdot f(q,\bar{p}) + h(q,\bar{p}), \tag{2}$$

where we have a vector of coordinates $q=(q_1,\ldots,q_d)$ with a corresponding vector of momenta $p=(p_1,\ldots,p_d)$, and an additional vector of coordinates $\bar{q}$, not appearing on the right-hand side of Eq. (2), with a corresponding vector of momenta $\bar{p}$. [Here we can take $q=(x,y)$, $p=(p_x,p_y)$, $\bar{q}=z$, and $\bar{p}=\delta$.]

The differential equations for the variables q and $\bar{p}$ form a system that is decoupled from the remaining variables. Let $Q(t,q,\bar{p})$ denote the unique solution of this system with the initial condition q at $t=0$, so explicitly:

$$\frac{dQ(t,q,\bar{p})}{dt} = f(Q(t,q,\bar{p}),\bar{p}), \qquad Q(0,q,\bar{p}) = q. \tag{3}$$

The momenta $\bar{p}$ are invariant, and may be treated as fixed parameters in the solution of Eq. (3).

If the solution $Q(t,q,\bar{p})$ of Eq. (3) is exactly known, then the solution of the entire system for Eq. (2) follows [6, 7].

Mixed-variable Generating Function

For each value of the independent variable t, we define a mixed-variable generating function F_t given by:

$$F_t(q,P,\bar{q},\bar{P}) = \bar{P}\cdot\bar{q} + P\cdot Q(t,q,\bar{P}) + \int_0^t h(Q(s,q,\bar{P}),\bar{P})ds. \tag{4}$$

The expression Eq. (4) yields a symplectic map of the form $(q,p,\bar{q},\bar{p})\mapsto(Q,P,\bar{Q},\bar{P})$, which is defined by the implicit relations:

$$Q_j = \frac{\partial F_t(q,P,\bar{q},\bar{P})}{\partial P_j}, \quad p_j = \frac{\partial F_t(q,P,\bar{q},\bar{P})}{\partial q_j}, \tag{5a}$$

$$\bar{Q}_k = \frac{\partial F_t(q,P,\bar{q},\bar{P})}{\partial \bar{P}_k}, \quad \bar{p}_k = \frac{\partial F_t(q,P,\bar{q},\bar{P})}{\partial \bar{q}_k}. \tag{5b}$$

Because of the simple, linear dependence on the momenta P appearing in Eq. (4), the relations Eq. (5) can be explicitly inverted [8].

We claim that the map $(q,p,\bar{q},\bar{p})\mapsto(Q,P,\bar{Q},\bar{P})$ describes the time evolution $0\mapsto t$ generated by the Hamiltonian Eq. (2). It is sufficient to show that the mixed-variable generating function F_t satisfies the following Hamilton-Jacobi equation [9]:

$$\frac{\partial F_t}{\partial t} = H\left(\frac{\partial F_t}{\partial P}, P, \frac{\partial F_t}{\partial \bar{P}}, \bar{P}\right), \tag{6a}$$

$$F_0(q,P,\bar{q},\bar{P}) = P\cdot q + \bar{P}\cdot\bar{q}, \tag{6b}$$

where the initial condition Eq. (6b) states simply that F_0 generates the identity map. This initial condition is easily checked by setting $t=0$ in Eq. (4). The left-hand side of Eq. (6a) becomes:

$$\begin{aligned}\frac{\partial F_t}{\partial t} &= P\cdot\frac{\partial Q}{\partial t}(t,q,\bar{P}) + h(Q(t,q,\bar{P}),\bar{P}) \\ &= P\cdot f(Q(t,q,\bar{P}),\bar{P}) + h(Q(t,q,\bar{P}),\bar{P}),\end{aligned} \tag{7}$$

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where in the second line we used Eq. (3). Likewise, the right-hand side of Eq. (6a) becomes:

$$\begin{aligned}&H\left(\frac{\partial F_t}{\partial P}, P, \frac{\partial F_t}{\partial \bar{P}}, \bar{P}\right)\\&= P\cdot f\left(\frac{\partial F_t}{\partial P}, \bar{P}\right) + h\left(\frac{\partial F_t}{\partial P}, \bar{P}\right)\\&= P\cdot f(Q(t,q,\bar{P}),\bar{P}) + h(Q(t,q,\bar{P}),\bar{P}).\end{aligned} \tag{8}$$

Clearly Eqs. (7) and (8) coincide.

Approximation using Symplectic Maps

Our goal is to obtain a representation of the map $(q,p,\bar{q},\bar{p}) \mapsto (Q,P,\bar{Q},\bar{P})$ that describes the time evolution generated by the Hamiltonian Eq. (2). This representation will agree with the underlying model (e.g., Eq. (1)) to the same order of approximation. This can be done by approximating the generating function Eq. (4), as follows.

We use the functions $f = (f_1,\dots,f_d)$ appearing in the Hamiltonian Eq. (2) to define a vector field L given by:

$$L = \sum_{k=1}^{d} f_k(q,\bar{p})\frac{\partial}{\partial q_k}. \tag{9}$$

The flow defined by Eq. (3) is given for each coordinate Q_k formally by:

$$Q_k(t,q,\bar{p}) = \exp(tL)q_k, \quad k = 1\dots,d \tag{10}$$

Using Eq. (10), the mixed-variable generating function Eq. (4) can then be expressed in the form:

$$F_t(q,P,\bar{q},\bar{P}) = \bar{P}\cdot\bar{q} + \sum_{k=1}^{d} P_k \exp(tL)q_k + t\,\mathrm{iex}(tL)h, \tag{11}$$

where iex denotes the integrated exponential function, defined by:

$$\mathrm{iex}(w) = \int_0^1 \exp(\tau w)d\tau = \sum_{m=0}^{\infty}\frac{w^m}{(m+1)!}. \tag{12}$$

Eq. (11) can easily be used to produce approximations of F_t (e.g., by expanding the differential operators). Applying Eq. (5) yields a symplectic approximation of the map for Eq. (2).

DIPOLE MODELS

The Lie generators Ω_M appearing in [4,5] take the general form Eq. (2). A symplectic approximation of the map Eq. (1) may be obtained by using Eq. (11) to obtain the time-one flow of the Hamiltonian $H = -\Omega_M$. The main results are summarized below.

Sector Dipoles

Hwang and Lee [4] give expression Eq. (1) for the fringe field map of a sector dipole, where the Lie generator takes the form:

$$\begin{aligned}\Omega_M &= xc_1 - xc_2 + p_xc_3 + (xp_x - yp_y)c_4\\&+ (x^2-y^2)c_5 - x^2c_6 + y^2c_7 - x^3c_8 + xy^2c_9\\&+ (x^2p_x - y^2p_x - 2xyp_y)c_{10} - y^2p_xc_{11} + y^4c_{12}\\&+ xc_{13} + (y^2-x^2)c_{14} + \left(\frac{xy^2}{2} - \frac{x^3}{3!}\right)c_{15}.\end{aligned} \tag{13}$$

Here we have simply numbered the coefficients as they appear in Eq. (35) of Ref. [4], so that for the entry fringe field:

$$c_1 = \frac{1}{\cos\theta_E}\frac{g}{\rho}K_1, \quad c_2 = \frac{\sec^3\theta_E\sin\theta_E}{2(1+\delta)}\frac{g^2}{\rho^2}K_0, \tag{14a}$$

$$c_3 = \frac{\sec^2\theta_E}{(1+\delta)}\frac{g^2}{\rho}K_0, \quad c_4 = \frac{1}{1+\delta}\frac{g}{\rho}K_1\frac{\sin\theta_E}{\cos^2\theta_E}, \tag{14b}$$

$$c_5 = \frac{1}{2}\frac{\tan\theta_E}{\rho}, \quad c_6 = \frac{1}{2}\frac{\sin^2\theta_E}{2\rho(1+\delta)\cos^3\theta_E}\frac{g}{\rho}K_1, \tag{14c}$$

$$c_7 = \frac{1}{2}\frac{\sec^3\theta_E}{(1+\delta)}\left[\frac{g}{2\rho^2}K_1 + (1+\sin^2\theta_E)\frac{g}{\rho^2}K_2\right], \tag{14d}$$

$$c_8 = \frac{1}{3!}\frac{\tan^3\theta_E}{2\rho^2(1+\delta)}, \quad c_9 = \frac{1}{2}\left[\frac{\tan\theta_E\sec^2\theta_E}{2\rho^2(1+\delta)}\right], \tag{14e}$$

$$c_{10} = \frac{1}{2(1+\delta)}\frac{\tan^2\theta_E}{\rho}, \quad c_{11} = \frac{1}{2\rho(1+\delta)}, \tag{14f}$$

$$c_{12} = \frac{1}{4!}\left[\frac{4}{\cos\theta_E} - \frac{8}{\cos^3\theta_E}\right]\frac{K_3}{\rho^2 g(1+\delta)}, \tag{14g}$$

$$c_{13} = \frac{\sin^2\theta_E}{2\cos^3\theta_E}\frac{g^2}{\rho R}K_4, \quad c_{14} = \frac{1}{2}\frac{\sin\theta_E}{\cos^3\theta_E}\frac{g}{\rho R}K_5, \tag{14h}$$

$$c_{15} = \frac{K_6/\rho R}{\cos^3\theta_E}. \tag{14i}$$

The definition of the gap parameter g, the bending radius ρ, the entry (exit) angle θ_E, and the field integrals K_j ($j = 0,\dots,6$) are as described in [4]. Note that we use y to denote the vertical coordinate.

The generator Eq. (13) was obtained by introducing an order parameter λ such that $(x,p_x,y,p_y,g) \sim O(\lambda)$ and retaining terms in the Hamiltonian through $O(\lambda^3)$. Using Eq. (11) to expand the corresponding mixed-variable generating function F through terms of $O(\lambda^3)$, we find:

$$\begin{aligned}F &= z\delta + P_x\left(x - c_3 + (c_{11}+c_{10})y^2 - c_{10}x^2\right)\\&+ P_y(y + 2c_{10}xy) + R,\end{aligned} \tag{15}$$

$$\begin{aligned}R &= (c_2 + c_3c_5 - c_{13})x + (c_{14} - c_5)x^2\\&+ \left(\frac{c_{15}}{6} + c_{10}c_5 + c_8\right)x^3 + (-c_{14} + c_5 - c_7)y^2\\&- c_{12}y^4 + \left(-\frac{1}{2}c_{15} + c_{10}c_5 - c_{11}c_5 - c_9\right)xy^2.\end{aligned} \tag{16}$$

The resulting symplectic map is:

$$\begin{aligned}X &= x - c_3 - c_{10}x^2 + (c_{10}+c_{11})y^2,\\Y &= y + 2c_{10}xy,\\P_x &= \left[(\partial_y Y)(p_x - \partial_x R) - (\partial_x Y)(p_y - \partial_y R)\right]/D,\\P_y &= \left[-(\partial_y X)(p_x - \partial_x R) + (\partial_x X)(p_y - \partial_y R)\right]/D,\end{aligned} \tag{17}$$

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where the polynomial appearing the denominator is:

$$D = 1 - 4c_{10}c_{11}y^2 - 4c_{10}^2(x^2 + y^2). \tag{18}$$

This map agrees with the $O(\lambda^2)$ Taylor map in eqs (40-43) of [4] when expanded through terms of the same order, except that a missing term proportional to K_5 in eq (43) is now included.

We obtain for the path-length coordinate $z = -v\Delta t$:

$$\begin{aligned} Z = z - \frac{1}{(1+\delta)}\big[&(c_2 + c_3c_5)x + (c_{10}c_5 + c_8)x^3 \\ &- c_7y^2 - c_{12}y^4 + (c_{10}c_5 - c_{11}c_5 - c_9)xy^2 \qquad (19)\\ &+P_x(-c_3 + (c_{11}+c_{10})y^2 - c_{10}x^2) + 2P_yc_{10}xy\big]. \end{aligned}$$

The corresponding map for the exit fringe field is obtained by changing the signs of c_3, c_4, c_{10}, and c_{11}.

Cartesian Dipoles

Lindberg and Borland [5] give the following expression for the Lie generator of the fringe field map of a combined-function Cartesian dipole:

$$\begin{aligned} \Omega_M = {}& c_1p_x + c_2x + c_3y^2 + c_4y^2 - c_5x^2 - c_6x^2 \qquad (20)\\ &+ c_7(p_yy - p_xx) - c_8p_xy^2 + c_9p_yxy - c_{10}p_x(x^2 - y^2) \\ &- c_{11}(3xy^2 + x^3) + c_{12}xy^2 - c_{13}x^3 - c_{14}y^4. \end{aligned}$$

Here we have simply numbered the coefficients as they appear in eq (60) of [5], so that for the entry fringe field:

$$c_1 = \frac{\sec^3\theta}{2(1+\delta)}\left(\frac{g^2K_0}{\rho}\right),$$

$$c_2 = \left[\frac{\tan^2\theta}{2}\left(\frac{g^2K_4}{R\rho}\right) - \tan\theta\left(1 - \frac{1}{2}\tan^2\theta\right)(g^2KI_1)\right],$$

$$\begin{aligned} c_3 = \Bigg[&-\left(\frac{\tan\theta}{\rho_+} - \frac{\tan\theta}{\rho_-}\right) + \frac{1+\sin^2\theta}{\cos^3\theta(1+\delta)}\left(\frac{gK_2}{\rho^2}\right) \\ &+ \left(1 + \frac{1}{2}\tan^2\theta\right)(gKI_0)\Bigg]\frac{1}{2}, \end{aligned}$$

$$c_4 = \tan\theta\left(\frac{gK_5}{R\rho}\right)\frac{\sec^2\theta}{2}, \quad c_5 = \tan\theta\left(\frac{gK_5}{R\rho}\right)\frac{1}{2},$$

$$c_6 = \left(1 - \frac{1}{2}\tan^2\theta\right)(gKI_0)\frac{1}{2}, \quad c_7 = \frac{\sec\theta}{1+\delta}(g^2KI_1),$$

$$c_8 = \frac{\sec^2\theta}{1+\delta}\left(\frac{1}{\rho_+} - \frac{1}{\rho_-}\right)\frac{1}{2}, \quad c_9 = \frac{\sec\theta}{1+\delta}\left(\frac{gK_5}{\rho R}\right),$$

$$c_{10} = \frac{\sec\theta}{1+\delta}\left(\frac{gK_5}{\rho R}\right)\frac{\sec^2\theta}{2}, \quad c_{11} = \frac{\tan\theta}{12}(K_+ - K_-),$$

$$c_{12} = \left(\frac{K_6}{R\rho}\right)\frac{3\sec^2\theta}{6}, \quad c_{13} = \left(\frac{K_6}{R\rho}\right)\frac{1}{6},$$

$$c_{14} = \frac{1+\sin^2\theta}{\cos^3\theta(1+\delta)}\left(\frac{K_3}{g\rho^2}\right)\frac{1}{6}.$$

The approximation scheme is the same as that described in the previous section, except that terms of the form $(gK_5/\rho)G$, where G is third order in the phase space variables, are retained.

Using Eq. (11) to expand the generating function F to the same order of approximation gives:

$$\begin{aligned} F = {}& z\delta + P_x\left(-c_1 + (1+c_7)x + c_{10}x^2 + (c_8 - c_{10})y^2\right) \\ &+ P_y(y - c_7y - c_9xy) + R, \qquad (22) \end{aligned}$$

$$\begin{aligned} R = {}& -c_2x + (c_5 + c_6)x^2 - (c_3 + c_4)y^2 + (c_{11} + c_{13})x^3 \\ &+ (3c_{11} - c_{12} + c_5c_8)xy^2 + c_{14}y^4. \qquad (23) \end{aligned}$$

The symplectic map for the transverse coordinates is:

$$\begin{aligned} X &= -c_1 + (1+c_7)x + c_{10}x^2 + (-c_{10} + c_8)y^2, \qquad (24a)\\ Y &= y - c_7y - c_9xy, \\ P_x &= \left[(\partial_yY)(p_x - \partial_xR) - (\partial_xY)(p_y - \partial_yR)\right]/D, \\ P_y &= \left[-(\partial_yX)(p_x - \partial_xR) + (\partial_xX)(p_y - \partial_yR)\right]/D, \end{aligned}$$

where the polynomial in the denominator is:

$$D = (1 - c_7 - c_9x)(1 + c_7 + 2c_{10}x) + 2c_9(c_8 - c_{10})y^2. \tag{24b}$$

Finally, we obtain for the path-length coordinate:

$$Z = \frac{\partial F}{\partial\delta}, \tag{24c}$$

where in the derivative it is important to include the δ-dependence of the c_j's.

CONCLUSION

We describe a method to represent dipole fringe field maps such as those described in [4, 5] in an explicit, symplectic form appropriate for particle tracking. Unlike previous approaches, this approach relies on a general result Eq. (11) for mixed-variable generating functions, and produces simple expressions defined by ratios of polynomials. The method described here can be applied to other fringe field models of similar form [7], as well as Hamiltonian systems (at most) linear in the momenta arising elsewhere, e.g. in optimal control theory.

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REFERENCES

[1] K. L. Brown, "A First- and Second-Order Matrix Theory for the Design of Beam Transport Systems and Charged Particle Spectrometers", SLAC Rep. No. 75, 1982.

ISBN: 978-3-95450-261-5 ISSN: 2673-7000 doi: 10.18429/JACoW-NAPAC2025-TUP040

[2] F. C. Iselin, “Lie Transformations and Transport Equations for Combined-Function Dipoles”, *Part. Accel.*, vol. 17, pp. 143–155, 1985.

[3] E. Forest, S. Leemann, and F. Schmidt, “Fringe Effects in MAD: Part I - Second Order Fringe in MAD-X for the Module PTC,” KEK Preprint 2005-109, Mar. 2006.

[4] K. Hwang and S. Y. Lee, “Dipole Fringe Field Thin Map for Compact Synchrotrons”, *Phys. Rev. Accel. Beams*, vol. 18, p. 122401, 2015. doi:10.1103/PhysRevSTAB.18.122401

[5] R. Lindberg and M. Borland, “Fringe field maps for symplectic models of general Cartesian dipoles”, *Phys. Rev. Accel. Beams*, vol. 26, p. 114001, 2023. doi:10.1103/PhysRevAccelBeams.26.114001

[6] P. Hagedorn, “A Short Note on Hamiltonians Linear in the Momenta”, *Celest. Mech.*, vol. 17, pp. 127–129, 1978. doi:10.1007/BF01371324

[7] C. Mitchell, K. Hwang, and D. Bruhwiler, “Explicit symplectification of nonlinear hard-edge fringe field maps”, in preparation.

[8] D. L. Bruhwiler, J. R. Cary, and S. Shasharina, “Symplectic Propagation of the Map, Tangent Map and Tangent Map Derivative through Quadrupole and Combined-Function Dipole Magnets without Truncation”, in *Proc. EPAC’98*, Stockholm, Sweden, Jun. 1998, paper THP41C, pp. 1171–1173.

[9] E. Hairer, C. Lubich, and G. Wanner, *Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations*, 2nd ed., Springer-Verlag, Berlin, 2006, Section VI.5, “Generating Functions”, Theorem 5.7.