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FAST SPIN TRACKING USING A MAGNUS EXPANSION

J. P. Devlin[*,1], E. Hamwi[1], G. H. Hoffstaetter[1,2]
[1]CLASSE, Cornell University, Ithaca, New York, USA
[2]Brookhaven National Laboratory, Upton, New York, USA

Abstract

Spin motion in particle accelerators obeys the Thomas-Bargmann-Michel-Telegdi (T-BMT) equation. Due to the structure of the T-BMT equation, the spin-transfer quaternion of a magnet is generally a nonlinear function of the entrance coordinates even if the phase-space motion is linear. This nonlinear function can be written as a Dyson expansion, for example as employed in the program Sprint, which normalized the first-order expansion of the spin-transfer quaternion. Alternatively, this nonlinear function can be written as a Magnus expansion. This paper points out that in cases where the phase-space coordinates change little, as is generally the case for accelerator elements, the Magnus expansion is a much more appropriate method to describe the nonlinear spin motion because this expansion terminates after the first term when the phase-space coordinates are constant. We will demonstrate, with several examples, that an approximation based on the Magnus expansion leads to very good agreement with time-consuming numerical integration, and to significantly better agreement than obtained with historical codes like Sprint.

INTRODUCTION

Despite the ubiquity of matrices in accelerator physics, the reality is that even the simplest magnetic field configurations lead to nonlinear transfer maps. This is exemplified by the fact that the motion through a drift space is nonlinear, which one can trace back to the square root in the relativistic energy-momentum relation [1]. As a consequence, for most magnets, the full transfer map can only be obtained through numerical integration. Nevertheless, matrices are useful in that they are fast and, in many cases, provide an adequate approximation of the transfer map. It would be useful if an analogous concept existed for spin motion, i.e., a method to track spins which avoided expensive numerical integration but still retained reasonable accuracy.

The equation of spin motion in a purely magnetic field is a special case of the Thomas-Bargmann-Michel-Telegdi (T-BMT) equation and takes the form [2, 3]

$$\begin{aligned} \boldsymbol{S}'(s) &= \left[\boldsymbol{\Omega}(z(s),s) + g\boldsymbol{e}_y\right] \times \boldsymbol{S}(s), \\ \boldsymbol{\Omega}(z,s) &= -\frac{q}{p_0}\frac{1+gx}{\sqrt{(1+p_z)^2 - p_x^2 - p_y^2}} \\ &\quad \times \left[(1+G\gamma)\boldsymbol{B}_\perp(z,s) + (1+G)\boldsymbol{B}_\parallel(z,s)\right]. \end{aligned} \tag{1}$$

Here, $\boldsymbol{B}_\perp$ (resp. $\boldsymbol{B}_\parallel$) represents the magnetic field perpendicular (resp. parallel) to the particle velocity, z contains the canonical coordinates, g is the horizontal curvature of the reference trajectory, and s denotes the arc length along the reference trajectory. The coordinate system and canonical coordinates used here are taken from Bmad, where the transverse momenta are normalized by the reference momentum p_0 and $p_z = (p - p_0)/p_0$ [4]. Given that the cross product is a linear operation, it is often useful to rewrite Eq. (1) in the form

$$\boldsymbol{S}'(s) = \mathbf{A}(z(s),s)\boldsymbol{S}(s), \tag{2}$$

where $\mathbf{A}(z,s)$ is a skew-symmetric matrix.

Even if the orbital motion is propagated using a matrix (i.e., z is a linear function of the initial coordinates z_0) and the magnetic field is a linear function of z, it is clear from Eq. (1) that $\boldsymbol{S}'$ is a nonlinear function of z_0. This is the first indication that the spin motion is quite complicated, especially in light of the fact that the linearity of z is merely an approximation. Another fact of note is that when $\mathbf{A}$ commutes with itself, i.e.,

$$[\mathbf{A}(z(s_1),s_1), \mathbf{A}(z(s_2),s_2)] = 0 \tag{3}$$

for any s_1 and s_2 with the matrix commutator $[\cdot,\cdot]$, the flow of Eq. (2) is given by

$$\mathbf{R}(z_0,s;s_0) = \exp\left(\int_{s_0}^{s} \mathbf{A}(z(\sigma),\sigma)\,\mathrm{d}\sigma\right). \tag{4}$$

The exponential will generally contain all orders of z_0 even if $\mathbf{A}(z,s)$ is approximated by a linear function of z.

PERTURBATION THEORY

Two ways to extend Eq. (4) to the case of noncommuting $\mathbf{A}$ are the Dyson expansion and the Magnus expansion. The leading-order Dyson expansion is

$$\mathbf{R}(z_0,s;s_0) \approx \mathbf{1} + \int_{s_0}^{s} \mathbf{A}(z(\sigma),\sigma)\,\mathrm{d}\sigma\,, \tag{5}$$

which is just the linear expansion of Eq. (4) [5]. The leading-order Magnus expansion is [6, 7]

$$\mathbf{R}(z_0,s;s_0) \approx \exp\left(\int_{s_0}^{s} \mathbf{A}(z(\sigma),\sigma)\,\mathrm{d}\sigma\right). \tag{6}$$

Of course, this is a good approximation when the noncommutativity of $\mathbf{A}$ perturbs the flow negligibly.

The Magnus expansion has some advantages over the Dyson expansion. To begin, note that when $\mathbf{A}(z,s)$ belongs to a Lie algebra $\mathfrak{g}$, the flow of Eq. (2) belongs to the corresponding Lie group G. The approximate flow provided by the leading-order Magnus expansion also belongs to G, while that of the Dyson expansion in general does not. In spin dynamics, the case of interest is $G = \mathrm{SO}(3)$ (although it is more efficient to use unit quaternions in numerical simulations) [8, 9]. For this reason, use of the Dyson expansion usually leads to an undesirable change in the length of spins.

* jpd285@cornell.edu

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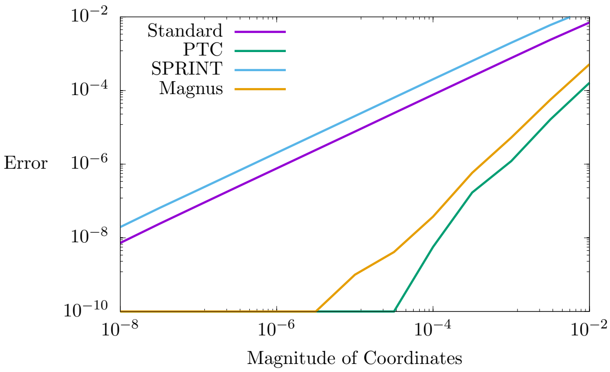

Figure 1: Comparison of spin-tracking methods for a sector bend with varying coordinate size.

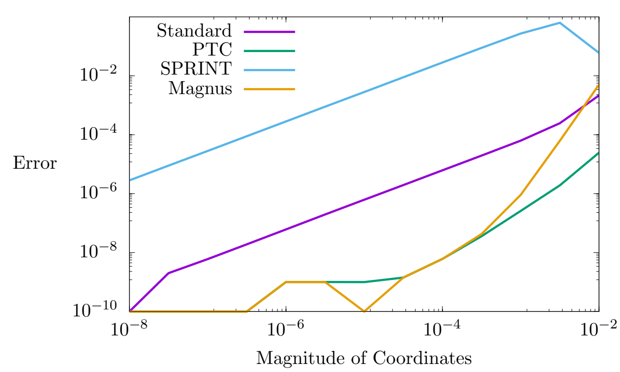

Figure 2: Comparison of spin-tracking methods for a quadrupole with varying coordinate size.

The program Sprint circumvented this problem by normalizing the result of the leading-order Dyson expansion [9, 10].

In the limit of a thin-lens element, the leading-order Magnus expansion is exact. The same cannot be said for the leading-order Dyson expansion, which requires infinitely many terms to reproduce the exact flow even when Eq. (3) is satisfied. Therefore, when the orbit changes only slightly inside a magnet so that $\mathbf{A}$ is nearly constant, the leading-order Magnus expansion will be a very good approximation.

In simulations, it was found that there was something to be gained by including the next term in the Magnus expansion, which leads to

$$\mathbf{R}(z_0, s; s_0) \approx \exp\left(\int_{s_0}^{s} \mathbf{A}(z(\sigma), \sigma)\, d\sigma + \frac{1}{2}\int_{s_0}^{s} d\sigma_1 \int_{s_0}^{\sigma_1} d\sigma_2\, [\mathbf{A}_1, \mathbf{A}_2]\right), \tag{7}$$

where $\mathbf{A}_i \equiv \mathbf{A}(z(\sigma_i), \sigma_i)$. As the matrix commutator is the Lie bracket of $\mathfrak{so}(3)$, this approximation enjoys the same advantages as the leading-order Magnus expansion [11]. The same is true of all additional corrections to the exponent, which involve iterated integrals of nested commutators [7]. The Magnus expansion can be understood as an order-by-order expansion of the exponent in the following manner: if $\mathbf{A}$ depends linearly on some parameter ε, then the exponent of Eq. (6) is linear in ε. Similarly, the exponent of Eq. (7) is quadratic in ε and each additional correction to the exponent adds one order in ε, owing to the presence of another commutator. In the linear approximation, the ordering parameter would be z_0.

APPLICATION TO SPIN TRACKING

The Magnus expansion was integrated into Bmad using the "Bmad standard" approximation for $z(s)$. For most magnets, this amounts to the paraxial approximation, i.e., propagation of the transverse coordinates by a p_z-dependent matrix. For bending magnets and quadrupoles, this leads to a magnetic field which is at most linear in the transverse coordinates, but the spin-precession vector remains nonlinear in the transverse coordinates according to Eq. (1). Hence, the exponent of Eq. (7) was truncated to second order in the transverse coordinates. This leads to

$$\mathbf{R}(z_0, s; s_0) \approx \exp\left(\int_{s_0}^{s} \mathbf{A}^{(2)}(z(\sigma), \sigma)\, d\sigma + \frac{1}{2}\int_{s_0}^{s} d\sigma_1 \int_{s_0}^{\sigma_1} d\sigma_2 \left[\mathbf{A}_1^{(1)}, \mathbf{A}_2^{(1)}\right]\right), \tag{8}$$

where $\mathbf{A}^{(i)}$ is the truncation of $\mathbf{A}$ to order i in the transverse coordinates. Due to the truncation, this is not equivalent to the true Magnus expansion of Eq. (7), but it nevertheless belongs to SO(3) because $\mathbf{A}^{(i)}$ is skew-symmetric for all i. This follows from the fact that $\mathbf{A}^{(i)}\boldsymbol{S} = \boldsymbol{W}^{(i)} \times \boldsymbol{S}$, where $\boldsymbol{W}^{(i)}$ is the truncation of the spin-precession vector.

Coordinates which do not change throughout the magnet were not included in the truncation, as doing so would sacrifice accuracy without simplifying the computation. In every case, all orders of p_z were retained because p_z does not change in magnetic elements. For the same reason, all orders of p_y were retained in the formulae for a pure bending magnet.

It should be noted that a similar method was used by D. P. Barber for tracking spins in SlickTrack [12]. However, that method truncated $z(s)$ and $\mathbf{A}(z(s), s)$ to first order in all coordinates (including p_z) and did not include the commutator in Eq. (7).

COMPARISON

To evaluate the effectiveness of the Magnus approximation, it was compared to other spin tracking methods available in Bmad. For single-element tests, the fourth-order Runge-Kutta method with adaptive step size was used as the baseline. The other methods included were the standard Bmad method, which uses Romberg integration; PTC, which uses Yoshida's sixth-order symplectic integrator; and the aforementioned Sprint method [13, 14]. Figure 1 shows the result for each tracking method using a 2-meter-long sector bend with a reference curvature and normalized field strength of 0.1 m^{-1}. The spin was initially vertical and the coordinates were

$$z_0 = M \times (1\,\text{m}, 2, 3\,\text{m}, 4, 5\,\text{m}, 6), \tag{9}$$

where M is the magnitude of the coordinates. The particle was a proton with a reference energy of 100 GeV. Here, the "error" is defined as the deviation (in the 2-norm) of

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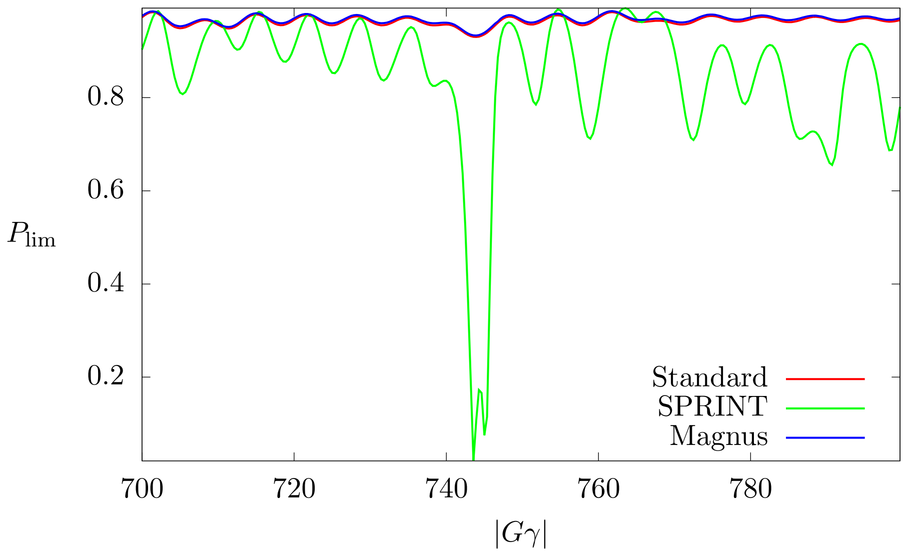

Figure 3: The maximum time-averaged polarization for a $\boldsymbol{J} = (2.5\sigma, 2.5\sigma, 0)$ particle in the Hadron Storage Ring.

the final spin from the Runge-Kutta result. Figure 2 shows the analogous result for a 2-meter-long quadrupole with a normalized field gradient of $0.1\ \mathrm{m}^{-2}$. For a fair comparison, fringe fields were turned off for these simulations, although they could easily be included using the Magnus expansion.

For the sector bend, if PTC and Runge-Kutta are considered to be reliable, it seems that the Magnus method is better than the standard Romberg integration for all coordinates from order 10^{-8} to 10^{-2}. In fact, the Romberg integration performed nearly as poorly as the Sprint (Dyson) approximation. For the quadrupole, the results were similar, although in this case the standard Romberg integration performed significantly better than the Sprint method. Additionally, it seems that the Magnus approximation is overtaken in accuracy by Romberg integration for coordinates on the order of 10^{-2}, although these coordinates are very large and not usually seen in simulations of polarized colliders.

To place the improvement over the Sprint method in the context of relevant simulations, the standard Romberg integration, Sprint method, and Magnus method were used to calculate the maximum time-averaged polarization P_{lim} for polarized helions in the Hadron Storage Ring (HSR) of the Electron-Ion Collider planned for construction at Brookhaven National Laboratory [15]. The particle actions $\boldsymbol{J}$ were 2.5σ in the horizontal and vertical planes, and longitudinal motion was ignored. Figure 3 shows that the Sprint method indicates a large dip in polarization, likely indicating the presence of a higher-order spin-orbit resonance, which does not appear with Romberg integration. On the other hand, the Magnus method agrees closely with Romberg integration. This is an unacceptable difference, as it is critical to understand whether the beam will be able to maintain its polarization during ramping through this energy range. The most likely cause for this difference is that, in addition to the relatively large particle amplitude, there are several misaligned quadrupoles in the HSR where the closed orbit deviates from the center of the magnet by as many as 4 cm, and as shown before, the Sprint method performs very poorly with large coordinates.

Finally, it is important to note that tracking polarized helions through the HSR using the Magnus method was more than twice as fast as with the standard Romberg integration. Therefore, the Magnus method can be used in necessary simulations to gain significant speed without sacrificing accuracy. The supported elements are solenoids, bending magnets (even with quadrupole fields), quadrupoles, sextupoles, and extension to other any element element with known (perhaps approximate) $\boldsymbol{z}(s)$, $\boldsymbol{B}(z, s)$, and $\boldsymbol{E}(z, s)$ is possible.

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REFERENCES

[1] J. D. Jackson, *Classical electrodynamics*. Wiley, 1999.

[2] L. H. Thomas, "The kinematics of an electron with an axis", *Phil. Mag. Ser. 7*, vol. 3, no. 13, pp. 1–22, 1927. doi:10.1080/14786440108564170

[3] V. Bargmann, L. Michel, and V. L. Telegdi, "Precession of the polarization of particles moving in a homogeneous electromagnetic field", *Phys. Rev. Lett.*, vol. 2, pp. 435–436, May 1959. doi:10.1103/PhysRevLett.2.435

[4] D. Sagan, "Bmad: A relativistic charged particle simulation library", *Nucl. Instrum. Methods Phys. Res. A*, vol. 558, no. 1, pp. 356–359, 2006. doi:10.1016/j.nima.2005.11.001

[5] F. J. Dyson, "The radiation theories of Tomonaga, Schwinger, and Feynman", *Phys. Rev.*, vol. 75, pp. 486–502, 1949. doi:10.1103/PhysRev.75.486

[6] W. Magnus, "On the exponential solution of differential equations for a linear operator", *Commun. Pure Appl. Math.*, vol. 7,

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no. 4, pp. 649–673, 1954.
doi:10.1002/cpa.3160070404

[7] S. Blanes, F. Casas, J. A. Oteo, and J. Ros, "The Magnus expansion and some of its applications", *Phys. Rep.*, vol. 470, no. 5, pp. 151–238, 2009. doi:10.1016/j.physrep.2008.11.001

[8] G. H. Hoffstaetter, *High-energy polarized proton beams, A modern view*. NY: Springer New York, 2006.

[9] M. Vogt, "Bounds on the maximum attainable equilibrium spin polarization of protons at high energy in HERA", Ph.D. thesis, Universität Hamburg, 2000.

[10] C. Weißbäcker, "Nichtlineare effekte der spindynamik in protonenbeschleunigern", MA thesis, Technische Universität Darmstadt, 1998.

[11] B. C. Hall, *Lie groups, Lie algebras, and representations, An elementary introduction*. Springer, 2015.

[12] D. P. Barber, "Spin resonance strengths, spin-orbit tracking and invariant spin fields with the SLIM formalism in storage rings with oscillating external magnetic fields", unpublished.

[13] E. Forest, F. Schmidt, and E. McIntosh, "Introduction to the polymorphic tracking code. Fibre bundles, polymorphic Taylor types and 'exact tracking'", KEK, Rep. 2002-3, 2002.

[14] H. Yoshida, "Construction of higher order symplectic integrators", *Phys. Lett. A*, vol. 150, no. 5, pp. 262–268, 1990. doi:10.1016/0375-9601(90)90092-3

[15] C. Montag et al., "The EIC accelerator: design highlights and project status", in *Proc. IPAC'24*, Nashville, TN, May 2024, pp. 214–217. doi:10.18429/JACoW-IPAC2024-MOPC67

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