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THIRD INTEGER RESONANT EXTRACTION TRANSIT TIME SIMULATION STUDIES

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Abstract

In this work, we present the investigation of transit time of particles in the non-linear third-integer resonant extraction process. Transit time is defined as the number of turns a particle takes to get extracted once it is in the unstable region in the phase space, i.e., outside the triangular separatrix in case of third-integer resonance. The study of transit time is important because transit time directly contributes to the beam response time during resonant extraction and thus knowing it apriori would be practically useful in designing of the extraction system. In this work, we shall investigate the analytical derivation of the transit time of particles (to the first order Kobayashi Hamiltonian) in different parts of the phase space distribution and compare against the analytical results. We also compare the simulation result of the transit time of particles (with higher statistics) for the static as well as dynamic extraction conditions cases, particularly in the context of resonant extraction parameters for Mu2e experiment at Fermilab.

HAMILTONIAN FORMALISM

The equation of a particle after every three turns in the presence of sextupole, and assuming that the horizontal tune goes very close to third-integer tune (29/3 in the case of Mu2e resonant extraction), turns out to be,

$$\Delta X_3 = \quad 6\pi\delta Q X'_0 + \frac{3}{2} S X_0 X'_0 \tag{1}$$

$$\Delta X'_3 = -6\pi\delta Q X'_0 + \frac{3}{4} S (X_0^2 - X_0'^2) \tag{2}$$

where X_3 and X'_3 are normalized phase space coordinates, δQ is the tune distance to a third-integer tune, and (X_0, X'_0) are the initial normalized phase space coordinates of the particle. Since the spill duration (43 ms) is much larger than the one-turn revolution time ($\approx 1.69\ \mu$s) of the particles, it is reasonable to consider three-turns as the basic unit of time instead of one-turn.

We can derive the Hamiltonian of this system (also known as Kobayashi hamiltonian [1]) by integrating the above equations of motion,

$$H = \underbrace{3\pi\delta Q\left(X^2 + X'^2\right)}_{\text{First term}} + \underbrace{\frac{S}{4}\left(3XX'^2 - X^3\right)}_{\text{Second term}} \tag{3}$$

The first term of the Hamiltonian represents the linear motion in the accelerator, and the second term, caused by the presence of the sextupole element, introduces non-linearity in the particle dynamics. It is worthy to note that this Hamiltonian is true to the first order and ignores higher order terms in δQ and S.

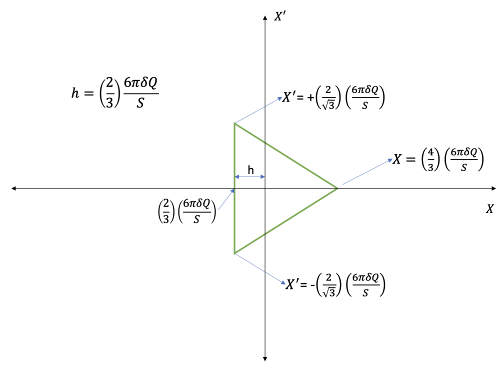

Figure 1: Triangular separatrix due to 3rd integer resonant excitation.

This non-linearity introduces a triangular stable region in the phase space (as shown in Fig. 1). Particles inside the stable region execute stable motion and are bound inside it, never venturing out. But the particles outside the stable region start to increase their amplitude with every turn, with their horizontal position eventually surpassing and jumping past the septum position. Once the particle is past the septum, it experiences an instantaneous horizontal kick to be sent to the extraction beam line and to the desired final location.

TRANSIT TIME EXPRESSION - STATIC CASE

One strategy [2] to derive the transit time of particles is to solve for the evolution of the particle's position as and when it reaches the electrostatic septum in the horizontal plane and solve back for the time step for the particle to reach the septum location. It is algebraically convenient to shift the coordinate system of the normalized phase space such that one of the vertices of the triangluar separatrix is in the origin. The translated Hamiltonian turns out to be $H_{\text{tran.}} = \frac{S}{4}(3hX^2 + 3h^3 + 3XX'^2 + 6\sqrt{3}XX'h - X^3 + 3X^2h + h^3)$.

We are now interested in the particle dynamics with the newly translated coordinates. We obtain them from the usual Hamilton's equations of motion,

$$\frac{\mathrm{d}X}{\mathrm{d}n} = \frac{\partial H}{\partial X'} = \frac{6SX}{4}\left(\sqrt{3}h + X'\right) \tag{4}$$

$$\tag{5}$$

Similarly, for the evolution of X', we have,

$$\frac{\mathrm{d}X'}{\mathrm{d}n} = -\frac{\partial H}{\partial X} = -\frac{3S}{4}\left(4hX + 2\sqrt{3}X'h + X'^2 - X^2\right) \tag{6}$$

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Because the linear and sextupole fields here are magnetic in nature, and since magnetic forces are conservative, the Hamiltonian here is a constant of motion. This implies that the Hamiltonian value for $H(X_0, X_0'; n)$ with some initial coordinates (X_0, X_0') at turn n must numerically be the same as the one at a later turn $H(X, X'; n + \Delta n)$.

Using the constancy of the Hamiltonian and eliminating X', we get the evolution equation of X as,

$$\frac{\mathrm{d}X}{\mathrm{d}n} = \frac{6SX}{4}\left(\sqrt{3}h + X'\right) \tag{7}$$

$$= \frac{S}{4}\left(6\sqrt{3}hX + \frac{6}{\sqrt{3}}X_0^2 + 6X_0X_0' - \frac{6}{\sqrt{3}}X^2\right) \tag{8}$$

$$= f(X) \qquad \text{(say)} \tag{9}$$

TRANSIT TIME EXPRESSION

In order to get the expression for transit time, we can simply invert equation (8) and find the number of turns taken by the particle to reach a certain X value.

We define transit time as the time taken by the particle to reach the horizontal position of where the septum is, X_{septum}. We can thus integrate equation (8) to get the transit time value,

$$T_{tt} = \int \mathrm{d}n = \int_{X_0}^{X_{\text{septum}}} \frac{1}{f(X)}\,\mathrm{d}X \tag{10}$$

$$= \frac{2}{\sqrt{3}S}\int_{X_0}^{X_{\text{septum}}} \frac{1}{-X^2 + 3hX + X_0^2 + \sqrt{3}X_0X_0'}\,\mathrm{d}X \tag{11}$$

Solving this integral, we get the transit time for a particle to be,

$$T_{tt} = \frac{2}{\sqrt{3}S}\frac{1}{\sqrt{9h^2 + 4(X_0^2 + \sqrt{3}X_0X_0')}} \times \left[\ln\left(\frac{NR}{DR}\right)\right]_{X_0}^{X_{\text{sept}}} \tag{12}$$

$$NR = -2X + 3h - \sqrt{9h^2 + 4(X_0^2 + \sqrt{3}X_0X_0')} \tag{13}$$

$$DR = -2X + 3h + \sqrt{9h^2 + 4(X_0^2 + \sqrt{3}X_0X_0')} \tag{14}$$

where we remind the reader that (X_0, X_0') is the initial coordinate of the particle, h is $(2/3)(6\pi\delta Q/S)$, and δQ is the horizontal tune distance of the beam from the resonant tune, and S is the sextupole strength.

STATIC TRANSIT TIME EXPRESSION

For Mu2e, the horizontal tune would be ramped from 9.650 to 9.666 in a particular ramp profile so as to result in uniform extraction rate. This would give us the profile of δQ at every time step during the extraction [3]. Plugging in the numbers, we get the analytically calculated transit time as shown in Fig. 2.

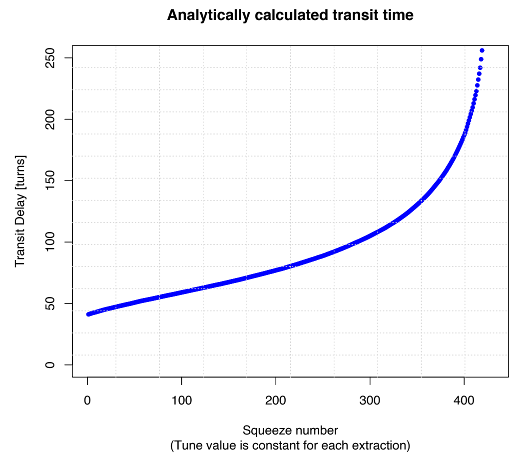

Figure 2: Analytical calculation of static transit time.

DYNAMIC TRANSIT TIME EXPRESSION

In the previous section, we reviewed the transit time expression for the static case, i.e., the condition of resonant extraction such as δQ and the sextupole strength S remains constant through the particle's transit time. However, in real life, these parameters are bound to change as the particle is traversing towards the septum position because the extraction condition is dynamically changing to ensure uniform extraction. These changing conditions imply that the size of the separatrix keeps on shrinking as the particle is transiting towards the septa, i.e., the distance of the centroid of the triangular separatrix from the closest base (denoted by h in Fig. 1) changes proportional to the change in δQ.

Since the separatrix would be moving *away* from a particle in the unstable region (the particle, too, would be moving away from the separatrix because of its ever increasing amplitude), we *add* the separatrix velocity to dX/dn (assuming this is the dominant effect),

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{\sqrt{3}S}{2}(3hX - X^2) + \frac{4\pi}{S}\frac{\mathrm{d}Q}{\mathrm{d}t} \tag{15}$$

Now we integrate the above equation (15) to get the transit time of particle in a dynamic condition:

$$T_{\text{dyn. TT}} = \int \mathrm{d}t = \int_{-X_0}^{-X_{\text{septum}}} \frac{1}{\frac{\sqrt{3}S}{2}(3hX - X^2) + \frac{4\pi}{s}\frac{\mathrm{d}Q}{\mathrm{d}t}}\,\mathrm{d}x \tag{16}$$

We note here that dQ/dt is independent of X.

Integrating the above result, we have the transit time function for the extraction in which conditions change dynamically,

$$T_{\text{dyn. TT}} = \frac{1}{3\sqrt{3}\pi\delta Q}\frac{1}{\sqrt{9h^2 + 4(X_0^2 + \sqrt{3}X_0X_0')}}\log\left|\frac{\mathrm{NR}_2}{\mathrm{DR}_2}\right| \tag{17}$$

$$\mathrm{NR}_2 = \left(\frac{2}{\sqrt{3}}\frac{X_{\text{sept}}}{h} - \frac{2}{3\delta Q}\frac{\mathrm{d}Q}{\mathrm{d}n}\right)\left(\frac{2}{\sqrt{3}}\frac{X_0}{h} + 2\sqrt{3} - \frac{1}{9\pi\delta Q^2}\frac{\mathrm{d}Q}{\mathrm{d}n}\right) \tag{18}$$

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$$\mathrm{DR}_2 = \left(\frac{2}{\sqrt{3}}\frac{X_0}{h} - \frac{2}{3\delta Q}\frac{\mathrm{d}Q}{\mathrm{d}n}\right)\left(\frac{2}{\sqrt{3}}\frac{X_{\mathrm{sept}}}{h} + 2\sqrt{3} - \frac{1}{9\pi\delta Q^2}\frac{\mathrm{d}Q}{\mathrm{d}n}\right) \tag{19}$$

Since we know the ideal tune ramp, we calculate dQ/dt by simply taking the difference in the tune values, which would give us the rate of change of tune. (It is to be noted that to get the real number of turns of transit time, a factor of 3 must be multiplied as the time unit of the equation of motion here is 3 turns.)

PARTICLE TRACKING

In order to check and validate the above derived expression for transit time, extensive numerical particle tracking was done with a huge number of particle sample size of about 4 million particles. To prepare the initial distribution of particles, we generate a Gaussian distribution of particles in both X as well as X', and the standard deviation of the distribution ($\sigma_X = \sqrt{\beta_0\epsilon_0}$) was calculated from the design emittance for Mu2e resonant extraction scheme [4], with a beta function value of 12 m and a normalized 95% emittance value ϵ_0 of 16π mm-mrad. After we generate the initial distribution, we run the tracking simulation at a constant horizontal tune of $\nu_x = 9.650$ in order to extract the tail of the beam so that the transit time study is not affected by the surge of halo that would get extracted in the first few hundred turns.

Particle tracking was done with the Mu2e resonant extraction scheme (approximated with a single virtual sextupole of equivalent strength of 500 T/m^2) with the septum located at 12 mm from the beam center. In order to an accurate estimate of the transit time, the tune 'squeeze' is done and the tracking is run for 1000 turns until all the particle *just outside* the separatrix is extracted. The first particle to arrive would be the particle closest to the vertex nearest to the septum.

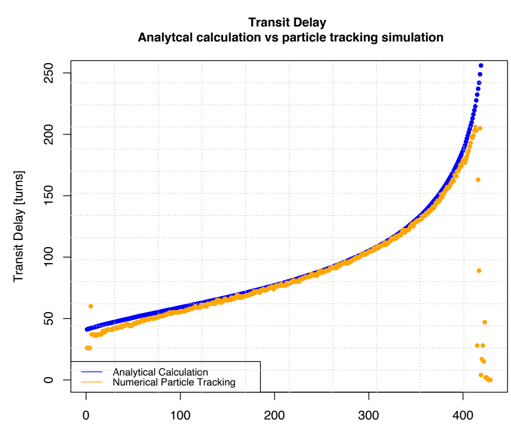

Figure 3: Comparison between analytically calculated transit time values vs. simulation (static case).

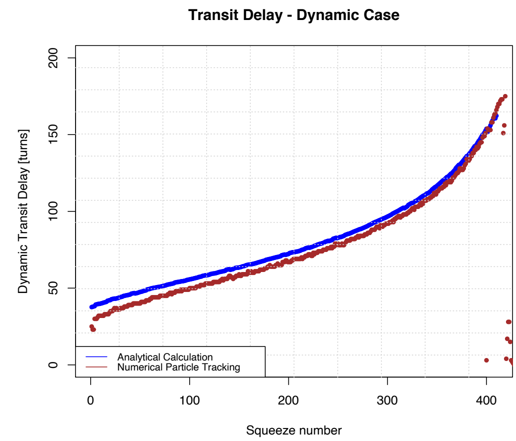

Figure 4: Comparison between analytically calculated dynamic transit time vs. particle tracking results.

CONCLUSION

We see in Figures 3 and 4 that there is a good agreement between the analytically predicted value and the simulation value. The slight discrepancy could be attributed to the fact that Kobayashi Hamiltonian is true only to first order effects but particle tracking contains *all* orders. We thus have validated the analytically derived transit time functions for both the static as well as the dynamic case, with a mean deviation of about 4.4% for the static case and about 7.3% for the dynamic case for 4 million particles.

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