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AN EXTENDED FROISSART-STORA FORMULA FOR CHANGING CROSSING SPEED

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Abstract

When the closed-orbit spin tune is ramped linearly through an isolated spin-orbit resonance, the asymptotic polarization loss is well-approximated by the Froissart-Stora formula. However, it is often observed in accelerator simulations that the crossing speed, defined as the slope of the amplitude-dependent spin tune with respect to the machine azimuth, changes at the moment of resonance crossing. For example, the behavior of the amplitude-dependent spin tune in the vicinity of a higher-order spin-orbit resonance can often be reasonably approximated by such a piecewise-linear function. In this paper, we derive an extension to the Froissart-Stora formula which describes the asymptotic polarization loss in the case of changing crossing speed. We then demonstrate that this formula provides a good estimate of the polarization lost when crossing a higher-order spin-orbit resonance in both a toy model and simulations of RHIC.

INTRODUCTION

In polarized hadron colliders, such as the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory, the primary obstacles in the way of high polarization are spin-orbit resonances [1]. At low energy, the dominant resonances are first-order spin-orbit resonances, for which it is well-known that the degree of depolarization encountered upon resonance crossing can be approximated using the Froissart-Stora formula [2]. At high energy, polarization can only be maintained if Siberian snakes are introduced to eliminate all first-order resonances, leaving higher-order resonances as the dominant effect. It is not well-known that the Froissart-Stora formula can be applied to some higher-order resonances as well [3]. In this paper, the Froissart-Stora formula is extended to apply to a larger class of higher-order resonances.

THE SINGLE-RESONANCE MODEL

First-order spin-orbit resonances are usually studied in the context of a toy model known as the single-resonance model (SRM) [4]. In this model, only one Fourier term is retained in the spin-precession vector, leading to the equation of motion:

$$\begin{aligned} \boldsymbol{S}'(\theta) &= [\boldsymbol{\Omega}_0(\theta) + \boldsymbol{\omega}(z(\theta),\theta)] \times \boldsymbol{S}(\theta), \\ \boldsymbol{\Omega}_0(\theta) &= \nu_0 \boldsymbol{n}_0(\theta), \\ \boldsymbol{\omega}(z(\theta),\theta) &= \epsilon[\cos(\kappa\theta+\phi)\boldsymbol{m}(\theta) + \sin(\kappa\theta+\phi)\boldsymbol{l}(\theta)]. \end{aligned} \tag{1}$$

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Here, ν_0 is the closed-orbit spin tune, $(\boldsymbol{m}, \boldsymbol{l}, \boldsymbol{n}_0)$ is a periodic coordinate system in which spins on the closed orbit rotate uniformly around $\boldsymbol{n}_0$ with frequency ν_0, ϵ is the resonance strength, ϕ is the resonance phase, and $\kappa = k_0 + \boldsymbol{k}\cdot\boldsymbol{Q}$, where $(k_0, \boldsymbol{k}) \in \mathbb{Z}^4$ and $\boldsymbol{Q}$ contains the orbital tunes.

This model's name derives from the fact that it contains a single spin-orbit resonance located at $\nu_0 = \kappa$. Froissart and Stora found that, in the case of a linear resonance crossing, i.e., $\nu_0(\theta) = \kappa+\alpha\theta$, the asymptotic polarization loss could be derived analytically [2]. With the initial condition $s_3(-\theta_0) = 1$ ($s_3 \equiv \boldsymbol{S}\cdot\boldsymbol{n}_0$), their result (the Froissart-Stora formula) is:

$$\lim_{\theta_0\to\infty} s_3(\theta_0) = 2\exp\left(-\frac{\pi\epsilon^2}{2\alpha}\right) - 1. \tag{2}$$

The Froissart-Stora formula can be generalized to higher-order resonances with the substitutions $s_3 \to J_S \equiv \boldsymbol{S}\cdot\boldsymbol{n}$ (the spin action) and $\alpha \to \mathrm{d}\nu/\mathrm{d}\theta$, where $\boldsymbol{n}$ is the invariant spin field (ISF) and ν is the amplitude-dependent spin tune (ADST) [3, 5, 6]. The problem with this idea is that the variation of ν with the reference energy can be very complicated so that ν does not vary linearly with θ even when the reference energy is ramped linearly.

In simulations of real accelerators, it is often found that ν is almost piecewise linear as a function of θ, with the slope changing at the moment of resonance crossing. Hence, one way to generalize the Froissart-Stora formula is to instead use:

$$\nu(\theta) = \begin{cases} \kappa + \alpha_1\theta, & \theta < 0 \\ \kappa + \alpha_2\theta, & \theta \geq 0. \end{cases} \tag{3}$$

After a calculation similar to that of Froissart and Stora with the initial condition $J_S(-\theta_0) = 1$, this leads to:

$$\begin{aligned} \lim_{\theta_0\to\infty} J_S(\theta_0) = {} & 2\exp\left[-\frac{\pi\epsilon^2}{8}\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right)\right] \\ & \times \left| \frac{\Gamma\left(\frac{1}{2}+\frac{i\epsilon^2}{8\alpha_2}\right)}{\Gamma\left(\frac{1}{2}+\frac{i\epsilon^2}{8\alpha_1}\right)} \cosh\left(\frac{\pi\epsilon^2}{8\alpha_2}\right) \right. \\ & \left. - \sqrt{\frac{\alpha_2}{\alpha_1}} \frac{\Gamma\left(1+\frac{i\epsilon^2}{8\alpha_2}\right)}{\Gamma\left(1+\frac{i\epsilon^2}{8\alpha_1}\right)} \sinh\left(\frac{\pi\epsilon^2}{8\alpha_2}\right) \right|^2 - 1. \end{aligned} \tag{4}$$

The derivation is to be published elsewhere.

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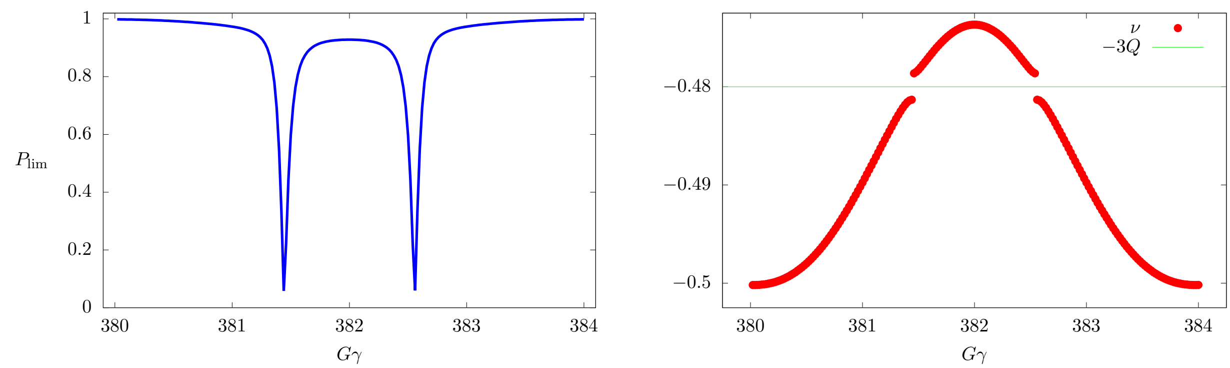

Figure 1: A pair of third-order resonances in the DRM with two Siberian snakes. Left: The maximum time-averaged polarization, computed by stroboscopic averaging [7]. Right: The ADST, computed by averaging the rotation angle around an ISF from stroboscopic averaging [6].

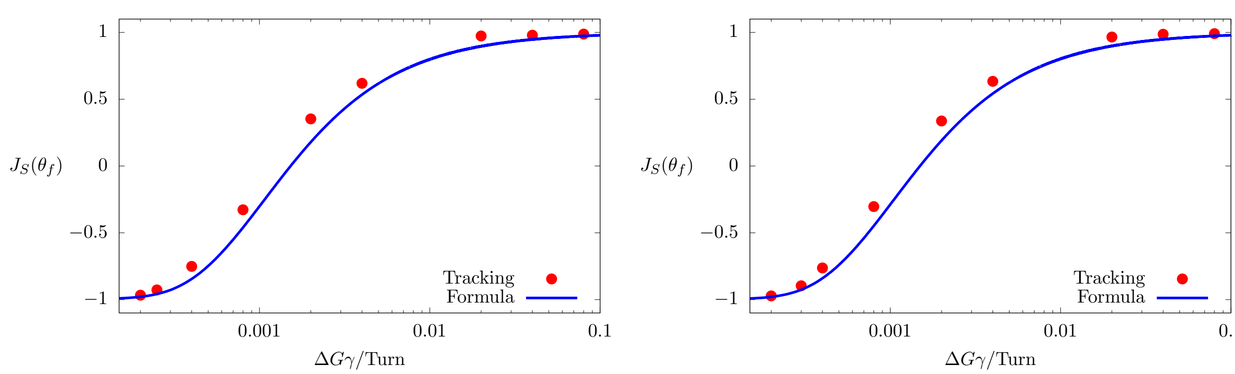

Figure 2: Comparison of the final spin action $J_S(\theta_f)$ from tracking through a higher-order resonance in the DRM with two Siberian snakes and the asymptotic spin action predicted by Eq. (4). Left: Spin action after crossing the left resonance in Fig. 1. Right: Spin action after crossing the right resonance in Fig. 1.

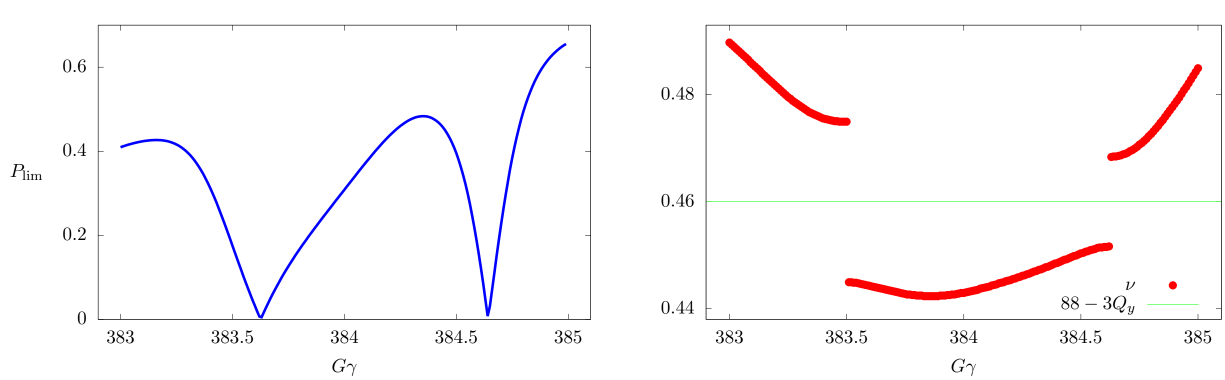

Figure 3: A pair of third-order resonances on the ellipse with normalized action $J_y \approx 32$ mm mrad between $G\gamma = 383$ and $G\gamma = 385$ in RHIC. Left: The maximum time-averaged polarization, computed by stroboscopic averaging [7]. Right: The ADST, computed by Fourier analysis of the turn-by-turn spin vector [8].

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THE DOUBLE-RESONANCE MODEL

A toy model which can be used to investigate higher-order resonances in accelerators with two Siberian snakes is the double-resonance model (DRM) [9]. An example of a pair of higher-order resonances in the DRM is shown in Fig. 1. While the slope of the ADST does not change drastically across the resonances, it does not remain the same. We will begin by tracking through the resonance on the left. To most nearly match the assumptions leading to Eq. (4), we start tracking at $G\gamma(\theta_i) = 378$ with $J_S(\theta_i) = 1$. The resonances have very little influence at θ_i and the ISF is almost exactly vertical. Hence, we simply use $\boldsymbol{S}(\theta_i) = \boldsymbol{e}_y$. We stop tracking at $G\gamma(\theta_f) = 382$, and we use Eq. (4) to approximate $J_S(\theta_f)$ with $\boldsymbol{n}(z(\theta_f), \theta_f)$ obtained by stroboscopic averaging [7,10]. The higher-order resonance strength was obtained by halving the spin-tune jump as in [3], and the slope parameter α_1 was obtained using the formula:

$$\alpha_1 = \frac{1}{2\pi} \frac{\mathrm{d}\nu}{\mathrm{d}(G\gamma)}\bigg|_{\text{pre-jump}} \left(\frac{\Delta G\gamma}{\text{Turn}}\right), \tag{5}$$

where $\Delta G\gamma/\text{Turn}$ is the change in $G\gamma$ per turn. To compute α_2, the pre-jump slope was replaced with the post-jump slope. The result is shown in Fig. 2. The qualitative agreement with Eq. (4) is excellent, although the tracking and the formula do not agree exactly. This disagreement is not surprising, as the assumptions leading to Eq. (4) cannot be exactly met. Firstly, there are two resonances influencing the spin motion, although our derivation assumed that there was only one. Secondly, we tracked through a finite range surrounding the resonance, and Eq. (4) is only fully correct in the limit of tracking from infinitely far before the resonance to infinitely far after the resonance.

We will now track through the resonance on the right. We start tracking at $G\gamma(\theta_i) = 382$ with $J_S(\theta_i) = 1$ and $\boldsymbol{n}(\theta_i)$ computed by stroboscopic averaging. We stop tracking at some θ_f where the influence of the resonance is almost completely absent so that $J_S(\theta_f) = S_y(\theta_f)$ to a very good approximation. We again use Eq. (4) to approximate $J_S(\theta_f)$. The result is is nearly identical to the result of tracking through the other resonance.

THE RELATIVISTIC HEAVY ION COLLIDER

In simulations of RHIC, there are higher-order resonances in which the slope of the ADST changes drastically during resonance crossing. A pair of such resonances is shown in Fig. 3. We will track through the resonance on the right in Fig. 3. We start tracking at $G\gamma(\theta_i) = 384.36$ with $J_S(\theta_i) = 1$ and $\boldsymbol{n}(z(\theta_i), \theta_i)$ computed by stroboscopic averaging. We stop tracking at $G\gamma(\theta_f) = 385$ and compute $J_S(\theta_f)$ by stroboscopic averaging. We again use Eq. (4) to approximate $J_S(\theta_f)$. The result is shown in Fig. 4. The disagreement is not very surprising, as the tracking begins at an energy with $P_{\text{lim}} \approx 45\%$ and ends at an energy with $P_{\text{lim}} \approx 65\%$. Thus, we are *not* tracking from far before an isolated resonance to far after an isolated resonance. However, Eq. (4) nevertheless provides a reasonable estimate of $J_S(\theta_f)$.

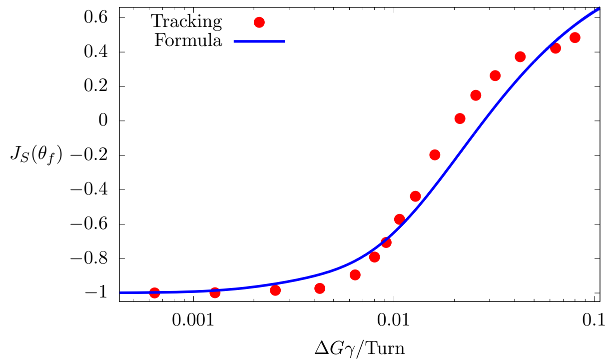

Figure 4: Comparison of the final spin action $J_S(\theta_f)$ from tracking through the right resonance in Fig. 3 and the asymptotic spin action predicted by Eq. (4).

CONCLUSION

Although it has regrettably not been widely recognized, it was shown more than twenty years ago that much of the polarization loss in high-energy colliders can be explained by higher-order spin-orbit resonance crossings, and the Froissart-Stora formula can be used together with the invariant spin field and the amplitude-dependent spin tune to predict the corresponding depolarization [3]. The usual Froissart-Stora formula suffers from limited applicability in this case because the slope of the amplitude-dependent spin tune often changes at the moment of resonance crossing. This work presented an extension to the Froissart-Stora formula which allows for such a change in slope and demonstrated its applicability.

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REFERENCES

[1] H. Huang *et al.*, "RHIC polarized proton operation", in *Proc. PAC'11*, New York, NY, USA, Mar.-Apr. 2011, paper MOOCN3, pp. 41–45. `https://www.jacow.org/proceedings/PAC2011/papers/moocn3.pdf`

[2] M. Froissart and R. Stora, "Depolarisation d'un faisceau de protons polarises dans un synchrotron", *Nucl. Instrum. Methods*, vol. 7, no. 3, pp. 297–305, 1960. `doi:10.1016/0029-554X(60)90033-1`

[3] G. H. Hoffstaetter and M. Vogt, "Strength of higher-order spin-orbit resonances", *Phys. Rev. E*, vol. 70, p. 056 501, 2004. `doi:10.1103/PhysRevE.70.056501`

[4] E. D. Courant and R. D. Ruth, "Acceleration of polarized protons in circular accelerators", BNL, Tech. Rep. 51270, 1980. `doi:10.2172/7034691`

ISBN: 978-3-95450-261-5 ISSN: 2673-7000 doi: 10.18429/JACoW-NAPAC2025-TUP009

[5] G. H. Hoffstaetter, *High-Energy Polarized Proton Beams, A Modern View*. Springer, 2006.

[6] M. Vogt, "Bounds on the maximum attainable equilibrium spin polarization of protons at high energy in HERA", Ph.D. dissertation, Universität Hamburg, 2000.

[7] K. Heinemann and G. H. Hoffstätter, "Tracking algorithm for the stable spin polarization field in storage rings using stroboscopic averaging", *Phys. Rev. E*, vol. 54, pp. 4240–4255, 1996. doi:10.1103/PhysRevE.54.4240

[8] D. P. Barber, J. A. Ellison, and K. Heinemann, "Quasiperiodic spin-orbit motion and spin tunes in storage rings", *Phys. Rev. Spec. Top. Accel. Beams*, vol. 7, p. 124 002, 2004. doi:10.1103/PhysRevSTAB.7.124002

[9] E. Hamwi, J. P. Devlin, G. H. Hoffstaetter, and D. P. Barber, "Modeling interference of two first-order resonances with two Siberian snakes using machine learning", in *Proc. IPAC'24*, Nashville, TN, May 2024, pp. 682–685. doi:10.18429/JACoW-IPAC2024-MOPS02

[10] D. Sagan, "A Superconvergent Algorithm for Invariant Spin Field Stroboscopic Calculations", in *Proc. IPAC'18*, Vancouver, Canada, Apr.-May 2018, pp. 145–148. doi:10.18429/JACoW-IPAC2018-MOPMF028