

Chromaticity Effects on Head-Tail Instabilities For Broad-band Impedance Using Two Particle Model, Vlasov Analysis and Simulations*

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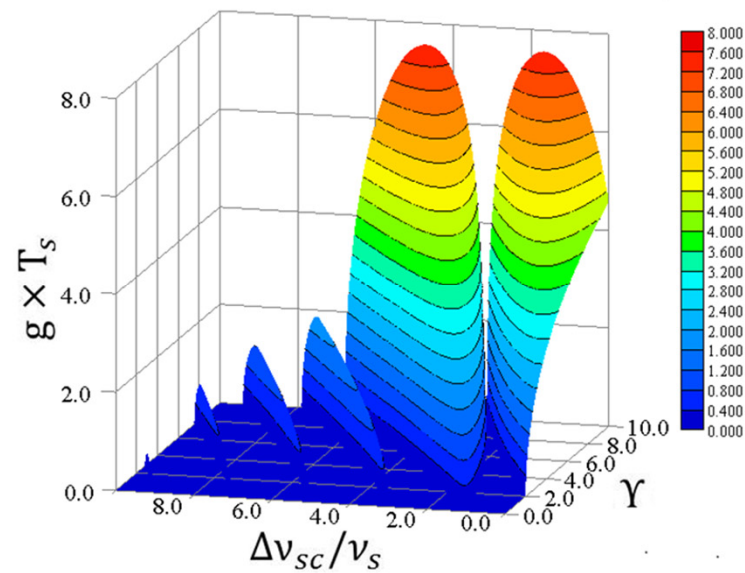
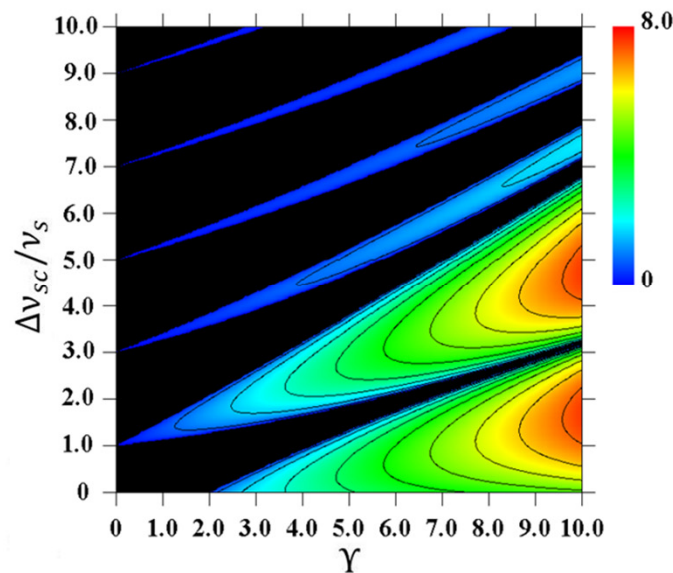
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Outline

- Motivations
- Two Particle Model with Chromaticity
- Vlasov Analysis with the Trapezoidal Model for the Constant Wake
- Vlasov Analysis with the LEP Impedance Model
- Simulations with TRANFT multi-particle code
- Findings and Conclusions

Motivations

- In 2016, we have published the paper titled “Two Particle Model for Studying the Effects of Space-Charge Force on Strong Head-Tail Instabilities”.



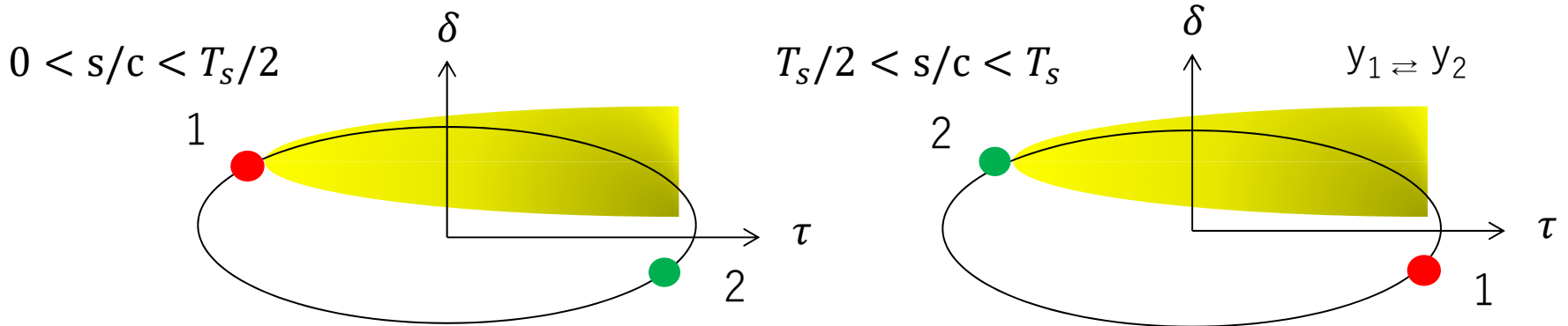
- The two particle model is a very effective tool to get a simple picture of essence of complicated phenomena.

Two Particle Model with Chromaticity

- The next step may be to include the chromaticity in this model.
- However, no exact solution of the two particle model with chromaticity (without space-charge) is known.
 - The chromaticity introduces the energy-error dependence to the betatron frequencies.
 - This small additional term, however, makes it very difficult to find exact solutions even for the free betatron oscillation.
 - Only the perturbation solution for a small chromaticity and weak wake field strength is presented in Chao's book.
- Thus, our next step is to find an exact solution of the two particle model with (any value of) chromaticity (without space-charge) to study effects of chromaticity on head-tail instabilities.

Two Particle Model with Chromaticity

- The premise of two particle model with chromaticity
 - Two macro-particles executing synchrotron and betatron oscillations.
 - Their synchrotron oscillations have equal amplitude, but opposite phases.



$$y_1'' + \left[\frac{\omega_\beta(\delta_1)}{c} \right]^2 y_1 = 0$$

$$y_2'' + \left[\frac{\omega_\beta(\delta_2)}{c} \right]^2 y_2 = \frac{Nr_0 W_0}{2\gamma C} y_1$$

$$\omega_\beta(\delta_{1,2}) = \omega_\beta \left(1 \mp \frac{\xi \omega_s \hat{z}}{c\eta} \cos \frac{\omega_s s}{c} \right)$$

Solution of Free Betatron Oscillation

- The first and key challenge is to find an exact solution of the free betatron oscillation:

- $$y_1'' + \left[\frac{\omega_\beta(\delta_1)}{c} \right]^2 y_1 = 0$$

- If you recognize that **this is a Hill's equation** with a periodic function $K(s)$ with the synchrotron oscillation period,

- $$y'' + K(s)y = 0$$

we can solve it by using “Twiss parameters” β and α :

- $$y_1(s) = a_1 \sqrt{\beta_1(s)} \cos \int_0^s \frac{ds}{\beta_1(s)}$$

Transfer Matrix

- The rest is just to follow Alex's book. We have the transfer matrix for the first half of synchrotron oscillation period as

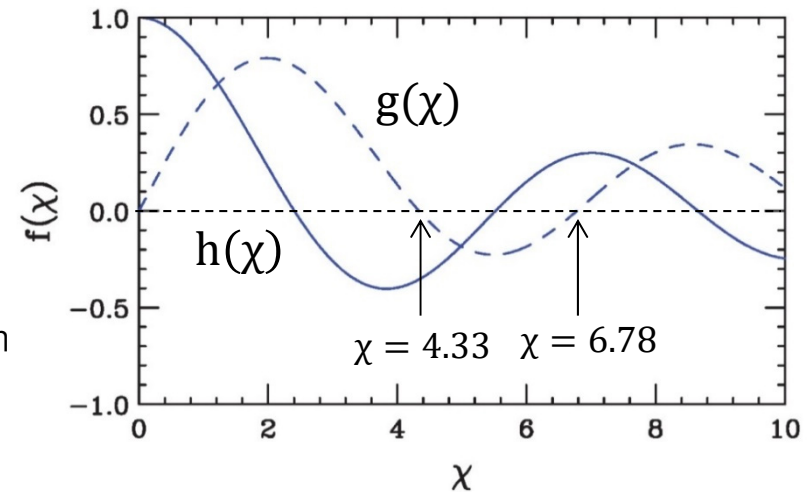
- $$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}_{s=\pi c/\omega_s} = \begin{bmatrix} 1 & 0 \\ i\Gamma & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}_{s=0}$$

- $\Gamma = \Upsilon \cdot f(\chi)$

- $\Upsilon = \frac{\pi N r_0 W_0 c^2}{4\gamma C \omega_\beta \omega_s}$ (dimensionless wake strength)

- $f(\chi) = h(\chi) + i g(\chi)$

- $\chi = 2 \frac{\xi \omega_\beta \hat{z}}{c\eta}$: the difference of the betatron phase advances between the head and the tail



Eigenvalues

- For the second half of synchrotron oscillation period,

- $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}_{s=2\pi c/\omega_s} = \begin{bmatrix} 1 & i\Gamma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}_{s=\pi c/\omega_s}$

- The total matrix is

- $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}_{s=2\pi c/\omega_s} = \begin{bmatrix} 1 & i\Gamma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ i\Gamma & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}_{s=0} = \begin{bmatrix} 1 - \Gamma^2 & i\Gamma \\ i\Gamma & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}_{s=0}$

- Look for eigenvalues

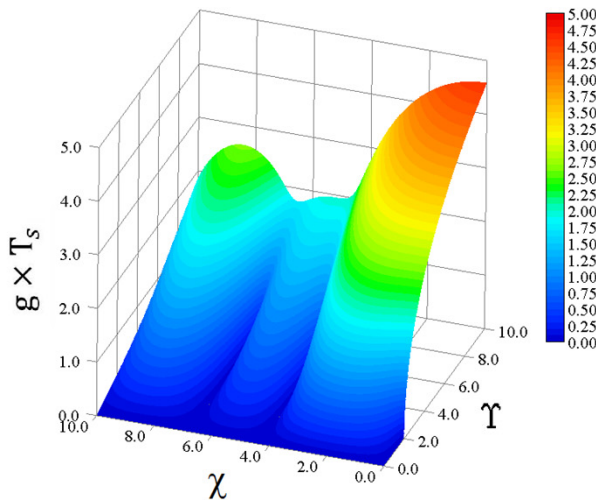
- $\begin{bmatrix} 1 - \Gamma^2 & i\Gamma \\ i\Gamma & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

- Eigenvalues

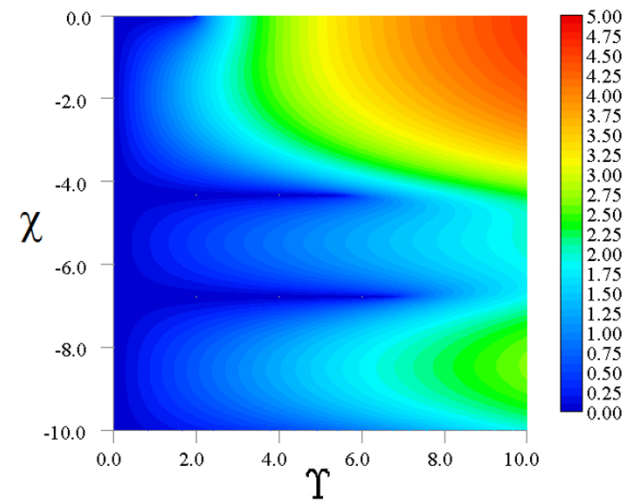
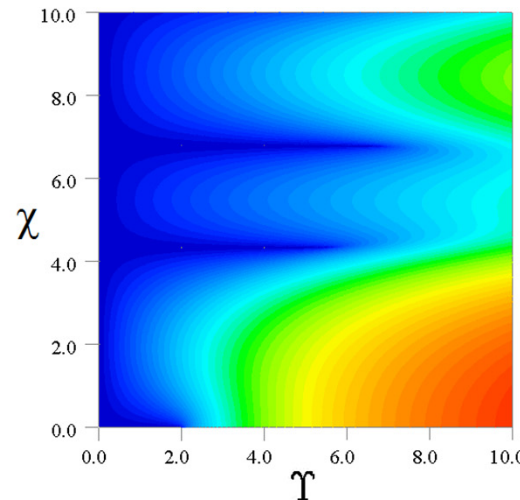
- $\lambda = 1 - \frac{\Gamma^2}{2} \pm \sqrt{\frac{\Gamma^2}{2} \cdot \left(\frac{\Gamma^2}{2} - 2\right)}.$

Universal Contour Plots for Growth Factor

- The growth factor over one synchrotron oscillation period:
 - $e^{gT_s} = |\lambda|$ or $g \times T_s = \log|\lambda|$
- The dimensionless growth factor is a function of the two dimensionless parameters Υ and χ :



Universal plot



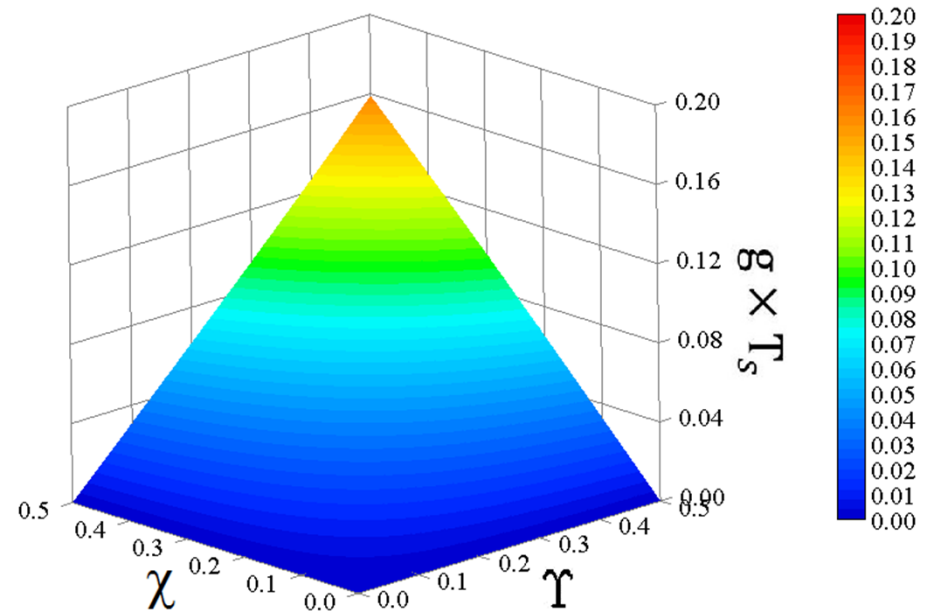
Symmetrical for positive and negative χ

Findings

- The two particle model provides the symmetrical behavior of the growth rate for positive and negative χ .
 - In the Vlasov equation analyses and real machines, slightly positive and negative values for the chromaticity are preferred for stability of beams for operation above and below transition.
 - This discrepancy arises since the \pm modes are treated equally in the two particle model, while they behave differently in real machines since the frequency of the pure dipole mode shifts always downward.
- Very roughly speaking, the real and the imaginary part of Γ are responsible for the strong head-tail instability (or TMCI) and the head-tail instability, respectively (as written in Chao's book).
 - Since both functions are continuous functions, a clear threshold for the strong head-tail instability is hard to identify.

For Small γ and χ

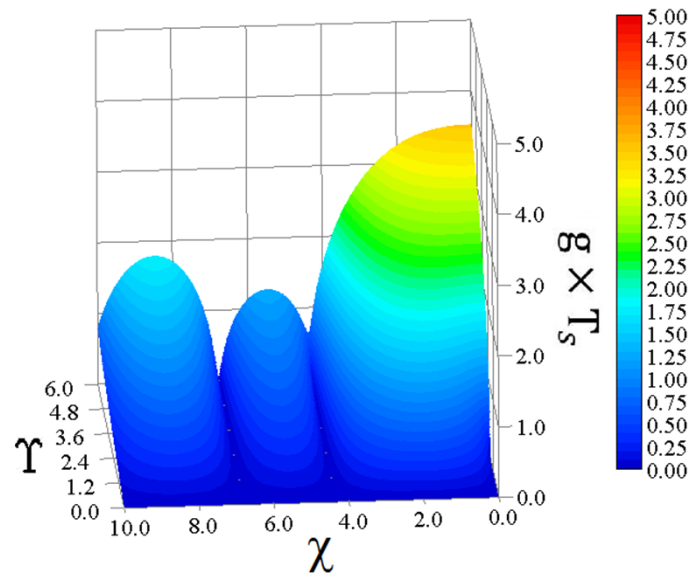
- Growth factor: $g \times T_s = \frac{2}{\pi} \gamma \chi$



- The chromaticity excites head-tail instabilities even at very small wake fields, and their growth rates are proportional to the chromaticity.

Possible Mitigation?

- The strong head-tail instability with a large growth rate appears mostly at χ below 4.
- Around the lines along $\chi = 4.33$ and 6.78 , head-tail instability is strongly suppressed even at a large Υ (strong wake fields), say $\Upsilon = 6$.
 - $\chi=4.33$ and 6.78 are roots of the function $g(\chi)$.



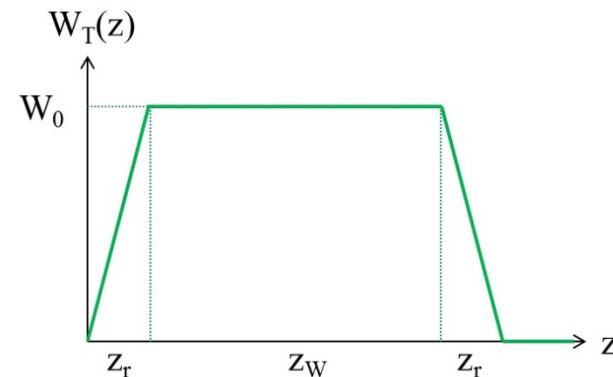
Vlasov Analysis

- Let us check if these findings are still cases in the Vlasov approach.
- The first challenge is the constant wake.
 - If we assume that the wake is constant forever behind a bunch, the impedance becomes singular at zero frequency:

- $Z_T(\omega) = W_0 \left[\frac{1}{\omega} - i\pi\delta(\omega) \right]$

- In reality, the wake has to be constant only inside a bunch and can be zero behind it.

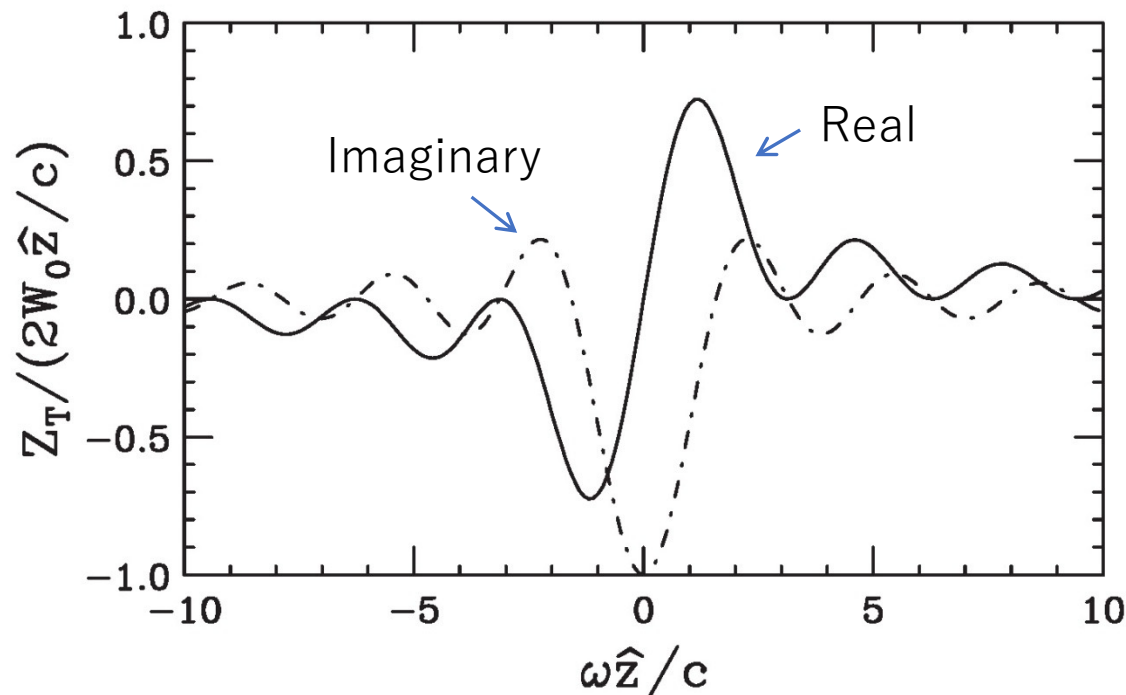
- The trapezoidal model



Impedance of Trapezoidal Wake Model

- By setting $Z_W = 2\hat{z}$ (the total bunch length in the two particle model),

- $$Z_T(\omega) = 2W_0\hat{z}/c \left[\frac{\sin^2 \omega\hat{z}/c}{\omega\hat{z}/c} - i \frac{\sin \omega\hat{z}/c \cdot \cos \omega\hat{z}/c}{\omega\hat{z}/c} \right]$$



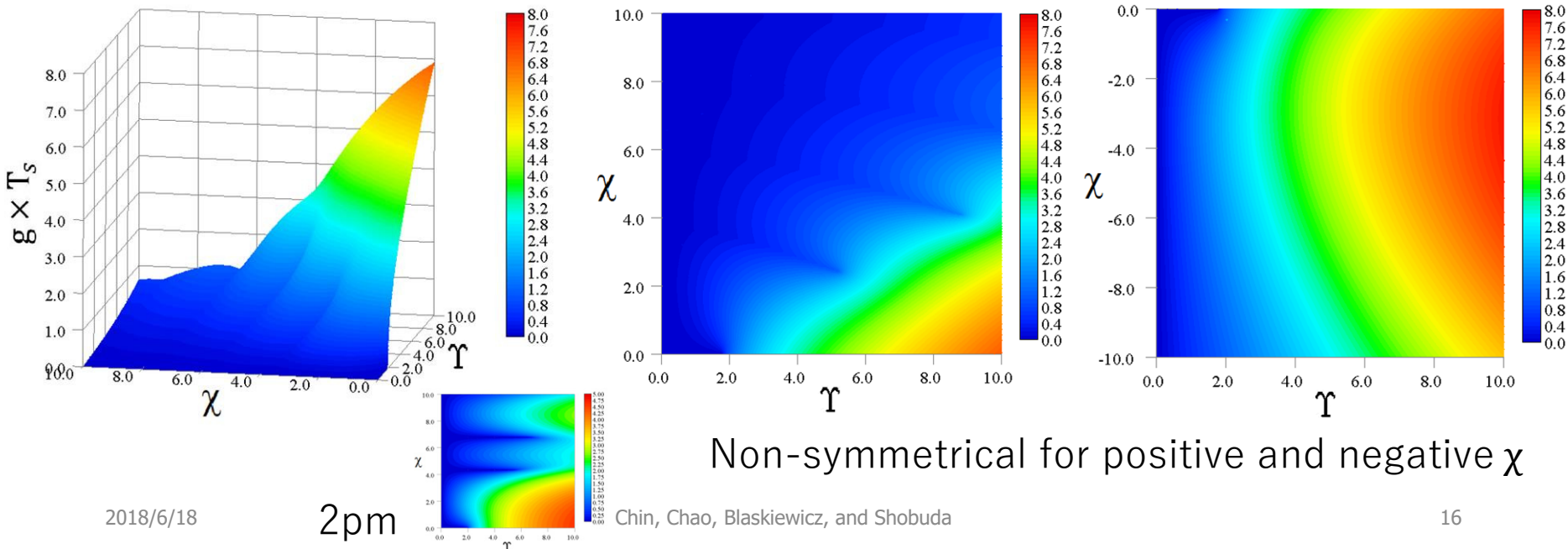
MOSCOW

- We found that the solutions of Vlasov eq. still scale as a function of the three dimensionless parameters, χ , Y , and the growth factor $g \times T_s$, as in the two particle model.
- The trapezoidal model was installed to MOSES code and it is named as “MOSCOW” (MOSES for the COntstant Wake).
- The question is how we define the head and the tail of a Gaussian bunch.
- From numerical calculations, we found that the bunch length \hat{z} of the two particle model and the rms bunch length, σ_z , of a Gaussian distribution should be equated as

$$\bullet \hat{z} = 2\sigma_z \quad \rightarrow \quad \chi = 2 \frac{\xi \omega_\beta}{c\eta} \hat{z} = 4 \frac{\xi \omega_\beta}{c\eta} \sigma_z$$

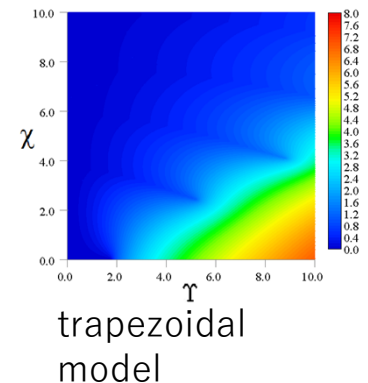
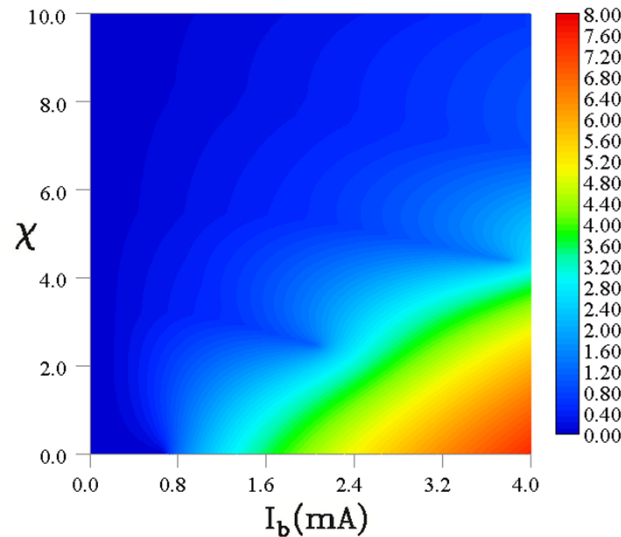
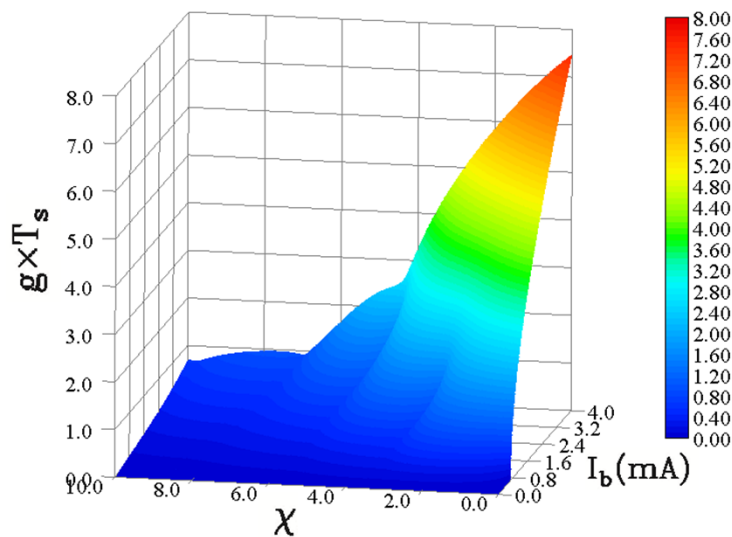
Contour Plots of Most Unstable Mode

- Remarkably similar to those of two particle model (2pm)
- The unstable region at small χ becomes smaller and there is no other significantly unstable region at large χ .
- The stable lines near $\chi = 4.33$ and 6.78 still exit, but they are tilted downward in the direction of stronger wake. They no longer constitute large stable areas around them.



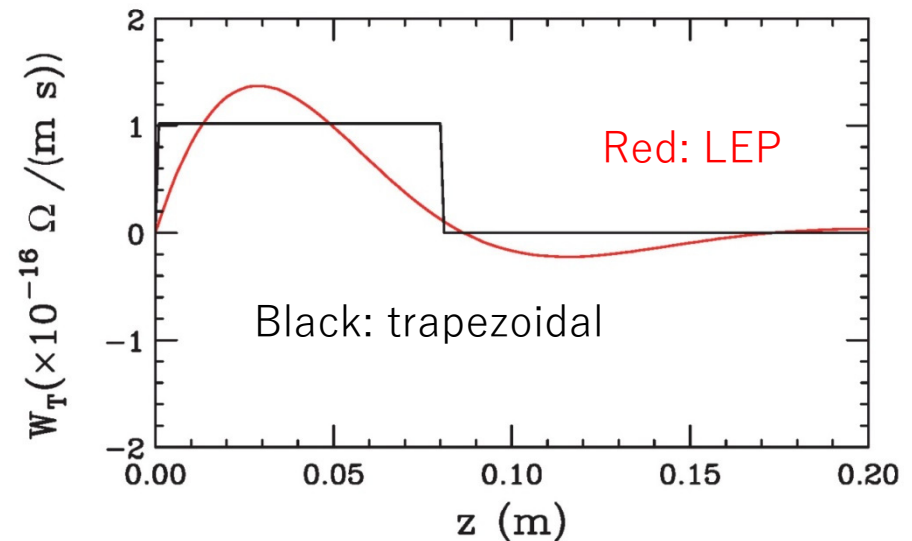
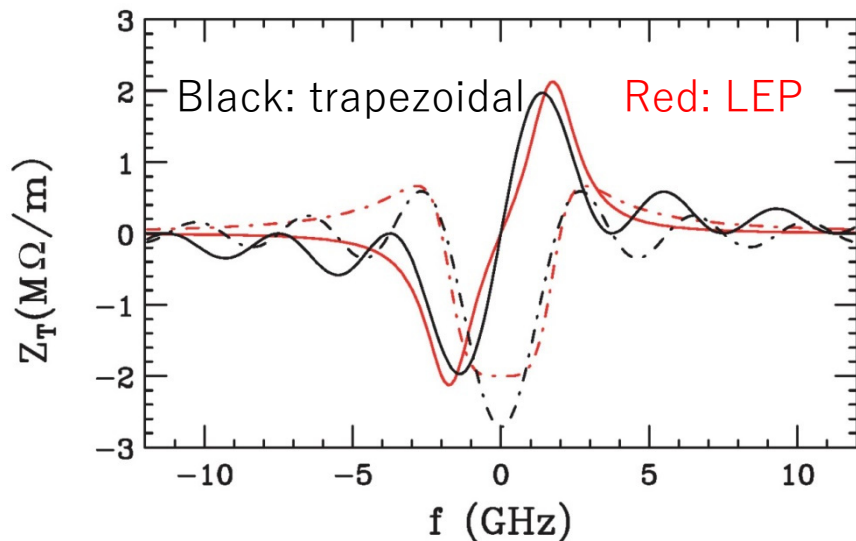
More Realistic Impedance Model

- Is the trapezoidal wake model a peculiar and oversimplified model?
- Let us try the LEP broadband resonator model with:
 - The resonator frequency = 2 GHz, the shunt impedance = 2M Ohm/m, and the Q-value = 1.
- Very similar to those of the trapezoidal model!

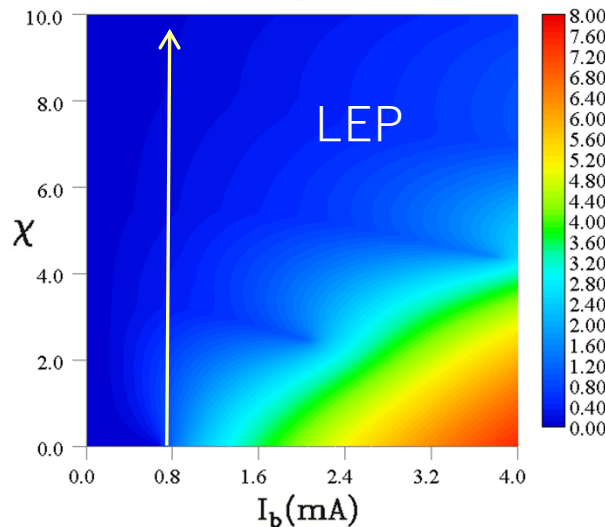
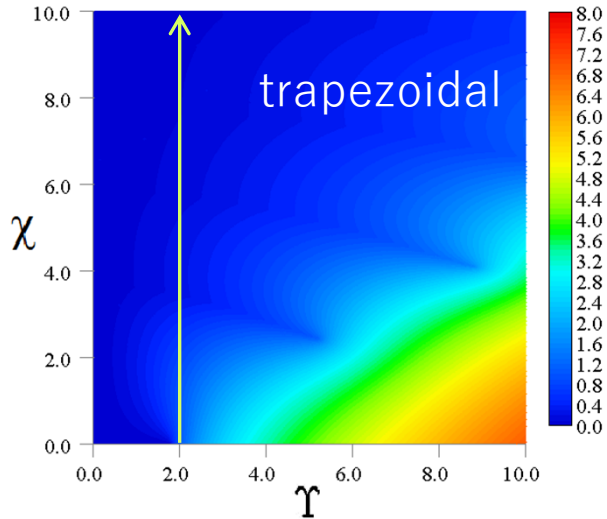


Reasons of the Similarity

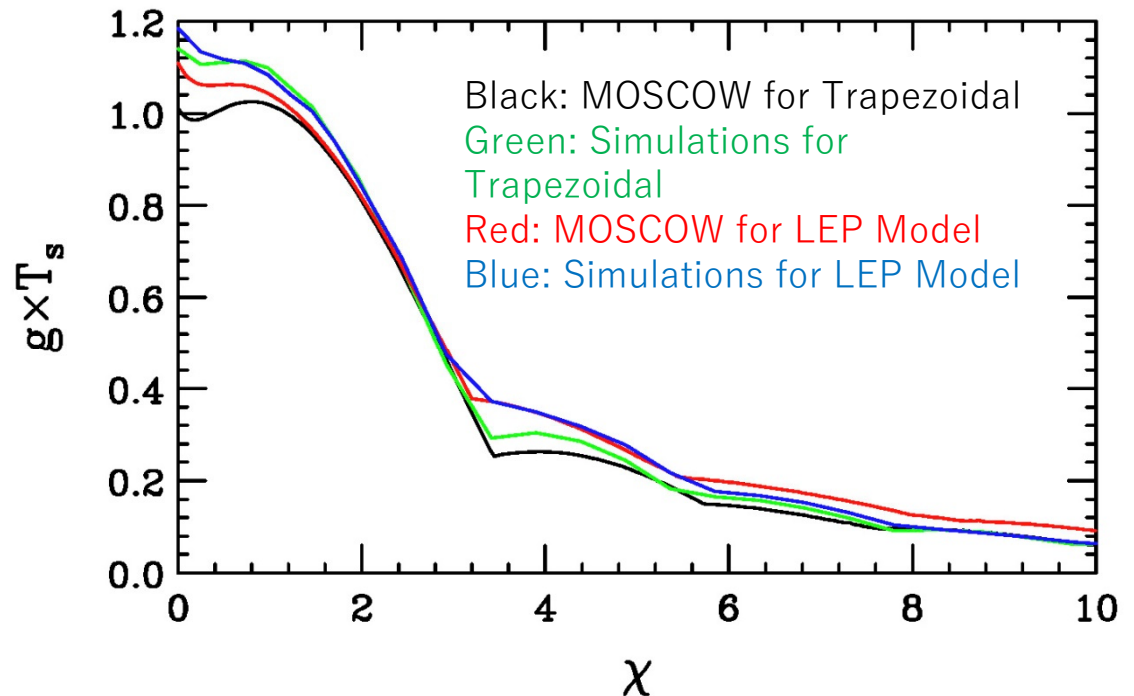
- The impedances are similar at low frequency
 - The difference exists mostly at high frequency.
 - The trapezoidal wake model needs more high frequency impedance components to realize the fast rise and fall.
 - They contribute little to lower-order head-tail mode behaviors, since their mode spectrum are concentrated at low frequency.



Simulations with Multi-Particle TRANFT Code (Not a 2 Particle Model Code)

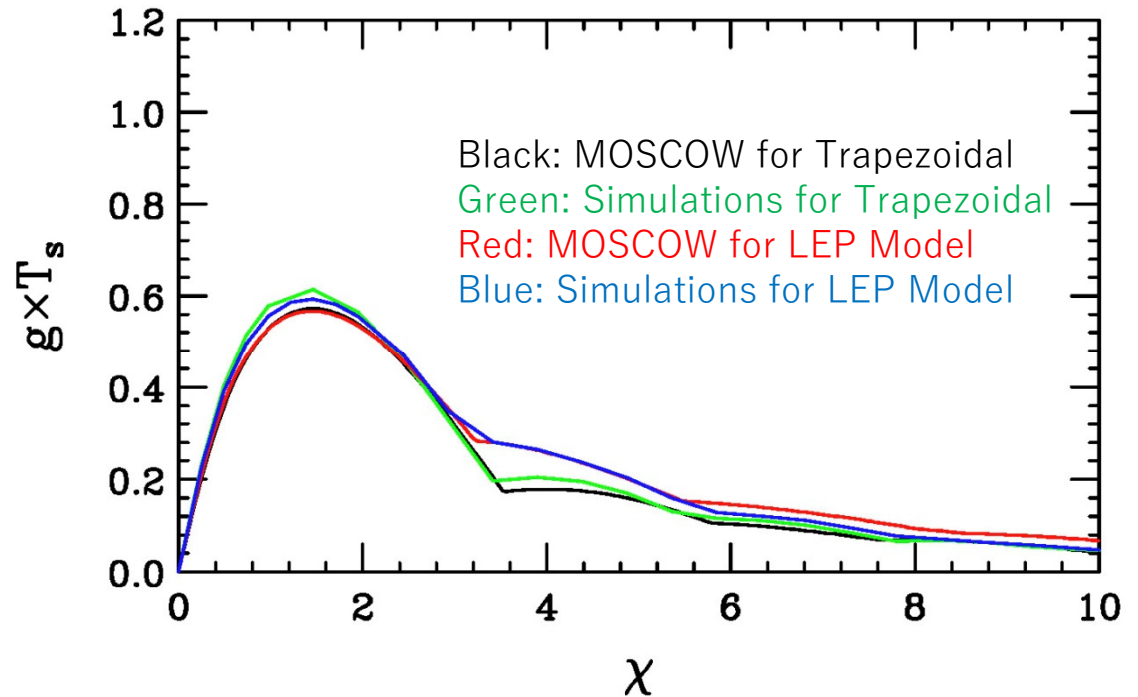
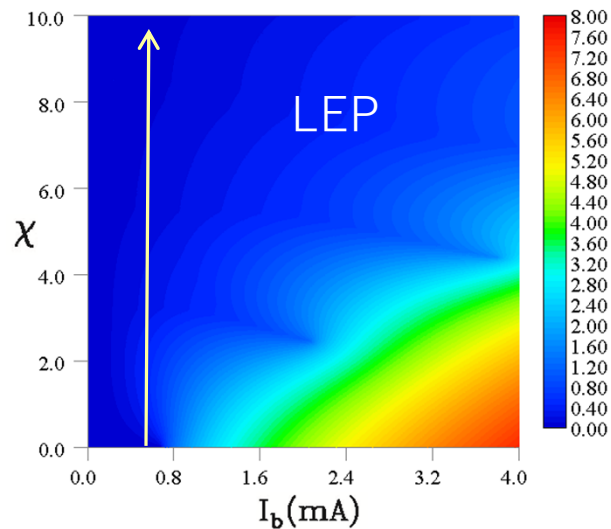
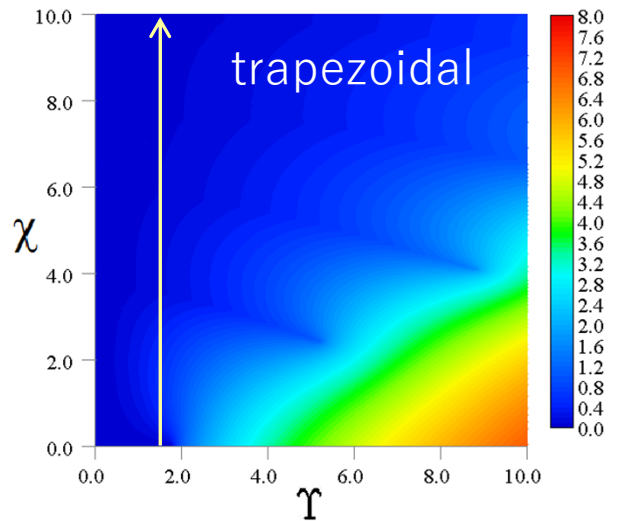


- Slightly above the TMCI threshold
- 10 wake field kicks per turn



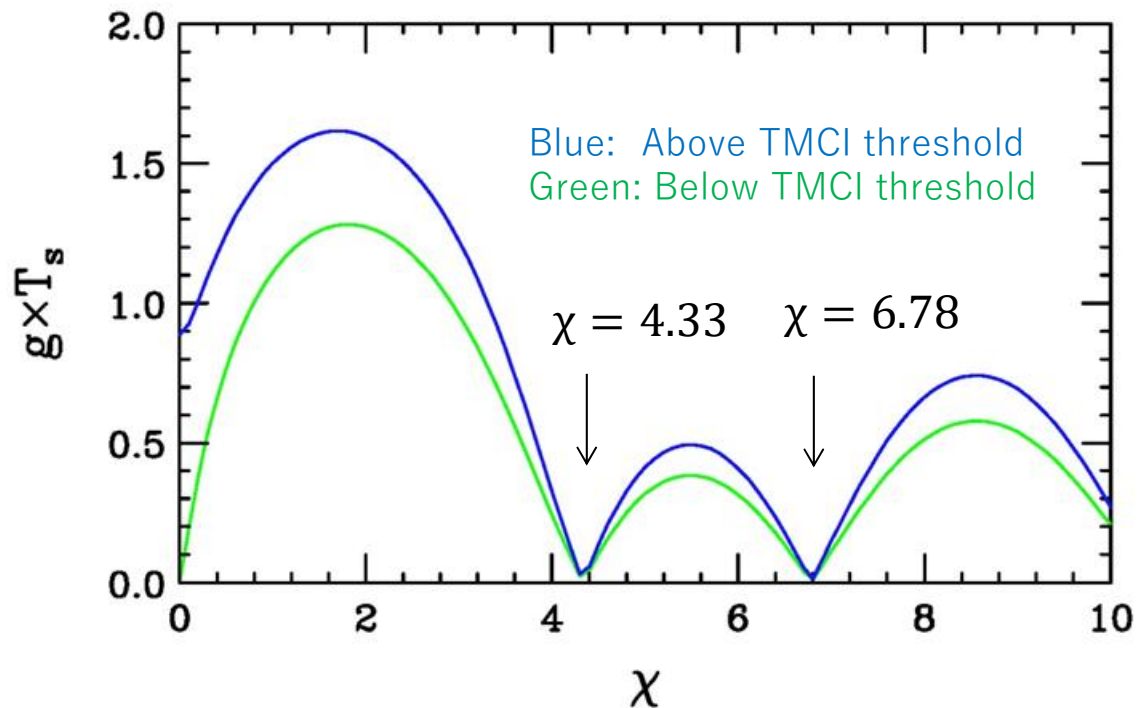
The growth factor = 1 means that the amplitude of dipole oscillation grows by a factor of $e = 2.72$ within one synchrotron oscillation period, which is only 11 turns.

Simulations Slightly Below TMCI Threshold



Growth Factor in the Analytical Two Particle Model

- Similar behaviors but not quite in accord quantitatively.



Findings and Conclusions 1

- Under the trapezoidal wake model, Vlasov analysis can produce the universal contour plots of the growth factor $g \times T_s$ as a function of Υ and χ .
 - It is similar to those in the two particle model.
- Even at $\Upsilon = 2$, slightly above the threshold value ($=1.8$) at zero chromaticity, the beam is very unstable at small χ .
 - It is unlikely that the unstable regions beyond $\Upsilon = 2$ are actually reachable in real machines.
 - That may explain why the chromaticity has not been an effective tool to mitigate TMCI in many machines.
- The growth factor changes a little when χ stays within a few.
 - It implies that the TMCI threshold can be hardly improved by increasing the chromaticity if it is within a reasonably attainable value (unless η is very small).

Findings and Conclusions 2

- The striking resemblance of the contour plots between the trapezoidal model and the LEP broadband resonator model demonstrates that the trapezoidal wake model can give quantitatively reasonable predictions, while it can significantly simplify the time domain analysis by having the constant wake potential within a bunch.
 - Supported by the simulation results
- It may be safe to say that the universal contour plots with trapezoidal wake model show overall accurate pictures on how the beam instabilities behave as a function of γ (or the bunch current) and χ (or the chromaticity).
- Our findings on the trapezoidal wake model will help to advance the instability theories further by its simplicity and good impedance behavior.