TU 12 : Modeling of mechanical systems - Part II

Lagrangian Dynamics

Master 1 - ISC, Robotics and Connected Objects

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D'Alembert principle

- Applied or impressed force : force exerted on a system
- Generalization of force exerted on a system : include D'Alembert or apparent forces resulting from the motion of the reference frame (force of inertia, $-m\mathbf{a}'$, centrifugal force, $-m\Omega \times (\Omega \times \mathbf{r}')$, Coriolis force, $-2m\Omega \times \mathbf{v}'$, Euler force, $-m\dot{\Omega} \times \mathbf{r}'$)
  $\Rightarrow$ this enables to consider the motion dynamics inside any moving reference frame, not only absolute reference frame (fixed frame)
- Use of Newton's law : $\mathbf{F_i} = \dot{\mathbf{p}}_\mathbf{i}$,
  $\mathbf{F_i}$ force exerted on particle i,
  $\mathbf{p_i}$ momentum of particle i.

D'Alembert principle

- Reminder of principle of virtual work : $\sum_i F_i \delta r_i = 0$

  This means that the vector of forces F_i is perpendicular to the **virtual** displacement vector δr_i, therefore cannot put the system into motion. The forces do not work.
- D'Alembert principle : generalization to dynamics

$$\boxed{\sum_i (F_i - \dot{p}_i)\delta r_i = 0} \quad (1)$$

$$\boxed{\sum_i (F_i + F_i^*)\delta r_i = 0} \quad (2)$$

  where F_i^* represent the D'Alembert or apparent forces due to the motion of the reference frame
- it is assumed that constraint forces are 0 (rigid bodies) or that constraint forces do not work (they are perpendicular to the constraint surface and the virtual displacement is tangent to this surface)

Transformation to generalized coordinates

- transformation of the N-body system to n independent generalized coordinates q_k :

$$\delta r_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j$$

  remember that no time variation δt involved since a virtual displacement only considers displacements of the coordinates.

$$\sum_i F_i \delta r_i = \sum_{i,j} F_i \frac{\partial r_i}{\partial q_j} \delta q_j = \sum_j Q_j \delta q_j$$

  with Q_j generalized forces
- Q_j do not necessarily have the dimensions of force, but the product $Q_j \delta q_j$ must have the dimensions of work.

Transformation to generalized coordinates

- we can transform the momentum term :

$$\sum_i \dot{p}_i \delta r_i = \sum_i m_i \ddot{r}_i \delta r_i = \sum_{i,j} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} \delta q_j$$
$$= \sum_{i,j} \left\{ \frac{d}{dt} \left(m_i \dot{r}_i \frac{\partial r_i}{\partial q_j} \right) - m_i \dot{r}_i \frac{d}{dt} \left(\frac{\partial r_i}{\partial q_j} \right) \right\} \delta q_j$$

Using $\frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$, and interchanging the order of differentiation w.r.t. t and q_j :

$$\sum_i \dot{p}_i \delta r_i = \sum_{i,j} \left\{ \frac{d}{dt} \left(m_i v_i \frac{\partial v_i}{\partial \dot{q}_j} \right) - m_i v_i \left(\frac{\partial v_i}{\partial q_j} \right) \right\} \delta q_j$$

Inserting into D'Alembert's law leads to $\sum_i (F_i - \dot{p}_i) \delta r_i =$

$$-\sum_j \left\{ \frac{d}{dt} \left(\frac{\partial}{\dot{q}_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right) - \frac{\partial}{q_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) - Q_j \right\} \delta q_j = 0$$

Transformation to generalized coordinates

identifying $\sum_i \frac{1}{2} m_i v_i^2$ with the kinetic energy T, it comes :

$$\sum_j \left[\left\{ \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right\} - Q_j \right] \delta q_j = 0$$

- for Cartesian coordinates, T is only function of velocities $(\dot{x}, \dot{y}, \dot{z})$, therefore $\frac{\partial T}{\partial q_j} = 0$.
- BUT in general $\frac{\partial T}{\partial q_j} \neq 0$, for example, if curvilinear coordinates are used, $v = \dot{r} e_r + r \dot{\theta} e_\theta$, $\frac{\partial T}{\partial r} \neq 0$.

If all n generalized coordinates are **independent**, we can get the basic Euler-Lagrange equations :

$$\forall j \in \{1 \dots n\}, \boxed{\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j} \qquad (3)$$

if m holonomic constraints apply, it is possible to reduce to $s = n - m$ independent coordinates.

Let us partition the generalized forces $Q_j = \sum_i F_i \frac{\partial r_i}{\partial q_j}$ into

- conservative velocity-independent forces that can be expressed in terms of the gradient of a scalar potential, $-\frac{\partial U}{\partial q_j}$, and
- excluded generalized forces Q_j^{EX} that contain non-conservative, velocity-dependent, and all forces not explicitly included in the potential U

$$Q_j = -\frac{\partial U}{\partial q_j} + Q_j^{EX}$$

Since $\frac{\partial U}{\partial \dot{q}_j} = 0$, we can write :

$$\sum_j \left[\left\{ \frac{d}{dt}\left(\frac{\partial (T-U)}{\partial \dot{q}_j} \right) - \frac{\partial (T-U)}{\partial q_j} \right\} - Q_j^{EX} \right] \delta q_j = 0$$

The standard definition of the Lagrangian is

$$L = T - U$$

$$\boxed{\sum_j \left[\left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} \right\} - Q_j^{EX} \right] \delta q_j = 0}$$

If all n generalized coordinates are **independent** :

$$\forall j \in \{1 \ldots n\}, \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = Q_j^{EX}} \qquad (4)$$

N.B. : holonomic constraint forces can be factored out of the generalized forces Q_j^{EX}

Lagrange equations from Hamilton's Principle

Hamilton's Principle states : "dynamical systems follow paths that minimize the time integral of the Lagrangian".
That is, the action functional S has a minimum value for the correct path of motion.

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt \tag{5}$$

Let us use the standard Lagrangian $L = T - U$, the Hamilton's principle can be written in terms of virtual infinitesimal displacement as :

$$\delta S = \delta \int_{t_1}^{t_2} Ldt = 0$$

According to variational calculus, a system of s independent generalized coordinates must satisfy the basic Euler-Lagrange equations : $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \quad j \in \{1 \ldots s\}$ (variation of L occurs between **definite end-positions**, and $Q_j^{EX} = 0$)

Constrained systems

Choice of generalized coordinates

- The generalized coordinates are not required to be orthogonal as is required using the vectorial Newtonian approach
- The secret to using generalized coordinates is to select coordinates that are perpendicular to the constraint forces so that they do not work
- If the constraints are rigid, then the constraint forces do not work in the direction of the constraint force, hence they do not contribute to the action integral

In case of m holonomic constraints $g_k(q_i, t) = 0$, we can derive the m kinematic constraints :

$$\sum_{j=1}^{n} \frac{\partial g_k}{\partial q_j}(q,t)dq_j + \frac{\partial g_k}{\partial t}dt = 0$$

Remember : total differential is not mandatory.
If g_k is scleronomic, $\frac{\partial g_k}{\partial t} = 0$.
Let us partition the excluded generalized forces into a Lagrange multiplier term plus a remainder force :

$$Q_j^{EX} = \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(q,t) + Q_j^{EXC} \tag{6}$$

We define an extended Lagrangian :

$$L'(q, \dot{q}, \lambda, t) = L(q, \dot{q}, t) + \sum_{k=1}^{m} \lambda_k g_k(q,t) \tag{7}$$

According to variational calculus, we can find the extremum of the extended Lagrangian using $\delta \int_{t_1}^{t_2} L' dt = 0$ using (see calculus of variations) :

$$\sum_{j=1}^{n} \left[\left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} \right\} - \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(q,t) - Q_j^{EXC} \right] \delta q_j = 0$$

The m λ_k can be chosen arbitrarily, let choose them such that the m first terms in the square brackets vanish, assuming the $s = n - m$ independent δq_j are $\delta q_{m+1} \dots \delta q_n$. It comes :

Lagrange multipliers approach

$$1 \le j \le m \Rightarrow \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} \right\} - \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(q,t) - Q_j^{EXC} = 0$$

$$\sum_{j=m+1}^{n} \left[\left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} \right\} - \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(q,t) - Q_j^{EXC} \right] \delta q_j = 0$$

Since $\delta q_{m+1} \ldots \delta q_n$ are independent, we have :

$$\forall j \in \{1 \ldots n\} \boxed{\left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} \right\} = \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(q,t) + Q_j^{EXC}} \quad (8)$$

the Lagrange multiplier approach can be used to solve the n equations plus the m holonomic equations of constraint, which determines the $n+m$ unknowns, which are the n coordinates and the m forces of constraint.

Generalized forces approach

- The two right-hand terms of force acting on the system are not absorbed into the scalar potential U component of the Lagrangian L.
- The Lagrange multiplier terms $\sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}$ account for the holonomic forces of constraint that are not included in the conservative potential or in the generalized forces Q_j^{EXC}.
- the generalized force $Q_j^{EXC} = \sum_{i=1}^{n} F_i \frac{\partial r_i}{\partial q_j}$ is the sum of the components in the q_j direction for all external forces that have not been taken into account by the scalar potential or the Lagrange multipliers.
- Q_j^{EXC} (non-conservative) contains non-holonomic constraint forces, including dissipative forces such as drag or friction.

Generalized forces approach : ex with spherical coordinates

- θ horizontal angle,
- ϕ vertical angle from z-axis.
- $OM = r\ e_r$
- $\dot{e}_r = \Omega \times e_r = (\dot{\theta} z + \dot{\phi} e_\theta)$

$$\delta OM = \delta r\ e_r + r \sin\phi \delta\theta\ e_\theta - r\delta\phi\ e_\phi$$

- $Q_r = F.\frac{\partial OM}{\partial r}$, $Q_\theta = F.\frac{\partial OM}{\partial \theta}$, $Q_\phi = F.\frac{\partial OM}{\partial \phi}$, assuming one body subject to force $F = F_r\ e_r + F_\theta\ e_\theta + F_\phi\ e_\phi$.

Unit vectors	F	δq	Q	$Q\delta q$
e_r	F_r	$\delta r\ e_r$	$F_r\ e_r$	$F_r \delta r$
e_θ	F_θ	$\delta\theta\ e_\theta$	$F_\theta\ r \sin\phi\ e_\theta$	$F_\theta r \sin\phi\ \delta\theta$
e_ϕ	F_ϕ	$\delta\phi\ e_\phi$	$-F_\phi r\ e_\phi$	$-F_\phi r \delta\phi$

F_θ and F_ϕ have the dimension of a torque.

Applying Euler-Lagrange equations to classical mechanics

1. Select a set of independent generalized coordinates
   - minimal set if possible
2. Partition the active forces
   - Conservative one-body forces (ex. gravitational forces)
   - Holonomic constraint forces, which provide algebraic relations that couple some of the generalized coordinates. Can be used to reduce the number of generalized coordinates, or to determine the holonomic constraint forces using the Lagrange multiplier approach.
   - Generalized forces provide a mechanism for introducing non-conservative and non-holonomic constraint forces into Lagrangian mechanics. Typically general forces are used to introduce dissipative forces.
3. Derive the Lagrangian
4. Derive the equations of motion, which are the same equations of motion as obtained using Newtonian mechanics

Exercises

1. Motion of a free particle, $U = 0$
2. Motion in a uniform gravitational field, $U = mgz$
3. Central forces : consider a mass m moving under the influence of a spherically-symmetric, conservative, attractive, inverse-square force, hence $U = -\frac{k}{r}$.
4. Disk rolling on an inclined plane with sufficient friction for rolling
5. Bead of mass m sliding along a frictionless straight horizontal wire constrained to rotate with constant angular velocity ω about a vertical axis