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Targeting multistable dynamics in a second-order memristor circuit

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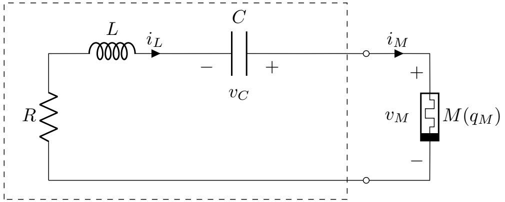
Outline of the talk

Introduction

- Memristors are candidates for the implementation of new analogue and non-boolean computational schemes for real time processing (attaining high bandwidths with reduced power consumption, in-memory computing)
- Memristor circuits display (extreme) multistability, i.e. coexistence of many different attractors
- Multistability is connected to the property that the state space of memristor circuits can be decomposed in a continuum of invariant manifolds
- Multistability control problem: steering the circuit dynamics through the attractors contained in different invariant manifolds
- Feedforward control design via pulse shaped independent voltage or current sources

Circuit description



- Linear circuit:

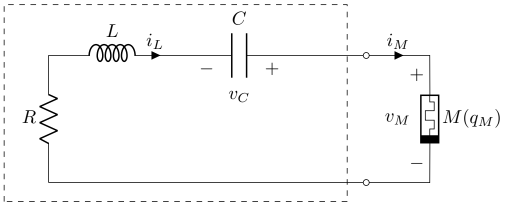
$$\begin{cases} \mathcal{D}v_C(t) = -\dfrac{1}{C} i_L(t) \\ \mathcal{D}i_L(t) = \dfrac{1}{L}\left(v_C(t) - Ri_L(t) - v_M(t)\right) \end{cases}$$

- Charge-controlled memristor:

$$\begin{cases} \mathcal{D}q_M(t) = i_M(t) \\ v_M(t) = \hat{\varphi}'(q_M) = \hat{\varphi}'(q_M) i_M(t) = M(q_M(t)) i_M(t) \end{cases}$$

- $\mathcal{D}$: time-derivative operator
  $\varphi_M = \hat{\varphi}(q_M)$: memristor characteristic
  $M(q_M)$: memristance

Circuit description



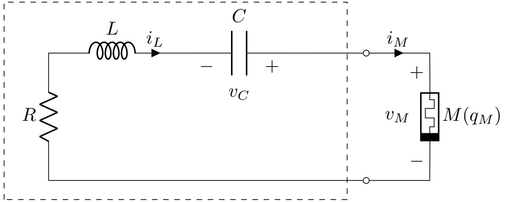
- Charge-flux characteristic:

$$\hat{\varphi}(q_M) = -s_0 q_M + \frac{s_1}{3} q_M^3 \ , \qquad s_0 > R, s_1 > 0$$

- The dynamics of the circuit is described by the third-order system

$$\Sigma : \quad \begin{cases} \mathcal{D}v_C(t) = -\dfrac{1}{C} i_L(t) \\ \mathcal{D}q_M(t) = i_L(t) \\ \mathcal{D}i_L(t) = \dfrac{1}{L}\big(v_C(t) + (s_0 - R) i_L(t) - s_1 q_M^2(t) i_L(t)\big) \end{cases}$$

Circuit description



- Charge-flux characteristic:

$$\hat{\varphi}(q_M) = -s_0 q_M + \frac{s_1}{3} q_M^3 \;, \qquad s_0 > R, s_1 > 0$$

- The dynamics of the circuit is described by the third-order system

$$\Sigma : \quad \begin{cases} \mathcal{D} v_C(t) = -\dfrac{1}{C} i_L(t) \qquad \Longleftrightarrow \mathcal{D}(q_M(t) + C v_C(t)) = 0 \\ \mathcal{D} q_M(t) = i_L(t) \\ \mathcal{D} i_L(t) = \dfrac{1}{L} \big(v_C(t) + (s_0 - R) i_L(t) - s_1 q_M^2(t) i_L(t) \big) \end{cases}$$

Circuit description

- The state space of Σ is foliated into infinitely many invariant manifolds of the form

$$\mathcal{M}(Q_0) = \left\{(v_C, q_M, i_L) : \; q_M(t) + C v_C(t) = Q_0 \ \forall t \geq t_0\right\},$$

$Q_0 \in \mathbb{R}$: index of the manifold

- The dynamics onto $\mathcal{M}(Q_0)$ is described by the second-order system

$$\Sigma_{\mathcal{M}(Q_0)} : \quad \begin{cases} \mathcal{D} q_M(t) & = i_L(t) \\ \mathcal{D} i_L(t) & = \dfrac{1}{LC}(Q_0 - q_M(t)) + \dfrac{s_0 - R}{L} i_L(t) \\ & \quad - \dfrac{s_1}{L} q_M^2(t) i_L(t) \,. \end{cases}$$

- $\Longrightarrow \quad \Sigma \equiv \{\Sigma_{\mathcal{M}(Q_0)}, Q_0 \in \mathbb{R}\}$

Circuit description

- The state space of Σ is foliated into infinitely many invariant manifolds of the form

$$\mathcal{M}(Q_0) = \left\{(v_C, q_M, i_L) : \; q_M(t) + Cv_C(t) = Q_0 \; \forall t \geq t_0\right\},$$

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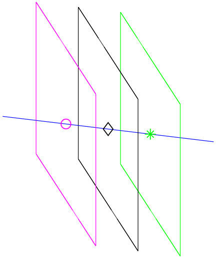
- $\Longrightarrow \quad \Sigma \equiv \{\Sigma_{\mathcal{M}(Q_0)}, Q_0 \in \mathbb{R}\}; \qquad Q_0 = q_M(t_0) + Cv_C(t_0)$

Circuit description

- The invariant manifolds $\mathcal{M}(Q_0)$ are planar surfaces



$Q_0 < \bar{Q}_0 < Q_0$

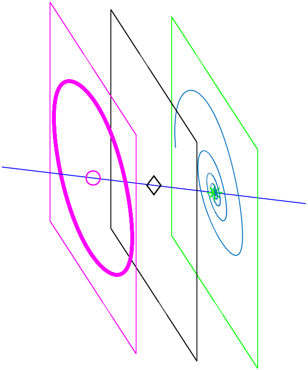
- For each Q_0, $\Sigma_{\mathcal{M}(Q_0)}$ has a unique equilibrium point at $(q_M, i_L) = (Q_0, 0) \in \mathcal{M}(Q_0)$
- The equilibrium point is asymptotically stable if $|Q_0| > \bar{Q}_0$ and unstable if $|Q_0| < \bar{Q}_0$ where

$$\bar{Q}_0 = \sqrt{\frac{s_0 - R}{s_1}}.$$

- At $Q_0 = \pm\bar{Q}_0$ the Jacobian has two pure imaginary eigenvalues $\implies$ Hopf bifurcation condition

Circuit description

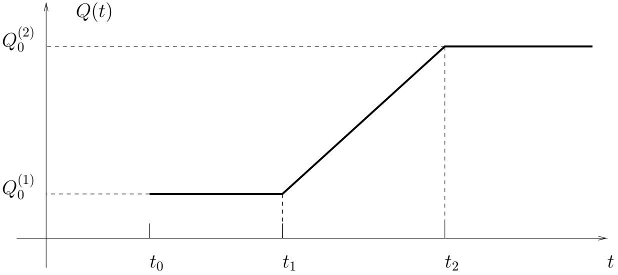
- A (supercritical) Hopf bifurcation (without parameters) scenario occurs at $Q_0 = \pm\bar{Q}_0$



- Dynamics on $\mathcal{M}(Q_0)$:
    - $|Q_0| > \bar{Q}_0$: convergence towards the equilibrium point $(Q_0, 0)$
    - $|Q_0| < \bar{Q}_0$: converge towards a unique limit cycle

Control problem formulation

- Design a control input with a pulse shape in order to move from the manifold $\mathcal{M}(Q_0^{(1)})$ to the manifold $\mathcal{M}(Q_0^{(2)})$ in a given time interval $[t_1, t_2]$ $(t_0 \leq t_1 < t_2 < +\infty)$, without modifying the attractors displayed on $\mathcal{M}(Q_0^{(1)})$ and $\mathcal{M}(Q_0^{(2)})$ in the uncontrolled case
- Desired behavior of the index $Q(t) \doteq q_M(t) + Cv_C(t)$



- Implement the control input via voltage or current sources

Control problem formulation

- Desired controlled dynamics

$$\begin{cases} \mathcal{D}Q(t) = u(t) \\ \mathcal{D}q_M(t) = i_L(t) \\ \mathcal{D}i_L(t) = \dfrac{1}{L}\left(\dfrac{1}{C}(Q(t) - q_M(t)) + (s_0 - R)i_L(t) - s_1 q_M^2(t) i_L(t)\right) \end{cases}$$

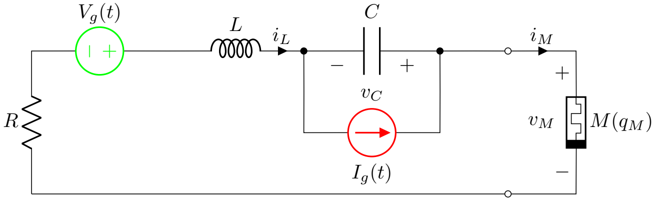
- Control input: $u(t) = 0$ for $t \in [t_0, t_1] \cup [t_2, +\infty)$ and

$$\int_{t_1}^{t_2} u(t)dt = Q_0^{(2)} - Q_0^{(1)} \ .$$

- Time behavior:
  - for $t \in [t_0, t_1]$, $Q(t) = Q_0^{(1)} \implies$ dynamics lies onto $\mathcal{M}(Q_0^{(1)})$
  - for $t \in [t_1, t_2]$, $Q(t) = Q_0^{(1)} + \int_{t_1}^{t} u(\tau)d\tau$
  - for $t \geq t_2$, $Q(t) = Q_0^{(2)} \implies$ dynamics lies onto $\mathcal{M}(Q_0^{(2)})$

Control problem formulation

- Controlled circuit implementation

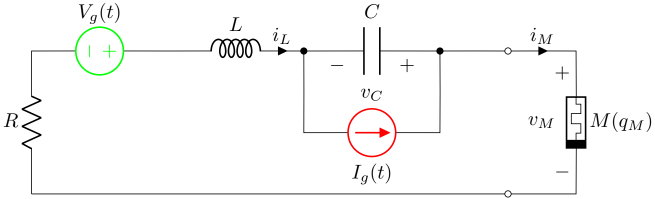


- Controlled circuit dynamics

$$\begin{cases} \mathcal{D}v_C(t) & = \dfrac{1}{C}\left(I_g(t) - i_L(t)\right) \\ \mathcal{D}q_M(t) & = i_L(t) \\ \mathcal{D}i_L(t) & = \dfrac{1}{L}\Big(V_g(t) + v_C(t) + (s_0 - R)i_L(t) - s_1 q_M^2(t) i_L(t)\Big) \end{cases}$$

Control problem formulation

- Controlled circuit implementation



- Controlled circuit dynamics

$$
\begin{cases}
\mathcal{D}v_C(t) &= \dfrac{1}{C}\left(I_g(t) - i_L(t)\right) \quad \Longleftrightarrow \quad \mathcal{D}Q(t) = I_g(t) \\
\mathcal{D}q_M(t) &= i_L(t) \\
\mathcal{D}i_L(t) &= \dfrac{1}{L}\left(V_g(t) + v_C(t) + (s_0 - R)i_L(t) - s_1 q_M^2(t) i_L(t)\right)
\end{cases}
$$

Design of pulse shaped sources

- Desired controlled system and controlled circuit comparison:
  - First equation

$$\mathcal{D}Q(t) = u(t) \quad \Longleftrightarrow \quad \mathcal{D}Q(t) = I_g(t)$$

  - Second equation

$$\mathcal{D}q_M(t) = i_L(t) \quad \Longleftrightarrow \quad \mathcal{D}q_M(t) = i_L(t)$$

  - Third equation

$$\mathcal{D}i_L(t) = \frac{1}{L}\left(\frac{1}{C}(Q(t) - q_M(t)) + (s_0 - R)i_L(t) - s_1 q_M^2(t) i_L(t)\right)$$
$$\Updownarrow$$
$$\mathcal{D}i_L(t) = \frac{1}{L}\left(V_g(t) + v_C(t) + (s_0 - R)i_L(t) - s_1 q_M^2(t) i_L(t)\right)$$

Design of pulse shaped sources

- From first equation:

$$I_g(t) = u(t)$$

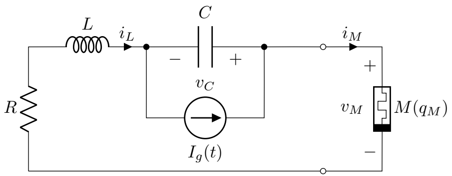
- From third equation:

$$\begin{aligned} V_g(t) &= \frac{1}{C}(Q(t) - q_M(t)) - v_C(t) = \\ &= \frac{1}{C}(q_M(t) + Cv_C(t) - q_M(t)) - v_C(t) = 0 \end{aligned}$$

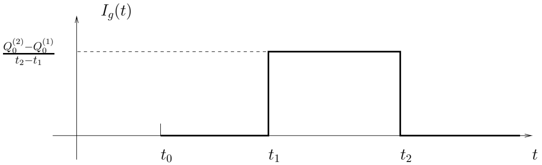
- Design of $V_g(t)$ and $I_g(t)$
  - $V_g(t) = 0$ for $t \geq t_0$
  - $I_g(t) = 0$ for $t \in [t_0, t_1] \cup [t_2, +\infty)$ and

$$\int_{t_1}^{t_2} I_g(t)dt = Q_0^{(2)} - Q_0^{(1)}$$

Design of pulse shaped sources



- Initial conditions: $Q(t_0) = q_M(t_0) + Cv_c(t_0) = Q_0^{(1)}$
- $I_g(t)$ behavior



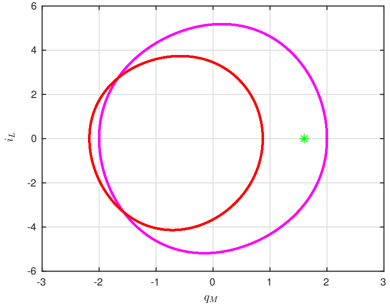
- The circuit dynamics remains on $\Sigma_{\mathcal{M}(Q_0^{(1)})}$ for $t \in [t_0, t_1]$, then for $t \in (t_1, t_2)$ it moves towards $\Sigma_{\mathcal{M}(Q_0^{(2)})}$, which is reached at $t = t_2$. The dynamics then remains onto $\Sigma_{\mathcal{M}(Q_0^{(2)})}$ for $t \geq t_2$.

Numerical examples

- Circuit parameters:

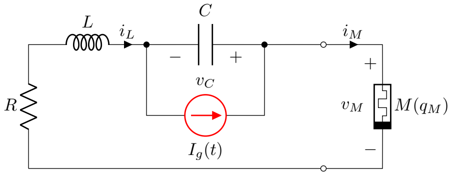
$$R = 0.4,\ C = 0.1,\ L = 1.5,\ s_0 = 0.7,\ s_1 = 1$$

- Attractors on $\mathcal{M}(Q_0)$ ($\bar{Q}_0 = 1$)
  - $|Q_0| > 1$: equilibrium point
  - $|Q_0| < 1$: limit cycle



- Example of attractors in the (q_M, i_L)- plane: $Q_0 = 0$ (magenta), $Q_0 = -0.65$ (red), $Q_0 = 1.6$ (green $*$)

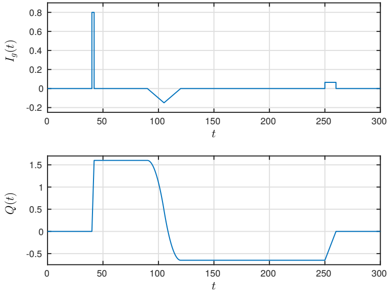
Numerical examples



- Initial conditions at $t_0 = 0$:

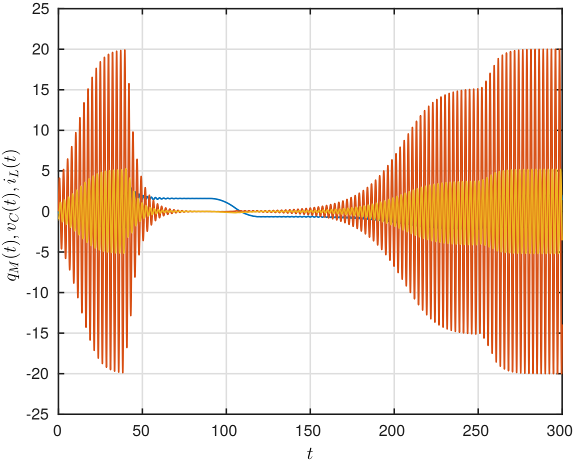
$$v_C(0) = 0,\ q_M(0) = 0,\ i_L(0) = -1 \ \longleftrightarrow\ Q(0) = 0$$

- Pulse programmed source $I_g(t)$ and corresponding index $Q(t)$



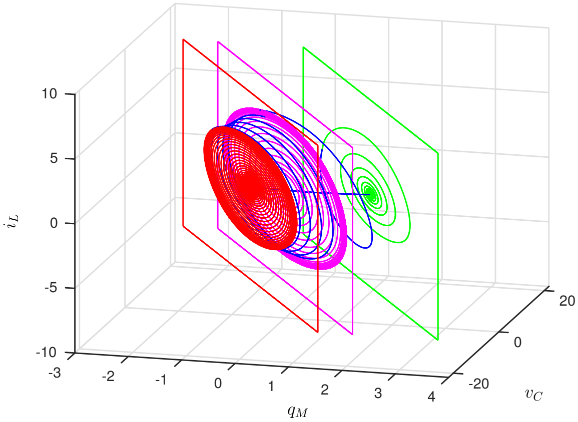
Numerical examples

Corresponding time evolution of the state space variables
(blue line for q_M, red for v_C, orange for i_L)



Numerical examples

Corresponding state-space trajectories
(magenta on $\mathcal{M}(0)$, green on $\mathcal{M}(1.6)$, red on $\mathcal{M}(-0.65)$)



Conclusion

- The problem of steering the dynamics of a second order memristor circuit from one invariant manifold to another (i.e., to switch among different attractors) has been considered
- The problem is solved via a feedforward control input with pulse shape generated by an independent current source
- The features of the pulse control input can be exploited to ensure a desired transient behavior
- Future research should consider more general classes of memristor circuits as well as the use of feedback control to modify the dynamics onto the invariant manifolds