TU 12 : Modeling of mechanical systems - Part II

Quaternions

Master 1 - ISC, Robotics and Connected Objects

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Complex numbers : reminder

- Real part and imaginary part :
  - Cartesian form : $z = a + i\ b$
  - Polar form : $z = |z|e^{i\theta} = |z|(\cos\theta + i\ \sin\theta)$

  with $i^2 = -1$, $\theta = arg\ z$, norm $|z| = \sqrt{a^2 + b^2}$
  $\theta = atan2(b, a)$, $(a, b) \neq (0, 0)$
- Conjugate of z noted $\overline{z}$: $\overline{z} = a - b\ i$, $\sqrt{z\overline{z}} = |z|$
- $z_1 z_2 = (a_1\ a_2 - b_1\ b_2) + (a_1\ b_2 + b_1\ a_2)\ i$
- Euler's formula useful to recalculate all trigonometry formulas :
  $e^{i\theta} = \cos\theta + i\ \sin\theta$, $e^{i(\theta_1+\theta_2)} = e^{i\theta_1}\ e^{i\theta_2}, \ldots$
  De Moivre's formula : $(\cos\theta + i\ \sin\theta)^n = \cos(n\theta) + i\ \sin(n\theta)$
- Rotation in the 2D plane (affix points) : $z' - z_0 = e^{i\theta}(z - z_0)$

Introduction to quaternions

Quaternions : extension of complex numbers introduced by Lord William Hamilton XIX century) composed of 4 scalar values, noted w, x, y, z ($\in \mathbb{R}$), making 1 real part and 3 imaginary parts, :

$$\begin{aligned} q &= w + x\,i + y\,j + z\,k \\ i^2 &= -1, j^2 = -1, k^2 = -1 \\ i\,j &= -j\,i = k \\ j\,k &= -k\,j = i \\ k\,i &= -i\,k = j \end{aligned}$$

The last three equations can be remembered by considering the three entities i, j, k as an orthonormal set of vectors, and replace the multiplication by the cross product.
Set of quaternions $(\mathbb{H}, +, \times)$: division ring or skew field, with multiplication being **non commutative.**

Why use quaternions ?

- sensor feedback from IMU (Inertial Mearurement Unit) :

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orientation:
  x: -0.628151834011
  y: 0.0210457909852
  z: -0.0200814530253
  w: -0.777546823025
```

- avoid gimbal lock of 3 DOF rotation due to the use of Euler angles.
  - singular alignment when the middle rotation aligns the axes of the first and last rotations $\Rightarrow$ loss of 1 DOF
  - afflicts every axis order at multiples of 90 deg.
  - ex. roll-pitch-yaw, pitch of ± 90 deg $\Rightarrow$ roll and yaw rotations have the same axis

Why use quaternions ?

- represent rotations in 3D space, ex. camera positioning in a scene, teleoperation/manipulation of objects for assembly planning.
  - ex [PhD thesis, Ulises Zaldivar Colado, Planification d'assemblage en environnement virtuel, 2009]
- use in kinematic control systems
  - ex. submarine exploration with tracking of transect, or rotation about a fixed point.
    [PhD thesis, Silvain Louis, Système robotisé semi-autonome pour l'observation des espèces marines, juillet 2018]

Quaternions : representation

In addition to the linear combination of 1, i, j and k :

- vector with the 4 scalar values : $q = (x\, y\, z\, w)$
- scalar and 3D vector : scalar value noted s and 3-component vector for the imaginary part, $q = (s, v)$, with $v = (x\, y\, z)$
- 4-4 matrix for the left multiplication of quaternions representing the linear application $p \longmapsto q\ p$

$$\mathcal{M}(q) = \begin{bmatrix} w & -x & -y & -z \\ x & w & -z & y \\ y & z & w & -x \\ z & -y & x & w \end{bmatrix}, \ q = w + i\ x + j\ y + k\ z$$

The matrix of 1 is the identity matrix and those of i, j, and k :

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- additions/subtractions
- scalar product ($\in \mathbb{R}$) :
  $< q_1|q_2 >=< q_2|q_1 >= x_1x_2 + y_1y_2 + z_1z_2 + w_1w_2$
  $< q_1|q_2 >= s_1s_2 + v_1.v_2$
- multiplication by scalar : $\lambda \in \mathbb{R}$, $\lambda q = (\lambda x, \lambda y, \lambda z, \lambda w)$
- multiplication : $q_1 \times q_2 = (s_1, v_1) \times (s_1, v_2)$ **noted** q_1q_2
  $\boxed{q_1q_2 = (s_1s_2 - v_1.v_2, s_1v_2 + s_2v_1 + v_1 \times v_2)}$
- identity element of $\times$: $(1\,0\,0\,0)$
- associative and distributive property of $\times$ w.r.t. $+$
- conjugate : $\overline{q} = (s, -v)$
  $\overline{q_1 + q_2} = \overline{q_1} + \overline{q_2}$,
  $\overline{q_1q_2} = \overline{q_2}\ \overline{q_1}$, $< q_1|q_2 >= \mathcal{R}eal(q_1\overline{q_2})$
- norm : $|q| = \sqrt{q\overline{q}} = \sqrt{w^2 + x^2 + y^2 + z^2}$,
  $|\overline{q}| = |q|$, $|q_1q_2| = |q_1|\ |q_2|$

Quaternions : definitions - properties - operations

- non-commutative property of $\times$:
  in general $q_1 \times q_2 \neq q_2 \times q_1$
  unless $v_1 \times v_2 = 0$ i.e. $v_1 = 0$, $v_2 = 0$ or $v_1\ //\ v_2$
- every element $\in \mathbb{H}^*$ is invertible : $q^{-1} = \overline{q}/|q|^2$
  $(qq^{-1} = q^{-1}q = 1)$
- unit quaternion, $|q_u| = 1$
  - $q_u^{-1} = \overline{q_u}$, $|q_u^{-1}| = 1$
  - $q_u = (\cos\theta,\ \sin\theta\ v)$, with $|v| = 1$
    $q_u = \cos\theta + v_u\ \sin\theta$, $v_u = (i\ j\ k)\ v$, $v = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}^T$
  - Euler formula : $q_u = e^{v_u\theta} = \cos\theta + v_u\ \sin\theta$
    $q_u^t = (\cos\theta + v_u\ \sin\theta)^t = e^{v_u t\theta} = \cos(t\theta) + v_u\ \sin(t\theta)$
    be careful, in general, $e^{p_u+q_u} \neq e^{p_u}e^{q_u}$ **(non commutative)**
- pure imaginary quaternion $q_i = (0, v)$
  - $q_i^{-1} = -q_i$
  - useful to represent a 3D vector and to consider 3D rotations

Quaternions and 3D rotation

Considering the application $S_q : p \longmapsto q\ p\ \overline{q}$,
with $q \neq 0$ being a unit quaternion
Theorems :

- S_q is a linear application
- S_q keeps the norm constant
- S_q transforms a real scalar into a real scalar
- S_q transforms a pure quaternion into a pure quaternion
- S_q, with $\boxed{q = (\cos(\theta/2),\ \sin(\theta/2)\ u)}$ represents the rotation about unit vector u of angle θ
- $S_{q_2}\ o\ S_{q_1} = S_{q_2 q_1}$, a rotation R_1 followed by a rotation R_2 can be represented by the multiplication of unit quaternions $q_2 q_1$.

Quaternions : exercises

1. Prove that a unit quaternion $q = w + x\ i + y\ j + z\ k$ can be written as $q = (\cos\theta, \sin\theta v)$ with $|v| = 1$, and give $\cos\theta$, $\sin\theta$, and the coordinates of v as functions of w, x, y, and z.
2. Prove the multiplication formula for quaternions using the representation (s, v) to start the calculus.
3. Prove the following using a 3D graphical scheme that decompose a 3D rotated vector about unit vector u of angle θ :

$$\boxed{v_{rot} = v\ \ \cos\theta + \sin\theta\ (u \times v) + (1 - \cos\theta)(u.v)u}$$

   give the matrix form of this equation $v_{rot} = [R]\ v$
4. prove the theorems related to application S_q.
   Use the result of rotated vector decomposition above for the representation of a 3D rotation by a unit quaternion.