TU 12 : Modeling of mechanical systems - Part II

Hamiltonian

Master 1 - ISC, Robotics and Connected Objects

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Generalized momentum

- The generalized momentum associated with the coordinate q_j is defined as :

$$\boxed{p_j = \frac{\partial L}{\partial \dot{q}_j}}$$

- p_j is called the conjugate momentum or canonical momentum to q_j
- q_j, p_j are conjugate, or canonical, variables
- if q_j is a spatial coordinate, p_j is a linear momentum
- if q_j is an angle, p_j is an angular momentum
- Example, if $L = \frac{1}{2}m\dot{x}^2$, $p = m\dot{x}$

Invariant transformations and Noether's Theorem

Noether's theorem : *For each symmetry of the Lagrangian, there is a conserved quantity*
Transformations where the equations of motion are invariant are called invariant transformations.
Variables that are invariant to a transformation are called cyclic variables
A cyclic coordinate is one that does not explicitly appear in the Lagrangian ($\frac{\partial L}{\partial q_j} = 0$).
The term cyclic is a natural name when the system has cylindrical or spherical symmetry.
In Hamiltonian mechanics a cyclic coordinate often is called an ignorable coordinate.

Invariant transformations and Noether's Theorem

Ignoring the external and Lagrange multiplier terms, we have :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(q,t) + Q_j^{EXC} = 0$$
$$\Rightarrow \dot{p}_j = \frac{\partial L}{\partial q_j}$$

If q_j is cyclic (does not appear in L), then $\frac{\partial L}{\partial q_j} = 0$, $\dot{p}_j = 0$.
$\Rightarrow$ if a generalized coordinate does not occur in the Hamiltonian, then the corresponding generalized momentum is conserved.

A cyclic coordinate has a **constant** corresponding momentum p_j for the Hamiltonian as well as for the Lagrangian.

Kinetic energy in generalized coordinates

In terms of fixed rectangular coordinates, the kinetic energy for N bodies, each having three degrees of freedom, is expressed as :

$$T(q,\dot{q},t) = \frac{1}{2}\sum_{\alpha=1}^{N}\sum_{i=1}^{n} m_\alpha \dot{x}_{\alpha,i}^2$$

As $x_{\alpha,i}$ depends on q_j and t : $x_{\alpha,i} = x_{\alpha,i}(q_j,t)$,

$$\begin{aligned}
\dot{x}_{\alpha,i} &= \sum_{j=1}^{s} \frac{\partial x_{\alpha,i}}{\partial q_j}\dot{q}_j + \frac{\partial x_{\alpha,i}}{\partial t} \\
T(q,\dot{q},t) &= \sum_{\alpha}\sum_{i,j,k} \frac{1}{2} m_\alpha \frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial q_k}\dot{q}_j\dot{q}_k \\
&+ \sum_{\alpha}\sum_{i,j} m_\alpha \frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial t}\dot{q}_j \\
&+ \sum_{\alpha}\sum_{i,j} \frac{1}{2} m_\alpha \left(\frac{\partial x_{\alpha,i}}{\partial t}\right)^2
\end{aligned}$$

Kinetic energy in generalized coordinates

We can write :

$$\boxed{T(q,\dot{q},t) = T_2(q,\dot{q},t) + T_1(q,\dot{q},t) + T_0(q,t)}$$

For scleronomic systems with velocity-independent potential :

$$\boxed{T = T_2 = \frac{1}{2}\sum_k \dot{q}_k p_k = \frac{1}{2}\dot{q}.p}$$

Proof :

- $T_2 = \sum_{j,k} a_{jk}\dot{q}_j\dot{q}_k, \quad \frac{\partial T_2}{\partial \dot{q}_m} = \sum_k a_{mk}\dot{q}_k + \sum_j a_{jm}\dot{q}_j$
- $\sum_m \dot{q}_m \frac{\partial T_2}{\partial \dot{q}_m} = \sum_{m,k} a_{mk}\dot{q}_k\dot{q}_m + \sum_{m,j} a_{jm}\dot{q}_j\dot{q}_m = 2\sum_{k,j} a_{kj}\dot{q}_k\dot{q}_j = 2T_2$
- $\sum_m \dot{q}_m \frac{\partial T_1}{\partial \dot{q}_m} = T_1$
- $\sum_m \dot{q}_m \frac{\partial T_0}{\partial \dot{q}_m} = 0$

Kinetic energy in generalized coordinates

Proof (continued) :

- when the transformed system is scleronomic, $\frac{\partial x_{\alpha,i}}{\partial t} = 0$, then $T = T_2$: the kinetic energy is a quadratic function of the generalized velocities $\dot{q}_j$
- if the potential U is velocity-independent, $p_m = \frac{\partial L}{\partial \dot{q}_m} = \frac{\partial T}{\partial \dot{q}_m} = \frac{\partial T_2}{\partial \dot{q}_m}$
- $\sum_m \dot{q}_m \frac{\partial T_2}{\partial \dot{q}_m} = 2T_2 = \sum_m \dot{q}_m p_m \Rightarrow T_2 = \frac{1}{2} p.\dot{q}.$

Generalized energy and the Hamiltonian

Jacobi's Generalized Energy is defined as :

$$\boxed{h(q,\dot{q},t) = \sum_j \left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) - L(q,\dot{q},t)} \qquad (1)$$

The Hamiltonian function is equal to the generalized energy expressed in terms of the conjugate variables (q_j, p_j) :

$$\boxed{H(q,p,t) = h(q,\dot{q},t) = \sum_j \dot{q}_j p_j - L(q,\dot{q},t) = p.\dot{q} - L} \qquad (2)$$

Generalized energy theorem :

- The total time derivative of the Lagrangian $L(q,\dot{q},t)$ is :

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j} \dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \frac{\partial L}{\partial t}$$

Generalized energy and the Hamiltonian

Generalized energy theorem (continued) :

- The Lagrangian for a conservative force is given by :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(q,t) + Q_j^{EXC}$$

- using both equations and replacing $\frac{\partial L}{\partial q_j}$ in the 1st one leads to :

$$\boxed{\frac{dH}{dt} = \sum_j \dot{q}_j \left[\sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j}(q,t) + Q_j^{EXC}\right] - \frac{\partial L(q,\dot{q},t)}{\partial t}}$$

- if the external non-potential forces are zero :

$$\boxed{\frac{dH}{dt} = -\frac{\partial L}{\partial t}} \tag{3}$$

Hamiltonian invariance

$$\boxed{\frac{dH}{dt} = -\frac{\partial L}{\partial t}}$$

- if the Lagrangian is **not** an explicit function of time, and
- if the external non-potential forces are zero,
- then
- both the Hamiltonian $H(q,p,t)$ and generalized energy $h(q,\dot{q},t)$ are constants of motion

Generalized energy and total energy

$$L(q, \dot{q}, t) = T_2(q, \dot{q}, t) + T_1(q, \dot{q}, t) + T_0(q, t) - U(q, t)$$

If the potential energy does not depend explicitly on $\dot{q}_j$, then

$$\begin{aligned}
p_j &= \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} \\
H(q, p, t) &= p.\dot{q} - L = \sum_j \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j - L \\
&= \sum_j \frac{\partial T_2}{\partial \dot{q}_j}\dot{q}_j + \sum_j \frac{\partial T_1}{\partial \dot{q}_j}\dot{q}_j + \sum_j \frac{\partial T_0}{\partial \dot{q}_j}\dot{q}_j - L \\
&= 2T_2 + T_1 - (T_2 + T_1 + T_0 - U) = T_2 - T_0 + U \\
&= (T + U) - (T_1 + 2T_0) \\
&= E - (T_1 + 2T_0)
\end{aligned}$$

Generalized energy and total energy

$$\boxed{H(q, p, t) = E - (T_1 + 2T_0)}$$

E : total energy $(T + U)$

- in general, $H(q, p, t) \neq E$
- if the transformation is scleronomic, $T_1 = T_0 = 0$, and if the potential energy does not depend explicitly on $\dot{q}_j$, $H(q, p, t) = E$.

  Hamiltonian mechanics plays a fundamental role in modern physics (quantum or statistical physics).

To go further :

- Hamilton-Jacobi equation : $\frac{\partial S}{\partial t} + H(q, p, t) = 0$, with action functional $S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$,
  obtained with principle of least action with variations at both ends of the path.
- Poisson bracket representation of Hamiltonian mechanics
- Hamilton equations of motion
- Liouville's theorem
- Hamilton-Jacobi theory