TU 12 : Modeling of mechanical systems - Part II

Fundamentals

Master 1 - ISC, Robotics and Connected Objects

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Fundamentals : Generalized coordinates

- Cartesian coordinates : (x, y, z) or (x_1, x_2, x_3)
- Cylindrical (ρ, θ, ϕ), spherical coordinates : (r, θ, ϕ)
- 3D : 3 coordinates to define the position of a single particle
- N particles $\rightarrow$ $3N$ coordinates
- notation : q_i for the generalized coordinates
  in general, they are assumed to be linearly independent (without constraints) : be careful with the definition of parameters !
- Conversion from Cartesian to generalized coordinates $(n = 3N)$:

$$\begin{aligned} q_1 &= q_1(x_1, x_2, x_3, ..., x_n, t) \\ q_2 &= q_2(x_1, x_2, x_3, ..., x_n, t) \\ &... \\ q_n &= q_{3N}(x_1, x_2, x_3, ..., x_n, t) \end{aligned}$$

Fundamentals : Generalized coordinates

- Inversely from Cartesian to generalized coordinates ($n = 3N$) :

$$
\begin{aligned}
x_1 &= x_1(q_1, q_2, q_3, ..., q_n, t) \\
x_2 &= x_2(q_1, q_2, q_3, ..., q_n, t) \\
&... \\
x_n &= x_n(q_1, q_2, q_3, ..., q_n, t)
\end{aligned}
$$

- inverse transformation not always possible
- one condition : Jacobian determinant $\neq 0$

$$
J(\frac{\partial(q_i)}{\partial(x_i)}) = \begin{pmatrix} \partial q_1/\partial x_1 & \partial q_1/\partial x_2 & ... & \partial q_1/\partial x_n \\ ... & ... & ... & \\ \partial q_n/\partial x_1 & \partial q_n/\partial x_2 & ... & \partial q_n/\partial x_n \end{pmatrix}
$$

- Exercise : give transformation equations from Cartesian to spherical coordinates, and link between volume elements (reminder : $dx_1 dx_2 dx_3 = |det(J(\partial x_i/\partial y_j))| dy_1 dy_2 dy_3$).

Fundamentals : Generalized velocity $\dot{q}_i$

- $x_i = x_i(q_i, t)$, $v_i = \frac{dx_i}{dt}$
- differentiation : $dx_i = \sum_k \frac{\partial x_i}{\partial q_k} dq_k + \frac{\partial x_i}{\partial t} dt$
- $\Rightarrow v_i = \sum_k \frac{\partial x_i}{\partial q_k} \dot{q}_k + \frac{\partial x_i}{\partial t}$
- Schwartz theorem : $\frac{\partial}{\partial \dot{q}_k} \frac{\partial x_i}{\partial t} = \frac{\partial}{\partial t} \frac{\partial x_i}{\partial \dot{q}_k}$
- $\frac{\partial x_i}{\partial \dot{q}_k} = 0$ (x_i does not depend on $\dot{q}_k$)

$$
\frac{\partial v_i}{\partial \dot{q}_j} = \sum_k \left(\frac{\partial}{\partial \dot{q}_j} \frac{\partial x_i}{\partial q_k} \right) \dot{q}_k + \sum_k \frac{\partial x_i}{\partial q_k} \frac{\partial \dot{q}_k}{\partial \dot{q}_j} = \sum_k \frac{\partial x_i}{\partial q_k} \delta_{kj} = \frac{\partial x_i}{\partial q_j}
$$

$$
\boxed{\frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j}} \tag{1}
$$

Fundamentals : constraints

- Degrees of freedom (DOF) = number of independent coord.
- constraint = relationship between the coordinates
- each constraint reduces by 1 the number of DOF
  $\Rightarrow$ coordinates no more independent
- a constraint that can be expressed in the form

  $$f(q_1, q_2, ...; t) = 0$$

  is called **holonomic**
- types of constraint : geometric, kinematic, integral

Fundamentals : type of constraints

- geometric – algebraic – constraints : **holonomic**
  algebraic relation in coordinate space (direct link between coordinates)

  $$f_k(q_1, q_2, ..., q_n; t) = 0, \quad k \leq m$$

- kinematic – differential – constraints : where constraints are expressed in terms of infinitesimal displacements :

  $$\sum_{j=1}^{n} \frac{\partial f_k}{\partial q_j} dq_j + \frac{\partial f_k}{\partial t} = 0$$

  - total differential of a function => integrable => holonomic
  - not a total differential => non holonomic (may be integrable after full problem resolution)

Fundamentals : type of constraints

- integral – isoperimetric – constraints : expressed in terms of direct integrals
  - examples : finding the max. volume bounded by a fixed area, the shape of a hanging rope of fixed length.
  - typically : finding the curve $y = y(x)$ such that the functional $F(y) = \int_{x_1}^{x_2} f(y, y'; x)dx$ has an extremum where the curve $y(x)$ satisfies boundary conditions $y(x_1) = a$ and $y(x_2) = b$ **with an integral constraint of the form :**

$$\int_{x_1}^{x_2} g(y, y'; x)dx = constant$$

    this constraint can be a fixed perimeter, surface, volume...
  - in general : geometric and holonomic

Fundamentals : constraints

- inequality constraints are not holonomic
- in general constraints that involve velocities and/or differentials of the coordinates are not holonomic
- scleronomic constraint : not explicitly time dependent (q_i depends on time but it is not an explicit dependence of the constraint)
- rheonomic constraint : explicitly time dependent (ex. deflating pneumatic tire)
- exercises :
  - express the constraints for a particle moving on the surface of an ellipsoid, of a elliptic paraboloid of axis z.
  - express the constraints of a stick whose one extremity leans on a vertical axis and the other on the horizontal ground. How many DOF ?
  - give constraints of a wheel rolling without slipping on the surface of a table (just rolling and pivoting motion), consider coordinates of its center and two angles of rotation.

Fundamentals : virtual displacement δx_i

- δx_i : infinitesimal, instantaneous displacement of coord. x_i, **consistent with any constraints** acting on the system. **Time is frozen**.
- Difference between δx_i and dx_i :

$$\begin{aligned} \delta x_i &= \sum_k \frac{\partial x_i}{\partial q_k} \delta q_k \\ dx_i &= \sum_k \frac{\partial x_i}{\partial q_k} dq_k + \frac{\partial x_i}{\partial t} dt \end{aligned}$$

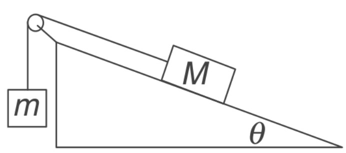
Fundamentals : virtual work and generalized force

- system subjected to applied forces F_i along x_i allowing the Cartesian coordinates to undergo virtual displacements δx_i

$$\begin{aligned} \delta W &= \sum_i F_i \delta x_i = \sum_i F_i \sum_k \frac{\partial x_i}{\partial q_k} \delta q_k \\ &= \sum_k \sum_i \left(F_i \frac{\partial x_i}{\partial q_k} \right) \delta q_k = \sum_k Q_k \delta q_k \end{aligned}$$

with generalized force $\boxed{Q_k = \sum_i F_i \frac{\partial x_i}{\partial q_k}}$

- Exercise : a particle is acted upon by a force $F(F_x, F_y)$. Generalized forces in polar coordinates ?
- Exercise : using virtual work show that equilibrium requires that $M = m/\sin\theta$ (no friction, inextensible string)



Fundamentals : del or nabla operator

Name	Partial derivative	Field	Action
Gradient	$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$	Scalar potential V	$\mathbf{E} = \nabla V$
Divergence	$\nabla . \equiv \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) .$	Vector field $\mathbf{E}$	$\nabla . \mathbf{E}$
Curl	$\nabla \times \equiv \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times$	Vector field $\mathbf{E}$	$\nabla \times \mathbf{E}$
Laplacian	$\nabla^2 = \nabla . \nabla \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$	Scalar potential V	$\nabla^2 V$
Euler-Lagrange	$\Lambda_j \equiv \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j} - \frac{\partial}{\partial q_j}$	Scalar Lagrangian L	$\Lambda_j L$
Canonical momentum	$p_j \equiv \frac{\partial}{\partial \dot{q}_j}$	Scalar Lagrangian L	$p_j = \frac{\partial L}{\partial \dot{q}_j}$