TU 12 : Modeling of mechanical systems - Part II
Calculus of variations

Master 1 - ISC, Robotics and Connected Objects

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Calculus of variations

- Euler's differential equation, applications
- Euler's integral equations
- Lagrange multipliers for holonomic constraints
- Constrained variational systems

Euler's differential equation

- objective : varying the functional $y(x)$ such that F reaches a stationary value, presumably an extremum :

$$F = \int_{x_1}^{x_2} f(y, y'; x) dx$$

- x : independent variable, y dependent variable, $y' = \frac{dy}{dx}$
- the quantity $f(y, y'; x)$ depends on y, y', and x
- all $y(x)$ have fixed values at x_1 and x_2
- introduction of a neighboring function $y(\epsilon, x)$ with $y(0, x) = y(x)$ being the function that yields the extremum for F,
  and a function $\eta(x)$ such that $\eta(x_1) = \eta(x_2) = 0$.
  Assuming $y(\epsilon, x)$ and $\eta(x)$ have continuous first derivatives

Euler's differential equation

$$\begin{aligned} y(\epsilon, x) &= y(0, x) + \epsilon\eta(x) \\ y'(\epsilon, x) &= \frac{dy(0, x)}{dx} + \epsilon\frac{d\eta}{dx} \end{aligned}$$

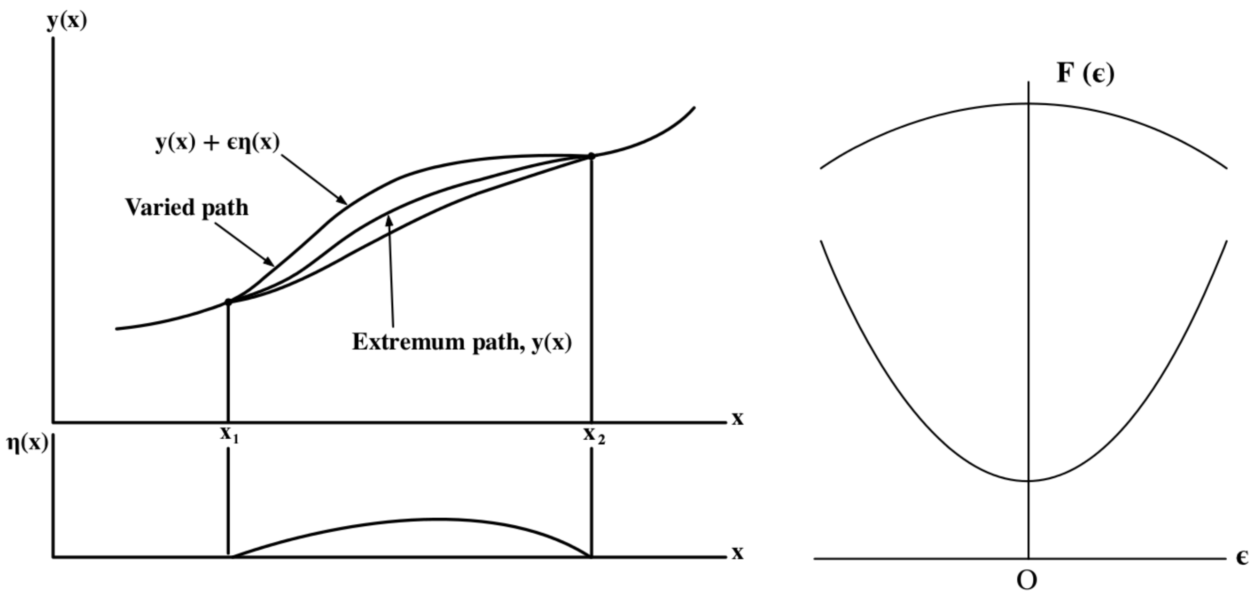
- the parametric family of curves F can be expressed as a function of ϵ :

$$F(\epsilon) = \int_{x_1}^{x_2} f\left[y(\epsilon, x), y'(\epsilon, x); x\right] dx$$

- condition that the integral has a stationary (extremum) value is that be independent of F to 1st order along the path giving the extremum value ($\epsilon = 0$) :

$$\left(\frac{dF}{d\epsilon}\right)_{\epsilon=0} = 0$$

Euler's differential equation



Euler's differential equation

$$
\begin{aligned}
\frac{dF}{d\epsilon} &= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \epsilon} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \epsilon} \right) dx = 0 \\
\frac{\partial y}{\partial \epsilon} &= \eta \qquad \frac{\partial y'}{\partial \epsilon} = \frac{d\eta}{dx}
\end{aligned}
$$

- integration by part of 2nd term in the integrand :

$$
\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{d\eta}{dx} dx = \left[\frac{\partial f}{\partial y'}\eta(x) \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) dx
$$

- it comes :

$$
\frac{dF}{d\epsilon} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) \eta(x) dx
$$

Euler's differential equation

$$\frac{dF}{d\epsilon} = 0 \Rightarrow \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) \eta(x) dx = 0$$

- $\eta(x)$ arbitrary function
- $\left(\frac{dF}{d\epsilon}\right)_{\epsilon=0} = 0, \Rightarrow y(0,x) = y(x),\ y'(0,x) = y'(x)$

Euler's differential equation

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0} \quad (1)$$

Be careful, we must have :

$$\boxed{\left[\frac{\partial f}{\partial y'}\eta(x)\right]_{x_1}^{x_2} = \frac{1}{\epsilon}\left[\frac{\partial f}{\partial y'}\delta y(x)\right]_{x_1}^{x_2} = 0} \quad (2)$$

which means that we can have $\delta y(x_1) = \delta y(x_2) = 0$, but also some $\frac{\partial f}{\partial y'}(y(x_i), y'(x_i), x_i) = 0$ instead of one $\delta y(x_i) = 0$.

Applications of Euler's equation

- Shortest distance between 2 points
- Brachistochrone problem : finding the frictionless motion path of a particle of mass m under uniform gravitational field having the minimum transit time between two points (zero initial velocity)
  A modern application : determination of the optimum shape of the low friction emergency chute that passengers slide down to evacuate a burning aircraft.
- Minimal travel cost : assuming the cost of flying an aircraft at height z is $\exp^{\kappa z}$ per unit distance, $\kappa > 0$, consider that the aircraft flies in the (x, z) plane from point $(-a, 0)$ to point $(a, 0)$ – $z = 0$ is ground level –. Find the optimal trajectory that minimizes the cost of the journey.

Functions with several independent variables

- Functional can depend on several independent variables :

$$F = \int_{x_1}^{x_2} f(y_i(x), y_i'(x); x)dx,\ i \in \{1, 2, 3..., N\}$$

- like the one-dimensional problem, define η_i as independent functions of x :

$$\begin{aligned}
y_i(\epsilon, x) &= y_i(0, x) + \epsilon\eta_i(x) \\
y_i'(\epsilon, x) &= \frac{dy_i(0, x)}{dx} + \epsilon\frac{d\eta_i}{dx} \\
\eta_i(x_1) &= \eta_i(x_2) = 0
\end{aligned}$$

- the requirement for an extremum becomes :

$$\begin{aligned}
\frac{dF}{d\epsilon} &= \int_{x_1}^{x_2} \sum_i^n \left(\frac{\partial f}{\partial y_i}\frac{\partial y_i}{\partial \epsilon} + \frac{\partial f}{\partial y_i'}\frac{\partial y_i'}{\partial \epsilon} \right) dx \\
\frac{dF}{d\epsilon} &= \int_{x_1}^{x_2} \sum_i^n \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} \right) \eta_i(x)dx = 0
\end{aligned}$$

Functions with several independent variables

- Variables $y_i(x)$ independent $\Rightarrow \eta_i(x)$ independent
- evaluating the above equation at $\epsilon = 0$ implies that each term in the bracket must vanish independently.
- Euler's differential equation becomes a set of N equations for the N independent variables :

$$\boxed{\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0} \tag{3}$$

- partial derivatives for the dependent variables y_i, y_i'
- the total derivative for the independent variable x
- Example : Fermat's principle (optics), *principle that the path taken between to points by a ray of light is the path that can be traversed in the least time* - $n_1 \sin\theta_1 = n_2 \sin\theta_2$.

Euler's integral equation

- when f does not depend explicitly on the independent variable x, that is, when $\frac{\partial f}{\partial x} = 0$

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \\
&= \frac{\partial f}{\partial y}y' + \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) - y'\frac{d}{dx}\frac{\partial f}{\partial y'} \\
&= y'\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) + \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right)
\end{aligned}$$

- when the Euler's equation is satisfied, the term times y' vanishes and it remains :

$$\frac{df}{dx} - \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

$$\Rightarrow \boxed{f - y'\frac{\partial f}{\partial y'} = constant} \qquad (4)$$

δ process : permutable with differentiation and integration

$$\begin{aligned}
y &= f(x), \quad y(\epsilon, x) = y(x) + \epsilon\eta(x) \\
\delta f &= f(\epsilon, x) - f(x) = \epsilon\eta(x)
\end{aligned}$$

Variation of derivative :

$$\begin{aligned}
\frac{d}{dx}\delta f &= \epsilon\frac{d\eta}{dx} \\
\delta\frac{df}{dx} &= \frac{df}{dx}(\epsilon, x) - \frac{df}{dx}(x) = \frac{df}{dx} + \epsilon\frac{d\eta}{dx} - \frac{df}{dx} = \epsilon\frac{d\eta}{dx}
\end{aligned}$$

Variation of definite integral :

$$\begin{aligned}
\delta\int_a^b f(x)dx &= \int_a^b f(\epsilon, x)dx - \int_a^b f(x)dx \\
&= \int_a^b (f(\epsilon, x)dx - f(x))dx = \int_a^b \delta f(x)dx
\end{aligned}$$

Treatment of constraint forces in variational calculus

- 3 major approaches
  - **generalized coordinate** approach : n generalized coordinates correlated with m holonomic constraint forces $\Rightarrow s = n - m$ DOF for the equations of motion
    embeds the m constraints forces into the choice of generalized coordinates and does not determine the constraint forces
  - **Lagrange multiplier** approach : exploits generalized coordinates but includes the m constraint forces into the Euler equations to determine both the constraint forces in addition to the n equations of motion
  - **Generalized forces** approach : introduces constraint and other forces explicitly – see next chapter dedicated to **Lagrangian dynamics**.

Treatment of constraint forces in variational calculus

Generalized coordinate approach : minimal set of generalized coordinates

- m holonomic equations of constraint $\Rightarrow$ can be used to transform the n coordinates into $s = n - m$ independent generalized coordinates $(q_i)_{i\in\{1\ldots s\}}$.
  can be cumbersome
- avoid the need of explicit treatment of the constraint forces
- $(q_i)_{i\in\{1\ldots s\}}$ are independent and verify the Euler equation :

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0$$

Treatment of constraint forces in variational calculus

Lagrange multipliers for holonomic constraints

- use q_i instead of y_i, and t instead of x
- use of virtual infinitesimal displacements (symbol δ)
- virtual displacement δq_i is imagined to be an instantaneous, infinitesimal, displacement of a coordinate q_i, not an actual displacement, in order to elucidate the local dependence of the function F on the coordinate (actual displacement dq_i occurs during time dt).
- Local dependence of any function $F(q_i, t)$ to virtual displacements of all n coordinates : $\delta F = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i} \delta q_i$
- F is stationary if $\delta F = 0$
- if δq_i variations are free, we need to have $\frac{\partial F}{\partial q_i} = 0, \quad \forall i$

Treatment of constraint forces in variational calculus

Lagrange multipliers for holonomic constraints

- In presence of auxiliary conditions, due to the m holonomic algebraic constraints for the n variables q_i, the related m equations can be written as :

$$g_k(q_i, t) = 0, \quad k \in \{1 \dots m\}$$

- in terms of differential equations with time frozen (equations must hold at any t) :

$$\delta g_k = \sum_{i=1}^{n} \frac{\partial g_k}{\partial q_i} \delta q_i = 0, \quad k \in \{1 \dots m\}$$

- using arbitrary undetermined factors $\lambda_k(t)$, multiplying each δg_k by λ_k, and adding to δF leads to :

$$\delta F(q_i, t) + \sum_{k=1}^{m} \lambda_k \delta g_k = 0$$

Lagrange multipliers for holonomic constraints

- using the decomposition with $\delta q_i,\ i \in \{1 \dots n\}$:

$$\sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_i} \right) \delta q_i = 0$$

- $(\delta q_i)_{i \in \{1 \dots n\}}$ are not independent variations since they are linked by the m constraints g_k,
- $\Rightarrow$ there are only $s = n - m$ independent variations $(\delta q_i)_{i \in \{1 \dots s\}}$
- we can choose the $(\lambda_k)_{k \in \{1 \dots m\}}$ so that the m variations $(\delta q_j)_{j \in \{s+1 \dots n\}}$ vanish and set the m linear equations w.r.t. (λ_k) :

$$\frac{\partial F}{\partial q_j} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_j} \delta q_j = 0, \quad j \in \{s+1 \dots n\}$$

Treatment of constraint forces in variational calculus

Lagrange multipliers for holonomic constraints

- From :

$$\sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_i} \right) \delta q_i = 0$$

- it remains :

$$\sum_{i=1}^{\mathbf{s}} \left(\frac{\partial F}{\partial q_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_i} \right) \delta q_i = 0$$

with $(\delta q_i)_{i \in \{1 \dots s\}}$ free independent variations, which leads to

$$\boxed{\frac{\partial F}{\partial q_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_i} = 0, \quad i \in \{1 \dots s\}} \tag{5}$$

Lagrange multipliers for holonomic constraints

- Finally, combining all equations we have :

$$\frac{\partial F}{\partial q_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_i} = 0, \quad i \in \{1 \ldots n\}$$

- which can be considered as obtained by the variational principle (λ_k constant or function of time if F explicitly depends on t) :

$$\delta(F + \sum_{k=1}^{m} \lambda_k g_k) = 0$$

so asking for the stationary value of $F + \sum_{k=1}^{m} \lambda_k g_k$

Lagrange multipliers for holonomic constraints

- Dropping the auxiliary equations of constraint, we can consider the $n + m$ unknowns, i.e. n q_i, and m λ_k, which leads to the system of $n + m$ equations :

$$\begin{aligned} \frac{\partial}{\partial q_i}\left(F + \sum_{k=1}^{m} \lambda_k g_k\right) &= 0, \quad i \in \{1 \ldots n\} \\ \frac{\partial}{\partial \lambda_j}\left(F + \sum_{k=1}^{m} \lambda_k g_k\right) &= 0, \quad j \in \{1 \ldots m\} \end{aligned}$$

- the last m equations are actually the algebraic equations of constraints $g_j(q_i) = 0$

Treatment of constraint forces in variational calculus

Lagrange multipliers for holonomic constraints

- F is a functional :

$$F = \int_{t_1}^{t_2} f(q_i(t), q_i'(t); t)dt, \; i \in \{1, \ldots N\}$$

  with $\delta q_j(t = t_1) = 0$, $\delta q_j(t = t_2) = 0$
- objective : determine $q_i(t)$ such that $\delta F = 0$ under conditions $g_k(q_i) = 0$

$$\begin{aligned}
\delta f(q_i, q_i', t) &= f(q_i(t,\epsilon), q_i'(t,\epsilon), t) - f(q_i(t), q_i'(t), t) \\
&= \sum_i \left(\frac{\partial f}{\partial q_i} \delta q_i + \frac{\partial f}{\partial q_i'} \delta q_i' \right) \\
&= \epsilon \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial q_i}{\partial \epsilon} + \frac{\partial f}{\partial q_i'} \frac{\partial q_i'}{\partial \epsilon} \right) \\
\delta F &= \delta \int_{x_1}^{x_2} f dt = \int_{x_1}^{x_2} \delta f dt
\end{aligned}$$

Treatment of constraint forces in variational calculus

Lagrange multipliers for holonomic constraints

$$\begin{aligned}
\delta F &= \int_{t_1}^{t_2} \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial q_i}{\partial \epsilon} + \frac{\partial f}{\partial q_i'} \frac{\partial q_i'}{\partial \epsilon} \right) \epsilon dt \\
&= \int_{t_1}^{t_2} \sum_i^n \left(\frac{\partial f}{\partial q_i} - \frac{d}{dx} \frac{\partial f}{\partial q_i'} \right) \epsilon \eta_i(t) dt \\
&= \int_{t_1}^{t_2} \sum_i^n \left(\frac{\partial f}{\partial q_i} - \frac{d}{dx} \frac{\partial f}{\partial q_i'} \right) \delta q_i dt
\end{aligned}$$

- m holonomic constraints :

$$\delta g_k = \sum_{i=1}^n \frac{\partial g_k}{\partial q_i} \delta q_i = 0 \;\; \Rightarrow \;\; \int_{t_1}^{t_2} g_k(qi) dt = 0$$

$$\Rightarrow \;\; \int_{t_1}^{t_2} \left(\sum_{i=1}^n \frac{\partial g_k}{\partial q_i} \delta q_i \right) dt = 0$$

Treatment of constraint forces in variational calculus

Lagrange multipliers for holonomic constraints

- multiplying by $\lambda_k(t)$: $\delta F + \sum_{k=1}^{m} \lambda_k \delta g_k = 0$

$$\int_{t_1}^{t_2} \sum_{i}^{n} \left(\frac{\partial f}{\partial q_i} - \frac{d}{dt}\frac{\partial f}{\partial q_i'} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_i} \right) \delta q_i dt \quad = \quad 0$$

- we made the same rationale as before to get for all i from 1 to n :

$$\boxed{\frac{\partial f}{\partial q_i} - \frac{d}{dt}\frac{\partial f}{\partial q_i'} + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k}{\partial q_i} = 0} \tag{6}$$

- we can consider a system of $n+m$ unknowns, including the λ_k if we use use $f^* = f + \sum_k \lambda_k g_k$
- the forces of constraint are given by the $\lambda_k \frac{\partial g_k}{\partial q_i}$ terms

Treatment of constraint forces in variational calculus

Lagrange multipliers still applicable with non holonomic constraints

- auxiliary conditions not given as algebraic relations but as differential equations
- left-hand side no longer exact differential

$$\overline{\delta g_k} = \sum_{i=1}^{n} A_{ki} \delta q_i = 0 \quad (holonomic : \delta g_k = \sum_{i=1}^{n} \frac{\partial g_k}{\partial q_i} \delta q_i = 0)$$

A_{ki} are given functions of the q_i, which cannot be considered as the partial derivatives of a function g_k.

- replace $\frac{\partial g_k}{\partial q_i}$ by A_{ik} in the equations (5) and (6)
- be careful with initial conditions : velocities are now restricted according to the non holonomic conditions :

$$\sum_{i=1}^{n} A_{ki} \dot{q}_i = 0, \quad k \in \{1 \ldots m\}$$

Lagrange multipliers with isoperimetric constraints

- Isoperimetric constraints given by an integral form :

$$G(q_i) = \int_{t_1}^{t_2} g(q_i, q_i'; t)dt = \ell$$

  with ℓ having a fixed value,
- where the objective is to find the $q_i(t)$ such that the functional $F(q_i) = \int_{t_1}^{t_2} f(q_i, q_i'; t)dt$ has an extremum, while satisfying boundary conditions $q_i(t_1) = a_i$ and $q_i(t_2) = b_i$
- both functionals can be combined to require :

$$\delta\left[F(q_i) + \lambda G(q_i)\right] = \delta \int_{t_1}^{t_2} [f + \lambda g]\, dt = 0$$

  to find an extremum path for the function $K(q_i, t, \lambda) = F(q_i, t) + \lambda G(q_i)$, $q_i(t)$ and λ being the variables.

Treatment of constraint forces in variational calculus

Lagrange multipliers with isoperimetric constraints

- Therefore the $q_i(t)$ must satisfy the differential equation :

$$\boxed{\frac{\partial f}{\partial q_i} - \frac{d}{dt}\frac{\partial f}{\partial q_i'} + \lambda\left[\frac{\partial g}{\partial q_i} - \frac{d}{dt}\frac{\partial g}{\partial q_i'}\right] = 0} \qquad (7)$$

- λ is a constant, independent of time
- boundary conditions : $q_i(t_1) = a_i$ and $q_i(t_2) = b_i$
- constraint : $G(q_i) = \ell$

Lagrange multipliers : examples

- Catenary, isoperimetric problem of determination of the shape of a uniform rope or chain of fixed length ℓ with minimization of the gravitational potential energy.
- Queen Dido problem, famous constrained isoperimetric legend is that of Dido, first Queen of Carthage. Problem : how to enclose the maximum area for a given perimeter ?
- Geodesic, defined as the shortest path between two fixed points for motion that is constrained to lie on a surface. Consider the geodesic constrained to follow the surface of a sphere of radius R.