MINIMUM JERK PATH GENERATION

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ABSTRACT

This paper presents a simple method of trajectory generation of robot manipulators based on an optimal control problem formulation. It was found recently that the jerk, the third derivative of position, of the desired trajectory, adversely affects the efficiency of the control algorithms and therefore should be minimized. Assuming joint position, velocity and acceleration to be constrained a cost criterion containing jerk is considered. Initially, the simple environment without obstacles and constrained by the physical limitations of the joint angles only is examined. For practical reasons, the free execution time has been used to handle the velocity and acceleration constraints instead of the complete bounded state variable formulation. The problem of minimizing the jerk along an arbitrary cartesian trajectory is formulated and given analytical solution, making this method useful for real world environments containing obstacles.

1. INTRODUCTION

During the last two decades several control methods for robotic manipulators have been developed[5,11,12]. Most of the already developed control methods indicate that the control of manipulators is achieved in several sequential stages. One of these stages is trajectory planning, where a desired trajectory with prespecified position, velocity and acceleration is necessary. In some of these methods, in order to calculate the input vector $\tau(k)$ of torques at instant k, the desired position at the instant $k+2$ is needed, making off–line trajectory generation necessary[5].

Several methods of trajectory planning have been developed in the past years[2,6,10,13,14,16,21,22]. These methods assume a prespecified path in cartesian space that can be subsequently transformed by inverse kinematics, in joint space as a set of points, each representing a specific configuration of the arm[1,15].

Earlier methods[10,14] assume that the preplanned path to be traveled is composed of straight line segments in Cartesian coordinates, connected by smooth arcs, while the other[2,13,16,21,22] assume a general smooth cartesian path. A real time technique for obstacle avoidances is presented in[6], where the path generation problem is being done in the lower level of a hierarchical control system.

A generalized version of the statement of the problem of trajectory planning is given in[21]: "Given a curve in the robot's joint space, the robot's dynamic properties, and the robot's characteristics, what set of signals to the actuators will drive the robot from its current state to a desired final state with a minimum cost?"

The actuator's characteristics represent mainly torque constraints which can be expressed in terms of bounds related to torque, rather than torque itself. For example, a trajectory under constrained joint position, velocity acceleration and jerk is a sufficient trajectory for the input torques. The last term, jerk, has recently been defined as a serious constraining factor for the trajectory planning problem. Some trajectory planning formulations simply report the resulting jerk from their methods[23], while others attempt time–optimal solutions taking into account jerk constraints, thus producing bounded jerk trajectories[22].

Experimental results in the Robotics and Automation Labs of Rensselaer Polytechnic Institute[8], have indicated that joint position errors are increasing when jerk increases. The present work yields desired trajectories, by minimizing a cost criterion containing jerk in a space free of obstacles. The results are interesting since they show the significance of this factor, and challenge the opinion of some researchers that jerk must always be continuous[3].

The free execution time of a manipulation task has been used to account for the velocity and acceleration constraints instead of the complete bounded state variable formulation[20]. This approach yields simpler and more practical expressions for trajectory planning.

Section 2 states the formulation of the problem in joint space. Sections 3 and 4 solve it for some cost criteria. In Section 5 the statement of the problem in cartesian space is presented and analytical solutions for two criteria are given. Section 6 presents the resulting desired trajectories and gives on–line results. Finally in section 7 further research in this direction is suggested.

2. STATEMENT OF THE PROBLEM

The problem under consideration is expressed as: "Starting from an initial point i, in the joint space, definedby the joint angles vector, q^i, with $\dot{q}^i = \ddot{q}^i = 0$, go to the final point f defined by q^f, and $\dot{q}^f = \ddot{q}^f = 0$, by minimizing a cost function $J(q, \dot{q}, \ddot{q}, \dddot{q}, T)$". The execution time T is free and is specified in the above cost function in order to satisfy the inequality constraints on the states.

Consequently, the joint position, velocity and acceleration for the i-th joint are expressed as state variables x_{i1}, x_{i2}, x_{i3} respectively, then

$$\dot{x}_{i1}(t) = \dot{q}_i(t) = x_{i2}(t)$$
$$\dot{x}_{i2}(t) = \ddot{q}_i(t) = x_{i3}(t)$$
$$\dot{x}_{i3}(t) = \dddot{q}_i\,(t) = u_i$$

where jerk $\dddot{q}_i$ is expressed as an input u_i to the system. Doing the same for all n joints of the manipulator, a system describing the position, velocity and acceleration of the manipulator is obtained, with the jerk as an input.

$$\begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \\ \dot{x}_{13} \\ \dot{x}_{21} \\ \vdots \\ \dot{x}_{n1} \\ \dot{x}_{n2} \\ \dot{x}_{n3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & & & & \\ 0 & 0 & 1 & & & 0 & \\ 0 & 0 & 0 & & & & \\ & & & \ddots & & & \\ & & & & 0 & 1 & 0 \\ & 0 & & & 0 & 0 & 1 \\ & & & & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ \vdots \\ x_{n1} \\ x_{n2} \\ x_{n3} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \dots & & \\ 0 & 0 & 0 & \dots & & \\ 1 & 0 & 0 & \dots & & \\ 0 & 0 & 0 & \dots & & \\ 0 & 0 & 0 & \dots & & \\ 0 & 1 & 0 & \dots & & \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ & & & \dots & 0 & 0 \\ & & & \dots & 0 & 0 \\ & & & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_n \end{bmatrix} \tag{2.1}$$

This system is obviously decoupled, consisting of n–subsystems of the form

$$\dot{x}_i = A_i x_i + B_i u_i \qquad i - 1, 2, \dots, n$$

where

$$x_i = [q_i \ \dot{q}_i \ \ddot{q}_i]^T \tag{2.2}$$

and u_i is the input (jerk) for the ith joint, and

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad B_i = [0 \ \ 0 \ \ 1]^T \tag{2.3}$$

with initial and final conditions

$$x_i(0) = [q_i(0) \ \ 0 \ \ 0]^T \qquad x_i(T) = [q_i(T) \ \ 0 \ \ 0]^T \tag{2.4}$$

where T represents the free execution time for the motion.

The current problem is a bounded state one, due to the physical limitations of the joints and actuators, as expressed by the equations

$$\begin{aligned} q_i(0) = q_{i\min} \le q_i(t) = x_{i1}(t) \le q_{i\max} = q_i(T) \\ |\dot{q}_i(t)| = |x_{i2}(t)| \le v_{i\max} \\ |\ddot{q}_i(t)| = |x_{i3}(t)| \le a_{i\max} \end{aligned} \tag{2.5}$$

In this analysis, a cost function containing only the jerk is assumed. Therefore, a control is sought that

$$u_i^* \in U \ni J(u)^* \le J(u) \forall u \in U$$

where U is the set of admissible controls.

At this point a proper selection of the minimizing criterion $J(u_i)$ is necessary. In this work, solutions were obtained and gave desired trajectories that were tested on-line, for two kinds of cost functions. The first was a "max" type cost function

$$J(u) = \max_{t \epsilon [0,T]} |u| \tag{2.6}$$

and the second an input energy type cost function, used mainly for comparison purposes, since an approach based on that has been given in [4]

$$J(u) = \int_0^T u^2 dt \tag{2.7}$$

Analytical solutions for these cost functions are presented in the following two sections respectively.

3. MINIMIZATION OF THE MAXIMUM JERK FOR JOINT SPACE

The following criterion minimizes the maximum value of the jerk over the interval $[0,T]$:

$$J(u_i) = \max_{t \epsilon [0,T]} |u_i(t)|$$

The above optimal control problem is not trivial. Therefore, a more detailed solution which is closely related to[18,19] is presented here. The solution of the problem without the position, velocity and acceleration constraints is as follows:

First, introduce the costate vector $p_i(t) = [p_{i1}(t) \ \ p_{i2}(t) \ \ p_{i3}(t)]^T$ and the Hamiltonian

$$H(x_i, p_i, u_i, t) = \max_{t \epsilon [0,T]} |u_i(t)| + p_i^T(t) \cdot A \cdot x_i(t) + p_i^T(t) \cdot B \cdot u_i(t)$$

Assuming that $u_i(t)$ is bounded by some value α_i:

$$\max_{t \epsilon [0,T]} |u_i(t)| = \alpha_i \qquad \alpha_i \epsilon \Re_0^+$$

the Hamiltonian is rewritten as

$$H(x_i, p_i, u_i, t) = \alpha_i + p_i^T(t) A x_i(t) + p_i^T(t) B u_i(t) \tag{3.1}$$

The Pontryagins minimum principle[17], expressed in terms of the optimal values indicated by *, yields

$$\begin{aligned} H(u_i^*, p_i^*, u_i^*, t) = \alpha_i + p_i^{*^T}(t) \cdot A \cdot x_i^*(t) + p_i^{*^T}(t) \cdot B \cdot u_i^*(t) \le \\ \alpha_i + p_i^{*^T}(t) \cdot A \cdot x_i^*(t) + p_i^{*^T}(t) \cdot B \cdot u_i(t) \end{aligned} \tag{3.2}$$

for some arbitrary admissible control $u_i(t)$.
The costate equation is

$$\dot{p}_i^*(t) = -\frac{\partial H}{\partial x_i} = -A^T p_i^*(t) \Longrightarrow p_i^*(t) = e^{-A^T t} p^*(0) \tag{3.3}$$

$$\text{with } e^{-A^T t} = \begin{bmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ \frac{t^2}{2} & -t & 1 \end{bmatrix} \tag{3.4}$$

and

$$p_i^*(0) = [p_{i1}(0) \ \ p_{i2}(0) \ \ p_{i3}(0)]^T \tag{3.5}$$

Substituting (3.5) and (3.4) into (3.3) and subsequently to (3.2) one obtains

$$\alpha_i + p_{i3}^*(t) u_i^*(t) \le \alpha_i + p_{i3}^*(t) \cdot u_i(t) \tag{3.6}$$

where

$$p_{i3}^*(t) = p_{i3}(0) - t p_{i2}(0) + \frac{t^2}{2} p_{i1}(0) \tag{3.7}$$

The minimum of the right side of (3.7) is achieved for

$$u^*(t) = -\alpha_i \cdot \operatorname{sgn} p_{i3}^*(t) \tag{3.8}$$

The case where $p_{i3}^*(t)$, given by eq. (3.7), has one or no real roots in $[0,T]$ give no physical results. In the case of two real roots in $[0,T]$ these lie at $t_1 = \frac{T}{4}$ and $t_2 = 3\frac{T}{4}$, between which $p_{i3}^*(0) < 0$.

Consider the solution of (3.8) and the resulting trajectories in the resulting time subintervals

I. $\left[0, t_1 = \frac{T}{4}\right]$

$$\begin{aligned} u_i^*(t) &= \alpha_i \\ \ddot{q}_i(t) &= \alpha_i t \\ \dot{q}_i(t) &= \alpha_i \frac{t^2}{2} \\ q_i(t) &= \alpha_i \frac{t^3}{6} \end{aligned} \tag{3.9}$$

II. $\left[t_1 = \frac{T}{4}, \ t_2 = \frac{3T}{4}\right]$

$$\begin{aligned} u_i^*(t) &= \alpha_i \\ \ddot{q}_i(t) &= -\alpha_i t + \frac{\alpha_i^T}{2} \\ \dot{q}_i(t) &= -\frac{\alpha_i}{2} t^2 + \frac{\alpha_i T}{2} t - \frac{\alpha_i T^2}{16} \\ q_i(t) &= -\frac{\alpha_i}{6} t^3 + \frac{\alpha_i T}{4} t^2 - \frac{\alpha_i T^2}{16} t + \frac{\alpha_i T^3}{192} \end{aligned} \tag{3.10}$$

III. $\left[t_2 = \frac{3T}{4}, T\right]$

$$\begin{aligned} u_i^*(t) &= \alpha_i \\ \ddot{q}_i(t) &= \alpha_i t - \alpha_i T \\ \dot{q}_i(t) &= \alpha_i \frac{t^2}{2} - \alpha_i T t + \alpha_i \frac{T^2}{2} \\ q_i(t) &= \alpha_i \frac{t^3}{6} - \alpha_i \frac{T}{2} t^2 + \alpha_i \frac{T^2}{2} t - \frac{13}{96} \alpha_i T^3 \end{aligned} \tag{3.11}$$

The above equations were derived by applying the initial conditions $q_i(0) = \dot{q}_i(0) = \ddot{q}_i(0) = 0$, assuming continuity of $q_i, \dot{q}_i$, and $\ddot{q}_i$ everywhere, and applying the final point conditions $\dot{q}_i(T) = \ddot{q}(T) = 0$. Applying the final point condition $q_i(T) = S_i$ one obtains

$$\alpha_i = 32 \frac{S_i}{T^3} \tag{3.12}$$

The set of equations (3.9) – (3.12) give a complete solution, and the corresponding trajectories have been plotted in figures 1–4, under the label "by minimizing max$|u|$".

A complete study of the current problem requires consideration of the velocity and acceleration constraints. However, if time T is free, the control $u(t)$ may be selected to avoid the joint velocity and acceleration of upper bounds $V_{i\,\max}$ and $\alpha_{i\,\max}$ respectively. The minimum execution time T may be obtained to satisfy these constraints.

$$\max_{t\epsilon[0,T]} |\ddot{q}_i(t)| = \frac{8S_i}{T^2} < a_{i\,\max} \Longrightarrow T > 2.828\left(\frac{S_i}{a_{i\,\max}}\right)^{\frac{1}{2}} \tag{3.13}$$

and

$$\max_{t\epsilon[0,T]} |\dot{q}_i(t)| = \frac{2S_i}{T} < v_{i\,\max} \Longrightarrow T > 2\left(\frac{S_i}{v_{i\,\max}}\right) \tag{3.14}$$

or, combining (3.13) and (3.14),

$$T > T_i = \max\left[2.828\left(\frac{S_i}{a_{i\,\max}}\right)^{\frac{1}{2}}, \; 2\left(\frac{S_i}{v_{i\,\max}}\right)\right] \tag{3.15}$$

and seeking a lower bound for an n–link manipulator

$$T > \max_{i=1,\ldots,n} T_i \tag{3.16}$$

This last equation (3.16) gives just a lower bound of execution time, to satisfy the velocity and acceleration constraints of the joint actuators. However, this is a gross estimation of the minimum execution time for a specific task, since the relation between velocity and acceleration constraints, and the torque of the actuators is position dependent.

The position constraints do not affect the execution time T, because if the initial and final values of the joint angles, obtained by inverse kinematics, are acceptable, the values of the intermediate joint position trajectory have to lie between these two values. This can be easily verified either from equations (3.9)(3.10)(3.11) where we see that $\dot{q}_i(t)\cdot S_i \geq 0 \; \forall \; t\epsilon[0,T]$, or from figure 1 where the joint position trajectory is plotted.

4. THE "MINIMUM ENERGY" PROBLEM FOR JOINT SPACE

The cost function (2.6) is to be minimized, for the system (2.2) with initial and final conditions (2.4) and under the inequality constraints (2.5).

The solution of the problem without the constraints, is as follows:

Define the costate vector $p_i(t) = [p_{i1}(t) \;\; p_{i2}(t) \;\; p_{i3}(t)]^T$ and the Hamiltonian

$$H(x_i, u_i, p_i, t) = u_i^2 + p_i^T A x_i + p_i^T B u_i \tag{4.1}$$

Applying Pontryagin's maximum principle[17]

$$\frac{\partial H}{\partial u_i} = 0 \Longrightarrow u_i^*(t) = -\frac{1}{2}B^T p_i^*(t) \tag{4.2}$$

and

$$\frac{\partial H}{\partial x} = -p_i^*(t) \Longrightarrow p_i^*(t) = e^{-A^T t} p_i^*(0) \tag{4.3}$$

where $p_i^*(0) = [\,p_{i1}(0) \;\;\; p_{i2}(0) \;\;\; p_{i3}(0)\,]^T$

In order to find $u_i^*(t)$ from (4.3) explicitly, the costate transition matrix $e^{-A^T t}$ is first calculated

$$e^{-A^T t} = \begin{bmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ \frac{t^2}{2} & -t & 1 \end{bmatrix} \tag{4.4}$$

Plugging (4.4) to (4.3) and subsequently to (4.2), the optimal jerk $u_i^*(t)$ is found in terms of $p_{i1}(0), p_{i2}(0), p_{i3}(0)$. The origin of the coordinate system is transferred to the initial point i, in order to simplify the resulting expressions.

Thus $\qquad q_i(0) = \dot{q}_i(0) = \ddot{q}_i(0) = 0$

Now the resulting equations do not give the absolute trajectory, but the differential one with respect to the initial joint position, i.e. $\quad q_i(T) = S_i\,,\; \dot{q}_i(T) = \ddot{q}_i(T) = 0$ where S_i represents the final angular increase for joint i. Using the above initial final point conditions, the joint position, velocity, acceleration and jerk trajectories are obtained.

$$q_i(T) = \left(\frac{6}{T^2}t^5 - \frac{15}{T}t^4 + 10t^3\right) * \frac{S_i}{T^3} \tag{4.5.a}$$

$$\dot{q}_i(t) = \left(\frac{30}{T^2}t^4 - \frac{60}{T}t^3 + 30t^2\right) * \frac{S_i}{T^3} \tag{4.5.b}$$

$$\ddot{q}_i(t) = \left(\frac{120}{T^2}t^3 - \frac{180}{T}t^2 + 60t\right) * \frac{S_i}{T^3} \tag{4.5.c}$$

$$u_i(t) = \dddot{q}_i\,(t) = \left(\frac{360}{T^2}t^2 - \frac{360}{T}t + 60\right) * \frac{S_i}{T^3} \qquad i = 1,\ldots n \tag{4.5.d}$$

Trajectories representing the above equations have been plotted in figures 1–4, under the label "by minimizing $\int_0^T u^2 dt$".

Investigation of the joint velocity and acceleration constraints yield the value of the minimum execution time.

Using (4.5) one can derive

$$\max_{t\epsilon[0,T]} |\ddot{q}_i(t)| = 5.7735\frac{S_i}{T^2} < a_{i\,\max} \Longrightarrow T > 2.4028\left(\frac{S_i}{a_{i\,\max}}\right)^{\frac{1}{2}} \tag{4.6}$$

and

$$\max_{t\epsilon[0,T]} |\dot{q}_i(t)| = \dot{q}_i\left(\frac{T}{2}\right) = \frac{15}{8}\frac{S_i}{T} < v_{i\,\max} \Longrightarrow T > 1.875\left(\frac{S^i}{v_{i\,\max}}\right) \tag{4.7}$$

or combining (4.6) and (4.7)

$$T > T_i = \max\left[2.4028\left(\frac{S_i}{A_{i\,\max}}\right)^{\frac{1}{2}}, \; 1.874\left(\frac{S_i}{V_{i\,\max}}\right)\right] \tag{4.8}$$

and seeking a lower bound for an n–link manipulator

$$T > \max_{i=1,n} T_i \tag{4.9}$$

The remarks stated in the preceding section for this time lower bound are also valid here.

In addition to that, for the reasons expressed before,the position constraints do not affect the execution time.

5. JERK MINIMIZATION IN CARTESIAN SPACE

The problems stated so far and their solutions, refer to the case of joint space constrained only by the physical limitations of the robot's joint angles and not including possible obstacles. However, in a realistic environment obstacles exist constraining the working space. Therefore, usually a sequence of set points are given representing a collision–free path specified by cartesian coordinates, orientation Euler angels and by manipulator configurations (left–right arm, up or down elbow, flipped or non–flipped wrist, etc.). This sequence of set points is connected properly (e.g. spline functions or straight line segments) finally giving a trajectory $r(\lambda)$ where λ is a parameter of the cartesian trajectory (e.g. the length of the trajectory covered up to a specific point).

In the general case, the robot arm has to follow a fixed path in cartesian space given by

$$r = r(\lambda) = [p_x(\lambda) \; p_y(\lambda) \; p_z(\lambda) \; \phi(\lambda) \; \theta(\lambda) \; \psi(\lambda)]^T \qquad 0 \leq \lambda \leq \lambda\max$$

where p_x, p_y, p_z represents the cartesian coordinates defining the position and ϕ, θ, ψ are the Euler angles defining the orientation of the end effector. The elements of the above vector can be calculated having the joint angles $q = [q_1 \; q_2 \; \cdots q_n]^T$ of the arm through

$$p_x = p_x(q) \quad p_y = p_y(q) \quad p_z = p_z(q)$$
$$\phi = \phi(q) \quad \theta = \theta(q) \quad \psi = \psi(q)$$

using the direct kinematic equations of the arm. The above elements can be put in a vector

$$\pi = \pi(q) = [p_x(q) \; p_y(q) \; p_z(q) \; \phi(q) \; \theta(q) \; \psi(q)]^T$$

Therefore, the constraint to follow a fixed cartesian trajectory is formulated as

$$\pi(q) = r(\lambda) \qquad 0 \leq \lambda \leq \lambda\max \tag{5.1}$$

Introducing again e.q. (2.1)

$$x = [q_1 \; \dot{q}_1 \; \ddot{q}_1 \; q_2 \; \dot{q}_2 \; \ddot{q}_2 \cdots q_n \; \dot{q}_n \; \ddot{q}_n] \tag{5.2}$$

and therefore (5.1) is changed to

$$\pi(x) = r(\lambda) \tag{5.1.a}$$

The dynamics of the manipulator are represented by

$$D(q)\ddot{q} + H(q,\dot{q}) + G(q) + R(\dot{q}) = \tau$$

where $D(q)$ is the inertia matrix
$H(q,\dot{q})$ are the Coriolis and Centrifugal terms
$G(q)$ is the Gravity term
$R(\dot{q})$ is the viscous friction term
$q,\dot{q},\ddot{q},\tau = 6\times 1$ vectors representing the joint position, velocity, acceleration and torque.
The dynamic contraints require that

$$\tau \leq \tau_{\max}$$

where $\tau_{\max}$ is a 6×1 vector containing the maximum torques that the actuators can apply. If we introduce also the state vector x as in (6.2), then

$$w(x) - \tau_{\max} \leq 0 \tag{5.3}$$

where $\quad w(x) = D(q)\ddot{q} + H(q,\dot{q}) + G(q) + R(\dot{q})$.
The system equation is

$$\dot{x} = Ax + Bu$$

where x is given at (5.2) A, B, can be seen at (2.1) and u is the desired jerk trajectory. The initial conditions of the system are

$$x_0 = \left[q_1^0 \;\; 0 \;\; 0 \;\; q_2^0 \;\; 0 \;\; 0 \ldots q_n^0 \;\; 0\right]^T \tag{5.4}$$

while the final condition is

$$\Psi(x(T)) = x(T) - x_f = 0 \tag{5.5}$$

where $\quad x_f = \left[q_1^f \; 0 \; 0 \; q_2^f \; 0 \; 0 \; \cdots q_n^f \; 0 \; 0\right]^T$

Therefore, the statement of the problem, expressed verbally in the introduction, can now be formulated mathematically as:

"Find the optimal control input u^* to the system

$$\dot{x} = Ax + Bu \qquad , x_0 \text{ given} \tag{5.6}$$

driving the system to the terminal manifold expressed as in eq. (5.5)

$$\Psi(x(T)) = 0 \tag{5.7}$$

under the contraints

$$\pi(x) = r(\lambda) \tag{5.8}$$

and

$$w(x) - \tau_m \leq 0 \tag{5.9}$$

by minimizing some cost function

$$J(u) = \int_0^T L(u)dt \tag{5.10}$$

of the input (jerk)."

The above problem is an optimal control problem which, in order to be solved, the inequality constraint (5.9) has to be equality by assigning

$$w(x) - \tau_m + f(s) = 0 \tag{5.9.a}$$

where $\quad f(s) = \sum_{i=1}^{n} b_i s^T e_i s \geq 0$

where

$$b_i = [0 \cdots \overset{i^{th}}{\overset{\downarrow}{1}} \cdots 0]^T$$

$$e_i = \begin{array}{c} i^{th} \\ \downarrow \; column \\ \begin{bmatrix} 0 & & & & & \\ & 0 & & & 0 & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & 0 & & & & \ddots \\ & & & & & & 0 \end{bmatrix} \end{array} \leftarrow i^{th} \; row$$

Doing so and introducing Lagrange multipliers the cost is given by

$$J'(\nu,\mu,x,s,\rho,\lambda,p,u) = \nu^T\Psi(x(\tau)) + \\ + \int_0^T \{\mu^T(w(x) - \tau_m + f(s)) + \rho^T(\pi(x) - r(\lambda)) + \\ + p^T(Ax + Bu - \dot{x}) + L(u)\}\, dt$$

Therefore, the first variation is

$$\delta J' = \delta\nu^T\Psi(x(\tau)) + \nu^T \cdot \delta\Psi(x(\tau)) \cdot \delta x|_T + \\ \int_0^T \left[\delta\mu^T(w(x) - \tau_m + f(s)) + \mu^T\left(\frac{\partial w}{\partial x}\cdot\delta x + \frac{\partial f}{\partial s}\delta s\right) + \right. \\ \delta\rho^T(\pi(x) - r(\lambda)) + \rho^T\left(\frac{\partial \pi}{\partial x}\delta x - \frac{\partial r}{\partial \lambda}\delta\lambda\right) + \\ \delta p^T(Ax + Bu - \dot{x}) + p^T(A\cdot\delta x \\ \left. + B\delta u - \delta\dot{x})\right] dt$$

and since

$$\delta J'(\nu^*,\mu^*,x^*,s^*,\rho^*,\lambda^*,p^*,u^*) = 0$$

it results that for L, w, π, f, r, twice continuously differentiable functions

$$\dot{x} = Ax^* + Bu^* \qquad , x_0 \tag{5.10.1}$$

$$\psi(x(\tau)) = x(\tau) - x_f = 0 \tag{5.10.2}$$

$$\pi(x^*) = r(\lambda^*) \tag{5.10.3}$$

$$w(x^*) - \tau_m + f(s^*) = 0 \tag{5.10.4}$$

$$p^*(T) = \nu^*(T) \tag{5.10.5}$$

$$\mu^{*^T}\frac{\partial w^*}{\partial x} + \rho^{*^T}\frac{\partial \pi^*}{\partial x} + p^{*^T}A + \dot{p}^{*^T} = 0 \tag{5.10.6}$$

$$\mu^{*^T}\cdot\frac{\partial f^*}{\partial s} = 0 \tag{5.10.7}$$

$$\rho^{*^T}\cdot\frac{\partial r^*}{\partial \lambda} = 0 \tag{5.10.8}$$

$$\frac{\partial L}{\partial u} + p^{*^T}B = 0 \tag{5.10.9}$$

The above problem is a two point boundary value problem which has to be solved numerically.

5.1 Minmax problem

Assuming that

$$\max_{t\epsilon[0,T]} \sum_{i=1}^{n} |u_i| = \alpha$$

where $\quad \alpha > 0$
then define as the cost criterion

$$J(u) = \int_0^T \frac{1}{T}\max \|u\|_2 dt = \alpha$$

The optimal input u^* cannot be found by (5.10.9) since L is not twice differentiable. Using Pontryagin's maximum principal as in Section 3, one obtains

$$J(\nu^*,\mu^*,p^*,\rho^*,\lambda^*,s^*,x^*,u^*) \leq J(\nu^*,\mu^*,p^*,\rho^*,\lambda^*,s^*,x^*,u)$$

which eventually gives

$$0 \leq \max_{t\epsilon[0,T]} \sum_{i=1}^{n} |u_i| + p^{*^T}Bu^* \leq \max_{t\epsilon[0,T]} \sum_{i=1}^{n} |u_i| + p^{*^T}Bu$$

and since

$$\alpha = \max_{t\epsilon[0,T]} \sum_{i=1}^{n} |u_i|$$

the minimum is obtained if

$$\alpha + p^{*^T}Bu^* = 0$$

5.2 Weighted input problem

If we assign as cost function

$$L(u) = ||u||_R^2 = u^T R u$$

then (5.10.9) gives

$$2Ru^* + p^{*^T} B = 0 \Rightarrow u^* = -\frac{1}{2} R^{-1} p^{*^T} B$$

Therefore, minimum energy problem given in[4] was treated in the n–th space. Notice the fact that the above formulation does not include joint angles constraints because it would introduce n more state variables complicating the solution more. This is suggested for further research.

6. ON–LINE RESULTS

Since numerical solutions for the cartesian space problem have not been obtained yet, in this section only the joint unconstrained space is considered in order to validate the assumption for the negative effect of jerk on the performance of a manipulator.

In order to plot the trajectories resulting from the previous analysis and validate the claims made, consider the following motion of the end effector of the PUMA–600, which was actually tested on the PUMA–600 robotic manipulator of the Robotics and Automation Labs of Rensselaer Polytechnic Institute.

Initial Condition i

$$\begin{aligned} p_i &= [0 \quad 0.4 \quad 0.2]^T \\ n_i &= [0 \quad 0 \quad -1]^T \\ o_i &= [-1 \quad 0 \quad 0]^T \\ a_i &= [0 \quad 1 \quad 0]^T \end{aligned}$$

Final Condition f

$$\begin{aligned} p_f &= [0.4 \quad 0.4 \quad -0.2]^T \\ n_f &= [0 \quad 0 \quad -1]^T \\ o_f &= [-1 \quad 0 \quad 0]^T \\ a_f &= [0 \quad 1 \quad 0]^T \end{aligned}$$

The minimum execution time for the case of minimum energy problem is found from (4.8), $T_2 = 1.375$ sec, while for the case of minmax problem is found from (3.15), $T_2 = 1.4667$ sec. Therefore, we selected $T = 1.8$ sec.

The trajectories and the errors of joint 2 were considered as representative and were plotted in Figures 1–5.

In order to compare the resulting "optimal" trajectories, two more trajectories were considered. The first plotted under the label "cartesian triangular acceleration profile", follows a straight line between the initial and final point, with a cartesian acceleration having a triangular profile, starting and finishing with zero cartesian velocity and acceleration, which for a configuration with nonsingular Jacobian are translated to zero joint velocity and acceleration for all joints; something desirable for the actuators. The second, plotted under the label "simple cartesian straight line", is a constant cartesian velocity straight line trajectory having, as a matter of fact, large jerk at the beginning and the end of the trajectory.

The trajectories were tested using a computed torque/PID controller and the position errors can be found in Figure 5. The superiority of the "optimal" trajectories is obvious, especially for the intervals where the other have large jerk. However, the performance of the trajectory with "cartesian triangular acceleration profile" is considered very satisfactory.

7. CONCLUSIONS

The error analysis of Section 6 demonstrated the significance of jerk and the optimality of our results. Jerk affects adversely the performance of the actuators and has to be minimum. The mathematical analysis and solution for the unconstrained joint space problem was given to given an insight of the problem and also to verify the assumptions on the significance of jerk.

However, due to the fact that in a realistic environment the robot has to follow an arbitrary cartesian trajectory, in order to satisfy the obstacle avoidance specifications, this problem was stated and theoretically solved. In this case of a general cartesian trajectory the optimal control problem was reduced to a two–point boundary value problem as expected. Further research could be directed towards the numerical solution of the resulting equations and a more detailed solution including the joint angles constraints which have not been considered here, using a formulation involving the complete bounded state variable approach developed in[20].

Additionally, other cost functions, as minimum time or functions of jerk should be considered for minimization.

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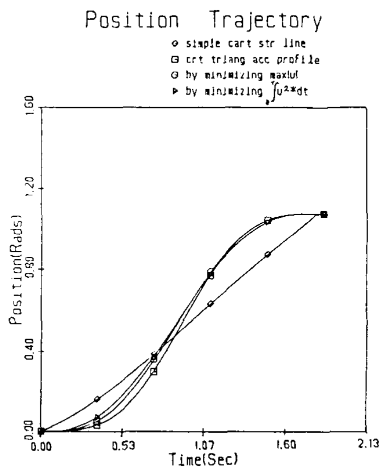

Figure 1. Differential Desired Position Trajectory wrt to Initial Angles for Joint 2

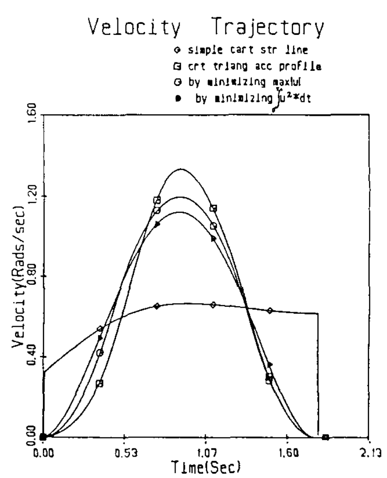

Figure 2. Desired Velocity Trajectory for Joint 2

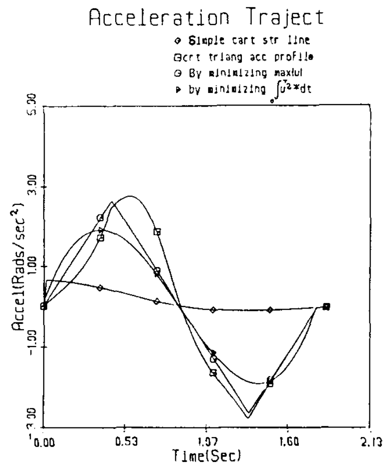

Figure 3. Desired Acceleration Trajectory for Joint 2

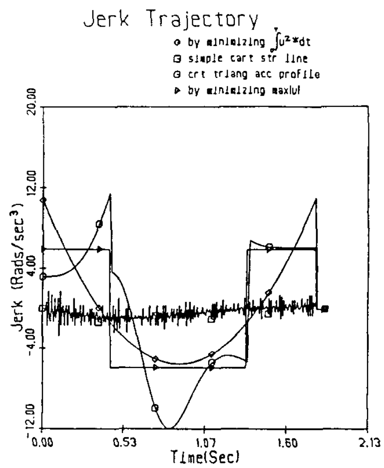

Figure 4. Resulting Jerk Trajectory for Joint 2

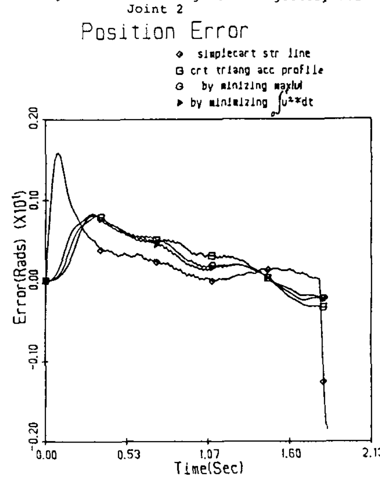

Figure 5. On-Line Position Errors for Joint 2