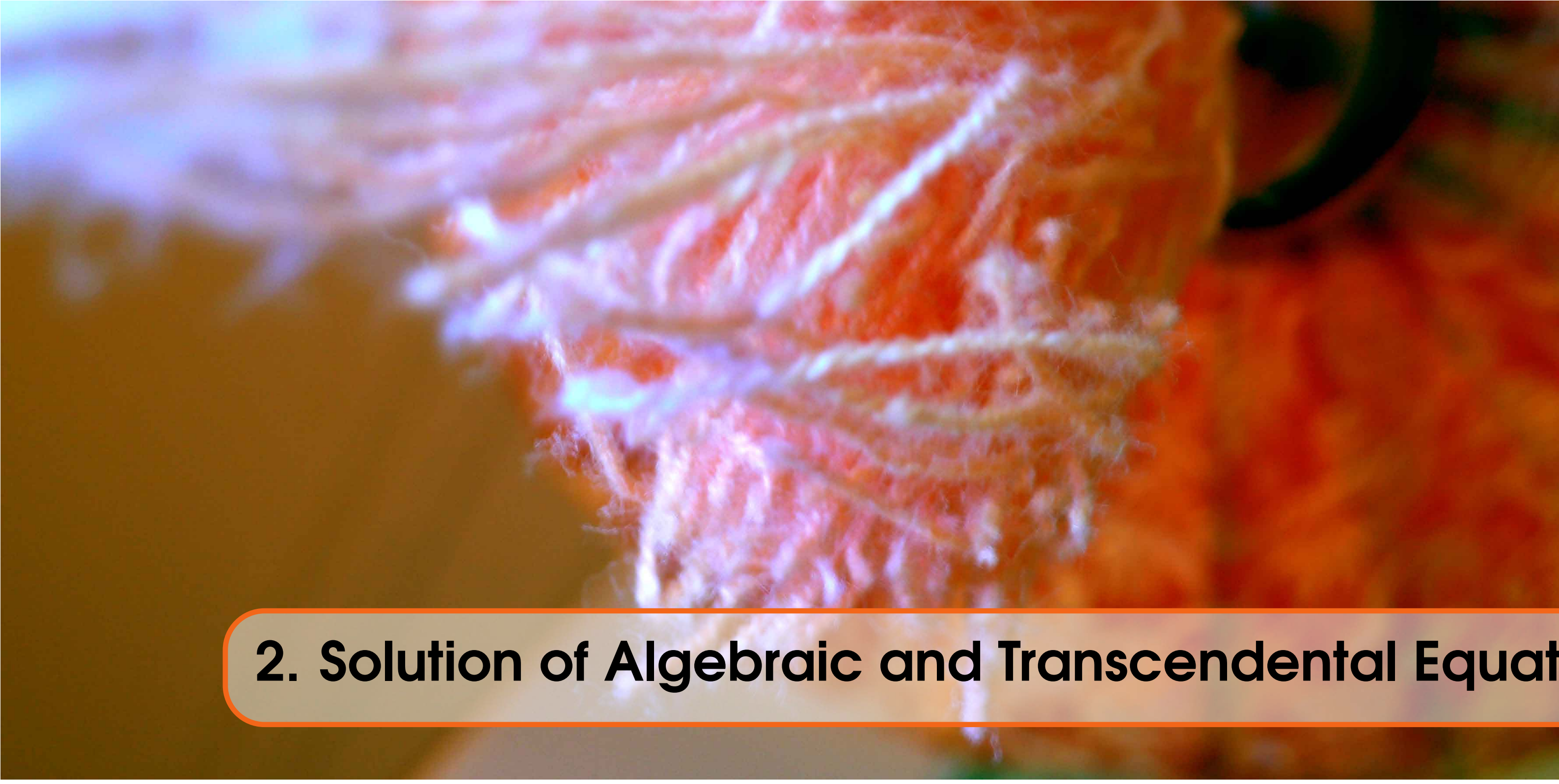

2. Solution of Algebraic and Transcendental Equat

2.1 Introduction to Direct and Iterative Methods

2.2 (Fixed point) Iteration Method

2.3 Order and Condition of Convergence

2.4 Secant and Newton-Raphson Methods for Simple Roots

2.5 Rates of Convergence

2.6 Solution of Algebraic and Transcendental Equations

The function of the form

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n \tag{2.1}$$

where n is a positive integer and $a_0, a_1, a_2, \ldots, a_n$ are constants, with $a_0 \neq 0$ and some of $a_1, a_2, \ldots, a_n$ may be zero is known as a polynomial in x with degree n. If we equate the above function $f(x)$ to zero then $f(x) = 0$ will represent an algebraic equation of degree n. i.e., $a_0x^n + a_1x^{n-1} + \ldots + a_n = 0$ represent an algebraic equation of degree n. When x in $f(x)$ occurs in an integral form i.e., x should not occur in denominator or with negative degree, then the function $f(x)$ is known as integral function and the equation $f(x) = 0$ is known as rational integral equation.

This rational integral equation is classified into two parts viz., (i) Algebraic or (ii) Transcendental equation according as

(i) $f(x)$ is purely a polynomial in x as in (2.1) or
(ii) contains some other functions (transcendental) such as trigonometric, logarithmic or exponential etc.

Example :

(a) $x^3 - 3x + 1 = 0, x^4 + 2x^3 - 3x^2 + 2x + 1 = 0$ are algebraic equations.
(b) $3x - \cos x - 1 = 0, x\log 10x - 1.2 = 0, 3x + \sin x - e^x = 0$ are transcendental equations.

The values of x which makes $f(x)$ zero are known as zero's (or) roots of the function $f(x)$.

Solution of an equation $f(x) = 0$ means we have to find its roots or zeros.

2.6.1 Properties of Equations

1. If $f(a)$ and $f(b)$ have opposite signs then one root of $f(x) = 0$ lies between a and b.
2. Every equation of an odd degree has atleast one real root whose sign is opposite to that of its last term.
3. Every equation of an even degree with last term negative, has alleast a pair of real roots one positive and other negative.
4. To find an equation whose roots are with opposite signs to those of the given equation, change x to $-x$.
5. To find an equation whose roots are reciprocals of the roots of the given equation, change x to $\frac{1}{x}$.

In this chapter we will study some of the following important methods of solving Algebraic and transcendental equations

1) Fixed Point Iteration : $x = g(x)$ method (or) Method of successive approximation.
2) Newton's method (or) Newton's Raphson method.

2.7 Fixed Point Iteration : $x = g(x)$ method or Method of Successive Approximation

Let $f(x) = 0$ be the given equation whose roots are to be determined. In this iteration method, first we write the given equation in the form $x = \phi(x)$.

Let $x = x_0$ be an initial approximation of the required root α, then the first approximation x_1 is given by $x_1 = \phi(x_0)$.

The second, third, etc. approximations are given by

$$\begin{aligned} x_2 &= \phi(x_1) \\ x_3 &= \phi(x_2) \\ x_4 &= \phi(x_3) \\ \dots & \quad \dots \\ \dots & \quad \dots \\ x_n &= \phi(x_{n-1}) \end{aligned}$$

Here x_n is the n^{th} iteration and the value of x_n gives the root of the given equation at the n^{th} iteration.

2.7.1 Sufficient Condition for Convergence of Iterations

Let $x = \alpha$ be a root of the equation $f(x) = 0$ which is equivalent to $x = \phi(x)$. Let I be any interval containing the root α. If $|\phi'(x)| < 1$ for all x in I, then the sequence of approximations $x_0, x_1, \dots, x_n$ will converge to the root α, provided the initial approximation x_0 is chosen in I.

Proof. Since α is a root of $f(x) = 0$.

$$x = \phi(x)$$

We have
$$\alpha = \phi(\alpha) \tag{2.2}$$

Let x_{n-1} and x_n be two consecutive approximations to α, we have

$$x_n = \phi(x_{n-1}) \tag{2.3}$$

Subtracting equations (2.3) and (2.2) we get,

$$x_n - \alpha = \phi(x_{n-1}) - \phi(\alpha) \tag{2.4}$$

Dividing throughout by $x_{n-1} - \alpha$, we get,

$$\frac{(x_n - \alpha)}{x_{n-1} - \alpha} = \frac{\phi\,(x_{n-1}) - \phi(\alpha)}{x_{n-1} - \alpha}$$

By mean value theorem, we have

$$\frac{\phi\,(x_{n-1}) - \phi(\alpha)}{x_{n-1} - \alpha} = \phi'(\xi) \text{ where } x_{n-1} < \xi < \alpha$$

$$(x_{n-1} - \alpha)\,\phi'(\xi) = \phi\,(x_{n-1}) - \phi(\alpha) \tag{2.5}$$

Substituting (2.5) in (2.4) we get,

$$x_n - \alpha = (x_n - 1 - \alpha)\,\phi'(\xi) \tag{2.6}$$

Let M be the maximum value of $\phi'(x)$ over the interval I including $x_0, x_1, \ldots, x_n, \alpha$, we have by (2.6)

$$|x_n - \alpha| \leq M\,|x_{n-1} - \alpha| \tag{2.7}$$

Replacing n by $n-1$, (2.7) gives

$$|x_{n-1} - \alpha| \leq M\,|x_{n-2} - \alpha| \tag{2.8}$$

$$\begin{aligned}
\therefore \qquad |x_n - \alpha| &\leq M\,|x_{n-1} - \alpha| \text{ gives}\\
|x_n - \alpha| &\leq M^2\,|x_{n-2} - \alpha| && [by(2.8)]\\
&\leq M^3\,|x_{n-3} - \alpha|\\
&\cdots\\
&\cdots\\
&\leq M^n\,|x_0 - \alpha|
\end{aligned}$$

Now if $M < 1$ over the interval I, and as n increases, the RHS becomes small and therefore $\text{Lim}\,|x^n - \alpha| = 0$ (when n, the number of iterations is large).

Hence the sequence of approximations converges to α provided $M < 1$ i.e., $|\phi'(x)| < 1$ for all x in I. ■

Note 2.7.1

1. The iteration process converges quickly if $|\phi'(x)| < 1$ where $x = \phi(x)$ is the given equation.
2. If $|\phi'(x)| > 1, |x_n - \alpha|$ will become infinitely large and hence this process will not converge.
3. This method is very useful to find the root of an equation given in the form of an infinite series.
4. Consider the equation $f(x) = x^3 + x - 1 = 0$

$$x = \frac{1}{1+x^2} = \phi(x)$$

Clearty

$$f(0) = -\text{ ve}$$
$$f(1) = +\text{ ve}$$

Hence the real root lies between 0 and 1 . Now

$$\phi'(x) = \frac{-2x}{(1+x^2)^2}$$

$|\phi'(x)| < 1$ in the interval $(0,1)$. For, if we take $x = 0.7$,

$$\left|\phi'(0.7)\right| = \frac{-1.4}{(1.49)^2} < 1$$

which shows that iteration will give better results. But if we write $f(x) = 0$ as $x = 1 - x^3 = \phi(x)$, then

$$\left|\phi'(x)\right| = 3x^2 > 1 \text{ for } x = 0.7$$

Here iteration will not work if we choose $x = 1 - x^3 = \phi(x)$.

5. From equation (2.7), we have the relation $|x_n - \alpha| \leq M|x_{n-1} - \alpha|$ where ' M ' is a constant. Hence the error at any stage is proportional to the error in the previous stage. We say that the convergence is linear.

■ **Example 2.1** Find the real root of a equation $x^3 + x^2 - 100 = 0$, using iteration method. ■

Solution.

Let $\quad f(x) = x^3 + x^2 - 100$

Now
$$\begin{aligned} f(3) &= (3)^3 + (3)^2 - 100 \\ &= 27 + 9 - 100 \\ &= -64(-ve) \\ f(4) &= (4)^3 + (4)^2 - 100 = -20(-\text{ve}) \end{aligned}$$

and $\quad f(5) = (5)^3 + (5)^2 - 100 = 50(+\text{ve})$

Hence a real root lies between 4 and 5 .

Now, $\quad x^3 + x^2 - 100 = 0$ can be written as

$$\begin{aligned} x^2(x+1) - 100 &= 0 \\ x^2 &= \frac{100}{x+1} \\ x &= \frac{10}{\sqrt{x+1}} \end{aligned}$$

Let
$$\phi(x) = \frac{10}{\sqrt{x+1}}$$

Now
$$\begin{aligned} \phi'(x) &= \frac{\frac{-10}{2\sqrt{x+1}}}{x+1} = \frac{-10}{2(x+1)^{3/2}} \\ &= -\frac{5}{(x+1)^{3/2}} \end{aligned}$$

Clearly,
$$\left|\phi'(x)\right| = \left|-\frac{5}{(x+1)^{3/2}}\right| < 1 \text{ in } (4,5)$$

For, when $x = 4.2$, $|\phi'(4.2)| = \left|\frac{5}{(5.2)^{3/2}}\right| < 1$.

Let the initial approximation be $x_0 = 4.2$. Now,

$$x_1 = \phi(x_0) = \frac{10}{\sqrt{(x_0+1)}} = \frac{10}{\sqrt{(4.2+1)}} = 4.38529$$
$$x_2 = \phi(x_1) = \frac{10}{\sqrt{(x_1+1)}} = \frac{10}{\sqrt{(4.38529+1)}} = 4.30919$$
$$x_3 = \phi(x_2) = \frac{10}{\sqrt{(x_2+1)}} = \frac{10}{\sqrt{(4.30919+1)}} = 4.33996$$
$$x_4 = \phi(x_3) = \frac{10}{\sqrt{(x_3+1)}} \equiv \frac{10}{\sqrt{(4.33996+1)}} = 4.32744$$
$$x_5 = \phi(x_4) = \frac{10}{\sqrt{(x_4+1)}} = \frac{10}{\sqrt{(4.32744+1)}} = 4.33252$$
$$x_6 = \phi(x_5) = \frac{10}{\sqrt{(x_5+1)}} = \frac{10}{\sqrt{(4.33252+1)}} = 4.33046$$
$$x_7 = \phi(x_6) = \frac{10}{\sqrt{(x_6+1)}} = \frac{10}{\sqrt{(4.33046+1)}} = 4.33129$$
$$x_8 = \phi(x_7) = \frac{10}{\sqrt{(x_7+1)}} = \frac{10}{\sqrt{(4.33129+1)}} = 4.33096$$
$$x_9 = \phi(x_8) = \frac{10}{\sqrt{(x_8+1)}} = \frac{10}{\sqrt{(4.33096+1)}} = 4.33109$$
$$x_{10} = \phi(x_9) = \frac{10}{\sqrt{(x_9+1)}} = \frac{10}{\sqrt{(4.33109+1)}} = 4.33104$$
$$x_{11} = \phi(x_{10}) = \frac{10}{\sqrt{(x_{10}+1)}} = \frac{10}{\sqrt{(4.33104+1)}} = 4.33106$$
$$x_{12} = \phi(x_{11}) = \frac{10}{\sqrt{(x_{11}+1)}} = \frac{10}{\sqrt{(4.33106+1)}} = 4.33105$$
$$x_{13} = \phi(x_{12}) = \frac{10}{\sqrt{(x_{12}+1)}} = \frac{10}{\sqrt{(4.33105+1)}} = 4.33105$$

Since the values of x_{12} and x_{13} are equal, the root is 4.33105 . ■

■ **Example 2.2** Find the negative root of the equation $x^3 - 2x + 5 = 0$, by using iteration method. ■

Solution. Given equation is

$$x^3 - 2x + 5 = 0 \tag{2.9}$$

We know that if α, β and γ are the roots of equation (2.9), then the equation whose roots are $-\alpha, -\beta, -\gamma$ is

is
$$x^3 + (-1)0x^2 + (-1)^2(-2x) + (-1)^3 5 = 0$$
$$x^3 - 2x - 5 = 0 \tag{2.10}$$

The negative root of the equation (2.9) is same as the positive root of (2.10).

Let $\quad f(x) = x^3 - 2x - 5$

Now $\quad f(1) = 1 - 2 - 5 = -6 = (-\text{ ve })$

$\quad f(2) = 8 - 4 - 5 = -1 = (-\text{ ve })$

and $\quad f(3) = 27 - 6 - 5 = 16 = (+\text{ ve })$

Hence the root lies between 2 and 3 . Equation (2.10) can be written as

$$x^3 = 2x+5$$

i.e.,
$$x = (2x+5)^{1/3}$$

i.e.,
$$\phi(x) = (2x+5)^{1/3}$$

Let
$$\phi'(x) = \frac{1}{3}(2x+5)^{-2/3}$$

$$|\phi'(x)| = \left|\frac{2}{3(2x+5)^{2/3}}\right|$$

In the interval $(2,3), |\phi'(x)| < 1$.

Therefore we can apply iteration method. Since the root is nearer to 2 , let the first approximation be $x_0 = 2$.

The successive approximation are as follows.

$$x_1 = \phi\,(x_0) = [2x_0+5]^{1/3} = [(2\times 2)+5]^{1/3} = 2.08008$$
$$x_2 = \phi\,(x_1) = [2x_1+5]^{1/3} = [(2\times 2.08008)+5]^{1/3} = 2.09235$$
$$x_3 = \phi\,(x_2) = [2x_2+5]^{1/3} = [(2\times 2.09235)+5]^{1/3} = 2.09422$$
$$x_4 = \phi\,(x_3) = [2x_3+5]^{1/3} = [(2\times 2.09422)+5]^{1/3} = 2.09450$$
$$x_5 = \phi\,(x_4) = [2x_4+5]^{1/3} = [(2\times 2.09450)+5]^{1/3} = 2.09454$$
$$x_6 = \phi\,(x_5) = [2x_5+5]^{1/3} = [(2\times 2.09454)+5]^{1/3} = 2.09455$$
$$x_7 = \phi\,(x_6) = [2x_6+5]^{1/3} = [(2\times 2.09455)+5]^{1/3} = 2.09455$$

Since the values of x_6 and x_7 are equal, the root is 2.09455 . Therefore the negative root of the given equation is -2.09455 .

■ **Example 2.3** Solve $e^x - 3x = 0$ using fixed point iteration. ■

Solution.

Let
$$\begin{aligned} f(x) &= e^x - 3x \\ f(0) &= e^0 - 0 = 1(+\text{ve}) \\ f(1) &= e^1 - 3(1) \\ &= -0.2817 = (-\text{ve}) \end{aligned}$$

$\therefore$ The root lies between 0 and 1 .

The given equation can be written as (or)

$$e^x = 3x$$

(or)
$$x = \frac{e^x}{3}$$

Let
$$\phi(x) = \frac{e^x}{3}$$

$$\phi'(x) = \frac{e^x}{3}$$

Here
$$\phi'(0) = \frac{e^0}{3} = \frac{1}{3} < 1$$

$$\phi'(1) = \frac{e}{3} < 1$$

$\therefore |\phi'(x)| < 1$ in the interval $[0,1]$.
Let $x_0 = 0.5$

$$x_1 = \phi(x_0) = \phi(0.5) = \frac{e^{0.5}}{3} = 0.5496$$
$$x_2 = \phi(x_1) = \phi(0.5496) = \frac{e^{0.5496}}{3} = 0.5775$$
$$x_3 = \phi(x_2) = \phi(0.5775) = \frac{e^{0.5775}}{3} = 0.5939$$
$$x_4 = \phi(x_3) = \phi(0.5939) = \frac{e^{0.5939}}{3} = 0.6037$$
$$x_5 = \phi(x_4) = \phi(0.6037) = \frac{e^{0.6037}}{3} = 0.6096$$
$$x_6 = \phi(x_5) = \phi(0.6096) = \frac{e^{0.6096}}{3} = 0.6132$$
$$x_7 = \phi(x_6) = \phi(0.6132) = \frac{e^{0.6132}}{3} = 0.6155$$
$$x_8 = \phi(x_7) = \phi(0.6155) = \frac{e^{0.6155}}{3} = 0.6168$$
$$x_9 = \phi(x_8) = \phi(0.6168) = \frac{e^{0.6168}}{3} = 0.6177$$
$$x_{10} = \phi(x_9) = \phi(0.6177) = \frac{e^{0.6177}}{3} = 0.6182$$
$$x_{11} = \phi(x_{10}) = \phi(0.6182) = \frac{e^{0.6182}}{3} = 0.6185$$
$$x_{12} = \phi(x_{11}) = \phi(0.6185) = \frac{e^{0.6185}}{3} = 0.6187$$
$$x_{13} = \phi(x_{12}) = \phi(0.6187) = \frac{e^{0.6187}}{3} = 0.6188$$
$$x_{14} = \phi(x_{13}) = \phi(0.6188) = \frac{e^{0.6188}}{3} = 0.6189$$
$$x_{15} = \phi(x_{14}) = \phi(0.6189) = \frac{e^{0.6189}}{3} = 0.6189$$

Since x_{14} and x_{15} are equal the required root is 0.6189. ■

■ **Example 2.4** Find the real root of the equation $\cos x = 3x - 1$, using iteration method. ■

Solution. Let

$$f(x) = \cos x - 3x + 1$$
$$f(0) = \cos 0 - 0 + 1 = 2 = (+\text{ ve })$$
$$f\left(\frac{\pi}{2}\right) = 0 - 3\frac{\pi}{2} + 1 = (-\text{ve})$$

$\therefore$ A root lies between 0 and $\frac{\pi}{2}$. The given equation can be written as

$$x = \frac{1}{3}(1 + \cos x)$$

Let $\phi(x) = \frac{1}{3}(1 + \cos x)$

$$\phi'(x) = -\frac{\sin x}{3}$$

Clearly,

$$\left|\phi'(x)\right| = \frac{|\sin x|}{3} < 1 \text{ in } \left(0, \frac{\pi}{2}\right).$$

Hence iteration method can be applied. Let the initial approximation be $x_0 = 0$.

The successive approximation are as follows:

$$\begin{aligned}
x_1 = \phi\left(x_0\right) &= \frac{1}{3}\left(1+\cos x_0\right) = \frac{1}{3}(1+\cos 0) = 0.66667 \\
x_2 = \phi\left(\iota_1\right) &= \frac{1}{3}\left(1+\cos x_1\right) = \frac{1}{3}(1+\cos 0.66667) = 0.59529 \\
x_3 = \phi\left(x_2\right) &= \frac{1}{3}\left(1+\cos x_2\right) = \frac{1}{3}(1+\cos 0.59529) = 0.60933 \\
x_4 = \phi\left(x_3\right) &= \frac{1}{3}\left(1+\cos x_3\right) = \frac{1}{3}(1+\cos 0.60933) = 0.60668 \\
x_5 = \phi\left(x_4\right) &= \frac{1}{3}\left(1+\cos x_4\right) = \frac{1}{3}(1+\cos 0.60668) = 0.60718 \\
x_6 = \phi\left(x_5\right) &= \frac{1}{3}\left(1+\cos x_5\right) = \frac{1}{3}(1+\cos 0.60718) = 0.60709 \\
x_7 = \phi\left(x_6\right) &= \frac{1}{3}\left(1+\cos x_6\right) = \frac{1}{3}(1+\cos 0.60709) = 0.60710 \\
x_8 = \phi\left(x_7\right) &= \frac{1}{3}\left(1+\cos x_7\right) = \frac{1}{3}(1+\cos 0.60710) = 0.60710
\end{aligned}$$

Since the values of x_7 and x_8 are equal, the required root is 0.60710 . ■

■ **Example 2.5** Use the iteration method to find a root of the equation

$$x = \frac{1}{2} + \sin x$$

■

Solution.

Let $$f(x) = \sin x - x + \frac{1}{2}$$

Now $$f(0) = \sin 0 - 0 + \frac{1}{2} = \frac{1}{2}(+\text{ve})$$

$$f(1) = \sin 1 - 1 + \frac{1}{2} = 0.84 - 0.5 = (+\text{ve})$$

and $$f(2) = \sin 2 - 2 + \frac{1}{2} = 0.909 - 1.5 = (-\text{ve})$$

Hence a root lies between 1 and 2. The given equation can be writtern as

$$x = \sin x + \frac{1}{2}$$

Let $$\phi(x) = \sin x + \frac{1}{2}$$

$$\left|\phi'(x)\right| = |\cos x| < 1 \text{ in } (1,2)$$

Hence iteration method can be applied. Let the initial approximation b_f $x_0 = 1$.

The successive approximations are as follows :

$$x_1 = \phi(x_0) = \sin(x_0) + \frac{1}{2} = \sin 1 + \frac{1}{2} = 0.8414 + 0.5 = 1.3414$$
$$x_2 = \phi(x_1) = \sin(x_1) + \frac{1}{2} = \sin(1.3414) + \frac{1}{2} = 0.9738 + 0.5 = 1.4738$$
$$x_3 = \phi(x_2) = \sin(x_2) + \frac{1}{2} = \sin(1.4738) + \frac{1}{2} = 0.9952 + 0.5 = 1.4953$$
$$x_4 = \phi(x_3) = \sin(x_3) + \frac{1}{2} = \sin(1.4953) + \frac{1}{2} = 0.9971 + 0.5 = 1.4972$$
$$x_5 = \phi(x_4) = \sin(x_4) + \frac{1}{2} = \sin(1.4972) + \frac{1}{2} = 0.9972 + 0.5 = 1.4972$$

Since x_4 and x_5 are equal, the required root is 1.4972 . ■

■ **Example 2.6** Solve by iteration method $2x - \log_{10} x = 7$ ■

Solution.

Let $f(x) = 2x - \log_{10} x - 7$

Now, $f(1) = 2(1) - \log_{10} 1 - 7 = 2 - 0 - 7 = -5 = -$ ve

$f(2) = 2(2) - \log_{10} 2 - 7 = 4 - 0.3010 - 7 = -3.3010 = -$ ve

$f(3) = 2(3) - \log_{10} 3 - 7 = 6 - 0.4771 - 7 = -1.4771 = -$ ve

and $f(4) = 4(2) - \log_{10} 4 - 7 = 8 - 0.602 - 7 = 0.3979 = +$ ve

Hence a real root lies between 3 and 4 . The given equation can be written as

$$x = \frac{1}{2}(\log_{10} x + 7)$$

Let $$\phi(x) = \frac{1}{2}(\log_{10} x + 7)$$

Now $$\phi'(x) = \frac{1}{2}\left[\frac{1}{x}\log_{10} e\right]$$

$$|\phi'(x)| < 1 \text{ for } x \text{ lies in } (3,4).$$

For, let $x = 3.7$

Then $$|\phi'(x)| = \frac{1}{2}\left[\frac{1}{3.7} \times 0.4343\right] < 1 \qquad [\because \log_{10} e = 0.4343]$$

Hence iteration method can be applied. Since $|f(4)| < |f(3)|$, the root is nearer to 4 .

Since the root is nearer to 4 , let the first approximation be $x_0 = 3.6$

$$x_1 = \phi(x_0) = \frac{1}{2}(\log_{10} x_0 + 7) = \frac{1}{2}(\log_{10} 3.6 + 7) = 3.77815$$
$$x_2 = \phi(x_1) = \frac{1}{2}(\log_{10} x_1 + 7) = \frac{1}{2}(\log_{10} 3.77815 + 7) = 3.78863$$
$$x_3 = \phi(x_2) = \frac{1}{2}(\log_{10} x_2 + 7) = \frac{1}{2}(\log_{10} 3.78863 + 7) = 3.78924$$
$$x_4 = \phi(x_3) = \frac{1}{2}(\log_{10} x_3 + 7) = \frac{1}{2}(\log_{10} 3.78924 + 7) = 3.78927$$

Since the difference between x_3 and x_4 is very small the root is 3.78927. ■

■ **Example 2.7** Find a positive root of $3x - \log_{10} x = 6$, using fixed iteration method. ■

Solution.

Let
$$\begin{aligned}
f(x) &= 3x - \log_{10} x - 6 \\
f(1) &= 3 - \log_{10} 1 - 6 \quad = -3(-\text{ve}) \\
f(2) &= 3(2) - \log_{10} 2 - 6 = -0.3010 = (-\text{ve}) \\
f(3) &= 3(3) - \log_{10} 3 - 6 = 3 - \log_{10} 3 = 3 - 0.4771 = 2.5229 = (+\text{ve})
\end{aligned}$$

$\therefore$ A root lies between 2 and 3.

The given equation can be written as

Let
$$\begin{aligned}
x &= \frac{1}{3}[6 + \log_{10} x] \\
\phi(x) &= \frac{1}{3}[6 + \log_{10} x] \\
\phi'(x) &= \frac{1}{3}\left[\frac{1}{x}\log_{10} x\right]
\end{aligned}$$

Clearly $|\phi'(x)| < 1$ in the interval $[2,3]$.

Take $x_0 = 2$

$$\begin{aligned}
x_1 = \phi(x_0) &= \frac{1}{3}[6 + \log_{10} x_0] = \frac{1}{3}[6 + \log_{10} 2] = 2.1003 \\
x_2 = \phi(x_1) &= \frac{1}{3}[6 + \log_{10} x_1] = \frac{1}{3}[6 + \log_{10} 2.1003] = 2.1074 \\
x_3 = \phi(x_2) &= \frac{1}{3}[6 + \log_{10} x_2] = \frac{1}{3}[6 + \log_{10} 2.1074] = 2.1079 \\
x_4 = \phi(x_3) &= \frac{1}{3}[6 + \log_{10} x_3] = \frac{1}{3}[6 + \log_{10} 2.1079] = 2.10795 \\
x_5 = \phi(x_4) &= \frac{1}{3}[6 + \log_{10} x_4] = \frac{1}{3}[6 + \log_{10} 2.10795] = 2.10795
\end{aligned}$$

Since x_4 and x_5 are equal the required root is 2.10795 ■

■ **Example 2.8** Solve the equation $x^2 - 2x - 3 = 0$ for the positive root by iteration method. ■

Solution.

Given $\quad x^2 - 2x - 3 = 0$

Let
$$f(x) = x^2 - 2x - 3 \tag{2.11}$$
$$\begin{aligned}
f(-2) &= (-2)^2 - 2(-2) - 3 \\
&= 4 + 4 - 3 = 5(+\text{ve}) \\
f(-1) &= (-1)^2 - 2(-1) - 3 \\
&= 1 + 2 - 3 = 0 \\
f(1) &= 1 - 2(1) - 3 = -4(-\text{ve})
\end{aligned}$$

Here
$$\begin{aligned}
f(-2) &= +\backslash\{ve\} \\
f(1) &= -\text{ve}
\end{aligned}$$

But
$$f(-1) = 0$$

$\therefore -1$ is a negative root of (2.11).

Here we have to find the positive root by iteration.

$\therefore$ We pre-assume that $-1\cdot 2$ is an approximate root of (1).

Rearranging (2.11), we get

$$x^2 = 2x+3$$

$$x = \sqrt{2x+3}$$

Take $$\phi(x) = \sqrt{2x+3}$$

Let $x_0 = -\mathbf{1\cdot 2}$

$$\begin{aligned}
x_1 = \phi(x_0) &= \sqrt{2x_0+3} &&= \sqrt{2(-1\cdot 2)+3} &&= 0.7746\\
x_2 = \phi(x_1) &= \sqrt{2x_1+3} &&= \sqrt{2(0\cdot 7746)+3} &&= 2.1329\\
x_3 = \phi(x_2) &= \sqrt{2x_2+3} &&= \sqrt{2(2\cdot 1329)+3} &&= 2.6955\\
x_4 = \phi(x_3) &= \sqrt{2x_3+3} &&= \sqrt{2(2\cdot 6955)+3} &&= 2.8967\\
x_5 = \phi(x_4) &= \sqrt{2x_4+3} &&= \sqrt{2(2\cdot 8967)+3} &&= 2.9654\\
x_6 = \phi(x_5) &= \sqrt{2x_5+3} &&= \sqrt{2(2\cdot 9654)+3} &&= 2.9884\\
x_7 = \phi(x_6) &= \sqrt{2x_6+3} &&= \sqrt{2(2.9884)+3} &&= 2.996\\
x_8 = \phi(x_7) &= \sqrt{2x_7+3} &&= \sqrt{2(2.996)+3} &&= 2.998\\
x_9 = \phi(x_8) &= \sqrt{2x_8+3} &&= \sqrt{2(2.998)+3} &&= 2.9993\\
x_{10} = \phi(x_9) &= \sqrt{2x_9+3} &&= \sqrt{2(2.999)+3} &&= 2.9998\\
x_{11} = \phi(x_{10}) &= \sqrt{2x_{10}+3} &&= \sqrt{2(2.9998)+3} &&= 2.9999\\
x_{12} = \phi(x_{11}) &= \sqrt{2x_{11}+3} &&= \sqrt{2(2\cdot 9999)+3} &&= 2.99999
\end{aligned}$$

Since the difference between x_{11} and x_{12} is very small the root is

$$2.9999 \simeq 3.000$$

■

■ **Example 2.9** Find a real root of the equation $x^3+x^2-1=0$ by iteration method. ■

Solution.

Let $$f(x) = x^3+x^2-1$$

Now $$f(0) = -1 = -\text{ve}$$

and $$f(1) = 1 = +\text{ve}$$

■

Hence a real root lies between 0 and 1 .

Now $x^3+x^2-1=0$ can be written as

$$x^2(x+1)-1=0$$
$$x^2=\frac{1}{x+1}$$
$$x=\frac{1}{\sqrt{x+1}}$$

Let
$$\phi(x)=\frac{1}{\sqrt{x+1}}$$

Therefore,
$$\phi'(x)=\frac{-\frac{1}{2\sqrt{x+1}}}{(x+1)}=-\frac{1}{2(x+1)^{\frac{3}{2}}}$$

Clearly
$$|\phi'(x)|=\left|\frac{1}{2(x+1)^{\frac{3}{2}}}\right|<1 \text{ in } (0,1)$$

For, when $x=0.5, |\phi'(0.5)|=\left|\frac{1}{2(1.5)^{\frac{3}{2}}}\right|<1$

Let the initial approximation be $x_0=0.5$. Now

$$\begin{aligned}
x_1=\phi(x_0) &= \frac{1}{\sqrt{x_0+1}} &&= \frac{1}{\sqrt{0.5+1}} &&= 0.81649\\
x_2=\phi(x_1) &= \frac{1}{\sqrt{x_1+1}} &&= \frac{1}{\sqrt{0.81649+1}} &&= 0.74196\\
x_3=\phi(x_2) &= \frac{1}{\sqrt{x_2+1}} &&= \frac{1}{\sqrt{0.74196+1}} &&= 0.75767\\
x_4=\phi(x_3) &= \frac{1}{\sqrt{x_3+1}} &&= \frac{1}{\sqrt{0.75767+1}} &&= 0.75427\\
x_5=\phi(x_4) &= \frac{1}{\sqrt{x_4+1}} &&= \frac{1}{\sqrt{0.75427+1}} &&= 0.75500\\
x_6=\phi(x_5) &= \frac{1}{\sqrt{x_5+1}} &&= \frac{1}{\sqrt{0.75500+1}} &&= 0.75485\\
x_7=\phi(x_6) &= \frac{1}{\sqrt{x_6+1}} &&= \frac{1}{\sqrt{0.75485+1}} &&= 0.75488
\end{aligned}$$

Since the difference between x_6 and x_7 is very small, the root of the given equation is 0.75488
. ■

■ **Example 2.10** Find the real root of the equation

$$1-x+\frac{x^2}{(2!)^2}-\frac{x^3}{(3!)^2}+\frac{x^4}{(4!)^2}-\ldots=0, \text{ using iteration method.}$$

■

Solution. The given equation can be written as

$$x=1+\frac{x^2}{(2!)^2}-\frac{x^3}{(3!)^2}+\frac{x^4}{(4!)^2}-\ldots$$

Let
$$\phi(x)=1+\frac{x^2}{(2!)^2}-\frac{x^3}{(3!)^2}+\frac{x^4}{(4!)^2}-\ldots$$

Omitting x^2 and higher powers of x, we get, $x=1$ approximately.

Therefore, let the initial approximation be $x_0 = 1$. Now

$$\begin{aligned}
x_1 = \phi(x_0) &= 1 + \frac{x_0^2}{(2!)^2} - \frac{x_0^3}{(3!)^2} + \frac{x_0^4}{(4!)^2} - \dots \\
&= 1 + \frac{1}{(2!)^2} - \frac{1}{(3!)^2} + \frac{1}{(4!)^2} - \dots \\
&= 1.2239 \\
x_2 = \phi(x_1) &= 1 + \frac{x_1^2}{(2!)^2} - \frac{x_1^3}{(3!)^2} + \frac{x_1^4}{(4!)^2} - \dots \\
&= 1 + \frac{(1.2239)^2}{(2!)^2} - \frac{(1.2239)^3}{(3!)^2} + \frac{(1.2239)^4}{(4!)^2} - \dots \\
&= 1.3274 \\
x_3 = \phi(x_2) &= 1 + \frac{x_2^2}{(2!)^2} - \frac{x_2^3}{(3!)^2} + \frac{x_2^4}{(4!)^2} - \dots \\
&= 1 + \frac{(1.3236)^2}{(2!)^2} - \frac{(1.3236)^3}{(3!)^2} + \frac{(1.3236)^4}{(4!)^2} - \dots \\
&= 1.3809
\end{aligned}$$

Similarly the following successive approximations can be obtained

$$\begin{aligned}
x_4 &= 1.409, \quad & x_5 &= 1.425 \\
x_6 &= 1.434 & x_7 &= 1.439
\end{aligned}$$

$\therefore$ The approximate root is 1.439 . ■

2.8 Practice Problem

Solve the following equations using iterative method.

1. $x^3 - x - 4 = 0$. [Ans. 1.7958]
2. $x^4 - x - 10 = 0$. [Ans. 1.8556]
3. $x^3 - 3x - 5 = 0$. [Ans. 2.2790]
4. $x^2 - 5x + 2 = 0$. [Ans. 0.4384]
5. $\cos x - x^2 - x = 0$. [Ans. 0.55001]
6. $\cos x - 3x + 2 = 0$. [Ans. 0.879]
7. Find the real root of $x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \dots$ [Ans. 0.443]

2.9 Introduction

An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$. where a's are constants ($a_0 \neq 0$) and n is a positive integer, is called a polynomial in x of degree n. The polynomial $f(x) = 0$ is called an algebraic equation of degree n. If $f(x)$ contains some other functions such as trigonometric, logarithmic, exponential etc., then $f(x) = 0$ is called a transcendental equation.

Definition 2.9.1 — Root. The value α of x which satisfies

$$f(x) = 0 \tag{2.12}$$

is called a root of $f(x) = 0$. Geometrically, a root of (2.12) is that value of x where the graph of $y = f(x)$ crosses the x-axis.

The process of finding the roots of an equation is known as the solution of that equation. This is a problem of basic importance in applied mathematics.

If $f(x)$ is a quadratic, cubic or a biquadratic expression, algebraic solutions of equations are available. But the need often arises to solve higher degree or transcendental equations for which no direct methods exist. Such equations can best be solved by approximate methods. In this chapter, we shall discuss some numerical methods for the solution of algebraic and transcendental equations.

2.9.1 Basic Properties of Equations

1. If $f(x)$ is exactly divisible by $x-\alpha$, then α is a root of $f(x)=0$.
2. Every equation of the nth degree has only n roots (real or imaginary).

Conversely if $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the roots of the ‘ nth degree equation $f(x)=0$, then where A is a constant.

$$f(x)=A(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n) \tag{2.13}$$

where A is a constant.

Note 2.9.1 If a polynomial of degree n vanishes for more than n values of x, it must be identically zero.

Note 2.9.2 Every equation of the odd degree has atleast one real root.

Note 2.9.3 If an equation of the nth degree has at the most p positive roots and at the most q negative roots, then it follows that the equation has at least $n-(p+q)$ imaginary roots.

2.10 Iterative Methods

The limitations of analytical methods for the solution of equations have necessitated the use of iterative methods. An iterative method begins with an approximate value of the root which is generally obtained with the help of Intermediate value property of the equation. This initial approximation is then successively improved iteration by iteration and this process stops when the desired level of accuracy is achieved. The various iterative methods begin their process with one or more initial approximations. Based on the number of initial approximations used, these iterative methods are divided into two categories: Bracketing Methods and Open-end Methods.

Bracketing methods begin with two initial approximations which bracket the root. Then the width of this bracket is systematically reduced until the root is reached to desired accuracy. The commonly used methods in this category are :

1. Graphical method
2. Bisection method
3. Method of False position.

Open-end methods are used on formulae which require a single starting value or two starting values which do not necessarily bracket the root. The following methods fall under this category :

1. Secant method
2. Iteration method
3. Newton-Raphson method

2.10.1 Rate of Convergence

Let $x_0, x_1, x_2, \cdots$ be the values of a root (α) of an equation at the 0th, 1st, 2nd, $\cdots$, iterations while its actual value is 3.5567. The values of this root calculated by three different methods, are as given below :

Root	1st method	2nd method	3rd method
x_0	5	5	5
x_1	5.6	3.8527	3.8327
x_2	6.4	3.5693	3.56834
x_3	8.3	3.55798	3.55743
x_4	9.7	3.55687	3.55672
x_5	10.6	3.55676	
x_6	11.9	3.55671	

The values in the 1st method do not converge towards the root 3.5567. In the 2nd and 3rd methods, the values converge to the root after 6th and 4th iterations respectively. Clearly 3rd met hod converges faster than the 2nd method. This fastness of convergence in any method is represented by its rate of convergence.

If e be the error then $e_i = \alpha - x_i = x_{i+1} - x_i$.

If $\frac{e_{i+1}}{e_i}$ is almost constant, convergence is said to be linear i.e. slow.

If $\frac{e_{i+1}}{e_i^p}$ is nearly constant, convergence is said to be of order p i.e. faster.

2.11 Secant Method

This method is an improvement over the method of false position as it does not require the condition $f(x_0)f(x_1) < 0$ of that method (Fig. 2.5). Here also the graph of the function $y = f(x)$ is approximated by a secant line but at each iteration, two most recent approximations to the root are used to find the next approximation. Also it is not necessary that the interval must contain the root.

Taking x_0, x_1 as the initial limits of the interval, we write the equation of the chord joining these as

$$y - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1)$$

Then the abscissa of the point where it crosses the x-axis $(y = 0)$ is given by

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

which is an approximation to the root. The general formula for successive approximations is, therefore, given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n), n \geq 1$$

2.11.1 Rate of Convergence

If at any iteration $f(x_n) = f(x_{n-1})$, this method fails and shows that it does not converge necessarily. This is a drawback of secant method over the method of false position which always converges. But if the secant method once converges, its rate of convergence is 1.6 which is faster than that of the method of false position.

■ **Example 2.11** Find a root of the equation $x^3 - 2x - 5 = 0$ using secant method correct to three decimal places. ■

Let $f(x)=x^3-2x-5$ so that

$$f(0)=(0)^3-2(0)-5=-5=-ve$$
$$f(1)=(1)^3-2(1)-5=-6=-ve$$
$$f(2)=(2)^3-2(2)-5=-1=-ve$$
$$f(3)=(3)^3-2(3)-5=16=+ve$$

$\therefore$ Taking initial approximations $x_0=2$ and $x_1=3$, by secant method, we have

$$x_{n+1}=x_n-\frac{x_n-x_{n-1}}{f(x_n)-f(x_{n-1})}f(x_n)$$

So we get,

$$x_2=x_1-\frac{x_1-x_0}{f(x_1)-f(x_0)}f(x_1)=3-\frac{3-2}{16+1}16=2.058823$$

Now $f(x_2)=-0.390799$

$$\therefore \quad x_3=x_2-\frac{x_2-x_1}{f(x_2)-f(x_1)}f(x_2)=2.081263$$

and

$$f(x_3)=-0.147204$$

Therefore,

$$x_4=x_3-\frac{x_3-x_2}{f(x_3)-f(x_2)}f(x_3)=2.094824$$

and

$$f(x_4)=0.003042$$

Therefore,

$$x_5=x_4-\frac{x_4-x_3}{f(x_4)-f(x_3)}f(x_4)=2.094549$$

Hence the root is 2.095 correct to 3 decimal places.

■ **Example 2.12** Find the root of the equation $xe^x=\cos x$ using the secant method correct to four decimal places. ■

Let $f(x)=\cos x-xe^x=0$. So that

$$f(0)=\cos 0-(0)e^{(0)}=1=+ve$$
$$f(1)=\cos(1)-(1)e^{(1)}=0.5403-2.7183=-2.17798=-ve$$

Taking the initial approximations $x_0=0, x_1=1$.

Then by secant method, we have

$$x_2=x_1-\frac{x_1-x_0}{f(x_1)-f(x_0)}f(x_1)=1+\frac{1}{3.17798}(-2.17798)=0.31467$$

Now $f(x_2)=0.51987$

Therefore

$$x_3 = x_2 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_2) = 0.44673$$

and

$$f(x_3) = 0.20354$$

Therefore

$$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} f(x_3) = 0.53171.$$

Repeating this process, the successive approximations are $x_5 = 0.51690, x_6 = 0.51775$, $x_7 = 0.51776$ etc.

Hence the root is 0.5178 correct to 4 decimal places.

2.12 Iteration Method or Fixed Point Method or Successive Approximation Method

To find the roots of the equation

$$f(x) = 0 \tag{2.14}$$

by successive approximations, we rewrite (2.14) in the form

$$x = \phi(x) \tag{2.15}$$

The roots of (2.14) are the same as the points of intersection of the straight line $y = x$ and the curve representing $y = \phi(x)$ Fig. 2.6 illustrates the working of the iteration method which provides a spiral solution.

Let $x = x_0$ be an initial approximation of the desired root α. Then the first approximation xi is given by $x_1 = \phi(x_0)$.

Now treating x_1 as the initial value, the second approximation is $x_2 = \phi(x_1)$.

Proceeding in this way, the nth approximation is given $x_n = \phi(x_{n-1})$.

2.12.1 Sufficient condition for convergence of iterations

Now it is not sure whether the sequence of approximations $x_1, x_2, \ldots, x_n$ always converges to the same number which is a root of (1) or not. As such, we have to choose the initial approximation x_0 suitably so that the successive approximations $x_1, x_2, \ldots, x_n$ converge to the root α. The following theorem helps in making the right choice of x_0.

Theorem 2.12.1 If

1. α be a root of $f(x) = 0$ which is equivalent to $x = \phi(x)$,
2. I, be any interval containing the point $x = \alpha$,
3. $|\phi'(x)| < 1$ for all x in I,

then the sequence of approximations $x_0, x_1, x_2, \ldots, x_n$ will converge to the root α provided the initial approximation x_0 is chosen in I.

Since α is a root of $x = \phi(x)$, we have $\alpha = \phi(\alpha)$.

If x_{n-1} and x_n be 2 successive approximations to α, we have $x_n = \phi(x_{n-1})$. Therefore,

$$x_n - \alpha = \phi(x_{n-1}) - \phi(\alpha) \tag{2.16}$$

By mean value theorem,

$$\frac{\phi\left(x_{n-1}\right)-\phi(\alpha)}{x_{n-1}-\alpha}=\phi'(\xi) \text{ where } x_{n-1}<\xi<\alpha$$

Hence (2.16) becomes $x_n-\alpha=\left(x_{n-1}-\alpha\right)\phi'(\xi)$.
If $\left|\phi'\left(x_i\right)\right|\leq k<1$ for all i, then

$$\left|x_n-\alpha\right|\leq k\left|x_{n-1}-\alpha\right| \tag{2.17}$$

i.e., Similarly

$$\left|x_{n-1}-\alpha\right|\leq k\left|x_{n-2}-\alpha\right|$$

That is

$$\left|x_n-\alpha\right|\leq k^2\left|x_{n-2}-\alpha\right|$$

Proceeding in this way,

$$\left|x_n-\alpha\right|\leq k^n\left|x_0-\alpha\right|$$

As $n\to\infty$, the R.H.S. tends to zero, therefore, the sequence of approximations converges to the root α.

Note 2.12.2 The smaller the value of $\phi'(x)$, the more rapid will be the convergence.

Note 2.12.3 This method of iteration is particularly useful for finding the real roots of an equation given in the form of an infinite series.

2.12.2 Acceleration of convergence

From (2.17), we have

$$\left|x_n-\alpha\right|\leq k\left|x_{n-1}-\alpha\right|, k<1.$$

It is clear from this relation that the iteration method is linearly convergent. This slow rate of convergence can be improved by using the following method :

2.12.3 Aitken's Δ^2 method

Let x_{i-1}, x_i, x_{i+1} be three successive approximations to the desired root α of the equation $x=\phi(x)$. Then we know that

$$\alpha-x_i=k\left(\alpha-x_{i-1}\right), \alpha-x_{i+1}=k\left(\alpha-x_i\right)$$

Dividing, we get $\frac{\alpha-x_i}{\alpha-x_{i+1}}=\frac{\alpha-x_{i-1}}{\alpha-x_i}$
whence

$$\alpha=x_{i+1}-\frac{\left(x_{i+1}-x_i\right)^2}{x_{i+1}-2x_i+x_{i-1}} \tag{2.18}$$

But in the sequence of successive approximations, we have

$$\begin{aligned}\Delta x_i&=x_{i+1}-x_i\\ \Delta^2 x_i&=\Delta\left(\Delta x_1\right)=\Delta\left(x_{i+1}-x_i\right)=\Delta x_{i+1}-\Delta x_i\\ &=x_{i+2}-x_{i+1}-\left(x_{i+1}-x_{\mathrm{i}}\right)=x_{i+2}-2x_{i+1}+x_i\end{aligned}$$

Therefore,

$$\Delta^2 x_{i-1} = x_{i+1} - 2x_i + x_{i-1}$$

Hence (2.18) can be writtern as

$$\alpha = x_{i+1} - \frac{(\Delta x_i)^2}{\Delta^2 x_{i-1}} \tag{2.19}$$

which yields successive approximations to the root α.

■ **Example 2.13** Find a real root of the equation $\cos x = 3x - 1$ correct to three decimal places using Iteration method. ■

We have $f(x) = \cos x - 3x + 1 = 0$,

$$f(0) = \cos 0 - 3(0) + 1 = 1 - 0 + 1 = 2 = +\text{ ve}$$
$$\text{and } f\left(\frac{\pi}{2}\right) = -3\left(\frac{\pi}{2}\right) + 1 = -3\left(\frac{3.14}{2}\right) = -3(1.57) + 1 = -4.71 + 3 = -3.71 = -\text{ ve}$$

$\therefore$ A root lies between 0 and $\frac{\pi}{2}$. Rewriting the given equation as $x = \frac{1}{3}(\cos x + 1) = \phi(x)$, we have

$$\phi'(x) = \frac{-\sin x}{3} \text{ and } |\phi'(x)| = \frac{1}{3}|\sin x| < 1 \text{ in } \left(0, \frac{\pi}{2}\right).$$

Hence the iteration method can be applied.

Since $|f(0)| < |f(1)|$ the root is near to 0, we can start with $x_0 = 0$. Then the successive approximations are,

$$x_1 = \phi(x_0) = \frac{1}{3}(\cos 0 + 1) = 0.6667$$
$$x_2 = \phi(x_1) = \frac{1}{3}(\cos 0.6667 + 1) = 0.5953$$
$$x_3 = \phi(x_2) = \frac{1}{3}(\cos 0.5953 + 1) = 0.6093$$
$$x_4 = \phi(x_3) = \frac{1}{3}(\cos 0.6093 + 1) = 0.6067$$
$$x_5 = \phi(x_4) = \frac{1}{3}(\cos 0.6067 + 1) = 0.6072$$
$$x_6 = \phi(x_5) = \frac{1}{3}(\cos 0.6072 + 1) = 0.6071$$

Hence x_5 and x_6 being almost the same, the root is 0.607 correct to 3 decimal places.

■ **Example 2.14** Using iteration method, find a root of the equation $x^3 + x^2 - 1 = 0$ correct to four decimal places. ■

We have $f(x) = x^3 + x^2 - 1 = 0$. Since,

$$f(0) = 0^3 + 0^2 - 1 = -1$$
$$\text{and } f(1) = 1^3 + 1^2 - 1 = 1,$$

a root lies between 0 and 1.

Rewriting the given equation as

$$x^3 + x^2 - 1 = 0 \quad \Rightarrow \quad x^2(x+1) - 1 = 0$$
$$x^2 = \frac{1}{(x+1)} \quad \Rightarrow \quad x = (x+1)^{\frac{-1}{2}} = \phi(x)$$

We have $\phi'(x) = -\frac{1}{2}(x+1)^{-3/2}$ and $|\phi'(x)| < 1$ for $x \in (0,1)$. Hence the iteration method can be applied.

Since $|f(1)| < |f(0)|$ the root is near to 1, we can starting with $x_0 = 0.75$, the successive approximations are

$$x_1 = \phi(x_0) = \frac{1}{\sqrt{(x_0+1)}} = 0.7559$$
$$x_2 = \phi(x_1) = \frac{1}{\sqrt{(0.7559+1)}} = 0.75466$$
$$x_3 = 0.75492,$$
$$x_4 = 0.75487,$$
$$x_5 = 0.75488$$

Hence x_4 and x_5 being almost the same, the root is 0.7548 correct to 4 decimal places.

Example 2.15 Apply iteration method to find the negative root of the equation $x^3 - 2x + 5 = 0$ correct to four decimal places. ∎

If α, β, γ are the roots of the given equation, then $-\alpha, -\beta, -\gamma$ are the roots of

$$(-x)^3 - 2(-x) + 5 = 0 \quad \Rightarrow \quad -(x^3 - 2x - 5 = 0)$$

$\therefore$ The negative root of the given equation is the negative of the positive root of

$$f(x) = x^3 - 2x - 5 = 0. \tag{2.20}$$

Since,

$$f(0) = 0^3 - 2(0) - 5 = -5 = -ve$$
$$f(1) = 1^3 - 2(1) - 5 = -6 = -ve$$
$$f(2) = 2^3 - 2(2) - 5 = -1 = -ve$$
$$f(3) = 3^3 - 2(3) - 5 = 16 = +ve$$

a root lies between 2 and 3 .

Rewriting Eq. (2.20)

$$x^3 - 2x - 5 = 0 \Rightarrow x^3 = 2x + 5 \Rightarrow x = (2x+5)^{\frac{1}{3}} = \phi(x)$$

We have $\phi'(x) = \frac{1}{3}(2x+5)^{\frac{-2}{3}}.2$ and $|\phi'(x)| < 1$ for $x \in (2,3)$. $\therefore$ The iteration method can be applied.

Since $|f(2)| < |f(3)|$ the root is near to 2, we can starting with $x_0 = 2$, the successive approximations are

$$x_1 = \phi(x_0) = (2x_0+5)^{\frac{1}{3}} = 2.08008$$
$$x_2 = \phi(x_1) = 2.09235,$$
$$x_3 = 2.09422$$
$$x_4 = 2.09450,$$
$$x_5 = 2.09454$$

Since x_4 and x_5 being almost the same, the root of (2.20) is 2.0945 correct to 4 decimal places.

Hence the negative root of the given equation is -2.0945.

■ **Example 2.16** Find a real root of $2x - \log_{10} x = 7$ correct to four decimal places using iteration method. ■

We have

$$\begin{aligned}
f(x) &= 2x - \log_{10} x - 7\\
f(0) &= 0 - \log_{10} 0 - 7 = 0 - \infty - 7 = -\infty = -ve\\
f(1) &= 2 - \log_{10} 1 - 7 = 2 - 0 - 7 = -5 = -ve\\
f(2) &= 4 - \log_{10} 2 - 7 = 4 - 0.0310 - 7 = -2.6989 = -ve\\
f(3) &= 6 - \log_{10} 3 - 7 = 6 - 0.4771 - 7 = -1.4471 = -ve\\
f(4) &= 8 - \log_{10} 4 - 7 = 8 - 0.602 - 7 = 0.398 = +ve
\end{aligned}$$

$\therefore$ A root lies between 3 and 4.

Rewriting the given equation as $x = \frac{1}{2}(\log_{10} x + 7) = \phi(x)$, we have

$$\phi'(x) = \frac{1}{2}\left(\frac{1}{x}\log_{10} e\right) \quad \left[\because \frac{d}{dx}\log_a x = \frac{1}{x \log a}\right]$$

$$\therefore \quad |\phi'(x)| < 1 \text{ when } 3 < x < 4 \quad [\because \log_{10} e = 0.4343]$$

Since $|f(4)| < |f(3)|$, the root is near to 4.

Hence the iteration method can be applied. Taking $x_0 = 3.6$, the successive approximations are

$$\begin{aligned}
x_1 &= \phi(x_0) = \frac{1}{2}(\log_{10} 3.6 + 7) = 3.77815\\
x_2 &= \phi(x_1) = \frac{1}{2}(\log_{10} 3.77815 + 7) = 3.78863\\
x_3 &= \phi(x_2) = \frac{1}{2}(\log 3.78863 + 7) = 3.78924\\
x_4 &= \phi(x_3) = \frac{1}{2}(\log 3.78924 + 7) = 3.78927
\end{aligned}$$

Hence x_3 and x_4 being almost equal, the root is 3.7892 correct to 4 decimal places.

■ **Example 2.17** Find the smallest root of the equation

$$1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \ldots = 0$$

. ■

Solution. Writing the given equation as

$$x = 1 + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \ldots = \phi(x)$$

Omitting x^2 and higher powers of x, we get $x = 1$ approximately.

Taking $x_0 = 1$, we obtain

$$\begin{aligned}
x_1 &= \phi(x_0) = 1 + \frac{1}{(2!)^2} - \frac{1}{(3!)^2} + \frac{1}{(4!)^2} - \frac{1}{(5!)^2} + \ldots = 1.2239\\
x_2 &= \phi(x_1) = 1 + \frac{(1.2239)^2}{(2!)^2} - \frac{(1.2239)^3}{(3!)^2} + \frac{(1.2239)^4}{(4!)^2} - \frac{(1.2239)^5}{(5!)^2} + \ldots = 1.3263
\end{aligned}$$

Similarly $x_3 = 1.38, x_4 = 1.409, x_5 = 1.425, x_6 = 1.434, x_7 = 1.439, x_8 = 1.442$.

The values of x_7 and x_8 indicate that the root is 1.44 correct to 2 decimal places. ■

■ **Example 2.18** Find the real root of the equation $x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \frac{x^{11}}{1320} + \ldots = 0.443$, correct to three decimal places using iteration method. ■

Solution. Rewrite the equation as a function $f(x)$:

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \frac{x^{11}}{1320} + \ldots$$

Calculate the derivative of $f(x)$, which we'll need for the iteration:

$$f'(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + \ldots$$

Choose an initial guess, say $x_0 = 1$.

■

2.13 Practice Problem

1. Find a root of the following equations, using the bisection method correct to three decimal places:
   (a) $x^3 - x - 1 = 0$
   (b) $x^3 - x^2 - 1 = 0$
   (c) $2x^3 + x^2 - 20x + 12 = 0$
   (d) $x^4 - x - 10 = 0$.
2. Evaluate a real root of the following equations by bisection method:
   (a) $x - \cos x = 0$
   (b) $e^{-x} - x = 0$
   (c) $e^x = 4\sin x$.
3. Find a real root of the following equations correct to three decimal places, by the method of false position:
   (a) $x^{-}5x + 1 = 0$
   (b) $x^3 - 4x - 9 = 0$
   (c) $x^6 - x^4 - x^3 - 1 = 0$
4. Using Regula falsi method, compute the real root of the following equations correct to three decimal places:
   (a) $xe^x = 2$
   (b) $\cos x = 3x - 1$
   (c) $xe^x = \sin x$
   (d) $x\tan x = -1$
   (e) $2x - \log x = 7$
   (f) $3x + \sin x = e^x$.
5. Find the fourth root of 12 correct to three decimal places by interpolation method.
6. Locate the root of $f(x) = x^{10} - 1 = 0$, between 0 and 1.3 using bisection method and method of false position. Comment on which method is preferable.
7. Find a root of the following equations correct to three decimal places by the method :
   (a) $x^3 + x^2 + x + 7 = 0$
   (b) $x - e^{-x} = 0$
   (c) $x\log_{10} x = 1.9$.
8. Use the iteration method to find a root of the equations to four decimal places:
   (a) $x^3 + x^2 - 100 = 0$

(b) $x^3 - 9x + 1 = 0$
(c) $x = \frac{1}{2} + \sin x$
(d) $\tan x = x$
(e) $e^x - 3x = 0$
(f) $2^x - x - 3 = 0$ which lies between $(-3, -2)$

9. Evaluate $\sqrt{30}$ by (i) secant method (ii) iteration method correct to four decimal places.
10. Find the root of the equation $2x = \cos x + 3$ correct to three decimal places using Iteration method.
11. Find the real root of the equation $x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \frac{x^{11}}{1320} + \ldots = 0.443$, correct to three decimal places using iteration method.

2.14 Newton-Raphson Method

Let x_0 be an approximate root of the equation $f(x) = 0$. If $x_1 = x_0 + h$ be the exact root then $f(x_1) = 0$.

$\therefore$ Expanding $f(x_0 + h)$ by Taylor's series $f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots\ldots = 0$.
Since h is small, neglecting h^2 and higher powers of h, we get $f(x_0) + hf'(x_0) = 0$
or

$$h = -\frac{f(x_0)}{f'(x_0)} \tag{2.21}$$

$\therefore$ A closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly starting with x_1, a still better approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general,

$$\mathbf{x_{n+1}} = \mathbf{x_n} - \frac{\mathbf{f}(\mathbf{x_n})}{\mathbf{f}'(\mathbf{x_n})} \quad (n = 0, 1, 2 \ldots) \tag{2.22}$$

which is known as the Newton-Raphson formula or Newton's iteration formula.

Note 2.14.1 Newton's method is useful in cases of large values of $f'(x)$ i.e., when the graph of $f(x)$ while crossing the x-axis is nearly vertical.

For if $f'(x)$ is small in the vicinity of the root, then by (2.21) h will be large and the computation of the root is slow or may not be possible. Thus this method is not suitable in those cases where the graph of $f(x)$ is nearly horizontal while crossing the x-axis.

Note 2.14.2 Geometrical interpretation. Let x_0 be a point near the root α of the equation $f(x) = 0$ (Fig. 2.7). Then the equation of the tangent at $A_0\,[x_0, f(x_0)]$ is

$$y - f(x_0) = f'(x_0)(x - x_0).$$

It cuts the x-axis at $x_1 = x_0 = \frac{f(x_0)}{f'(x_0)}$. which is a first approximation to the root α. If A_1 is the point corresponding to x_1 on the curve, then the tangent at A_1 will cut the x-axis at x_2 which is nearer to α and is, therefore, a second approximation to the root. Repeating this process, we approach the root α quite rapidly. Hence the method consists in replacing the part of the curve between the point A_0 and the x-axis by means of the tangent to the curve at A_0.

Note 2.14.3 Newton's method is generally used to improve the result obtained by other methods. It is applicable to the solution of both algebraic and transcendental equations.

2.14.1 Convergence of Newton-Raphson Method

Newton's formula converges provided the initial approximation x_0 is chosen sufficiently close to the root.

If it is not near the root, the procedure may lead to an endless cycle. A bad initial choice will lead one astray. Thus a proper choice of the initial guess is very important for the success of the Newton's method.

Comparing (2.22) with the relation $x_{n+1} = \phi(x_n)$ of the iteration method, we get

$$\phi(x_n) = x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

In general, $\phi(x) = x - \frac{f(x)}{f'(x)}$ which gives $\phi'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}$.

Since the iteration method (\$2.10) converges if $|\phi'(x)| < 1$

$\therefore$ Newton's formula will converge if $|f(x)f''(x)| < |f'(x)|^2$ in the interval considered.

Assuming $f(x), f'(x)$ and $f''(x)$ to be continuous, we can select a small interval in the vicinity of the root α, in which the above condition is satisfied. Hence the result.

Newtons method converges conditionally while Regula-falsi method always converges. However when once Newton-Raphson method converges, it converges faster and is preferred.

2.14.2 Newton's method has a quadratic convergence

Suppose x_n differs from the root α by a small quantity ε_n so that

$$x_0 = \alpha + \varepsilon_n \text{ and } x_{n+1} = \alpha + \varepsilon_{n+1}$$

Then (2.22) becomes $\alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \frac{f(\alpha+\varepsilon_n)}{f'(\alpha+\varepsilon_n)}$

i.e.

$$\begin{aligned}
\varepsilon_{n+1} &= \varepsilon_n - \frac{f(\alpha+\varepsilon_n)}{f'(\alpha+\varepsilon_n)} \\
&= \varepsilon_n - \frac{f(\alpha) + \varepsilon_n f'(\alpha) + \frac{1}{2!}\varepsilon_n^2 f''(\alpha) + \ldots\ldots}{f'(\alpha) + \varepsilon_n f''(\alpha) + \ldots\ldots} \text{ by Taylor's expansion.} \\
&= \varepsilon_n - \frac{\varepsilon_n f'(\alpha) + \frac{1}{2}\varepsilon_n{}^2 f''(\alpha) + \ldots\ldots}{f'(\alpha) + \varepsilon_n f''(\alpha) + \ldots\ldots} = \frac{\varepsilon_n{}^2}{2}\frac{f''(\alpha)}{f'(\alpha)}. \quad [\because f(\alpha) = 0]
\end{aligned}$$

This shows that the subsequent error at each step, is proportional to the square of the previous error and as such the convergence is quadratic. Thus Newton-Raphson method has second order convergence.

■ **Example 2.19** Find the positive root of $x^4 - x = 10$ correct to three decimal places, using Newton-Raphson method. ■

Let $f(x) = x^4 - x - 10$.

so that

$$\begin{aligned}
f(0) &= 0^4 - 0 - 10 = -10 = -ve \\
f(1) &= 1^4 - 1 - 10 = -10 = -ve, \\
f(2) &= 2^4 - 2 - 10 = 16 - 2 - 10 = 4 = +ve.
\end{aligned}$$

$\therefore$ A root of $f(x) = 0$ lies between 1 and 2.

Let us take $x_0 = 2$.

Also $f'(x) = 4x^3 - 1$.

Newton-Raphson's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2.23}$$

Putting $n = 0$, the first approximation x_1 is given by

$$\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} \\
&= 2 - \frac{4}{(4 \times 2^3) - 1} = 2 - \frac{4}{31} = 1.871
\end{aligned}$$

Putting $n = 1$ in (2.23), the second approximation is

$$\begin{aligned}
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 1.871 - \frac{f(1.871)}{f'(1.871)} \\
&= 1.871 - \frac{(1.871)^4 - (1.871) - 10}{4(1.871)^3 - 1} \\
&= 1.871 - \frac{0.3835}{25.199} = 1.856
\end{aligned}$$

Putting $n = 2$ in (2.23), the third approximation is

$$\begin{aligned}
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 1.856 - \frac{(1.856)^4 - (1.856) - 10}{4(1.856)^3 - 1} \\
&= 1.856 - \frac{0.010}{24.574} = 1.856
\end{aligned}$$

Here $x_2 = x_3$. Hence the desired root is 1.856 correct to three decimal places.

■ **Example 2.20** Find by Newton's method, the real root of the equation $3x = \cos x + 1$, correct to four decimal places. ■

Let

$$\begin{aligned}
f(x) &= 3x - \cos x - 1 \\
f(0) &= -2 = -ve \\
f(1) &= 3 - 0.5403 - 1 = 1.4597 = +ve.
\end{aligned}$$

So a root of $f(x) = 0$ lies between 0 and 1. It is nearer to 1 . Let us take $x_0 = 0.6$.

Also

$$f'(x) = 3 + \sin x$$

$\therefore$ Newton's iteration formula gives

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n} \\ &= \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \end{aligned} \tag{2.24}$$

Putting $n = 0$, the first approximation x_1 is given by

$$\begin{aligned} x_1 &= \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{(0.6)\sin(0.6) + \cos(0.6) + 1}{3 + \sin(0.6)} \\ &= \frac{0.6 \times 0.5729 + 0.82533 + 1}{3 + 0.5729} = 0.6071 \end{aligned}$$

Putting $n = 1$ in (2.24), the second approximation is

$$\begin{aligned} x_2 &= \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = \frac{0.6071 \sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)} \\ &= \frac{0.6071 \times 0.57049 + 0.8213 + 1}{3 + 0.57049} = 0.6071 \end{aligned}$$

Here $x_1 = x_2$. Hence the desired root is 0.6071 correct to four decimal places.

■ **Example 2.21** Using Newton's iterative method, find the real root of $x \log_{10} x = 1.2$ correct to five decimal places. ■

Let

$$\begin{aligned} & f(x) = x \log_{10} x - 1.2 \\ & f(0) = 0 - 1.2 = -\text{ ve}, \\ & f(1) = 0 - 1.2 = -\text{ ve}, \\ & f(2) = 2\log_{10} 2 - 1.2 = 0.59794 = -\text{ ve} \\ \text{and} \quad & f(3) = 3\log_{10} 3 - 1.2 = 1.4314 - 1.2 = 0.23136 = +\text{ ve}. \end{aligned}$$

So a root of $f(x) = 0$ lies between 2 and 3 . Let us take $x_0 = 2$.

Also

$$f'(x) = \log_{10} x + x \cdot \frac{1}{x} \log_{10} e = \log_{10} x + 0.43429 \qquad \left[\log_{10} e = 0.43429 \text{ and } \frac{d}{dx} \log_a(x) = \frac{1}{x \log(a)}\right]$$

$\therefore$ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n \log_{10} x_n - 1.2}{\log_{10} x_n + 0.43429} = \frac{0.43429 x_n + 1.2}{\log_{10} x_n + 0.43429} \tag{2.25}$$

Putting $n = 0$, the first approximation is

$$\begin{aligned} x_1 &= \frac{0.43429 \times x_0 + 1.2}{\log_{10} x_0 + 0.43429} = \frac{0.43429 \times 2 + 1.2}{\log_{10} 2 + 0.43429} \\ &= \frac{0.86858 + 1.2}{0.30103 + 0.43429} = 2.81 \end{aligned}$$

Similarly putting $n = 1,2,3,4$ in (1), we get

$$x_2 = \frac{0.43429 \times 2.81 + 1.2}{\log_{10} 2.81 + 0.43429} = 2.741$$
$$x_3 = \frac{0.43429 \times 2.741 + 1.2}{\log_{10} 2.741 + 0.43429} = 2.74064$$
$$x_4 = \frac{0.43429 \times 2.741 + 1.2}{\log_{10} 2.74064 + 0.43429} = 2.74065$$
$$x_5 = \frac{0.43429 \times 2.74065 + 1.2}{\log_{10} 2.74065 + 0.43429} = 2.74065$$

Here $x_4 = x_5$. Hence the required root is 2.74065 correct to five decimal places.

2.14.3 Some Deductions from Newton-Raphson Formula

Theorem 2.14.4 We can derive the following useful results from the Newton's iteration formula :

1. Iterative formula to find $1/N$ is $\mathbf{x_{n+1} = x_n\,(2 - Nx_n)}$.
2. Iterative formula to find $\sqrt{N}$ is $\mathbf{x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)}$.
3. Iterative formula to find $1/\sqrt{N}$ is $\mathbf{x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)}$.
4. Iterative formula to find $\sqrt[k]{N}$ is $\mathbf{x_{n+1} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{{x_n}^{k-1}}\right)\Big]}$.

(1) Let $x = 1/N$ or $1/x - N = 0$.
Taking $f(x) = 1/x - N$, we have $f'(x) = -x^{-2}$.
Then Newton's formula gives

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(1/x_n - N)}{-{x_n}^{-2}} = x_n + \left(\frac{1}{x_n} - N\right)x_n^2 \\ &= x_n + x_n - Nx_n^2 = x_n\,(2 - Nx_n) \end{aligned}$$

(2) Let $x = \sqrt{N}$ or $x^2 - N = 0$.
Taking $f(x) = x^2 - N$, we have $f'(x) = 2x$.
Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{{x_n}^2 - N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

(3) Let $x = \frac{1}{\sqrt{N}}$ or $x^2 - \frac{1}{N} = 0$.
Taking $f(x) = x^2 - \frac{1}{N}$, we have $f'(x) = 2x$.
Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{{x_n}^2 - \frac{1}{N}}{2x_n} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$$

(4) Let $x = \sqrt[k]{N}$ or $x^k - N = 0$.
Taking $f(x) = x^k - N$, we have $f'(x) = kx^{k-1}$.
Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n \frac{{x_n}^k - N}{k{x_n}^{k-1}} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right].$$

■ **Example 2.22** Evaluate the following (correct to four decimal places) by Newton's iteration method:

1. $\frac{1}{31}$
2. $\sqrt{5}$
3. $\frac{1}{\sqrt{14}}$
4. $\sqrt[3]{24}$
5. $(30)^{\frac{-1}{5}}$

■

(1). Since, Iterative formula to find $1/N$ is $\mathbf{x_{n+1} = x_n\,(2-Nx_n)}$.
Taking $N = 31$ in the above formula, we get

$$x_{n+1} = x_n\,(2-31x_n)$$

Since an approximate value of $1/31 = 0.03$, we take $x_0 = 0.03$. Then

$$\begin{aligned}
x_1 &= x_0\,(2-31x_0) = 0.03(2-31\times 0.03) = 0.0321\\
x_2 &= x_1\,(2-31x_1) = 0.0321(2-31\times 0.0321) = 0.032257\\
x_3 &= x_2\,(2-31x_2) = 0.032257(2-31\times 0.032257) = 0.03226
\end{aligned}$$

Since $x_2 = x_3$ upto 4 decimal places, we have $\frac{1}{31} = 0.0323$.

(2). Since, Iterative formula to find $\sqrt{N}$ is $\mathbf{x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)}$.
Taking $N = 5$ in the above formula, we get

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right).$$

Since an approximate value of $\sqrt{5} = 2$, we take $x_0 = 2$. Then

$$\begin{aligned}
x_1 &= \frac{1}{2}\left(x_0 + \frac{5}{x_0}\right) = \frac{1}{2}\left(2 + \frac{5}{2}\right) = 2.25\\
x_2 &= \frac{1}{2}\left(x_1 + \frac{5}{x_1}\right) = 2.2361\\
x_3 &= \frac{1}{2}\left(x_2 + \frac{5}{x_2}\right) = 2.2361
\end{aligned}$$

Since $x_2 = x_3$ upto 4 decimal places, we have $\sqrt{5} = 2.2361$.

(3). Since Iterative formula to find $\frac{1}{\sqrt{N}}$ is $\mathbf{x_{n+1}} = \frac{1}{2}\left(\mathbf{x_n} + \frac{1}{\mathbf{Nx_n}}\right)$.
Taking $N = 14$ in the above formula, we get

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{1}{14x_n}\right].$$

Since an approximate value of $\frac{1}{\sqrt{14}} = \frac{1}{\sqrt{16}} = \frac{1}{4} = 0.25$, we take $x_0 = 0.25$,
Then

$$\begin{aligned}
x_1 &= \frac{1}{2}\left[x_0 + \frac{1}{14x_0}\right] = \frac{1}{2}\left[0.25 + \frac{1}{14\times 0.25}\right] = 0.26785\\
x_2 &= \frac{1}{2}\left[x_1 + \frac{1}{14x_1}\right] = \frac{1}{2}\left[0.26785 + \frac{1}{14\times 0.26785}\right] = 0.2672618\\
x_3 &= \frac{1}{2}\left[x_2 + \frac{1}{14x_2}\right] = \frac{1}{2}\left[0.2672618 + \frac{1}{14\times 0.2672618}\right] = 0.2672612
\end{aligned}$$

Since $x_2 = x_3$ upto 4 decimal places, we take $\frac{1}{\sqrt{14}} = 0.2673$.

(4). Since Iterative formula to find $\sqrt[k]{N}$ is $\mathbf{x_{n+1}} = \frac{1}{\mathbf{k}}\left[(\mathbf{k}-1)\mathbf{x_n} + \frac{\mathbf{N}}{\mathbf{x_n}^{\mathbf{k}-1}}\right]$.
Taking $N = 24$ and $k = 3$, in the above formula, we get

$$x_{n+1} = \frac{1}{3}\left[2x_n + \frac{24}{x_n}\right].$$

Since an approximate value of $(24)^{\frac{1}{3}} = (27)^{\frac{1}{3}} = 3$, we take $x_0 = 3$. Then

$$x_1 = \frac{1}{3}\left(2x_0 + \frac{24}{x_0^2}\right) = \frac{1}{3}\left(6 + \frac{24}{9}\right) = 2.88889$$
$$x_2 = \frac{1}{3}\left(2x_1 + \frac{24}{x_1^2}\right) = \frac{1}{3}\left[(2\times 2.88889) + \frac{24}{2.88889^2}\right] = 2.88451$$
$$x_3 = \frac{1}{3}\left(2x_2 + \frac{24}{x_2^2}\right) = \frac{1}{3}\left[2\times 2.88451 + \frac{24}{2.88451^2}\right] = 2.8845$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(24)^{\frac{1}{3}} = 2.8845$.

(5). Since Iterative formula to find $\sqrt[k]{N}$ is $\mathbf{x_{n+1}} = \frac{1}{\mathbf{k}}\left[(\mathbf{k}-1)\mathbf{x_n} + \frac{\mathbf{N}}{\mathbf{x_n}^{\mathbf{k}-1}}\right]$.
Taking $N = 30$ and $k = -5$, in the above formula, we get

$$x_{n+1} = \frac{1}{-5}\left(6x_n + \frac{30}{x_n^{-6}}\right) = \frac{x_n}{5}\left(6 - 30x_n^5\right)$$

Since an approximate value of $(30)^{\frac{-1}{5}} = (32)^{\frac{-1}{5}} = \frac{1}{2}$, we take $x_0 = \frac{1}{2}$, Then

$$x_1 = \frac{x_0}{5}\left(6 - 30x_0^5\right) = \frac{1}{10}\left(6 - \frac{30}{2^5}\right) = 0.50625$$
$$x_2 = \frac{x_1}{5}\left(6 - 30x_1^5\right) = \frac{0.50625}{5}\left[6 - 30(0.50625)^5\right] = 0.506495$$
$$x_3 = \frac{x_2}{5}\left(6 - 30x_2^5\right) = \frac{0.506495}{5}\left[6 - 30(0.506495)^5\right] = 0.506496$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(30)^{\frac{-1}{5}} = 0.5065$.

2.15 Practice Problem

1. Find by Newton-Raphson method, a root of the following equations correct to 3 decimal places :
   (a) $x^3 - 3x + 1 = 0$
   (b) $x^3 - 2x - 5 = 0$
   (c) $x^3 - 5x + 3 = 0$
   (d) $3x^3 - 9x^2 + 8 = 0$.
2. Using Newton's iterative method, find a root of the following equations correct to 4 decimal places :
   (a) $x^4 + x^3 - 7x^2 - x + 5 = 0$ which lies between 2 and 3.
   (b) $x^5 - 5x^2 + 3 = 0$
3. Find the negative root of the equation $x^3 - 21x + 3500 = 0$ correct to 2 decimal places by Newton's method.
4. Using Newton-Raphson method, find a root of the following equations correct to 3 decimal places :

(a) $x^2+4\sin x=0$
(b) $x\sin x+\cos x=0$ or $x\tan x+1=0$
(c) $e^x=x^3+\cos 25x$ which is near 4.5 .
(d) $x\log_{10} x=12.34$, start with $x_0=10$.
(e) $\cos x=xe^x$
(f) $10^x+x-4=0$

5. The equation $2e^{-x}=\frac{1}{x+2}+\frac{1}{x+1}$ has two roots greater than -1 . Calculate these roots correct to five decimal places.
6. The bacteria concentration in a reservoir varies as $C=4e^{-2t}+e^{-0.1t}$. Using Newton Raphson method, calculate the time required for the bacteria concentration to be 0.5 .
7. Use Newton's method to find the smallest root of the equation $e^x\sin x=1$ to four places of decimal.
8. The current i in an electric circuit is given by $i=10e^{-t}\sin 2\pi t$ where t is in seconds. Using Newton's method, find the value of t correct to 3 decimal places for $i=2$amp.
9. Find the iterative formulae for finding $\sqrt{N}, \sqrt[3]{N}$ where N is a real number, using Newton-Raphson formula. Hence evaluate : (a) $\sqrt{15}$. (b) $\sqrt{21}$ (c) the cube-root of 17 to three places of decimal.
10. Develop an algorithm using N.R. method, to find the fourth root of a positive number N. Hence find $\sqrt[4]{32}$.
11. Evaluate the following (correct to 3 decimal places) by using the Newton-Raphson method.
    (a) $1/18$
    (b) $1/\sqrt{15}$
    (c) $(28)^{-1/4}$.