Module-I

Algebraic, Transcendental and Linear System of Equations

Definition 1.0.1 (Solution or Root)

The solution, or root, of an equation is any value or set of values that can be substituted into the equation to make it a true statement.

A number ξ is a solution of $f(x) = 0$ if $f(\xi)$ is a root or a zero of $f(x) = 0$.

Geometrically, a root of the equation is the value of x at which the graph of $y = f(x)$ intersects the $x-$axis.

Example 1.0.2

- For example, $x + 2 = 15$ is an equation then the solution is $x = 13$
- $y^2 = 4$ has two values that will make the statement true, namely 2 and -2.

Definition 1.0.3 (Linear equations)

Linear equations are those equations that are of the first order. These equations are defined for lines in the coordinate system.
Linear equations are also first-degree equations as it has the highest exponent of variables as 1. Some of the examples of such equations are as follows:
Linear equations in 1 variable $2y = 8$ and linear equations in 2 variable $2x - 3 = 0$.

Definition 1.0.4 (Solving multivariable equations)

Equations with two unknowns are represented by the general formula $ax+by = c$; where a, b, and c are constants and x and y are variables. The solution of this type of equation would be the ordered pair of x and y that makes the equation true.

Example 1.0.5

For example, the solution set for the equation $x + y = 7$ would contain all the pairs of values for x and y that satisfy the equation, such as $(2, 5)$, $(3, 4)$, $(4, 3)$, etc.

Definition 1.0.6 (Unknowns and linear equations)

Suppose, this year, Lynn is twice as old as Ruthie, but two years ago, Lynn was three times as old as Ruthie. Two equations can be written for this problem. If one lets x = Lynn's age and y = Ruthie's age, then the two equations relating the unknown ages would be $x = 2y$ and $x - 2 = 3(y - 2)$. The relationships can be rewritten in the general format for linear equations to obtain,

$$x - 2y = 0$$
$$2x - 3y = -4$$

The solution of this system of equations will be any ordered pair that makes both equations true. This system has only solution, the ordered pair of $x = 8$ and $y = 4$, and is thus called consistent.

Definition 1.0.7 (Systems in three or more variables)

A system of linear equations (or linear system) is a collection of one or more linear equations involving the same set of variables.
For example,

$$\begin{aligned} 3x + 2y - z &= 1 \\ 2x - 2y + 4z &= -2 \\ -x + \frac{1}{2}y - z &= 0 \end{aligned}$$

is a system of three equations in the three variables x, y, z. A solution to a linear system is an assignment of values to the variables such that all the equations are simultaneously satisfied. A solution to the system above is given by

$$x = 1, y = -2, z = -2.$$

Definition 1.0.8 (Algebraic equation)

A polynomial equation of the form

$$f(x) = p_n(x) = a_0x^{n-1} + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

is called an Algebraic equation. For example, $x^4 - 4x^2 + 5 = 0, 4x^2 - 5x + 7 = 0; 2x^3 - 5x^2 + 7x + 5 = 0$ are algebraic equations.

Definition 1.0.9 (Transcendental equation)

An equation which contains polynomials, trigonometric functions, logarithmic functions, exponential functions etc., is called a Transcendental equation. For example, $tanx - e^x = 0; sinx - xe^{2x} = 0; xe^x = cosx$ are transcendental equations.

Definition 1.0.10 (Continuous Function)

Function is said to be convergence if it approaching a limit more and more closely as a variable of the function increases or decreases or as the number of terms of the series increases.

Example 1.0.11

The function $y = 1/x$ converges to zero as x increases. Although no finite value of x will cause the value of y to actually become zero, the limiting value of y is zero because y can be made as small as desired by choosing x large enough.

$$i.e. \sum_{x=0}^{\infty} \frac{1}{x} \approx 0.$$

Similarly, for any value of x between (but not including) -1 and $+1$, the series $1 + x + x^2 + \cdots + x^n$ converges toward the limit $1/(1-x)$ as n, the number of terms, increases. The interval $-1 < x < 1$ is called the range of convergence of the series; for values of x outside this range, the series is said to diverge.

The graphical method of solving a system of equations in three variables involves plotting the planes that are formed when graphing each equation in the system and then finding the intersection point of all three planes. The single point where all three planes intersect is the unique solution to the system.

Example 1.0.12 (Unique solution)

This is a set of linear equations, also known as a linear system of equations, in three variables:

$$3x + 2y - z = 6 \tag{1}$$

$$2x + 2y + z = 3 \tag{2}$$

$$x + y + z = 4 \tag{3}$$

The solution to this system of equations is: $x = 1, y = 2, z = 1$

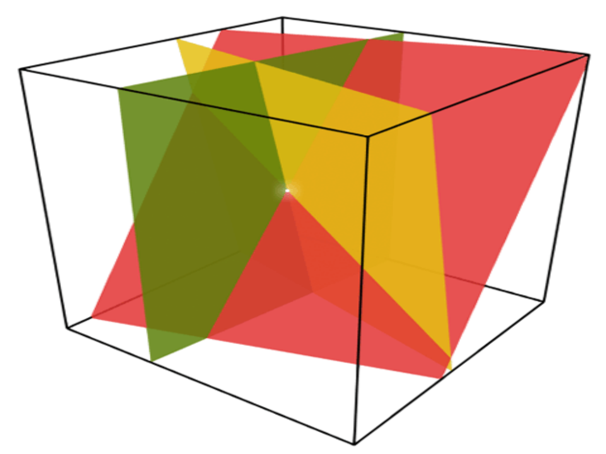


Figure 1: This images shows a system of three equations in three variables. The intersecting point (white dot) is the unique solution to this system.

Example 1.0.13 (Infinite number of solutions)

For example, consider this system of equations:

$$\begin{aligned} 2x + y - 3z &= 0 \\ 4x + 2y - 6z &= 0 \\ x - y + z &= 0 \end{aligned}$$

First, multiply the first equation by -2 and add it to the second equation:

$$\begin{aligned} -2(2x + y - 3z) + (4x + 2y - 6z) &= 0 + 0 \\ (-4x + 4x) + (-2y + 2y) + (6z - 6z) &= 0 \\ 0 &= 0 \end{aligned}$$

We do not need to proceed any further. The result we get is an identity, $0 = 0$, which tells us that this system has an infinite number of solutions.

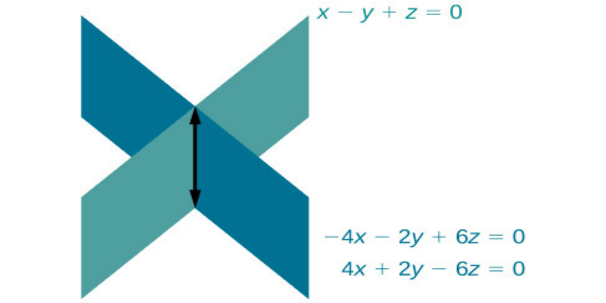


Figure 2: Two equations represent the same plane, and these intersect the third plane on a line.

Example 1.0.14 (No solution)

For example, consider the system of equations

$$\begin{aligned} x - 3y + z &= 4 \\ -x + 2y - 5z &= 3 \\ 5x - 13y + 13z &= 8 \end{aligned}$$

Using the elimination method for solving a system of equation in three variables, notice that we can add the first and second equations to cancel x:

$$\begin{aligned} (x - 3y + z) + (-x + 2y - 5z) &= 4 + 3 \\ (x - x) + (-3y + 2y) + (z - 5z) &= 7 \Rightarrow -y - 4z = 7 \end{aligned}$$

Next, multiply the first equation by -5, and add it to the third equation:

$$\begin{aligned} -5(x - 3y + z) + (5x - 13y + 13z) &= -5(4) + 8 \\ (-5x + 5x) + (15y - 13y) + (-5z + 13z) &= -20 + 8 \\ 2y + 8z &= -12 \end{aligned}$$

Cont...

Now, notice that we have a system of equations in two variables:

$$y - 4z = 7$$
$$2y + 8z = -12$$

We can solve this by multiplying the top equation by 2, and adding it to the bottom equation:

$$\begin{aligned} 2(-y - 4z) + (2y + 8z) &= 2(7) - 12 \\ (-2y + 2y) + (-8z + 8z) &= 14 - 12 \\ 0 &= 2 \end{aligned}$$

The final equation $0 = 2$ is a contradiction, so we conclude that the system of equations in inconsistent, and therefore, has no solution.

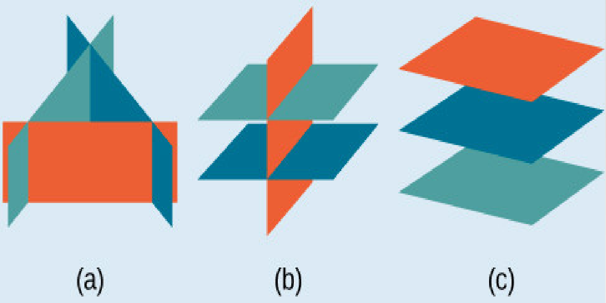


Figure 3: All three figures represent three-by-three systems with no solution. (a) The three planes intersect with each other in three different parallel lines, which do not intersect at a common point. (b) Two of the planes are parallel and intersect with the third plane, but not with each other. (c) All three planes are parallel, so there is no point of intersection.

Definition 1.0.15 (Properties of Algebraic Equation)

- Every algebraic equation of $n-$th degree, where $n-$ is a positive integer, has n and only n roots.
- Complex roots occur in pairs. That is, if $(a + ib)$ is a root of $f(x) = 0$, then $(a - ib)$ is also a root of this equation.
- If $x = a$ is a root of $f(x) = 0$, a polynomial of degree n, then $(x - a)$ is a factor of $f(x)$. On dividing $f(x)$ by $(x - a)$ we obtain a polynomial of degree $(n - 1)$.
- Descartes rule of signs: The number of positive roots of an algebraic equation $f(x) = 0$ with real coefficient cannot exceed the number of changes in sign of the coefficients in the polynomial $f(x) = 0$. Similarly, the number of negative roots of $f(x)$ can not exceed the number of changes in the sign of the coefficients of $f(-x) = 0$.

Properties of Algebraic Equation (Cont...)

- For example, consider an equation

$$x^3 - 3x^2 + 4x - 5 = 0$$

  As there are three changes in sign, also, the degree of the equation is three, and hence the given equation will have all the three positive roots.

- Intermediate value property: If $f(x)$ is a real valued continuous function in the closed interval $a \leq x \leq b$. If $f(a)$ and $f(b)$ have opposite signs, then the graph of the function $y = f(x)$ crosses the x-axis at least once; that is $f(x)$ has at least one root ξ such that $a < \xi < b$.

Example 1.0.16

The equation

$$8x^3 - 12x^2 - 2x + 3 = 0$$

has three real roots. Find the intervals each of unit length containing each one of these roots.

Solution

x	-2	-1	0	1	2	3
f(x)	-105	-15	3	-3	15	105

From the table, we find that the equation $f(x) = 0$ has roots in the intervals $(-1, 0), (0, 1)$ and $(1, 2)$. The exact roots are $-0.5, 0.5$ and 1.5.

Example 1.0.17

Obtain an interval which contains a root of the equation

$$f(x) = \cos x - xe^x = 0.$$

Solution

We prepare table of the value of th function $f(x)$ for various values of x.

x	0	0.5	1	1.5	2
f(x)	1	0.0532	-2.1780	-6.6518	-15.1942

From the table, we find that the equation $f(x) = 0$ has at least one root in the intervals $(0.5, 1)$. The exact roots corrected to ten decimal places is -0.5177573637 .

Definition 1.1.1 (Diagonally Dominant Matrix)

A square matrix is called diagonally dominant if

$$|A_{i,i}| \geq \sum_{j=1,\, j\neq i}^{n} |A_{ij}|$$

A is called strictly diagonally dominant if

$$|A_{i,i}| > \sum_{j=1,\, j\neq i}^{n} |A_{ij}|$$

for all i.

Example 1.1.2

Find the solution to the following system of equations using the Gauss-Seidel method.

$$5x_1 - x_2 + 2x_3 = 12 \tag{4}$$

$$3x_1 + 8x_2 - 2x_3 = -25 \tag{5}$$

$$x_1 + x_2 + 4x_3 = 6 \tag{6}$$

The coefficient matrix

$$[A] = \begin{bmatrix} 5 & -1 & 2 \\ 3 & 8 & -2 \\ 1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is diagonally dominant as

$$|a_{11}| = 5 \qquad 5 \geq |a_{12}| + |a_{13}| = |-1| + |2| = 3$$
$$|a_{22}| = 8 \qquad 8 \geq |a_{21}| + |a_{23}| = |3| + |-2| = 5$$
$$|a_{33}| = 4 \qquad 4 \geq |a_{31}| + |a_{32}| = |1| + |1| = 2$$

and the inequality is strictly greater than for at least one row. Hence, the solution should converge using the Gauss-Seidel method.

Since, no initial values are given, let us assume $x_1 = 0$, $x_2 = 0$ and $x = 0$ arbitrarily. Find the x_1, x_2 and x_3 values form Eqs (4), (5) and (6) correspondingly.

$$x_1 = \frac{12 + x_2 - 2x_3}{5} \tag{7}$$

$$x_2 = \frac{-25 - 3x_1 + 2x_3}{8} \tag{8}$$

$$x_3 = \frac{6 - x_1 - x_2}{4} \tag{9}$$

I^{st} Iteration:

Let us assume $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$ in Eq (7), (8) and (9), Then form Eq.(7) we have

$$x_1 = \frac{12 + x_2 - 2x_3}{5} = \frac{12 + (0) - 2(0)}{5} = \frac{12}{5} = 2.4$$

Now the recent $x_1 = 2.4$. So, substitute $x_1 = 2.4$, $x_2 = 0$ and $x_3 = 0$ in Eq.(8), then we have,

$$x_2 = \frac{-25 - 3x_1 + 2x_3}{8} = \frac{-25 - 3(2.4) + 2(0)}{8} = -4.025$$

Now substitute the recent value of $x_1 = 2.4$, $x_2 = -4.025$ and $x_3 = 0$ in Eq.(9) we have

$$x_3 = \frac{6 - x_1 - x_2}{4} = \frac{6 - (2.4) - (-4.025)}{4} = 1.90625$$

2^{nd} Iteration:

Now the recent value of $x_1 = 2.4$, $x_2 = -4.025$ and $x_3 = 1.90625$. Substitute these values in Eq.(7), we get

$$x_1 = \frac{12 + x_2 - 2x_3}{5} = \frac{12 + (-4.025) - 2(1.90625)}{5} = 0.8325.$$

Substitute the recent value of $x_1 = 0.8325$, $x_2 = -4.025$ and $x_3 = 1.90625$ in Eq.(8), we get

$$x_2 = \frac{-25 - 3x_1 + 2x_3}{8} = \frac{-25 - 3(0.8325) + 2(1.90625)}{8} = -2.960625.$$

Substitute the recent values of $x_1 = 0.8325$, $x_2 = -2.960625$ and $x_3 = 1.90625$ in Eq.(9), we get

$$x_3 = \frac{6 - x_1 - x_2}{4} = \frac{6 - (0.835) - (-2.960625)}{4} = 2.03203125$$

3^{nd} Iteration:

Now the recent value of $x_1 = 0.8325$, $x_2 = -2.960625$ and $x_3 = 2.03203125$. Substitute these values in Eq.(7), we get

$$x_1 = \frac{12 + x_2 - 2x_3}{5} = \frac{12 + (-2.960625) - 2(2.03203125)}{5} = 0.9950625.$$

Substitute the recent value of $x_1 = 0.9950625$, $x_2 = -2.960625$ and $x_3 = 2.03203125$ in Eq.(8), we get

$$x_2 = \frac{-25 - 3x_1 + 2x_3}{8} = \frac{-25 - 3(0.9950625) + 2(2.03203125)}{8} = -2.99014$$

Substitute the recent values of $x_1 = 0.9950625$, we get $x_2 = -2.990140625$ and $x_3 = 2.03203125$ in Eq.(9)

$$x_3 = \frac{6 - x_1 - x_2}{4} = \frac{6 - (0.9950625) - (-2.990140625)}{4} = 1.99876953125$$

Three iteration values are tabulated below.

	x_1	x_2	x_3
I^{st} iteration	2.4	-4.025	1.90625
II^{nd} iteration	0.8325	-2.960625	2.03203125
III^{rd} iteration	0.9950625	-2.9901406525	1.9987953125

From the table one can conclude the value of $x_1 = 1$, $x_2 = -3$ and $x_3 = 2$.

Example 1.1.3

Find the solution to the following system of equations using the Gauss-Seidel method.

$$\begin{aligned} 12x_1 + 3x_2 - 5x_3 &= 1 \\ x_1 + 5x_2 + 3x_3 &= 28 \\ 3x_1 + 7x_2 + 13x_3 &= 76. \end{aligned}$$

Use $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ as the initial guess and conduct three iterations.

The coefficient matrix

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is diagonally dominant as

$$\begin{aligned} |a_{11}| &= 12 \qquad & 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8 \\ |a_{22}| &= 5 \qquad & 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4 \\ |a_{33}| &= 13 \qquad & 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10 \end{aligned}$$

and the inequality is strictly greater than for at least one row. Hence, the solution should converge using the Gauss-Seidel method.

Rewriting the equations, we get

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$
$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$
$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}.$$

Assuming an initial guess of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$

Iteration I

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12} = \frac{1 - 3(0) + 5(1)}{12} = 0.50000$$
$$x_2 = \frac{28 - x_1 - 3x_3}{5} = \frac{28 - (0.50000) - 3(1)}{5} = 4.9000$$
$$x_3 = \frac{76 - 3x_1 - 7x_2}{13} = \frac{76 - 3(0.50000) - 7(4.9000)}{13} = 3.0923$$

Iteration II

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12} = \frac{1 - 3(4.9000) + 5(3.0923)}{12} = 0.14679$$
$$x_2 = \frac{28 - x_1 - 3x_3}{5} = \frac{28 - (0.14679) - 3(3.0923)}{5} = 3.7153$$
$$x_3 = \frac{76 - 3x_1 - 7x_2}{13} = \frac{76 - 3(0.14679) - 7(3.7153)}{13} = 3.8118$$

Iteration III

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12} = \frac{1 - 3(3.7153) + 5(3.8118)}{12} = 0.742758$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5} = \frac{28 - (0.742758) - 3(3.8118)}{5} = 3.164368$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13} = \frac{76 - 3(0.742758) - 7(3.164368)}{13} = 3.9708$$

Iteration	x_1	x_2	x_3
1	0.50000	4.9000	3.0923
2	0.14679	3.7153	3.8118
3	0.74275	3.1644	3.9708
4	0.94675	3.0281	3.9971

From the table we can approximate the solution $x_1 = 1$, $x_2 = 3$ and $x_3 = 4$.

Example 1.1.4

Given the system of equations

$$\begin{aligned} 3x_1 + 7x_2 + 13x_3 &= 76 \\ x_1 + 5x_2 + 3x_3 &= 28 \\ 12x_1 + 3x_2 - 5x_3 &= 1 \end{aligned}$$

find the solution using the Gauss-Seidel method. Use $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ as the initial guess.

Rewriting the equations, we get

$$x_1 = \frac{76 - 7x_2 - 13x_3}{3}$$
$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$
$$x_3 = \frac{1 - 12x_1 - 3x_2}{-5}$$

Assuming an initial guess of $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ the next six iterative values are given in the table below.

Iteration	x_1	x_2	x_3
1	21.000	0.80000	50.680
2	–196.15	14.421	–462.30
3	1995.0	–116.02	4718.1
4	–20149	1204.6	–47636

You can see that this solution is not converging and the coefficient matrix is not diagonally dominant. The coefficient matrix

$$[A] = \begin{bmatrix} 3 & 7 & 13 \\ 1 & 5 & 3 \\ 12 & 3 & -5 \end{bmatrix}$$

is not diagonally dominant as

$$|a_{11}| = |3| = 3 \ngeq |a_{12}| + |a_{13}| = |7| + |13| = 20$$

Hence, the Gauss-Seidel method may or may not converge. However, it is the same set of equations as the previous example and that converged. The only difference is that we exchanged first and the third equation with each other and that made the coefficient matrix not diagonally dominant.

Therefore, it is possible that a system of equations can be made diagonally dominant if one exchanges the equations with each other. However, it is not possible for all cases. For example, the following set of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ 2x_2 + 3x_2 + 4x_3 &= 9 \\ x_1 + 7x_2 + x_3 &= 9 \end{aligned}$$

cannot be rewritten to make the coefficient matrix diagonally dominant.

Convert any non-singular square matrix to a strictly diagonally dominant one using only elementary row operations.

Elementary row operations

1. Swap the positions of two of the rows.
2. Multiply one of the rows by a nonzero scalar.
3. Add or subtract the scalar multiple of one row to another row.

Example 1.1.5

Solve the following system of equation using Gauss-Seidel method

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 10 \\ x_1 - 2x_2 + 3x_3 &= 5 \\ 3x_1 + 2x_2 - 4x_3 &= 3 \end{aligned}$$

The given system of equations is not diagonally dominant. Multiply $2nd$ equation by 2. Then the system become diagonally dominant.

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 10 \\ 2x_1 - 4x_2 + 6x_3 &= 10 \\ 3x_1 + 2x_2 - 4x_3 &= 3 \end{aligned}$$

Solution 1.1.6

2,-1,3

Example 1.1.7

Solve the following system of equation using Gauss-Seidel method

$$
\begin{aligned}
2x + y + z &= 8 \\
x + 3y + z &= 6 \\
3x + 2y + 4z &= 16
\end{aligned}
$$

To convert it into a diagonally dominant system, you can multiply the second equation by 2:

$$
\begin{aligned}
2x + y + z &= 8 \\
2x + 6y + 2z &= 12 \\
3x + 2y + 4z &= 16
\end{aligned}
$$

Example 1.1.8

Solve the following system of equation using Gauss-Seidel method

$$\begin{aligned} 4x - 2y + z &= 7 \\ 2x + 5y + 2z &= 1 \\ 3x + y + 6z &= 4 \end{aligned}$$

Multiply the second equation by 2:

$$\begin{aligned} 4x - 2y + z &= 7 \\ 4x + 10y + 4z &= 2 \\ 3x + y + 6z &= 4 \end{aligned}$$

Example 1.1.9

Solve the following system of equation using Gauss-Seidel method

$$
\begin{aligned}
3x + 2y - z &= 5 \\
2x - 4y + z &= -3 \\
x + 3y + 2z &= 9
\end{aligned}
$$

Multiply the third equation by 2:

$$
\begin{aligned}
3x + 2y - z &= 5 \\
2x - 4y + z &= -3 \\
2x + 6y + 4z &= 18
\end{aligned}
$$

Example 1.1.10

Solve the following system of equation using Gauss-Seidel method

$$5x + 2y + z = 7$$
$$3x - 4y - 2z = 1$$
$$2x + y - 3z = 4$$

Multiply the first equation by 2 and the second equation by 3:

$$10x + 4y + 2z = 14$$
$$9x - 12y - 6z = 3$$
$$2x + y - 3z = 4$$

Example 1.1.11

Solve the following system of equation using Gauss-Seidel method

$$
\begin{aligned}
4x + 3y - z &= 9 \\
2x - y + z &= 3 \\
3x + 2y + 5z &= 2
\end{aligned}
$$

Multiply the second equation by 3:

$$
\begin{aligned}
4x + 3y - z &= 9 \\
6x - 3y + 3z &= 9 \\
3x + 2y + 5z &= 2
\end{aligned}
$$

Example 1.1.12

Find the solution to the following system of equations using the Gauss-Seidel method.

$$
\begin{aligned}
3x + y - z &= 7 \\
3x + 4y - 6z &= 8 \\
x - 4y + 2z &= -4
\end{aligned}
$$

You can rearrange your system of equations as

$$\begin{aligned} 3x + y - z &= 7 \\ x - 4y + 2z &= -4 \\ 3x + 4y - 6z &= 8 \end{aligned}$$

Now the first and second rows are diagonally dominant.

$$\begin{aligned} 3x + y - z &= 7 \\ x - 4y + 2z &= -4 \\ 3y - 5z &= 1 \end{aligned}$$

$R_3 \Rightarrow R_3 - R_1$

Now the system is diagonally dominant.

Example 1.1.13

Find the solution to the following system of equations using the Gauss-Seidel method.

$$\begin{aligned} 4x + 2y - z &= 7 \\ -2x + 5y + 2z &= -9 \\ 3x - y + 6z &= 15 \end{aligned}$$

$R_2 \Rightarrow R_1 + 2R_2$ gives

$$\begin{aligned} 4x + 2y - z &= 7 \\ 9y + 3z &= -11 \\ 3x - y + 6z &= 15 \end{aligned}$$

$R_3 \Rightarrow 3R_3 - R_1$ gives

$$\begin{aligned} 4x + 2y - z &= 7 \\ 0x + 9y + 3z &= -11 \\ 0x - 7y + 9z &= 6 \end{aligned}$$

Example 1.2.1

Solve the system of equations

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ -6x_1 + x_2 - x_3 &= 3 \\ 3x_1 + x_2 + x_3 &= 4 \end{aligned}$$

using Gaussian elimination method (GEM) with partial pivoting and backward substitution.

Solution: Consider the augmented matrix of the given system of equations

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ -6 & 1 & -1 & 3 \\ 3 & 1 & 1 & 4 \end{array}\right]$$

Step: 1 Find the largest absolute value in the first column and shift the row into first row. Among $\{|1|, |-6|, |3|\}$, R_2 has the largest absolute value $|-6| = 6$. So, bring R_2 into first row.

$$R_2 \longleftrightarrow R_1 \Rightarrow \left[\begin{array}{ccc|c} -6 & 1 & -1 & 3 \\ 1 & -1 & 1 & 2 \\ 3 & 1 & 1 & 4 \end{array}\right]$$

Step: 2 To obtain echelon form, let us make first element of R_2 into 0 using R_1.

$$R_2 + \left(\frac{1}{6}\right) R_1 \longrightarrow R_2 \Rightarrow \left[\begin{array}{ccc|c} -6 & 1 & -1 & 3 \\ 0 & \frac{-5}{6} & \frac{5}{6} & \frac{5}{2} \\ 3 & 1 & 1 & 4 \end{array}\right]$$

Step: 3 To obtain echelon form, let us make first element of R_3 into 0 using R_1.

$$R_3 + \left(\frac{1}{2}\right) R_1 \longrightarrow R_3 \Rightarrow \left[\begin{array}{ccc|c} -6 & 1 & -1 & 3 \\ 0 & \frac{-5}{6} & \frac{5}{6} & \frac{5}{2} \\ 0 & \frac{3}{2} & \frac{1}{2} & \frac{11}{2} \end{array}\right]$$

Step: 4 Now, compare the absolute value of second element of R_2 and R_3. (i.e.) $\left|\frac{-5}{6}\right|, \left|\frac{3}{2}\right|$. Since absolute value of second element of 3^{rd} row is maximum. Swap R_3 and R_2. Then, we have,

$$R_3 + \left(\frac{1}{2}\right) R_1 \longrightarrow R_3 \Rightarrow \left[\begin{array}{ccc|c} -6 & 1 & -1 & 3 \\ 0 & \frac{3}{2} & \frac{1}{2} & \frac{11}{2} \\ 0 & \frac{-5}{6} & \frac{5}{6} & \frac{5}{2} \end{array}\right]$$

Step: 5 To obtain echelon form, let us make second element of R_3 into 0 using R_2 (Since first element of R_2 is zero).

$$R_3 + \left(\frac{5}{9}\right) R_2 \longrightarrow R_3 \Rightarrow \left[\begin{array}{ccc|c} -6 & 1 & -1 & 3 \\ 0 & \frac{3}{2} & \frac{1}{2} & \frac{11}{2} \\ 0 & 0 & \frac{10}{9} & \frac{50}{9} \end{array}\right]$$

Step: 6 Now, let us convert the augmented matrix into equations.

$$-6x_1 + 1x_2 - 1x_3 = 3 \tag{10}$$

$$\frac{3}{2}x_2 + \frac{1}{2}x_3 = \frac{11}{2} \tag{11}$$

$$\frac{10}{9}x_3 = \frac{50}{9} \tag{12}$$

Step: 7 Using backward substitution to solve the equations.
From Eq.(12), we have,

$$\begin{aligned} \frac{10}{9}x_3 &= \frac{50}{9} \\ x_3 &= \frac{50}{9} \times \frac{9}{10} \\ x_3 &= 5 \end{aligned}$$

Substitute $x_3 = 5$ in Eq.(11), we have,

$$\begin{aligned} \frac{3}{2}x_2 + \frac{1}{2}x_3 &= \frac{11}{2} \\ \frac{3}{2}x_2 + \frac{1}{2}(5) &= \frac{11}{2} \\ \frac{3}{2}x_2 &= \frac{11}{2} - \frac{5}{2} = 3 \\ x_2 &= 3\left(\frac{2}{3}\right) \\ x_2 &= 2 \end{aligned}$$

Substitute $x_2 = 2$ and $x_3 = 5$ in Eq.(10), we have,

$$\begin{aligned} -6x_1 + x_2 - x_3 &= 3 \\ -6x_1 + (2) - (5) &= 3 \\ -6x_1 &= 3 + 3 = 6 \\ x_1 &= -1 \end{aligned}$$

So, the roots are $x_1 = -1$, $x_2 = 2$ and $x_3 = 5$.

Example 1.2.2

Let us use the GEM with partial pivoting to solve the following system:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 5 \\ 4x_1 - 6x_2 &= -2 \\ -2x_1 + 7x_2 + 2x_3 &= 9. \end{aligned}$$

$$\left[\begin{array}{rrr|r} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right]$$

$$R_1 \longleftrightarrow R_2 \left[\begin{array}{rrr|r} 4 & -6 & 0 & -2 \\ 2 & 1 & 1 & 5 \\ -2 & 7 & 2 & 9 \end{array}\right]$$

$$R_2 \to R_2 - (1/2)R_1, R_3 \to R_3 + (1/2)R_1 \left[\begin{array}{ccc|c} 4 & -6 & 0 & -2 \\ 0 & 4 & 1 & 6 \\ 0 & 4 & 2 & 8 \end{array}\right]$$

$$R_3 \Rightarrow R_3 - R_2 \left[\begin{array}{ccc|c} 4 & -6 & 0 & -2 \\ 0 & 4 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Thus $x_3 = 2$.
Then $4x_2 + x_3 = 6$ gives $x_2 = 1$, and $4x_1 - 6x_2 = -2$ gives $x_1 = 1$.
Hence $x_1 = 1, x_2 = 1, x_3 = 2$ is the (unique) solution of the given linear system.

Example 1.2.3

Solve the system equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ 4x_1 + 3x_2 + 4x_3 &= 8 \\ 9x_1 + 3x_2 + 4x_3 &= 7 \end{aligned}$$

using Gauss elimination method with partial pivoting and backward substitution method.

Solution

- $R_1 \longleftrightarrow R_1$
- $R_2 \rightarrow R_2 - 4R_3$, $R_3 \rightarrow R_1 - 9R_3$
- $R_3 \rightarrow -6R_2 + R_3$
- $x_1 = -\frac{1}{5}$, $x_2 = 4$ and $x_3 = -\frac{4}{5}$

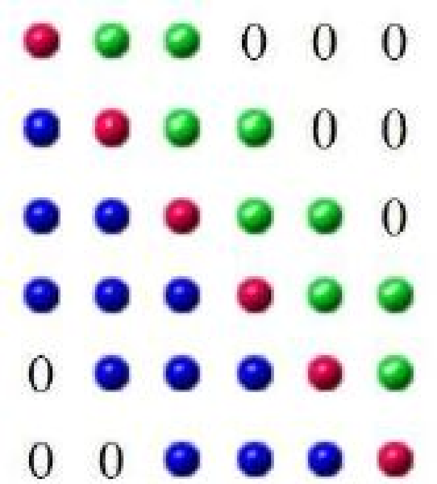
Banded matrix

A band matrix is a sparse matrix whose non-zero entries are confined to a diagonal band, comprising the main diagonal and zero or more diagonals on either side.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 6 & 2 & 5 \\ 0 & 7 & 3 \end{bmatrix}$$

Banded matrix

The matrix can be symmetric, having the same number of sub- and super- diagonals. If a matrix has only one sub- and one super-diagonal, we have a tridiagonal matrix etc. The number of super-diagonals is called the upper bandwidth (two in the example), and the number of sub-diagonals is the lower bandwidth (three in the example). The total number of diagonals, six in the example, is the bandwidth.



A tridiagonal matrix is a band matrix that has nonzero elements only on the main diagonal, the first diagonal below this, and the first diagonal above the main diagonal.

Considering a 4 X 4 Matrix $\begin{bmatrix} a_1 & b_1 & 0 & 0 \\ c_2 & a_2 & b_2 & 0 \\ 0 & c_3 & a_3 & b_3 \\ 0 & 0 & c_4 & a_4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

- Thomas' algorithm, also called TriDiagonal Matrix Algorithm (TDMA) is essentially the result of applying Gaussian elimination to the tridiagonal system of equations.
- A system of simultaneous algebraic equations with nonzero coefficients only on the main diagonal, the lower diagonal, and the upper diagonal is called a tridiagonal system of equations.

Consider a tridiagonal system of N equations with N unknowns, $u_1, u_2, u_3, \cdots, u_N$ as given below:

$$\begin{bmatrix} b_1 & c_1 & & & & & \\ a_1 & b_2 & c_2 & & & & \\ & a_3 & b_3 & c_3 & & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & & & a_N & b_N \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ \vdots \\ d_{N-1} \\ d_N \end{bmatrix}$$

Consider the system of linear simultaneous algebraic equations given by

$$\mathbf{A}\mathbf{x} = b$$

where $\mathbf{A}$ is a tridiagonal matrix, $x = [x_1, x_2, \ldots, x_n]^T$ and $b = [b_1, b_2, \ldots, b_n]^T$. Hence, we consider a 4×4 tridiagonal system of equations given by

$$\begin{bmatrix} a_{12} & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{31} & a_{32} & a_{33} \\ 0 & 0 & a_{41} & a_{42} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \tag{13}$$

Equation (13) can be written as

$$\begin{aligned} a_{12}x_1 + a_{13}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_2 + a_{32}x_3 + a_{33}x_4 &= b_3 \\ a_{41}x_3 + a_{42}x_4 &= b_4 \end{aligned} \tag{14}$$

The system of equations given by Eq. (14) is solved using Thomas Algorithm which is described in three steps as shown below:

Step 1: Set $y_1 = a_{12}$ and compute

$$y_i = a_{i2} - \frac{a_{i1}a_{(i-1)3}}{y_{i-1}} \quad i = 2, 3, \ldots, n$$

Step 2: Set $z_1 = \frac{b_1}{a_{12}}$ and compute

$$z_i = \frac{b_i - a_{i1}z_{i-1}}{y_i} \quad i = 2, 3, \ldots, n$$

Step 3: $x_i = z_i - \frac{a_{i3}x_{i+1}}{y_i} \quad i = n-1, n-2, \ldots, 1$, where $x_n = z_n$

Example 1.3.1

Solve the following equations by Thomas Algorithm.

$$
\begin{aligned}
3x_1 - x_2 &= 5\\
2x_1 - 3x_2 + 2x_3 &= 5\\
x_2 + 2x_3 + 5x_4 &= 10\\
x_3 - x_4 &= 1
\end{aligned}
$$

Here

$$\begin{bmatrix} 3 & -1 & 0 & 0 \\ 2 & -3 & 2 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 10 \\ 1 \end{bmatrix}$$

$$[a_2, a_3, a_4] = [2, 1, 1]$$

$$[b_1, b_2, b_3, b_4] = [3, -3, 2, -1]$$

$$[c_1, c_2, c_3] = [-1, 2, 5]$$

$$[d_1, d_2, d_3, d_4] = [5, 5, 10, 1]$$

Step 1: Set $y_1 = b_1 = 3$ and compute $y_i = b_i - \dfrac{a_i c_{i-1}}{y_{i-1}}, i = 2, 3, \ldots, n$

$$i = 2, y_2 = b_2 - \frac{a_2 c_1}{y_1} = -3 - \frac{2(-1)}{3} = -\frac{7}{3}$$

$$i = 3, y_3 = b_3 - \frac{a_3 c_2}{y_2} = 2 - \frac{1 \times 2}{-\frac{7}{3}} = \frac{20}{7}$$

$$i = 4, y_4 = b_4 - \frac{a_4 c_3}{y_3} = -1 - \frac{1.5}{20} = -\frac{55}{20}$$

Step 2: Set $z_1 = \frac{d_1}{b_1} = \frac{5}{3}$ and compute $z_i = \frac{d_i - a_i z_{i-1}}{y_i}, i = 2, 3, \ldots, n$

$$i = 2, z_2 = \frac{d_2 - a_2 z_1}{y_2} = \frac{5 - 2 \times \frac{5}{3}}{-\frac{7}{3}} = -\frac{5}{7}$$

$$i = 3, z_3 = \frac{d_3 - a_3 z_2}{y_3} = \frac{10 - 1\left(-\frac{5}{7}\right)}{\frac{20}{7}} = \frac{75}{20}$$

$$i = 4, z_4 = \frac{d_4 - a_4 z_3}{y_4} = \frac{1 - 1 \times \frac{75}{20}}{-\frac{55}{20}} = 1$$

Step 3: Set $x_n = z_n$; i.e., $x_4 = z_4 = 1$ and compute $x_i = z_i - \frac{c_i x_{i+1}}{y_i}; i = n-1, n-2, \ldots, 1$

$$i = 3, x_3 = z_3 - \frac{c_3 x_4}{y_3} = \frac{75}{20} - \frac{5 \times 1}{\frac{20}{7}} = 2$$

$$i = 2, x_2 = z_2 - \frac{c_2 x_3}{y_2} = -\frac{5}{7} - \frac{2 \times 2}{-\frac{7}{3}} = 1$$

$$i = 1, x_1 = z_1 - \frac{c_1 x_2}{\ } = \frac{5}{\ } - \frac{(-1) \times 1}{\ } = 2$$

Example 1.3.2

Solve the following set of tridiagonal set of algebraic equations using Thomas's method.

$$
\begin{aligned}
x_1 + 4x_2 &= 10 \\
2x_1 + 10x_2 - 4x_3 &= 7 \\
x_2 + 8x_3 - x_4 &= 6 \\
x_3 - 6x_4 &= 4
\end{aligned}
$$

Here

$$\begin{bmatrix} 1 & 4 & 0 & 0 \\ 2 & 10 & -4 & 0 \\ 0 & 1 & 8 & -1 \\ 0 & 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ 6 \\ 4 \end{bmatrix}$$

$$[a_2, a_3, a_4] = [2, 1, 1]$$

$$[b_1, b_2, b_3, b_4] = [1, 10, 8, -6]$$

$$[c_1, c_2, c_3] = [4, -4, -1]$$

$$[d_1, d_2, d_3, d_4] = [10, 7, 6, 4]$$

Step 1: Set $y_1 = b_1 = 1$ and compute $y_i = b_i - \dfrac{a_i c_{i-1}}{y_{i-1}}, i = 2, 3, \ldots, n$

$$i = 2, y_2 = b_2 - \frac{a_2 c_1}{y_1} = 10 - \frac{2 \times 4}{1} = 2$$

$$i = 3, y_3 = b_3 - \frac{a_3 c_2}{y_2} = \frac{8 - 1(-4)}{2} = 8 + 2 = 10$$

$$i = 4, y_4 = b_4 - \frac{a_4 c_3}{y_3} = -6 - \frac{1 \times (-1)}{10} = \frac{-60 + 1}{10} = -\frac{59}{10}$$

Step 2: Set $z_1 = \frac{d_1}{b_1} = \frac{10}{1} = 10$ and compute $z_i = \frac{d_i - a_i z_{i-1}}{y_i}, i = 2, 3, \ldots, n$

$$i = 2, z_2 = \frac{d_2 - a_2 z_1}{y_2} = \frac{7 - 2.10}{2} = -\frac{13}{2}$$

$$i = 3, z_3 = \frac{d_3 - a_3 z_2}{y_3} = \frac{6 - 1(-13/2)}{10} = \frac{6 + 13/2}{10} = \frac{25}{20}$$

$$i = 4, z_4 = \frac{d_4 - a_4 z_3}{y_4} = \frac{4 - 1 \times 25/20}{-59/10} = -\frac{55}{118}$$

Step 3: Set $x_n = z_n$; i.e., $x_4 = z_4 = -\frac{55}{118} = -0.466$ and compute $x_i = z_i - \frac{c_i x_{i+1}}{y_i}; i = n-1, n-2, \ldots, 1$

$$i = 3, x_3 = z_3 - \frac{c_3 x_4}{y_3} = \frac{25}{20} - \frac{(-1)(-55/118)}{10} = 1.203$$

$$i = 2, x_2 = z_2 - \frac{c_2 x_3}{y_2} = -\frac{13}{2} - \frac{(-4)1.203}{2} = -4.094$$

$$i = 1, x_1 = z_1 - \frac{c_1 x_2}{y_1} = 10 - \frac{4(-4.094)}{1} = 26.376$$