Module-IV

Numerical Differentiation and Integration

Example 4.1.1

Given a cubic polynomial with following data points

x	0	1	2	3
$f(x)$	5	6	3	8

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 0$.

The forward difference table is:

x	y	Δ	Δ^2	Δ^3
0	5			
		1		
1	6		-4	
		-3		12
2	3		8	
		5		
3	8			

To find the derivative at $x = 0$, taking $x_0 = 0$ and applying the relation:

$$\left.\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right] \tag{48}$$

From table $h = 1$,
Substituting forward difference table values in equation (48), we get

$$\left.\frac{dy}{dx}\right]_{x=0} = \frac{1}{1}\left[1 - \frac{(-4)}{2} + \frac{12}{3}\right] = 7$$

Also,

$$\left.\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right] \tag{49}$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=0} = \frac{1}{1^2}\left[-4 - 12 + 0\right] = -16$$

Example 4.1.2

Given a polynomial with following data points:

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
$f(x)$	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.1$ and $x = 1.5$.

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	7.989						
		0.414					
1.1	8.403		-0.036				
		0.378		0.006			
1.2	8.781		-0.030		-0.002		
		0.358		0.004		0.001	
1.3	9.129		-0.026		-0.001		0.002
		0.322		0.003		0.003	
1.4	9.451		-0.023		0.002		
		0.299		0.005			
1.5	9.750		-0.018				
		0.281					
1.6	10.031						

To find the derivative at $x = 1.1$, taking $x_0 = 1.1$ and applying the relation:

$$\left.\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right] \tag{50}$$

From given data $h = 0.1$.
Substituting forward difference table value in equation (50), we get

$$\begin{aligned}\left.\frac{dy}{dx}\right]_{x=x_0} &= \frac{1}{0.1}\left[0.378 - \frac{(-0.030)}{2} + \frac{0.004}{3} - \frac{(-0.001)}{4} + \frac{0.003}{5} - \cdots\right] \\ &= 3.9518.\end{aligned}$$

Also,

$$\left.\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=1.1} = \frac{1}{(0.1)^2}\left[-0.03 - 0.004 + \frac{11}{12}(-0.001) - \frac{5}{6}(0.003)\right] = -3.74$$

To find the derivative at $x = 1.5$, taking $x_n = 1.5$ and applying the relation:

$$\left.\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \cdots\right] \tag{51}$$

From given data $h = 0.1$.
Substituting the forward difference table value in equation (51), we get

$$\left.\frac{dy}{dx}\right]_{x=1.5} = \frac{1}{0.1}\left[0.299 + \frac{(-0.023)}{2} + \frac{0.003}{3} + \frac{(-0.001)}{4} + \frac{0.001}{5}\right] = 2.8845$$

Also,

$$\left.\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\right] \tag{52}$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=1.5} = \frac{1}{(0.1)^2}\left[-0.023 + 0.003 + \frac{11}{12}(-0.001) + \frac{5}{6}(0.001)\right] = -2.0083.$$

Trapezoidal rule

Trapezoidal rule to evaluate $\int_a^b f(x)dx$, where the function $y = f(x)$ is given as discrete set of points (x_i, y_i), $i = 0, 1, 2, 3, \cdots, n$, is given by

$$\int_a^b f(x)dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n]$$

Simpson's one-third rule

Simpson's one-third $\left(\frac{1}{3}\right)$ rule to evaluate $\int_a^b f(x)dx$, where the function $y = f(x)$ is given as discrete set of points (x_i, y_i), $i = 0, 1, 2, 3, \cdots, n$, is given by

$$\int_a^b f(x)dx = \frac{h}{3}\left[\begin{array}{l} (y_0 + y_n) \\ +4(y_1 + y_3 + \cdots + y_{n-1}) \\ +2(y_2 + y_4 + \cdots + y_{n-2}) \end{array}\right].$$

Simpson's three-eighths rule

Simpson's three-eighths $\left(\frac{3}{8}\right)$ rule to evaluate $\int_a^b f(x)dx$, where the function $y = f(x)$ is given as discrete set of points (x_i, y_i), $i = 0, 1, 2, 3, \cdots, n$, is given by

$$\int_a^b f(x)dx = \frac{3h}{8}\left[\begin{array}{l} (y_0 + y_n) \\ +3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-2} + y_{n-1}) \\ +2(y_3 + y_6 + \cdots + y_{n-3}) \end{array}\right].$$

Example 4.2.1

Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ using

(a) Trapezoidal rule taking $h = \frac{1}{5}$

(b) Simpson's $\frac{1}{3}$ rule taking $h = \frac{1}{4}$

(c) Simpson's $\frac{3}{8}$ rule taking $h = \frac{1}{6}$

(a). To solve $\int_0^1 \frac{1}{1+x^2} dx$ using trapezoidal rule. Taking

$$h = \frac{1}{5} = 0.2, n = \frac{b-a}{h} = \frac{1-0}{0.2} = 5$$

$\therefore$ Dividing the interval $(0, 1)$ into 5 equal parts for the function $f(x) = \frac{1}{1+x^2}$

x	0	0.2	0.4	0.6	0.8	1
$y = f(x)$	1	0.96	0.86	0.74	0.61	0.5

By trapezoidal rule,

$$\begin{aligned}\int_0^1 \frac{1}{1+x^2} dx &= \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4) + y_5] \\ &= \frac{0.2}{2}[1 + 2(0.96 + 0.86 + 0.74 + 0.61) + 0.5] \\ &= 0.784.\end{aligned}$$

$\therefore \int_0^1 \frac{1}{1+x^2} dx = 0.784$ using trapezoidal rule.

(b). To solve $\int_0^1 \frac{1}{1+x^2}dx$ using Simpson's $\frac{1}{3}$ rule.
Taking

$$h = \frac{1}{4} = 0.25, n = \frac{b-a}{h} = \frac{1-0}{0.25} = 4$$

$\therefore$ Dividing the interval $(0, 1)$ into 4 equal parts for the function $f(x) = \frac{1}{1+x^2}$.

x	0	0.25	0.5	0.75	1
$y = f(x)$	1	0.94	0.8	0.64	0.5

By Simpson's $\frac{1}{3}$ rule,

$$\begin{aligned}\int_0^1 \frac{1}{1+x^2}dx &= \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)] \\ &= \frac{0.25}{3}[(1 + 0.5) + 4(0.94 + 0.64) + 2(0.8)] \\ &= 0.7850\end{aligned}$$

$\therefore \int_0^1 \frac{1}{1+x^2}dx = 0.7850$ using Simpson's $\frac{1}{3}$ rule.

(c). To solve $\int_0^1 \frac{1}{1+x^2}dx$ using Simpson's $\frac{3}{8}$ rule.
Taking

$$h = \frac{1}{6}, n = \frac{b-a}{h} = \frac{1-0}{\frac{1}{6}} = 6$$

$\therefore$ Dividing the interval into 6 equal parts for the function $f(x) = \frac{1}{1+x^2}$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y = f(x)$	1	0.97	0.9	0.8	0.69	0.59	0.5

By Simpson's $\frac{3}{8}$ rule

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$
$$= \left(\frac{3}{8}\right)\left(\frac{1}{6}\right)[(1 + 0.5) + 3(0.97 + 0.9 + 0.69 + 0.59) + 2(0.8)]$$
$$= 0.7844$$

$\therefore \int_0^1 \frac{1}{1+x^2}dx = 0.7844$ using Simpson's $\frac{3}{8}$ rule.

Example 4.2.2

Evaluate $\int_0^{\frac{\pi}{2}} \sin x dx$ using

a. Trapezoidal rule
b. Simpson's $\frac{1}{3}$ rule
c. Simpson's $\frac{3}{8}$ rule, taking $n = 6$.

(a). Taking

$$n = 6, h = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{6} = \frac{\pi}{12}$$

$\therefore$ Dividing the interval $\left(0, \frac{\pi}{2}\right)$ into 6 equal parts for the function $f(x) = \sin x$.

x	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	$\frac{\pi}{2}$
$y = f(x)$	0	0.2588	0.5	0.7071	0.866	0.9659	1

To solve $\int_0^{\frac{\pi}{2}} \sin x dx$ using trapezoidal rule

$$\int_0^{\frac{\pi}{2}} \sin x dx$$

$$= \frac{h}{2}[y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5) + y_6]$$

$$= \left(\frac{\pi}{12}\right)\left(\frac{1}{2}\right)[0 + 2(0.2588 + 0.5 + 0.7071 + 0.866 + 0.9659) + 1]$$

$\therefore \int_0^{\frac{\pi}{2}} \sin x dx = 0.9943$ using trapezoidal rule.

(b). To solve $\int_0^{\frac{\pi}{2}} \sin x dx$ using Simpson's $\frac{1}{3}$ rule

$$\int_0^{\frac{\pi}{2}} \sin x dx$$
$$= \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$
$$= \left(\frac{\pi}{12}\right)\left(\frac{1}{3}\right)[(0+1) + 4(0.2588 + 0.7071 + 0.9659) + 2(0.5 + 0.866)]$$

$\therefore \int_0^{\frac{\pi}{2}} \sin x dx = 1.000004$ using Simpson's $\frac{1}{3}$ rule.

(c). To solve $\int_0^{\frac{\pi}{2}} \sin x dx$ using Simpson's $\frac{3}{8}$ rule

$$\int_0^{\frac{\pi}{2}} \sin x dx$$
$$= \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)]$$
$$= \left(\frac{\pi}{12}\right)\left(\frac{3}{8}\right)[(0+1) + 3(0.2588 + 0.5 + 0.866 + 0.9659) + 2(0.7071)]$$

$\therefore \int_0^{\frac{\pi}{2}} \sin x dx = 1.00004$ using Simpson's $\frac{3}{8}$ rule.

Gauss Quadrature 2-point formula

Gauss Quadrature 2-point formula for $I = \int_a^b f(x)dx$, is given by

$$I = \frac{b-a}{2}\left[w_1 f(x_1) + w_2 f(x_2)\right] \tag{53}$$

where,

$$x_1 = \frac{b-a}{2} z_1 + \frac{b+a}{2}$$
$$x_2 = \frac{b-a}{2} z_2 + \frac{b+a}{2}$$
$$w_1 = w_2 = 1$$
$$z_1 = \frac{-1}{\sqrt{3}}; z_2 = \frac{1}{\sqrt{3}}.$$

Gauss Quadrature 3-point formula

Gauss Quadrature 3-point formula for $I = \int_a^b f(x)dx$, is given by

$$I = \frac{b-a}{2}\left[w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)\right] \tag{54}$$

where,

$$x_1 = \frac{b-a}{2} z_1 + \frac{b+a}{2}$$
$$x_2 = \frac{b-a}{2} z_2 + \frac{b+a}{2}$$
$$x_3 = \frac{b-a}{2} z_3 + \frac{b+a}{2}$$
$$w_1 = \frac{5}{9}, \;\; w_2 = \frac{8}{9}, \;\; w_3 = \frac{5}{9}$$
$$z_1 = -\sqrt{\frac{3}{5}}, \;\; z_2 = 0, \;\; z_3 = \sqrt{\frac{3}{5}}$$

Example 4.3.1

Use Gauss-Legendre two-point formula to evaluate

$$I = \int_{-2}^{2} e^{\frac{-x}{2}} dx$$

Given data: $a = -2$, $b = 2$, and $f(x) = e^{\frac{-x}{2}}$

Gauss Quadrature 2-point formula:

$$I = \frac{b-a}{2}\left[w_1 f(x_1) + w_2 f(x_2)\right] \tag{55}$$

where,

$$x_1 = \frac{b-a}{2} z_1 + \frac{b+a}{2}; \qquad x_2 = \frac{b-a}{2} z_2 + \frac{b+a}{2}$$

$$w_1 = 1; \qquad w_2 = 1$$

$$z_1 = \frac{-1}{\sqrt{3}}; \qquad z_2 = \frac{1}{\sqrt{3}}.$$

$$x_1 = \frac{b-a}{2}z_1 + \frac{b+a}{2} = \frac{2-(-2)}{2}\left(\frac{-1}{\sqrt{3}}\right) + \frac{2-2}{2} = \frac{-2}{\sqrt{3}}$$

$$f(x_1) = e^{\frac{-x_1}{2}} = e^{-\left(\frac{\left[\frac{-2}{\sqrt{3}}\right]}{2}\right)}$$

$$x_2 = \frac{b-a}{2}z_2 + \frac{b+a}{2} = \frac{2-(-2)}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{2-2}{2} = \frac{2}{\sqrt{3}}$$

$$f(x_2) = e^{\frac{-x_2}{2}} = e^{-\left(\frac{\left[\frac{2}{\sqrt{3}}\right]}{2}\right)}$$

$$I = \frac{b-a}{2}\left[w_1 f(x_1) + w_2 f(x_2)\right]$$

$$= \frac{2-(-2)}{2}\left[1 \times e^{-\left(\frac{\left[\frac{-2}{\sqrt{3}}\right]}{2}\right)} + 1 \times e^{-\left(\frac{\left[\frac{2}{\sqrt{3}}\right]}{2}\right)}\right] = 4.6854.$$

Example 4.3.2

Use Gauss-Legendre three-point formula to evaluate

$$I = \int_2^4 (x^4 + 1)dx \tag{56}$$

Given data: $a = 2$, $b = 4$, and $f(x) = (x^4 + 1)$

Gauss Quadrature 3-point formula:

$$I = \frac{b-a}{2}\left[w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)\right] \tag{57}$$

where,

$$x_1 = \frac{b-a}{2}z_1 + \frac{b+a}{2} \qquad x_2 = \frac{b-a}{2}z_2 + \frac{b+a}{2} \qquad x_3 = \frac{b-a}{2}z_3 + \frac{b+a}{2}$$

$$w_1 = \frac{5}{9}, \qquad w_2 = \frac{8}{9}, \qquad w_3 = \frac{5}{9}$$

$$z_1 = -\sqrt{\frac{3}{5}}, \qquad z_2 = 0, \qquad z_3 = \sqrt{\frac{3}{5}}$$

$$x_1 = \frac{b-a}{2} z_1 + \frac{b+a}{2};\ x_2 = \frac{b-a}{2} z_2 + \frac{b+a}{2};\ x_3 = \frac{b-a}{2} z_3 + \frac{b+a}{2}$$

$$x_1 = \frac{4-2}{2}\left(-\sqrt{\frac{3}{5}}\right) + \frac{4+2}{2} = 2.2254$$

$$x_2 = \frac{4-2}{2}(0) + \frac{4+2}{2} = 3$$

$$x_3 = \frac{4-2}{2}\left(\sqrt{\frac{3}{5}}\right) + \frac{4+2}{2} = 3.7746$$

$$f(x_1) = f(2.2254) = (2.2254^4 + 1) = 25.5263$$

$$f(x_2) = f(3) = (3^4 + 1) = 82$$

$$f(x_3) = f(3.7746) = (3.7746^4 + 1) = 203.9942$$

$$
\begin{aligned}
I &= \frac{b-a}{2}\left[w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)\right] \\
&= \left(\frac{4-2}{2}\right)\left[\left(\frac{5}{9}\right)(25.5263) + \left(\frac{8}{9}\right)(82) + \left(\frac{5}{9}\right)(203.9942)\right] \\
&= 200.4014
\end{aligned}
$$

Example 4.3.3

Use Gauss-Legendre two-point formula to evaluate $I = \int_{-1}^{1} e^x dx$

Given data: $a = -1$ $b = 1$ and $f(x) = e^x$ Using Gauss Quadrature 2-Point Formula:

$$I = \frac{b-a}{2}\left[w_1 f(x_1) + w_2 f(x_2)\right] \tag{58}$$

where,

$$w_1 = 1; \qquad w_2 = 1$$

$$z_1 = \frac{-1}{\sqrt{3}}; \qquad z_2 = \frac{1}{\sqrt{3}}.$$

$$x_1 = \frac{b-a}{2}z_1 + \frac{b+a}{2} = \frac{1-(-1)}{2}\left(\frac{-1}{\sqrt{3}}\right) + \frac{1-1}{2} = \frac{-1}{\sqrt{3}}$$

$$x_2 = \frac{b-a}{2}z_2 + \frac{b+a}{2} = \frac{1-(-1)}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{1-1}{2} = \frac{1}{\sqrt{3}}$$

Gauss Quadrature 2-point formula:

$$\begin{aligned} I &= \frac{b-a}{2}\left[w_1 f(x_1) + w_2 f(x_2)\right] \\ &= \frac{1-(-1)}{2}\left[1 \times e^{-\left(\frac{-1}{\sqrt{3}}\right)} + 1 \times e^{\left(\frac{1}{\sqrt{3}}\right)}\right] = 2.3427. \end{aligned}$$

Example 4.3.4

Use Gauss-Legendre three-point formula to evaluate $I = \int_{-1}^{1} e^x dx$

Given data: $a = -1$ $b = 1$ and $f(x) = e^x$ $x = \frac{b-a}{2}z + \frac{b+a}{2} = z$ Using Gauss Quadrature 3-Point Formula:

$$I = \frac{b-a}{2}\left[w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)\right] \tag{59}$$

where,

$$x_1 = \frac{b-a}{2}z_1 + \frac{b+a}{2} \quad x_2 = \frac{b-a}{2}z_2 + \frac{b+a}{2} \quad x_3 = \frac{b-a}{2}z_3 + \frac{b+a}{2}$$

$$w_1 = \frac{5}{9}, \quad w_2 = \frac{8}{9}, \quad w_3 = \frac{5}{9}$$

$$z_1 = -\sqrt{\frac{3}{5}}, \quad z_2 = 0, \quad z_3 = \sqrt{\frac{3}{5}}$$

Gauss Quadrature 3-point formula:

$$x_1 = \frac{b-a}{2} z_1 + \frac{b+a}{2} = \frac{1-(-1)}{2}\left(-\sqrt{\frac{3}{5}}\right) + \frac{1+(-1)}{2} = -0.7745$$

$$f(x_1) = e^{(-0.7746)} = 0.4609$$

$$x_2 = \frac{b-a}{2} z_2 + \frac{b+a}{2} = \frac{1-(-1)}{2}(0) + \frac{1+(-1)}{2} = 0$$

$$f(x_2) = e^{(0)} = 1$$

$$x_3 = \frac{b-a}{2} z_3 + \frac{b+a}{2} = \frac{1-(-1)}{2}\left(\sqrt{\frac{3}{5}}\right) + \frac{1+(-1)}{2} = 0.7745$$

$$f(x_3) = e^{(0.7746)} = 2.1697$$

$$I = \frac{b-a}{2}\left[w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)\right]$$

$$= \frac{1-(-1)}{2}\left[\frac{5}{9} \times (0.4609) + \frac{8}{9} \times (1) + \frac{5}{9} \times (2.1697)\right] = 2.3503.$$

There are two primary problems with Newton-Coates methods.

1. They are unsuitable for large intervals since high degree formulas are required and the coefficients of the formulas are hard to find.
2. They are based on interpolating polynomials and high degree polynomials oscillate over large intervals.

Example 4.3.3

Use Gauss-Legendre two-point formula to evaluate $I = \int_{-1}^{1} e^x dx$

Given data: $a = -1$ $b = 1$ and $f(x) = e^x$ Using Gauss Quadrature 2-Point Formula:

$$I = \frac{b-a}{2}\left[w_1 f(x_1) + w_2 f(x_2)\right] \tag{58}$$

where,

$$w_1 = 1; \qquad w_2 = 1$$

$$z_1 = \frac{-1}{\sqrt{3}}; \qquad z_2 = \frac{1}{\sqrt{3}}.$$

$$x_1 = \frac{b-a}{2} z_1 + \frac{b+a}{2} = \frac{1-(-1)}{2}\left(\frac{-1}{\sqrt{3}}\right) + \frac{1-1}{2} = \frac{-1}{\sqrt{3}}$$

$$x_2 = \frac{b-a}{2} z_2 + \frac{b+a}{2} = \frac{1-(-1)}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{1-1}{2} = \frac{1}{\sqrt{3}}$$

Gauss Quadrature 2-point formula:

$$
\begin{aligned}
I &= \frac{b-a}{2}\left[w_1 f(x_1) + w_2 f(x_2)\right] \\
&= \frac{1-(-1)}{2}\left[1 \times e^{-\left(\frac{-1}{\sqrt{3}}\right)} + 1 \times e^{\left(\frac{1}{\sqrt{3}}\right)}\right] = 2.3427.
\end{aligned}
$$

Example 4.3.4

Use Gauss-Legendre three-point formula to evaluate $I = \int_{-1}^{1} e^x dx$

Given data: $a = -1$ $b = 1$ and $f(x) = e^x$ $x = \frac{b-a}{2}z + \frac{b+a}{2} = z$ Using Gauss Quadrature 3-Point Formula:

$$I = \frac{b-a}{2}\left[w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)\right] \tag{59}$$

where,

$$x_1 = \frac{b-a}{2}z_1 + \frac{b+a}{2} \qquad x_2 = \frac{b-a}{2}z_2 + \frac{b+a}{2} \qquad x_3 = \frac{b-a}{2}z_3 + \frac{b+a}{2}$$

$$w_1 = \frac{5}{9}, \qquad w_2 = \frac{8}{9}, \qquad w_3 = \frac{5}{9}$$

$$z_1 = -\sqrt{\frac{3}{5}}, \qquad z_2 = 0, \qquad z_3 = \sqrt{\frac{3}{5}}$$

Gauss Quadrature 3-point formula:

$$x_1 = \frac{b-a}{2}z_1 + \frac{b+a}{2} = \frac{1-(-1)}{2}\left(-\sqrt{\frac{3}{5}}\right) + \frac{1+(-1)}{2} = -0.7745$$

$$f(x_1) = e^{(-0.7746)} = 0.4609$$

$$x_2 = \frac{b-a}{2}z_2 + \frac{b+a}{2} = \frac{1-(-1)}{2}(0) + \frac{1+(-1)}{2} = 0$$

$$f(x_2) = e^{(0)} = 1$$

$$x_3 = \frac{b-a}{2}z_3 + \frac{b+a}{2} = \frac{1-(-1)}{2}\left(\sqrt{\frac{3}{5}}\right) + \frac{1+(-1)}{2} = 0.7745$$

$$f(x_3) = e^{(0.7746)} = 2.1697$$

$$I = \frac{b-a}{2}\left[w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)\right]$$

$$= \frac{1-(-1)}{2}\left[\frac{5}{9}\times(0.4609) + \frac{8}{9}\times(1) + \frac{5}{9}\times(2.1697)\right] = 2.3503.$$

There are two primary problems with Newton-Coates methods.

1. They are unsuitable for large intervals since high degree formulas are required and the coefficients of the formulas are hard to find.
2. They are based on interpolating polynomials and high degree polynomials oscillate over large intervals.

Example 4.4.1

Using Romberg Method compute $\int_0^{\pi} \sin x dx$ at $R_{4,4}$.

By Trapezoidal rule:

$$
\begin{aligned}
R_{1,1} &= \frac{b-a}{2}[f(a)+f(b)] \\
&= \frac{\pi}{2}\left[\sin 0 + \sin \pi\right] = 0 \\
R_{2,1} &= \frac{b-a}{4}\left[f(a) + 2f\left(a + \frac{(b-a)}{2}\right) + f(b)\right] \\
&= \frac{\pi}{4}\left[\sin 0 + 2\sin\frac{\pi}{2} + \sin\pi\right] = 1.57079633
\end{aligned}
$$

$$R_{3,1}=\frac{b-a}{8}\left[f(a)+2\left(f\left(a+\frac{(b-a)}{4}\right)+f\left(a+\frac{2(b-a)}{4}\right)+f\left(a+\frac{3(b-a)}{4}\right)\right)+f(b)\right]$$

$$=\frac{\pi}{8}\left[\sin 0+2\left(\sin\frac{\pi}{4}+\sin\frac{2\pi}{4}+\sin\frac{3\pi}{4}\right)+\sin\pi\right]=1.89611890$$

$$R_{4,1}=\frac{b-a}{16}\left[f(a)+2\left(\begin{array}{c} f\left(a+\frac{(b-a)}{8}\right)+f\left(a+\frac{2(b-a)}{8}\right)+f\left(a+\frac{3(b-a)}{8}\right)\\ +f\left(a+\frac{4(b-a)}{8}\right)+f\left(a+\frac{5(b-a)}{8}\right)+f\left(a+\frac{6(b-a)}{8}\right)\\ +f\left(a+\frac{7(b-a)}{8}\right)\end{array}\right)+f(b)\right]$$

$$=\frac{\pi}{16}\left[\sin 0+2\left(\begin{array}{c}\sin\frac{\pi}{8}+\sin\frac{2\pi}{8}+\sin\frac{3\pi}{8}\\ +\sin\frac{4\pi}{8}+\sin\frac{5\pi}{8}+\sin\frac{6\pi}{8}\\ +\sin\frac{7\pi}{8}\end{array}\right)+\sin\pi\right]=1.97423160$$

In general,

$$R_{i,j}=\frac{b-a}{2n}\left[f(a)+2\left(\sum_{i=1}^{n-1}f(a+ih)\right)+f(b)\right],\ n=1,2,4,8,\cdots$$

$$R_{2,2} = R_{2,1} + \frac{R_{2,1} - R_{1,1}}{3} = 2.09439$$

$$R_{3,2} = R_{3,1} + \frac{R_{3,1} - R_{2,1}}{3} = 2.00455976$$

$$R_{4,2} = R_{4,1} + \frac{R_{4,1} - R_{3,1}}{3} = 2.000026917$$

$$R_{3,3} = R_{3,2} + \frac{R_{3,2} - R_{3,1}}{15} = 1.99857073$$

$$R_{4,3} = R_{4,2} + \frac{R_{4,2} - R_{3,2}}{15} = 1.99998313$$

$$R_{4,4} = R_{4,3} + \frac{R_{4,3} - R_{3,3}}{63} = 2.0000$$

Example 4.4.2

The vertical distance in meters covered by a rocket from $t = 8$ to $t = 30$ seconds is given by

$$x = \int_{8}^{30} \left(2000 \log\left[\frac{140000}{140000 - 2100t}\right] - 9.8t\right) dt$$

Use Romberg's rule to find the distance covered. Use the 1, 2,3 and 4-segment trapezoidal rule results.

Solutions:

$$R_{1,1} = 11868$$
$$R_{1,2} = 11266$$
$$R_{1,3} = 11113$$
$$R_{1,4} = 11074$$

1^{st} order extrapolation values:

$$R_{2,1} = R_{1,2} + \frac{R_{2,1} - R_{1,1}}{3} = 11266 + \frac{11266 - 11868}{3} = 11065$$
$$R_{2,2} = R_{1,3} + \frac{R_{1,3} - R_{1,2}}{3} = 11113 + \frac{11113 - 11266}{3} = 11062$$
$$R_{2,3} = R_{1,4} + \frac{R_{1,4} - I_{1,3}}{3} = 11074 + \frac{11074 - 11113}{3} = 11061$$

2^{nd} order extrapolation values

$$R_{3,1} = R_{2,2} + \frac{R_{2,2} - R_{2,1}}{15} = 11062 + \frac{11062 - 11065}{15} = 11062$$
$$R_{3,2} = I_{2,3} + \frac{R_{2,3} - R_{2,2}}{15} = 11061 + \frac{11061 - 11062}{15} = 11061$$

3^{rd} order extrapolation values

$$R_{4,1} = R_{3,2} + \frac{R_{3,2} - R_{3,1}}{63} = 11061 + \frac{11061 - 11062}{63} = 11061m$$

		First Order	Second Order	Third Order
1-segment	11868			
		11065		
2-segment	11266		11062	
		11062		11061
3-segment	11113		11061	
		11061		
4-segment	11074			

h_i		First Order	Second Order	Third Order
h	I_1			
		$I_1' = I_2 + \frac{I_2 - I_1}{3}$		
$h/2$	I_2		$I_1'' = I_2' + \frac{I_2' - I_1'}{3}$	
		$I_2' = I_3 + \frac{I_3 - I_1}{3}$		$I_1''' = I_2'' + \frac{I_2'' - I_1''}{3}$
$h/4$	I_3		$I_2'' = I_3' + \frac{I_3' - I_1'}{3}$	
		$I_3' = I_4 + \frac{I_4 - I_1}{3}$		
$h/8$	I_4			

Example 4.4.3

Find the value of $\int_0^8 x^2 dx$ at $I_{4,4}$ using Romberg Integration.

$$h = \frac{b-a}{2} = \frac{8-0}{2} = 4$$

x	f_0	f_1	f_2
y	0	4	8

$$I_1 = \frac{h}{2}\left[f_0 + 2f_1 + f_2\right] = \frac{4}{2}\left[0^2 + 2(4^2) + 8^2\right] = \frac{4}{2}[96] = 192$$

$$h = \frac{b-a}{4} = \frac{8-0}{4} = 2$$

x	f_0	f_1	f_2	f_3	f_4
y	0	2	4	6	8

$$
\begin{aligned}
I_2 &= \frac{h}{2}\left[f_0 + 2(f_1 + f_2 + f_3) + f_4\right] \\
&= \frac{2}{2}\left[0^2 + 2(2^2 + 4^2 + +6^2) + 8^2\right] = 176
\end{aligned}
$$

$$
h = \frac{b-a}{8} = \frac{8-0}{8} = 1
$$

x	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
y	0	1	2	3	4	5	6	7	8

$$
\begin{aligned}
I_3 &= \frac{h}{2}\left[f_0 + 2\left(f_1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7\right) + f_8\right] \\
&= \frac{1}{2}\left[0 + 2(1 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2) + 8^2\right] \\
&= 172
\end{aligned}
$$

$$h = \frac{b-a}{16} = \frac{8-0}{16} = 0.5$$

x	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
y	0	0.5	1	1.5	2	2.5	3	3.5	4
x	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}	
y	4.5	5	5.5	6	6.5	7	7.5	8	

$$I_4 = \frac{h}{2}\left[f_0 + 2\begin{pmatrix} f_1 + f_2 + f_3 + f_4 \\ +f_5 + f_6 + f_7 + f_8 \\ +f_9 + f_{10} + f_{11} + f_{12} \\ +f_{13} + f_{14} + f_{15} \end{pmatrix} + f_{16}\right]$$

$$= \frac{h}{2}\left[0 + 2\begin{pmatrix} 0.5^2 + 1^2 + 1.5^3 + 2^2 \\ +2.5^5 + 3^2 + 3.5^2 + 4^2 \\ +4.5^2 + 5^2 + 5.5^2 + 6^2 \\ +6.5^2 + 7^2 + 7.5^2 \end{pmatrix} + 8^2\right]$$

$$= 171$$

h_i		**First Order**	**Second Order**	**Third Order**
h	I_1			
		$I_1' = I_2 + \frac{I_2 - I_1}{3}$		
$h/2$	I_2		$I_1'' = I_2' + \frac{I_2' - I_1'}{3}$	
		$I_2' = I_3 + \frac{I_3 - I_1}{3}$		$I_1''' = I_2'' + \frac{I_2'' - I_1''}{3}$
$h/4$	I_3		$I_2'' = I_3' + \frac{I_3' - I_1'}{3}$	
		$I_3' = I_4 + \frac{I_4 - I_1}{3}$		
$h/8$	I_4			

h_i		**First Order**	**Second Order**	**Third Order**
h	192			
		$I_1' = I_2 + \frac{I_2 - I_1}{3}$ $= 176 + \frac{176-192}{3}$ $= 170.66$		
$h/2$	176		$I_1'' = I_2' + \frac{I_2' - I_1'}{3}$ $= 170.66 + \frac{170.66-170.66}{3}$ $= 170.66$	
		$I_2' = I_3 + \frac{I_3 - I_1}{3}$ $= 172 + \frac{172-192}{3}$ $= 170.66$		$I_1''' = I_2'' + \frac{I_2'' - I_1''}{3}$ $= 170.66 + \frac{170.66-170.66}{3}$ $= 170.66$
$h/4$	172		$I_2'' = I_3' + \frac{I_3' - I_1'}{3}$ $= 170.66 + \frac{170.66-170.66}{3}$ $= 170.66$	
		$I_3' = I_4 + \frac{I_4 - I_1}{3}$ $= 171 + \frac{171-192}{3}$ $= 170.66$		
$h/8$	171			

So, $\int_0^8 x^2 dx \approx 170.66$.

Find the approximate value of $\int_0^8 (x+1)dx$ using Romberg method at $I_{4,4}$.