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ERRORS

ABSOLUTE ERROR

Let *A* be an exact number and *a* an approximation of *A.* The absolute error Δ of the approximation *a*, denoted as Δa, is the absolute value of the difference between the corresponding exact number and its approximation

$$D = | A - a |$$

ABSOLUTE ERROR. EXAMPLE

Determine the absolute error of the approximation *3,14* of π

ABSOLUTE VS RELATIVE ERROR

Example

Determine the absolute error of the approximation *3,14* of π

Exact number $A=\pi$

Approximation $a=3,14$

Absolute error of a $\Delta=|\pi-3,14|$

Cannot be computed BUT approximated as:

$$|3,141592653\ldots - 3,14| \approx 0,001592653\ldots$$

ABSOLUTE ERROR. EXAMPLE

We are looking for a root of $f(x)$, that is, x_0 so that $f(x_0)=0$. The best obtained result corresponds to $x_a=2{,}34803$ so that $f(x_a)=10^{-5}$

What is the absolute error of $f(x_a)$?

ABSOLUTE VS RELATIVE ERROR

Example

We are looking for a root of $f(x)$, i.e. x_0 so that $f(x_0)=0$. The best result obtained so far is $x_a=2,34803$ that satisfies $f(x_a)=10^{-5}$

What is the absolute error of $f(x_a)$?

$A=f(x_0)=0 \quad a=f(x_a)=10^{-5} \quad \Delta=|0-10^{-5}|=10^{-5}$

ABSOLUTE ERROR. EXAMPLE

We are looking for a root of $f(x)$, that is, x_0 so that $f(x_0)=0$. The best obtained result corresponds to $x_a=2{,}34803$ so that $f(x_a)=10^{-5}$

What is the absolute error of x_a?

ABSOLUTE VS RELATIVE ERROR

Example

We are looking for a root of $f(x)$, i.e. x_0 so that $f(x_0)=0$. The best result obtained so far is $x_a=2,34803$ that satisfies $f(x_a)=10^{-5}$

What is the absolute error of x_a?

$A=x_0 \quad a=2,34803 \qquad \Delta=|x_0-2,34803|$

The error Δx_a **cannot** be expressed in a different way since we do not know x_0

ABSOLUTE ERROR BOUND

A bound Δ_a of the absolute error $\Delta a=|A-a|$ is any number that delimits the error, that is, any number satisfying

$$\Delta = |A - a| \leq \Delta_a$$

That expression defines an interval around the unknown *A*

$$A \in [a - \Delta_a, a + \Delta_a]$$

i.e.

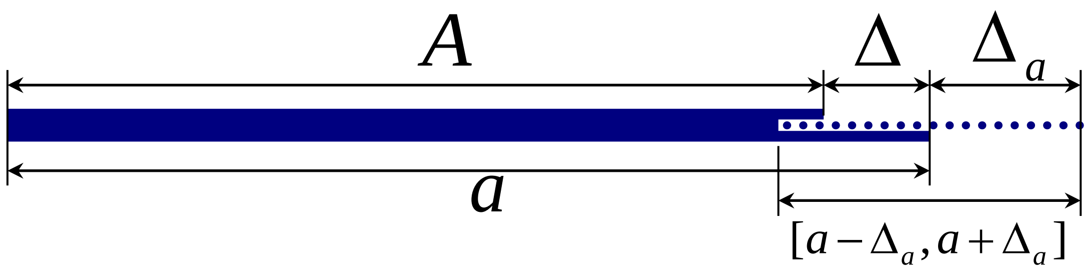
$$a - \Delta_a \leq A \leq a + \Delta_a$$

what is expressed as

$$A \approx a \pm \Delta_a$$

ABSOLUTE ERROR BOUND

Meaning of $A = a \pm \Delta_a$



$$|A-a| = \Delta \leq \Delta_a$$

$$a-\Delta_a \leq A \leq a+\Delta_a \quad \Rightarrow \quad A = a \pm \Delta_a$$

The smaller the bound Δ_a, the better, as the interval will be shorter $[a-\Delta_a, a+\Delta_a]$ and the error is narrower Δ

ABSOLUTE ERROR BOUND. EXAMPLE

Approximante $A=1/3$ with $a=0,33$

- Is $\Delta_a=0,001$ an upper bound of the absolute error?

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ABSOLUTE ERROR BOUND. EXAMPLE

Approximate $A=1/3$ with $a=0,33$

- Is $\Delta_a=0,001$ an upper bound of the absolute error?
- Is it $\Delta_a=0,004$?

ABSOLUTE ERROR BOUND. EXAMPLE

Approximate $A=1/3$ with $a=0{,}33$

- Is $\Delta_a=0{,}001$ an upper bound of the absolute error?

- Is it $\Delta_a=0{,}004$?

- Is it $\Delta_a = 0{,}00334\overset{\frown}{3}$?

ABSOLUTE ERROR BOUND. EXAMPLE

Approximate *A=1/3* with *a=0,33*

- Is $\Delta_a=0,001$ an upper bound of the absolute error?
- Is it $\Delta_a=0,004$?
- Is it $\Delta_a = 0,00334\widehat{3}$?
- *What is better?*

ABSOLUTE VS RELATIVE ERROR

Example

Approximate *A=1/3* with *a=0,33*

- Is $\Delta_a=0,001$ an upper bound of the absolute error?

NO because $\Delta = |A-a| > \Delta_a$

$$|1/3-0,33| = 0,0033\widehat{3} > 0,001$$

Is it $\Delta_a=0,004$? YES

$$\Delta = |A-a| \leq \Delta_a \qquad \Delta = 0,0033\widehat{3} \leq 0,004$$

Is it $\Delta_a = 0,00334\widehat{3}$? *YES BETTER* $0,0033\widehat{3} \leq 0,00334\widehat{3}$

RELATIVE ERROR

The relative error *δ* of an approximation *a* is the ratio between the absolute error and the absolute value of the exact value *A*

$$\delta \quad \text{ó} \quad \delta a = \frac{|A - a|}{|A|} = \frac{\Delta a}{|A|}$$

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RELATIVE ERROR. EXAMPLES

Obtain the relative error:

- A=5,35   a=5,4
- A=624,05 a=624

ABSOLUTE VS RELATIVE ERROR

Error Relativo

The relative error δ is the ratio between the absolute error and the absolute value of the exact number A

$$\delta \quad \text{ó} \quad \delta a = \frac{|A-a|}{|A|} = \frac{\Delta a}{|A|}$$

Examples

A=5,35 a=5,4 $\Delta = 0{,}05 \quad \delta = \dfrac{0{,}05}{5{,}35} = 0{,}0093$

A=624,05 a=624 $\Delta = 0{,}05 \quad \delta = \dfrac{0{,}05}{624{,}05} = 8{,}0122 \cdot 10^{-5}$

RELATIVE ERROR BOUND

A bound δ_a of the relative error $\Delta a/|A|$ is any number not less than that error, i.e.

$$\delta = \frac{\Delta}{|A|} \leq \delta_a$$

RELATIVE ERROR BOUND

A bound δ_a of the relative error $\Delta a/|A|$ is any number not less than that error, i.e.

$$\delta = \frac{\Delta}{|A|} \leq \delta_a$$

In addition

If $\Delta = |A|\delta$ and $\delta \leq \delta_a$ then $\Delta \leq |A|\delta_a$

which leads $\Delta_a = |A|\delta_a$

The bound of relative error by the exact value is a bound of the absolute error.

RELATIONSHIP BETWEEN RELATIVE AND ABSOLUTE BOUNDS

In practice, the value A is often unknown, so the relationship you want to use is the relative error boundary times the absolute value of the approximation, which is actually an absolute error boundary

$$\Delta_a = |a|\delta_a$$

That is, $a-|a|\delta_a \leq A \leq a+|a|\delta_a$, and $A = a(1\pm\delta_a)$

what is expressed $A \approx a \pm \delta_a\%$ with δ_a in %

To be demonstrated in other cases...

RELATIVE ERROR BOUNDS. EXAMPLES

$$A = 5{,}35$$

$$a = 5{,}4$$

1. $\Delta_a = 0{,}06$

2. $\Delta_a = 0{,}051$

3. $\Delta_a = 0{,}054$

ABSOLUTE VS RELATIVE ERROR

Examples of bounds $\delta_a=\Delta_a/|a|$

$$A = 5{,}35 \qquad \Delta = |A - a| = |5{,}35 - 5{,}4| = 0{,}05$$

$$a = 5{,}4 \qquad \delta = \frac{\Delta}{|A|} = \frac{0{,}05}{5{,}35} = 0{,}0093\ldots$$

1. $\Delta_a = 0{,}06 \qquad \delta_a = \dfrac{\Delta_a}{|a|} = \dfrac{0{,}06}{5{,}4} = 0{,}01\widehat{1} \geq \delta$

2. $\Delta_a = 0{,}051 \qquad \delta_a = \dfrac{\Delta_a}{|a|} = \dfrac{0{,}051}{5{,}4} = 0{,}009\widehat{4} \geq \delta$

3. $\Delta_a = 0{,}054 \qquad \delta_a = \dfrac{\Delta_a}{|a|} = \dfrac{0{,}054}{5{,}4} = 0{,}01 \geq \delta$

ERROR BOUNDS. EXERCISE

A rectangular image measures 8.5 centimeters in length. The actual length of the image is 8.1 centimeters.

What is the absolute error of the image length?

What is the relative error of the image length?

DECIMAL DECOMPOSITION

A positive real number *A* can be expressed as the following finite (or infinite) sum

$$A = \alpha_m 10^m + \alpha_{m-1} 10^{m-1} + \ldots + \alpha_{m-n+1} 10^{m-n+1} + \ldots$$

where $m \in \mathbf{Z}$ $\quad \alpha_i \in \{0,1,2,\ldots,9\}$ $\quad \alpha_m \neq 0$

This sum is called the decimal form, and it is said that $\boldsymbol{\alpha_i}$ are digits, being α_m the most significant digit and *m* is the highest power of 10 for *A*

DECIMAL APPROXIMATION

It is called the decimal approximation of a positive real number *A* to the next finite-sum decimal form

$$a = \beta_m 10^m + \beta_{m-1} 10^{m-1} + \ldots + \beta_{m-n+1} 10^{m-n+1} \quad (\beta_m \neq 0)$$

Example, decimal form of*A*=*π* and its approximation *3,142*

$$\pi = 3{,}1415\ldots = 3\cdot 10^0 + 1\cdot 10^{-1} + 4\cdot 10^{-2} + 1\cdot 10^{-3} + 5\cdot 10^{-4} + \ldots$$

$$a = 3{,}142 = 3\cdot 10^0 + 1\cdot 10^{-1} + 4\cdot 10^{-2} + 2\cdot 10^{-3}$$

SIGNIFICANT DIGITS

Any non-zero digit α_i is significant

Any digit $\alpha_i=0$ is significant if α_{i+1} and α_{i-1} are significant

The rest of the zero digits

- In decimal form there are not $\alpha_i=0$ before the most significant digit $\alpha_m \neq 0$

$$0{,}04030 = 4\cdot 10^{-2} + 0\cdot 10^{-3} + 3\cdot 10^{-4} + 0\cdot 10^{-5}$$

- The zeros after the last non-zero digit $\alpha_i \neq 0$ will be considered significant if it is interesting for any reason (interpretation, accuracy, etc.)

$$600 = 6\cdot 10^{2} + 0\cdot 10^{1}$$

EXACT DIGITS

The exact digits of an approximation *a* in decimal form are the maximum number *n* of significant digits

$$\beta_m, \beta_{m-1}, \ldots \beta_{m-n+1}$$

satisfying

$$\Delta = | A - a | \leq (1/2) \cdot 10^{m-n+1}$$

Then we say that the approximation *a* has *n* exact digits.

EXACT DIGITS

Interpretation of the condition

$$\Delta = | A - a | \leq (1/2) \cdot 10^{m-n+1}$$

The approximation *a* has the first *n* exact digits if the absolute error of *a* does not exceed half a unit at the n-th place counting from left to right, and those first *n* exact digits are

$$\beta_m, \beta_{m-1}, \ldots \beta_{m-n+1}$$

EXACT DIGITS. EXAMPLE

If $A=3{,}25$ and $a=3{,}29$

How many exact digits does *a* have?

SIGNIFICANT DIGITS AND EXACT DIGITS

Example

If *A=3,25* and *a=3,29*

How many exact digits do has *a*?

Set the expressions of 5 that bound the tightest as possible the absolute error:

$$\Delta = | A - a | = 0{,}04 \qquad a = 3 \cdot 10^{m=0} + 2 \cdot 10^{-1} + 9 \cdot 10^{-2}$$

$$0{,}005 < 0{,}04 \le 0{,}05$$

Let us calculat the value of n for both bounds:

SIGNIFICANT DIGITS AND EXACT DIGITS

Example

If *A=3,25* and *a=3,29*

How many exact digits do has *a*?

$$\Delta = | A - a | = 0{,}04 \qquad a = 3 \cdot 10^{m=0} + 2 \cdot 10^{-1} + 9 \cdot 10^{-2}$$

$$0{,}005 < 0{,}04 \leq 0{,}05$$

$$\Delta > (1/2) \cdot 10^{m-n+1} = 0{,}005$$

$$-2 = m - n + 1 \Rightarrow n = 3$$

from below *n=3*

$$\Delta \leq (1/2) \cdot 10^{m-n+1} = 0{,}05$$

$$-1 = m - n + 1 \Rightarrow n = 2$$

from above *n=2*

Ejemplo

If *A=3,25* and *a=3,29*

How many exact digits do has *a*?

$$\Delta = | A - a | = 0{,}04 \qquad a = 3\cdot 10^{m=0} + 2\cdot 10^{-1} + 9\cdot 10^{-2}$$

$$\cancel{0{,}005} < 0{,}04 \leq 0{,}05$$

$$\cancel{\Delta > (1/2)\cdot 10^{m-n+1} = 0{,}005}$$

$$\cancel{-2 = m - n + 1 \Rightarrow n = 3}$$

$$\Delta \leq (1/2)\cdot 10^{m-n+1} = 0{,}05$$

$$-1 = m - n + 1 \Rightarrow n = 2$$

~~From below *n*=3 NO~~ from above *n*=2 YES

EXACT DIGITS. EXAMPLE

If $A=3,25$ and $a=3,29$

How many exact digits does a have?

And if $A=3,25$ and $a=3,31$?

Example

If *A=3,25* and *a=3,31*?

$$0{,}05 < 0{,}06 \leq 0{,}5$$

$$0{,}5 \cdot 10^{-1} < 0{,}06 \leq 0{,}5 \cdot 10^{0}$$

$$-1 = m - n + 1 \qquad 0 = m - n + 1$$

$$m = 0 \quad n = 2 \qquad m = 0 \quad n = 1$$

SIGNIFICANT DIGITS AND EXACT DIGITS

Example

If *A=3,25* and *a=3,31*?

$$\cancel{0,05} < 0,06 \le 0,5$$

$$0,5 \cdot 10^{-1} < 0,06 \le 0,5 \cdot 10^{0}$$

n=1

$$\cancel{-1 = m - n + 1} \qquad 0 = m - n + 1$$

$$\cancel{m = 0 \quad n = 2} \qquad m = 0 \quad n = 1$$

SIGNIFICANT DIGITS AND EXACT DIGITS

Example

If *A=3,25* and *a=3,31*?

$$\cancel{0,05} < 0,06 \leq 0,5$$

$$0,5 \cdot 10^{-1} < 0,06 \leq 0,5 \cdot 10^{0}$$

n=1

$$\cancel{-1 = m - n + 1 \qquad m = 0 \quad n = 2} \qquad\qquad 0 = m - n + 1 \qquad m = 0 \quad n = 1$$

If *A=3,23* and *a=3,29*?

SIGNIFICANT DIGITS AND EXACT DIGITS

Example

If A=3,25 and a=3,31? $\cancel{0{,}05} < 0{,}06 \le 0{,}5$

$$\cancel{0{,}5 \cdot 10^{-1}} < 0{,}06 \le 0{,}5 \cdot 10^{0}$$

n=1

$$\cancel{-1 = m - n + 1} \qquad 0 = m - n + 1$$

$$\cancel{m = 0 \quad n = 2} \qquad m = 0 \quad n = 1$$

If A=3,23 and a=3,29? $\cancel{0{,}05} < 0{,}06 \le 0{,}5 = 0{,}5 \cdot 10^{0}$

n=1 $\qquad m = 0 \quad 0 = m - n + 1$

If A=3,25 and a=3,3?

SIGNIFICANT DIGITS AND EXACT DIGITS

Example

If A=3,25 and a=3,31?

$$\cancel{0{,}05} < 0{,}06 \le 0{,}5$$

$$\cancel{0{,}5 \cdot 10^{-1}} < 0{,}06 \le 0{,}5 \cdot 10^{0}$$

n=1

$$\cancel{-1 = m - n + 1} \qquad 0 = m - n + 1$$

$$\cancel{m = 0 \quad n = 2} \qquad m = 0 \quad n = 1$$

If A=3,23 and a=3,29?

$$\cancel{0{,}05} < 0{,}06 \le 0{,}5 = 0{,}5 \cdot 10^{0}$$

n=1

$$m = 0 \quad 0 = m - n + 1$$

If A=3,25 and a=3,3?

$$\cancel{0{,}005} < 0{,}05 \le 0{,}05 = 0{,}5 \cdot 10^{-1}$$

n=2

$$m = 2 \quad 1 = m - n + 1$$

If A=700,8 and a=700?

SIGNIFICANT DIGITS AND EXACT DIGITS

Example

If A=3,25 and a=3,31?

$$\cancel{0,05} < 0,06 \leq 0,5$$

$$\cancel{0,5 \cdot 10^{-1}} < 0,06 \leq 0,5 \cdot 10^{0}$$

n=1

$$\cancel{-1 = m - n + 1} \qquad 0 = m - n + 1$$

$$\cancel{m = 0 \quad n = 2} \qquad m = 0 \quad n = 1$$

If A=3,23 and a=3,29?

$$\cancel{0,05} < 0,06 \leq 0,5 = 0,5 \cdot 10^{0}$$

n=1

$$m = 0 \quad 0 = m - n + 1$$

If A=3,25 and a=3,3?

$$\cancel{0,005} < 0,05 \leq 0,05 = 0,5 \cdot 10^{-1}$$

n=2

$$m = 0 \quad -1 = m - n + 1$$

If A=700,8 and a=700?

$$\cancel{0,5} < 0,8 \leq 5 = 0,5 \cdot 10^{1}$$

n=2

$$m = 2 \quad 1 = m - n + 1$$

RELATIONSHIP BETWEEN RELATIVE ERROR AND EXACT DIGITS. BOUNDING THEOREM

If an approximation *a>0* has *n* exact digits, its relative error satisfies

$$\delta \leq \frac{1}{\beta_m}\left(\frac{1}{10}\right)^{n-1}$$

where β_m is the first significant digit (the most significant digit) of *a*

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RELATIONSHIP BETWEEN RELATIVE ERROR AND EXACT DIGITS. EXAMPLE

The aim is to approximate $\sqrt{2} = 1.4142135623...$

How many digits are needed so that the relative error does not exceed *0,1%*?

RELATION BETWEEN RELATIVE ERROR AND EXACT DIGITS

Example 1

We want to approximate $\sqrt{2} = 1,4142135623\ldots$

How many digits are needed so that the relative error is not greater that 0,1%?

- Calculate first digit $\beta_m = \beta_0 = 1$
- We want $\delta_a = 0,001$
- Apply the bounding theorem

$$\delta_a = \frac{1}{\beta_m}\left(\frac{1}{10}\right)^{n-1} \qquad 0,001 = \frac{1}{1}\left(\frac{1}{10}\right)^{n-1} \qquad 10^{-3} = 10^{1-n}$$

- And clear n: $-3 = 1 - n$ $\quad n=4$

RELATIONSHIP BETWEEN RELATIVE ERROR AND EXACT DIGITS. EXAMPLE

The aim is to approximate e $=2.718281828\ldots$

How many digits are needed so that the relative error does not exceed *0,05%*?

RELATION BETWEEN RELATIVE ERROR AND EXACT DIGITS

Example 2

We want to approximate e =*2.718281828…*

How many digits are needed to haver a relative error not greater than *0,05%*?

- Calculate first digit $\beta_m=\beta_0=2$
- Target: $\delta_a=0{,}0005$
- Apply the bounding theorem

$$\delta_a = \frac{1}{\beta_m}\left(\frac{1}{10}\right)^{n-1} \quad 0{,}0005 = \frac{1}{2}\left(\frac{1}{10}\right)^{n-1} \quad 0{,}5\cdot 10^{-3} = 0{,}5\cdot 10^{1-n}$$

- *And clear n* $\quad -3 = 1 - n \quad$ *n=4*

ABSOLUTE ERROR OF THE SUM

The sum of the absolute errors is the bounds of the absolute error of the sum of approximations

- Exact Value $S = A_1 + A_2 + \ldots + A_n$
- Approximation $s = a_1 + a_2 + \ldots + a_n$

$$S - s = A_1 - a_1 + A_2 - a_2 + \ldots + A_n - a_n$$

$$S - s = (\pm\Delta a_1) + (\pm\Delta a_2) + \ldots + (\pm\Delta a_n)$$

$$| S - s | \leq | \pm\Delta a_1 | + | \pm\Delta a_2 | + \ldots + | \pm\Delta a_n |$$

$$\Delta s \leq \Delta_s = \Delta a_1 + \Delta a_2 + \ldots + \Delta a_n$$

ABSOLUTE ERROR OF THE REST

As *-a* approximates *-A* with the same absolute error that approximates *A*

$$|A-a|=\Delta a$$

$$|(-A)-(-a)|=|-(A-a)|=|A-a|=\Delta a$$

Then the sum of errors is also a bound for the subtraction

$$S=A_1-A_2=A_1+(-A_2)$$
$$s=a_1-a_2=a_1+(-a_2)$$

$$\Delta s\le\Delta_r=\Delta a_1+\Delta a_2$$

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RELATIVE ERROR OF THE SUM

The maximum relative error bounds of the terms of a sum limits the relative error of that sum if all the terms are of the same sign

- Exact Value $S = A_1 + A_2 + \ldots + A_n$
- Approximation $s = a_1 + a_2 + \ldots + a_n$
- If $\Delta_s = \Delta a_1 + \Delta a_2 + \ldots + \Delta a_n$

then $\Delta_s \leq \Delta_{a_1} + \Delta_{a_2} + \ldots + \Delta_{a_n}$

$$\delta s = \frac{\Delta s}{|S|} \leq \frac{\Delta_s}{|S|} \leq \frac{\Delta_{a_1} + \Delta_{a_2} + \ldots + \Delta_{a_n}}{|A_1 + A_2 + \ldots + A_n|}$$

RELATIVE ERROR OF THE SUM

$$\delta s = \frac{\Delta s}{|S|} \le \frac{\Delta_s}{|S|} \le \frac{\Delta_{a_1} + \Delta_{a_2} + \ldots + \Delta_{a_n}}{|A_1 + A_2 + \ldots + A_n|} =$$

$$= \frac{|A_1|\delta_{a_1} + |A_2|\delta_{a_2} + \ldots + |A_n|\delta_{a_n}}{|A_1 + A_2 + \ldots + A_n|} \le$$

$$\le \delta^* \frac{|A_1| + |A_2| + \ldots + |A_n|}{|A_1 + A_2 + \ldots + A_n|} = \delta^*$$

where $\delta^* = \max(\delta_{a_1}, \delta_{a_2}, \ldots, \delta_{a_n})$

Conclusion $\delta s \le \delta_s = \max(\delta_{a_1}, \delta_{a_2}, \ldots, \delta_{a_n})$

ABSOLUTE ERROR OF A FUNCTION

The absolute error of a function tends to the absolute error of its variable by the absolute value of its derivative when the absolute error of the variable tends to zero

$$\Delta x \to 0 \Rightarrow \Delta f \to \Delta x \,| f'|$$

Demonstration

We assume an approximation to x and its absolute error $\Delta x = | x - a |$

By definition of derivative $f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x) - f(x \pm \Delta x)}{\pm \Delta x}$

ABSOLUTE ERROR OF A FUNCTION

The absolute error of a function tends to the absolute error of its variable by the absolute value of its derivative when the absolute error of the variable tends to zero

If we calculate its absolute value

$$\left| f'(x) \right| = \left| \lim_{\Delta x \to 0} \frac{f(x) - f(x \pm \Delta x)}{\pm \Delta x} \right| = \lim_{\Delta x \to 0} \frac{\left| f(x) - f(a) \right|}{\Delta x}$$

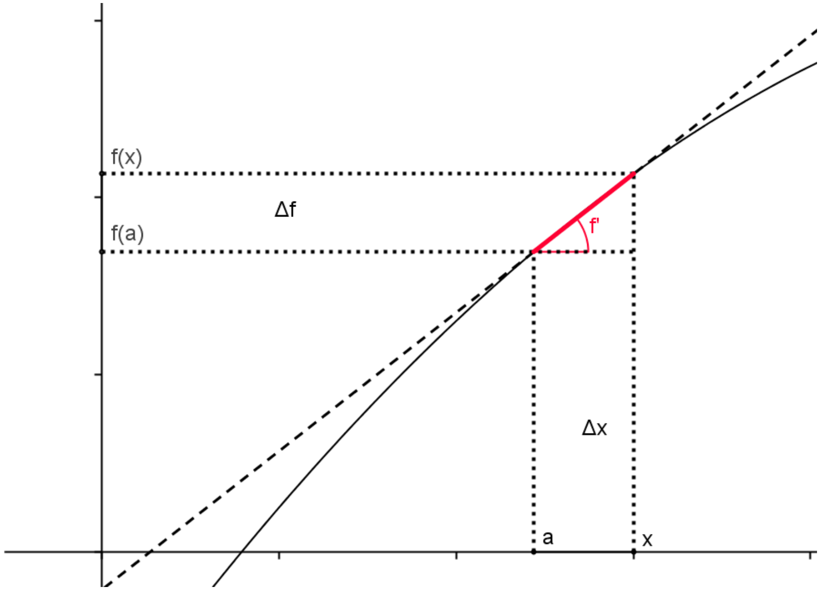
$$\left| f'(x) \right| = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} \qquad | f' | \approx \frac{\Delta f}{\Delta x} \qquad \Delta f \approx \Delta x \, | f' |$$

ABSOLUTE ERROR OF A FUNCTION

The absolute error of a function tends to the absolute error of its variable by the absolute value of its derivative when the absolute error of the variable tends to zero

Interpretation

$$\Delta f \approx \Delta x \, | f'|$$



ABSOLUTE ERROR OF A LOGARITHM FUNCTION

The absolute error of the natural logarithm tends to the absolute error of its variable when the absolute error of this variable tends to *0*

$$f(x) = \ln(x) \qquad \Delta f(x) \approx \Delta x \cdot \left| f'(x) \right|$$

$$f'(x) = \frac{1}{x} \qquad \Delta \ln(x) \approx \frac{\Delta x}{|x|} = \delta x$$

The absolute error of the natural logarithm is bound by the bound of the relative error of *x*

$$\Delta \ln(x) \approx \frac{\Delta x}{|x|} \leq \frac{\Delta_x}{|x|} = \delta_x \qquad \delta_x = \Delta_{\ln(x)}$$

ABSOLUTE ERROR OF A ROOT FUNCTION

The absolute error of a square root tends to the absolute error of its variable divided by twice the value of the function, when the absolute error of this variable tends to *0*

$$f(x) = \sqrt{x} \qquad \Delta f(x) \approx \Delta x \cdot \left| f'(x) \right|$$

$$f'(x) = \frac{1}{2\sqrt{x}} \qquad \Delta\sqrt{x} \approx \frac{\Delta x}{2\left|\sqrt{x}\right|}$$

Example

$$\sqrt{(23 \pm 1{,}09)} \approx \sqrt{23} \pm \left(1{,}09 \times \frac{1}{2\left|\sqrt{23}\right|} \right) \approx$$

$$\approx \pm (4{,}796 \pm 0{,}11364) \approx \pm (4{,}796 \pm 2{,}37\%)$$

RELATIVE PRODUCT ERROR

The relative error of the product is bounded by the sum of the relative errors of the factors

- Exact Value $P = A_1 \cdot A_2 \cdot \ldots \cdot A_n$
- Approximation $p = a_1 \cdot a_2 \cdot \ldots \cdot a_n$

$$\ln(p) = \ln(a_1) + \ln(a_2) + \ldots + \ln(a_n)$$

$$\Delta_{\ln(p)} \leq \Delta_{\ln(a_1)} + \Delta_{\ln(a_2)} + \ldots + \Delta_{\ln(a_n)}$$

$$\delta_p \leq \delta_{a_1} + \delta_{a_2} + \ldots + \delta_{a_n}$$

RELATIVE ERROR OF THE QUOTIENT

The relative error of the quotient is bounded by the sum of the relative errors of the dividend and the divisor

- Exact Value $C = A_1 / A_2$
- Approximation $c = a_1 / a_2$

$$\ln(c) = \ln(a_1) - \ln(a_2)$$

$$\Delta_{\ln(c)} \leq \Delta_{\ln(a_1)} + \Delta_{\ln(a_2)}$$

$$\delta_c \leq \delta_{a_1} + \delta_{a_2}$$

OPERATIONS WITH ERRORS. EXAMPLE

If $A_1 = 5 \pm 0{,}25$,

$A_2 = 2 \pm 0{,}1$,

$A_3 = 4 \pm 0{,}2$,

Obtain
$$\frac{A_3\,(A_1 + A_2)}{A_3 - A_2}$$

OPERATING WITH ERRORS

Example

If $A_1 = 5 \pm 0{,}25$,
$A_2 = 2 \pm 0{,}1$,
$A_3 = 4 \pm 0{,}2$,

Calculate
$$\frac{A_3\,(A_1 + A_2)}{A_3 - A_2}$$

OPERATING WITH ERRORS

Example

If $A_1=5\pm 0,25$,
$A_2=2\pm 0,1$,
$A_3=4\pm 0,2$,

Calculate $$\frac{A_3\,(A_1+A_2)}{A_3 - A_2}$$

$$\frac{(4\pm 0{,}2)[(5\pm 0{,}25)+(2\pm 0{,}1)]}{(4\pm 0{,}2)-(2\pm 0{,}1)}=\frac{(4\pm 0{,}2)(7\pm 0{,}35)}{(2\pm 0{,}3)}=$$

$$=\frac{(4\pm 5\%)(7\pm 5\%)}{(2\pm 15\%)}=\frac{(28\pm 10\%)}{(2\pm 15\%)}=14\pm 25\%=14\pm 3{,}5$$

OPERATIONS WITH ERRORS. EXAMPLE

If $A = 1 \pm 0{,}02$ $\;B = -5 \pm 0{,}05$ and $C = 6 \pm 0{,}03$

Obtain the roots of $Ax^2 + Bx + C$

OPERATING WITH ERRORS

Example

If $A = 1 \pm 0{,}02 \quad B = -5 \pm 0{,}05 \quad C = 6 \pm 0{,}03$

Calculate the roots of $Ax^2 + Bx + C$

OPERATING WITH ERRORS

Example

If $A = 1 \pm 0{,}02 \quad B = -5 \pm 0{,}05 \quad C = 6 \pm 0{,}03$
Calculate the roots of $Ax^2 + Bx + C$

$A = 1 \pm 2\%$

$B = -5 \pm 1\%$

$C = 6 \pm 0{,}5\%$

OPERATING WITH ERRORS

Example

If $A = 1 \pm 0{,}02 \quad B = -5 \pm 0{,}05 \quad C = 6 \pm 0{,}03$
Calculate the roots of $Ax^2 + Bx + C$

$A = 1 \pm 2\%$
$B = -5 \pm 1\%$
$C = 6 \pm 0{,}5\%$

$AC = 6 \pm 2{,}5\% = 6 \pm 0{,}15$
$4AC = 24 \pm 2{,}5\% = 24 \pm 0{,}6$
$B^2 = BB = 25 \pm 2\% = 25 \pm 0{,}5$

OPERATING WITH ERRORS

Example

If $A = 1 \pm 0{,}02 \quad B = -5 \pm 0{,}05 \quad C = 6 \pm 0{,}03$
Calculate the roots of $Ax^2 + Bx + C$

$A = 1 \pm 2\%$
$B = -5 \pm 1\%$
$C = 6 \pm 0{,}5\%$

$AC = 6 \pm 2{,}5\% = 6 \pm 0{,}15$
$4AC = 24 \pm 2{,}5\% = 24 \pm 0{,}6$
$B^2 = BB = 25 \pm 2\% = 25 \pm 0{,}5$
$B^2 - 4AC = 1 \pm 1{,}1$

$$\sqrt{B^2 - 4AC} = \sqrt{1} \pm \left(\frac{1{,}1}{2\left|\sqrt{1}\right|}\right) = 1 \pm 0{,}55$$

OPERATING WITH ERRORS

Example

If $A = 1 \pm 0{,}02 \quad B = -5 \pm 0{,}05 \quad C = 6 \pm 0{,}03$

Calculate the roots of $Ax^2 + Bx + C$

$$A = 1 \pm 2\% \qquad AC = 6 \pm 2{,}5\% = 6 \pm 0{,}15$$

$$B = -5 \pm 1\% \qquad 4AC = 24 \pm 2{,}5\% = 24 \pm 0{,}6$$

$$C = 6 \pm 0{,}5\% \qquad B^2 = BB = 25 \pm 2\% = 25 \pm 0{,}5$$

$$B^2 - 4AC = 1 \pm 1{,}1$$

$$2A = 2 \pm 2\% \qquad \sqrt{B^2 - 4AC} = \sqrt{1} \pm \left(\frac{1{,}1}{2\left|\sqrt{1}\right|}\right) = 1 \pm 0{,}55$$

$$\frac{-B + \sqrt{B^2 - 4AC}}{2\mathrm{A}} = \frac{6 \pm 0{,}60}{2 \pm 0{,}04} = \frac{6 \pm 10\%}{2 \pm 2\%} = 3 \pm 12\% = 3 \pm 0{,}36$$

OPERATING WITH ERRORS

Example

If $A = 1 \pm 0{,}02 \quad B = -5 \pm 0{,}05 \quad C = 6 \pm 0{,}03$

Calculate the roots of $Ax^2 + Bx + C$

$$A = 1 \pm 2\% \qquad AC = 6 \pm 2{,}5\% = 6 \pm 0{,}15$$

$$B = -5 \pm 1\% \qquad 4AC = 24 \pm 2{,}5\% = 24 \pm 0{,}6$$

$$C = 6 \pm 0{,}5\% \qquad B^2 = BB = 25 \pm 2\% = 25 \pm 0{,}5$$

$$B^2 - 4AC = 1 \pm 1{,}1$$

$$2A = 2 \pm 2\% \qquad \sqrt{B^2 - 4AC} = \sqrt{1} \pm \left(\frac{1{,}1}{2\left|\sqrt{1}\right|}\right) = 1 \pm 0{,}55$$

$$\frac{-B + \sqrt{B^2 - 4AC}}{2A} = \frac{6 \pm 0{,}60}{2 \pm 0{,}04} = \frac{6 \pm 10\%}{2 \pm 2\%} = 3 \pm 12\% = 3 \pm 0{,}36$$

$$\frac{-B - \sqrt{B^2 - 4AC}}{2A} = \frac{4 \pm 0{,}60}{2 \pm 0{,}04} = \frac{4 \pm 15\%}{2 \pm 2\%} = 2 \pm 17\% = 2 \pm 0{,}34$$