Discrete Mathematics
Practice Class 3
20-02-2024

Problem 1. Consider the graph with adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(i) Compute R(v) for every vertex v. Check the results with Magrada (Menu **Basic calculations**, Option **Reachable vertices from**).

(ii) Fill in the blanks: Vertex 3 reaches vertices_________________________ and vertex 4 reaches___________________.

(iii) Using the sets R(v) built the reachable matrix R. Check the matrix using Magrada (Menu **Algorithms** Option **Warshall, Final result)**.

(iv) Using the reachable matrix R, compute the access matrix Q.

(v) Using the access matrix Q compute the sets Q(v), for all the vertices v.

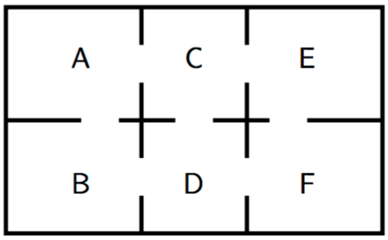
(vi) Fill in the blanks: The vertices that reach to vertex 1 are________________. The vertices that reach to vertex 3 are______________________.

(vii) Compute the connected components using both studied methods. Check the connected components using Magrada (Menu **Basic calculations**, Option **Connected components**).

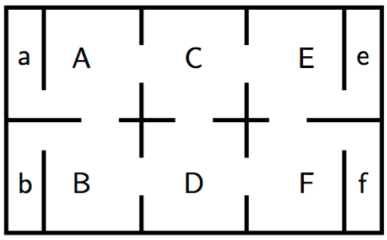
(viii) Is this graph connected? Why? Check the answer using Magrada (Menu **Basic calculations,** Option **Graph**, **Connected).**

Problem 2. Can I traverse the interior doors of the house without repeats for the following houses? In order to answer the question, represent the situation using a graph. Identify the graph problem to be solved and use the appropriate theorem to answer the question. Then write down the path (labelled by rooms) using the corresponding algorithm or explain why there is no such path. Check the result with Magrada, the corresponding algorithm is in Menu **Algorithms**.

(i)



(ii)



In the next practice we shall focus on the following idea: Is it possible to draw two graphs that appear distinct but have the same underlying structure?

Definition.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two undirected graphs. A function $f: V_1 \longrightarrow V_2$ is called a graph **isomorphism** if

(a) f is one-to-one and onto (surjective), and

(b) for all $a, b \in V_2$, $\{a, b\} \in E_1$ if and only if $\{f(a), f(b)\} \in E_2$.

When such a function exists, G_1 and G_2 are called **isomorphic graphs**.

The vertex correspondence of a graph isomorphism preserves adjacencies. Since which pairs of vertices are adjacent and which are not is the only essential property of an undirected graph, in this way the structure of the graphs is preserved.

For directed graphs there is a similar definition.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two directed graphs. A function $f: V_1 \longrightarrow V_2$ is called a graph **isomorphism** if

(a) f is one-to-one and onto, and

(b) for all $a, b \in V_2$, $(a, b) \in E_1$ if and only if $(f(a), f(b)) \in E_2$.

Example 1.

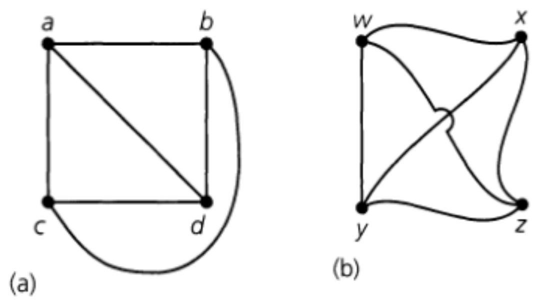


Figure 1. Isomorphic graphs.

For the graphs in parts (a) and (b) of Fig. 1 the function f defined by

$f(a) = w$,

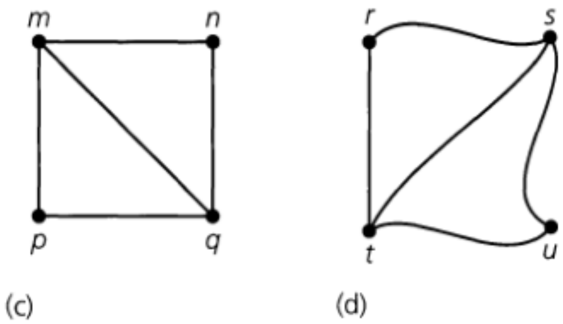
$f(b)= x$,

$f(c) = y$,

$f(d) = z$

provides an isomorphism.

Example 2.



For the graphs in parts (c) and (d) of Fig. 2 we need to be a little more careful. The function g defined by

$g(m) = r$, $g(n) = s$, $g(p) = t$, $g(q) = u$

is one-to-one and onto (for the given vertex sets). However, although {m, q} is an edge in the graph of part (c), {g(m), g(q)} = {r, u} is not an edge in the graph of part (d). Consequently, the function g does not define a graph isomorphism. To maintain the correspondence of edges, we consider the one-to-one onto function h where

$$h\{m) = s,\ h(n) = r,\ h(p) = u,\ h(q) = t.$$

In this case we have the edge correspondences

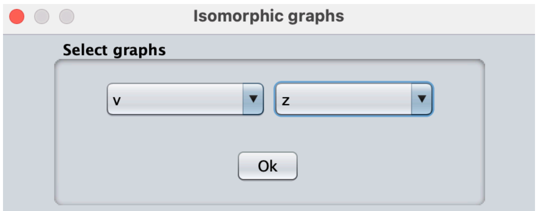
$$\begin{aligned}
\{m, n\} &\longleftrightarrow \{h(m), h(n)\} = \{s, r\},\\
\{m, p\} &\longleftrightarrow \{h(m), h(p)\} = \{s, u\},\\
\{p, q\} &\longleftrightarrow \{h(p), h(q)\} = \{u, t\},\\
\{m, q\} &\longleftrightarrow [h(m), h(q)\} = \{s, t\},\\
\{n, q\} &\longleftrightarrow \{h(n), h(q)\} = \{r, t\},
\end{aligned}$$

so h is a graph isomorphism.

Finally, since the graph in part (a) of Fig. 1 has six edges and that in part (c) has only five edges, these two graphs cannot be isomorphic.

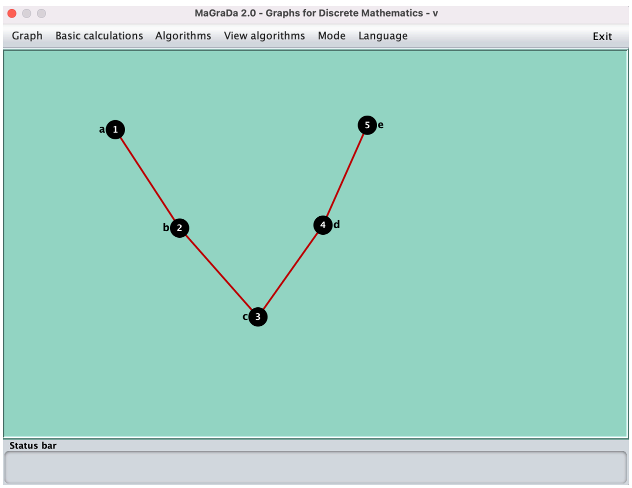
Isomorphic graphs in Magrada.

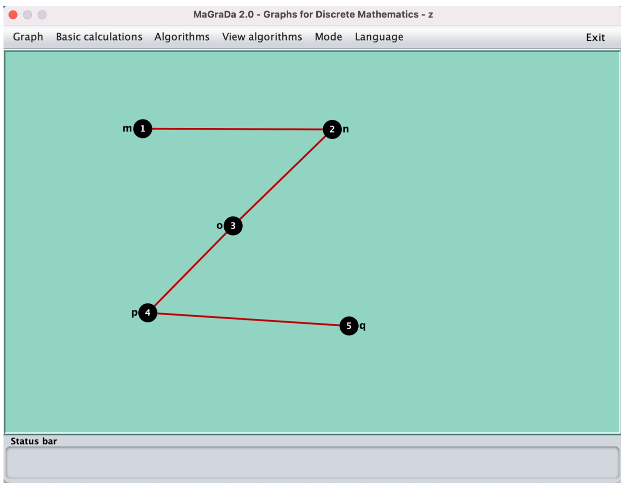
With Magrada we can study whether two graphs are isomorphic using option **Isomorphic graphs** in menu **Basic calculations**.To do this, from **text mode** or from the **graphic mode**, Magrada asks two unweighted graphs previously introduced. So, Magrada will present a screen like this:

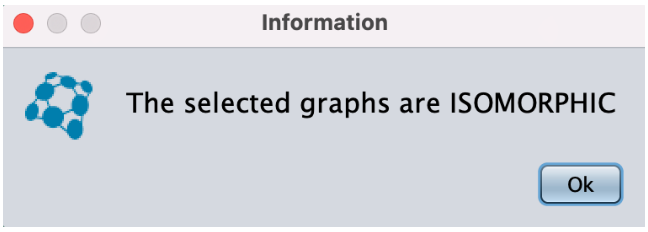


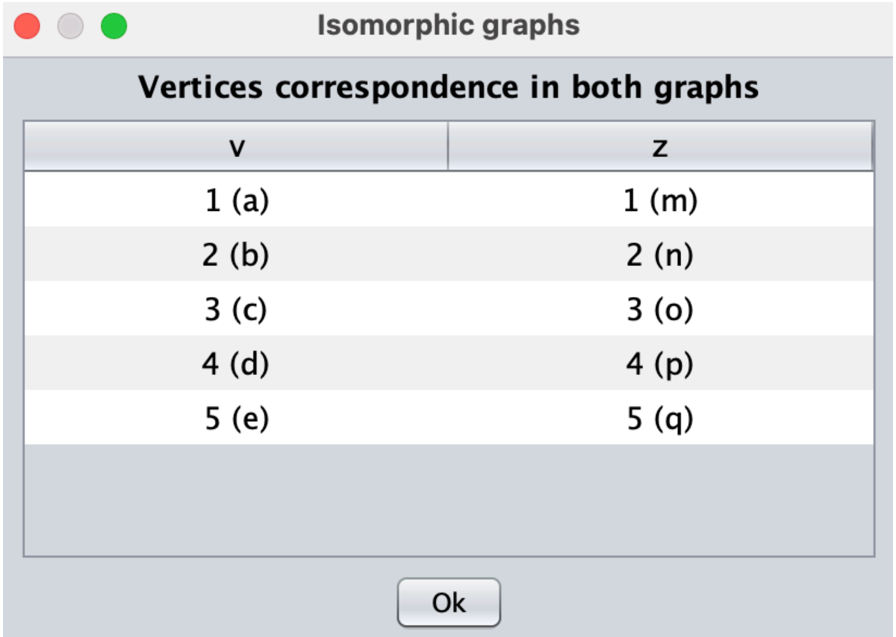
Then you have to select two graphs. If the graphs are not isomorphic, Magrada will indicate it with a message. If instead they are isomorphic, it will give the bijection.

Example.









V	Z
1 (a)	1 (m)
2 (b)	2 (n)
3 (c)	3 (o)
4 (d)	4 (p)
5 (e)	5 (q)

Problem 3. For the pair of graphs in next figure, determine whether or not the graphs are isomorphic. Check the answer using Magrada.