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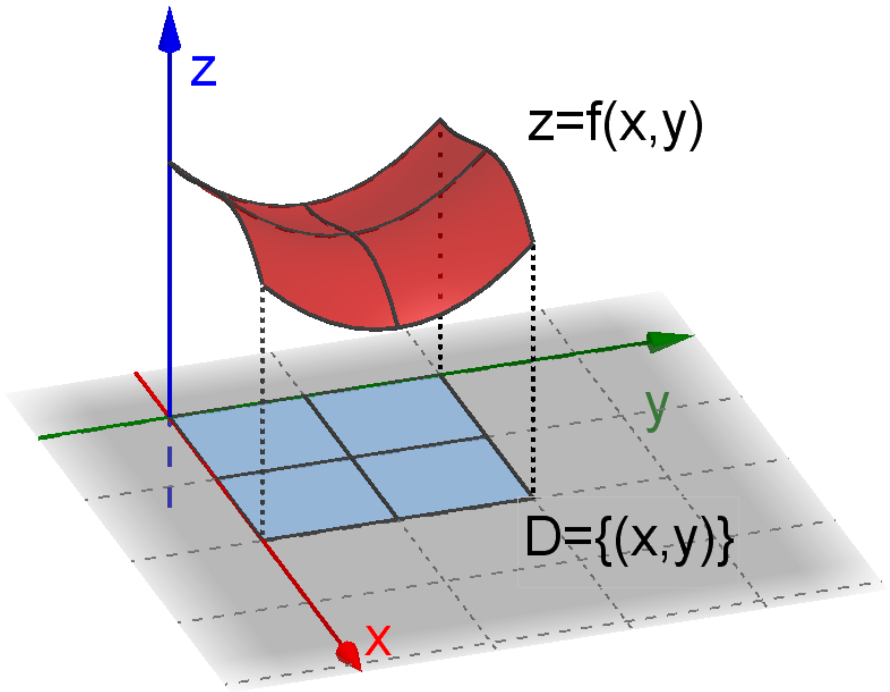
INTEGRAL CALCULUS. APPLICATIONS (II)

DOUBLE INTEGRALS

A bi-variate function $f(x, y)$ represents a surface in the space, whose projection onto the XY plane is its domain D

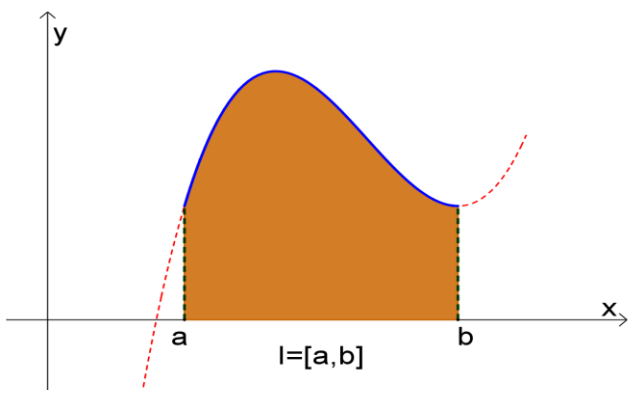
$$f(x,y): D \subseteq \mathbb{R}^2 \to \mathbb{R}$$
$$(x,y) \to z$$

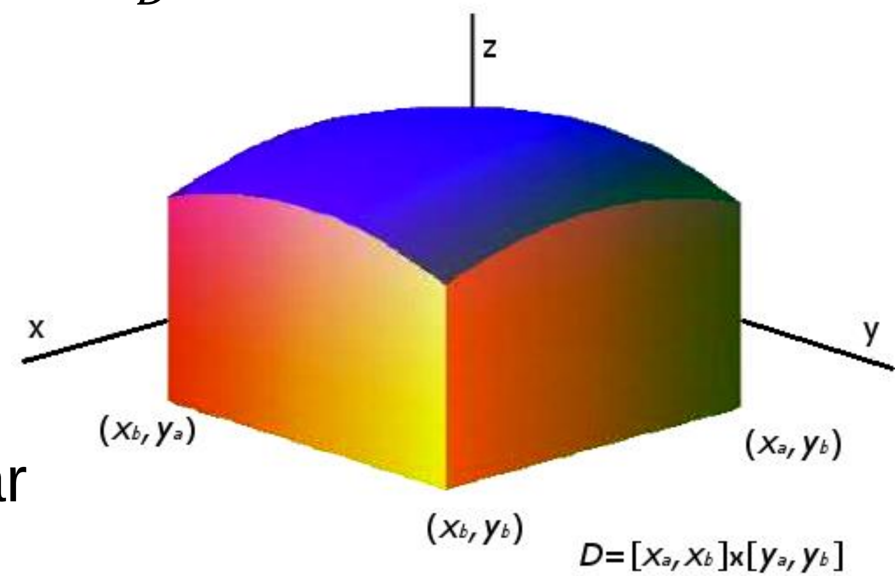


DOUBLE INTEGRALS

While a simple integral defined on an interval I=[a, b] is the area between a function $f(x)$ and the X axis, a double integral in a domain $D = [x_a, x_b] \times [y_a, y_b]$ is the volume between the function $f(x, y)$ and the XY plane

$$S = \int_a^b f(x)dx \qquad\qquad V = \iint_D f(x, y)dA$$





where $dA = dxdy$ is a rectangular differential of size $dx \times dy$

DOUBLE INTEGRALS OVER RECTANGLES

Let $f(x, y)$ be defined in the rectangle *D* of the XY plane
$D = [x_a, x_b] \times [y_a, y_b] = \{(x, y) \in \mathbb{R}^2 \;\; x_a \le x \le x_b \, , y_a \le y \le y_b\}$

If $P \; = \; \{[x_{i-1}, x_i] \times [y_{j-1}, y_j]\}$ is a partition into subrentangles of D with $x_a = x_0 \le x_i \le x_n = x_b$ and $y_a = y_0 \le y_j \le y_m = y_b$

then the double integral of f on *D*, denoted by $\iint_D \; f(x, y) dA$, is defined as

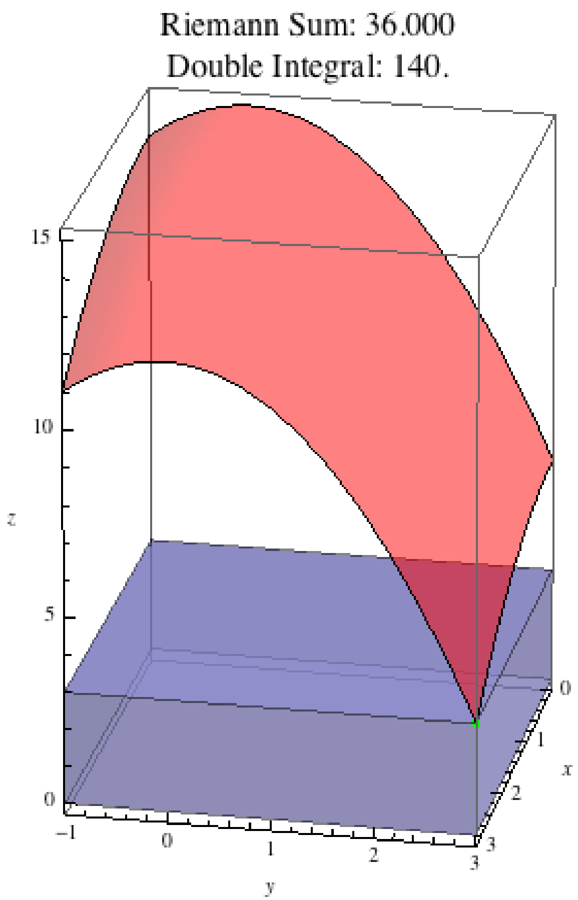
$$\iint_D \; f(x, y) dA = \lim_{\|P\| \to 0} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_i) \Delta A_{ij}$$

where

$\Delta A_{ij} = \Delta x_i \times \Delta y_j$ is the area of the subrectangle ij of the partition *P* of *D*, and $\|P\|$ is the maximum length of the diagonal of the subrectangles

DOUBLE INTEGRALS

INTERPRETATION OF THE DOUBLE INTEGRAL AS VOLUME



DOUBLE INTEGRALS

The solution of a double integral, by definition, is a complex calculation, since it is the result of the limit of a double sum that can be approximated by double Riemann sums.

A new concept, the **iterated integral** will simplify the calculus by means of succesive single integrals.

DOUBLE INTEGRALS

Recall that iterated derivatives are those that result from deriving first with respect to one variable, and then with respect to the other, considering in both cases that the other variable is constant.

In a similar way, we can define iterated integrals as iterated antiderivatives, that is, integrals with respect to one variable, first, and then with respect to the other, considering in both cases the variable with respect to which it is not integrated as a constant

$$\iint f(x,y)dxdy = \int\left(\int f(x,y)dx\right)dy$$

ITERATED INTEGRALS

Iterated integrals have the same property as iterated derivatives with respect to the order in which variables are chosen, i.e., it does not matter which variable is integrated first.

In the case of defined iterated integrals, this property is reflected in the following equality:

$$\int_{y_a}^{y_b}\int_{x_a}^{x_b} f(x,y)\,dx\,dy = \int_{x_a}^{x_b}\int_{y_a}^{y_b} f(x,y)\,dy\,dx$$

DOUBLE INTEGRALS

Fubini's theorem

Let $f(x,y)$: $\mathbb{R}^2 \rightarrow \mathbb{R}$ be a real function continuous in the rectangle $D = [x_a, x_b] \times [y_a, y_b]$, then:

$$\iint_D f(x,y)dA = \int_{y_a}^{y_b} \int_{x_a}^{x_b} f(x,y)dx\, dy = \int_{x_a}^{x_b} \int_{y_a}^{y_b} f(x,y)dy\, dx$$

The calculation of the double integral is solved by the iterated integral, which is the successive evaluation of two simple integrals

DOUBLE INTEGRALS – EXAMPLE

Calculate the iterated integral $\int_1^4 \int_2^7 dx\, dy$

DOUBLE INTEGRAL– EXAMPLE

Calculate the iterated integral $$\int_1^4 \int_2^7 dx\, dy$$

$$\int_1^4 \left(\int_2^7 dx \right) dy = \int_1^4 [x]_2^7\, dy = \int_1^4 (7-2)\, dy =$$

$$[(7-2)y]_1^4 = \big((7-2)4 - (7-2)1\big) = (7-2)(4-1) = 5 \cdot 3 = 15$$

DOUBLE INTEGRAL– EXAMPLE

Calculate the iterated integral $$\int_1^4 \int_2^7 dx\, dy$$

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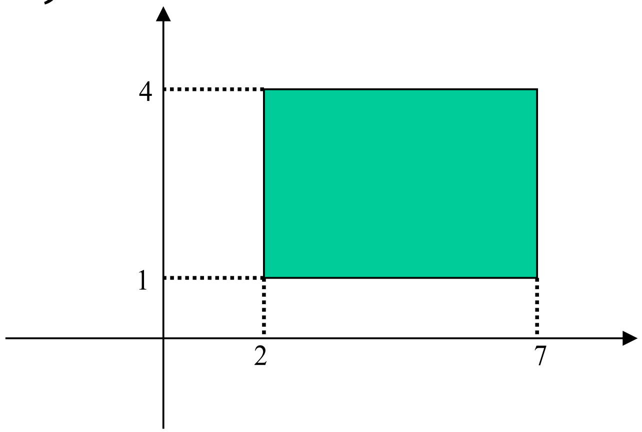
$$[(7-2)y]_1^4 = \big((7-2)4 - (7-2)1\big) = (7-2)(4-1) = 5 \cdot 3 = 15$$

The result is coincident with the **area of the rectanble** $[2,7]\times[1,4]$



DOUBLE INTEGRAL– EXAMPLE

Calculate the double integral of
$f(x,y) = k$ con $D = [2,7]\times[1,4]$

$$\iint_D f(x,y)\,dA$$

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DOUBLE INTEGRAL– EXAMPLE

Calculate the double integral of $f(x, y) = k$ con $D = [2, 7]\times[1, 4]$

$$\iint_D f(x, y)\, dA$$

$$\iint_D k\, dA = \int_1^4 \int_2^7 k\, dx\, dy = \int_1^4 [kx]_2^7 dy = [[kx]_2^7 y]_1^4 =$$

$$= [(7k - 2k)y]_1^4 = [(7 - 2)ky]_1^4 = \big((7 - 2)4k - (7 - 2)k\big) =$$

$$= (7 - 2)(4 - 1)k = 5 \cdot 3 \cdot k$$

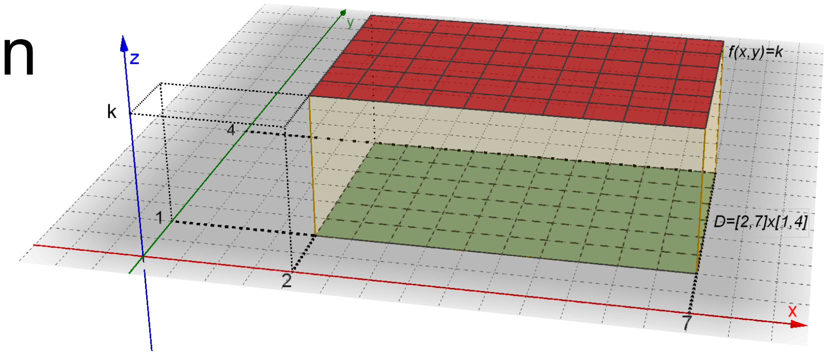
DOUBLE INTEGRAL– EXAMPLE

Calculate the double integral of $f(x,y) = k$ con $D = [2,7]\times[1,4]$

$$\iint_D f(x,y)\,dA$$

$$\iint_D k\,dA = \int_1^4\int_2^7 k\,dx\,dy = [[kx]_2^7 y]_1^4 = (7-2)(4-1)k = 5\cdot 3\cdot k$$

The result is the volume of an Octohedron of height k and base $[2,7]\times[1,4]$



k is the difference in height between the surfaces that represent $f(x,y)$ in $z = k$ and D in $z = 0$

DOUBLE INTEGRAL– EXAMPLE

Calculate the volume between the functions $f(x, y) = 3$ and $g(x, y) = 1$ in the domain

$$D = [2, 7] \times [1, 4]$$

DOUBLE INTEGRAL– EXAMPLE

Calculate the volume between the functions
$f(x,y) = 3$ and $g(x,y) = 1$ in $D = [2,7]\times[1,4]$

$$V_f = \iint_D f(x,y)\,dA \qquad V_g = \iint_D g(x,y)\,dA$$

$$V = V_f - V_g = \iint_D f(x,y)dA - \iint_D g(x,y)dA$$

$$V = \int_1^4 \int_2^7 3\,dx\,dy - \int_1^4 \int_2^7 1\,dx\,dy =$$

$$= (7-2)(4-1)3 - (7-2)(4-1)1 =$$
$$= (7-2)(4-1)(3-1) = 5 \cdot 3 \cdot 2$$

DOUBLE INTEGRAL– EXAMPLE

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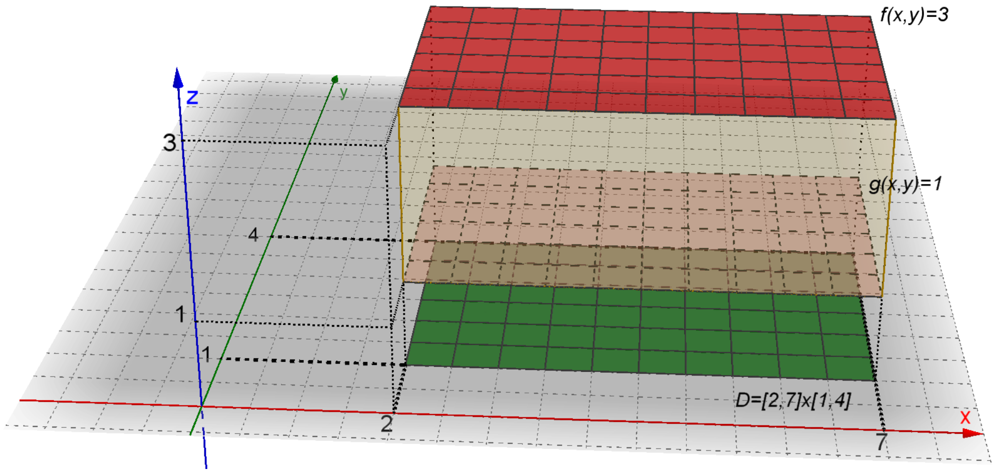


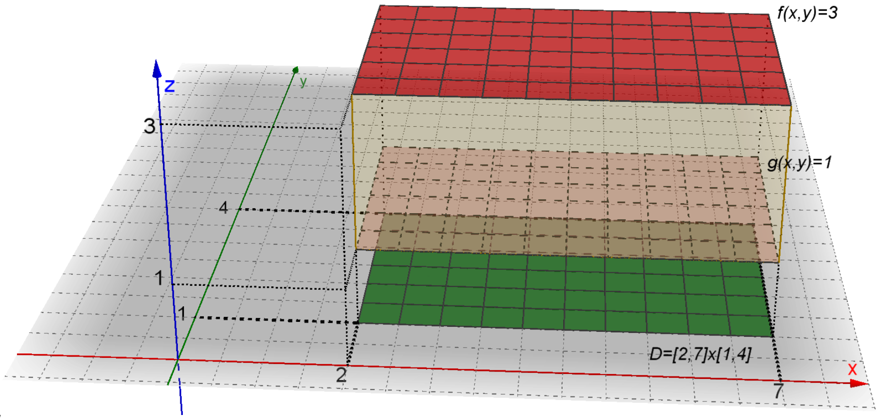
DOUBLE INTEGRAL– EXAMPLE

Calculate the volume between the functions
$f(x,y) = 3$ and $g(x,y) = 1$ in $D = [2,7]\times[1,4]$

$$\int_1^4 \int_2^7 3\,dx\,dy - \int_1^4 \int_2^7 1\,dx\,dy = (7-2)(4-1)(3-1) = 5 \cdot 3 \cdot 2$$

Same result as in :

$$\int_1^2 \int_1^4 \int_2^7 dx\,dy\,dz$$



Or triple integral $\iiint_D dx\,dy\,dz$

In the domain $D = [2,7]\times[1,4]\times[1,2]$

MULTIPLE INTEGRALS

Generalization of multiple integral and its Domain

As we have seen in the previous examples, the domain of a multiple integral can be an interval, if it is defined in $\mathbb{R}$, a surface, if set to $\mathbb{R}^2$, a volume, if set to $\mathbb{R}^3$, and in general, a hypervolume if it is set to $\mathbb{R}^n$

The defined multiple integral is the calculation of a surface, volume, or hypervolume that lies between the curve, surface, volume, or hypervolume that defines the function, and the axis, plane, space, or hyperspace on which the domain is defined

MULTIPLE INTEGRALS

Generalization of multiple integral and its Domain

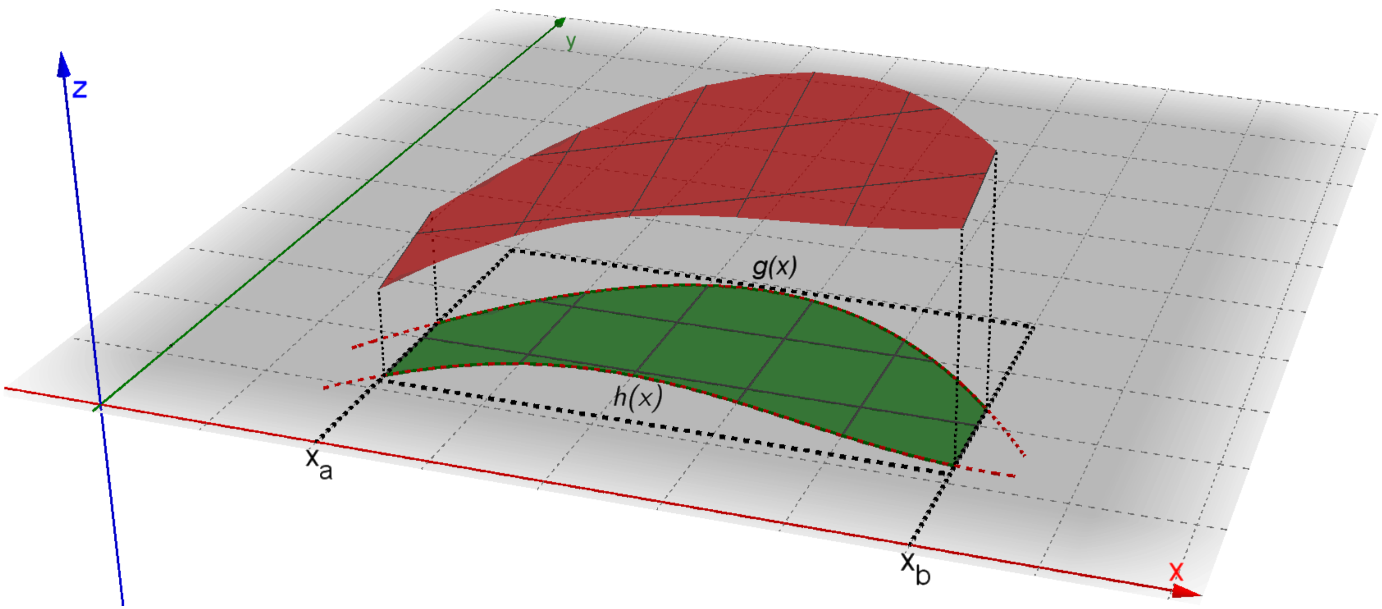
The surface, volume, or hypervolume of the domain itself can be calculated by integrating the constant 1 or unit function, i.e., only the differentials of each of its dimensions.

The integration of the unit function (only the differentials) can also be interpreted as the contained surface between two curves, the contained volume between two surfaces, the contained hypervolume between two volumes, etc.

MULTIPLE INTEGRALS

Integration limits in multiple integrals

The domain is not necessarily a line, rectangle, octahedron, etc. It can be a region or shape defined by functions.



MULTIPLE INTEGRALS

Integration limits in multiple integrals

Depending on which variables are used to define these functions, different strategies can be followed.

There are two special cases to highlight. These are cases in which one of the variables is used to delimit the other variable by means of functions, and the region is then said to be bounded by transversal sections.

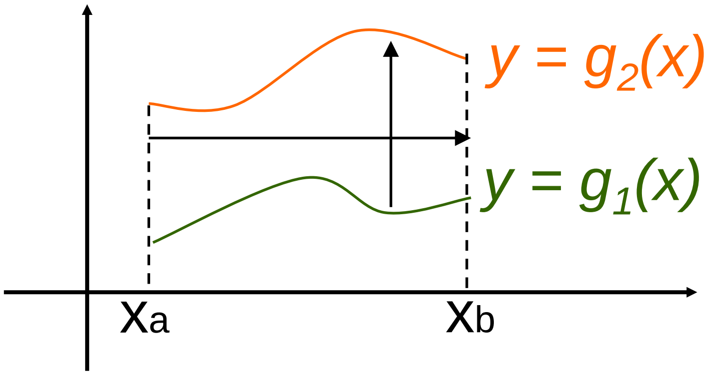
MULTIPLE INTEGRALS

Integration limits in multiple integrals

Vertical transversal sections

The region R is bounded by the functions $g_1(x)$ and $g_2(x)$ in the interval $[x_a, x_b]$. If R is described by

$$R:\quad x_a \le x \le x_b \qquad g_1(x) \le y \le g_2(x)$$



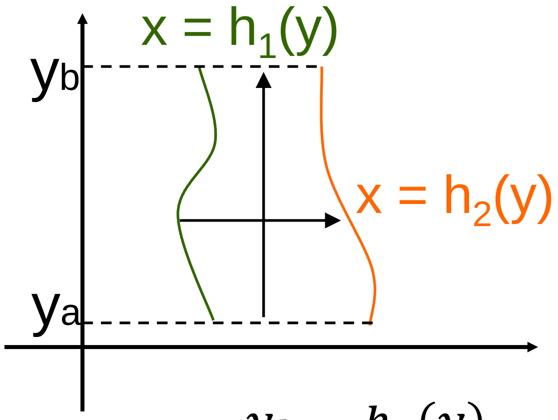
$$\iint_R f(x,y)\,dA = \int_{x_a}^{x_b}\int_{g_1(x)}^{g_2(x)} f(x,y)dy\,dx$$

Integration limits in multiple integrals

Horizontal transversal sections

The region R is limited by the functions $h_1(y)$ and $h_2(y)$ in the interval $[y_a, y_b]$. If R is described by

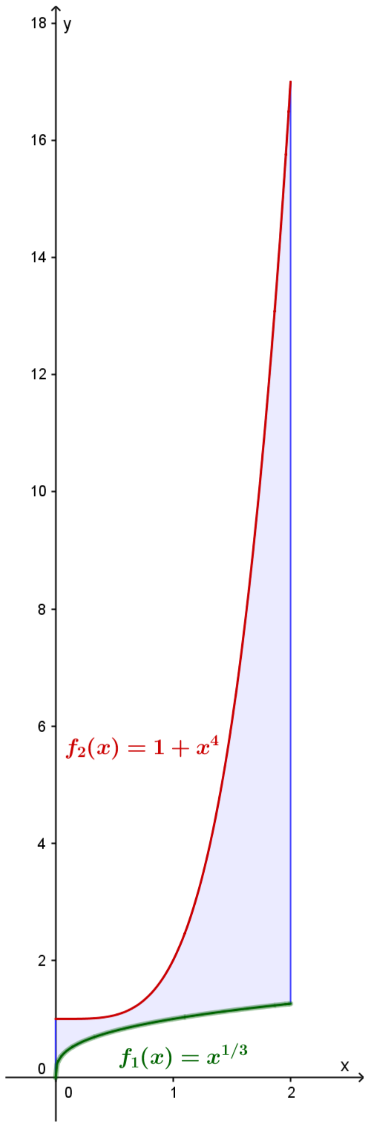
$$R\text{:}\ \ h_1(y) \leq x \leq h_2(y) \quad y_a \leq y \leq y_b$$



$$\iint_R f(x,y)\, dA = \int_{y_a}^{y_b} \int_{h_1(y)}^{h_2(y)} f(x,y) dx\, dy$$

MULTIPLE INTEGRALS – EXAMPLE

Calculate the area of a region bounded by function $f_1(x) = \sqrt[3]{x}$ and $f_2(x) = 1 + x^4$ in the interval $[0,2]$

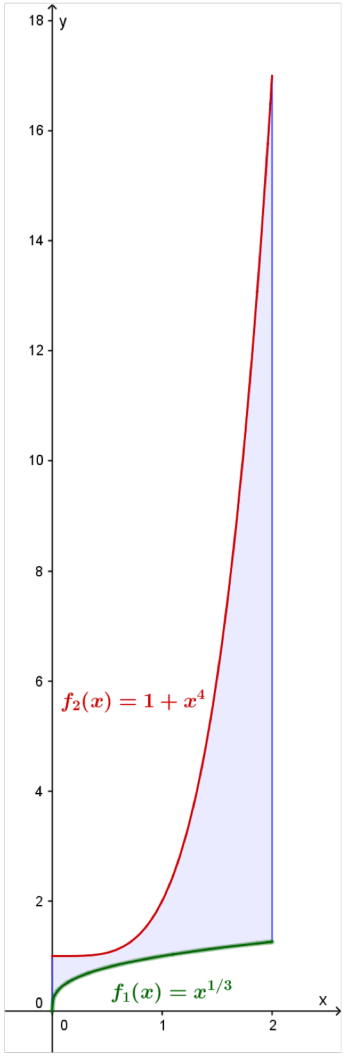


MULTIPLE INTEGRALS– EXAMPLE

Calculate the area of the region bounded by the graphs of the functions $f_1(x) = \sqrt[3]{x}$ y $f_2(x) = 1 + x^4$ in the interval $[0,2]$

$$R: [0, 2]\times[f_1(x), f_2(x)] \qquad S = \iint_R dA =$$

$$= \int_0^2 \int_{f_1(x)}^{f_2(x)} dy\, dx = \int_0^2 \left(\int_{x^{1/3}}^{1+x^4} dy \right) dx = \int_0^2 F(x) dx$$



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$$F(x) = \int_{x^{\frac{1}{3}}}^{1+x^4} dy = \left[y \right]_{\sqrt[3]{x}}^{1-x^4} = 1 + x^4 - x^{1/3}$$

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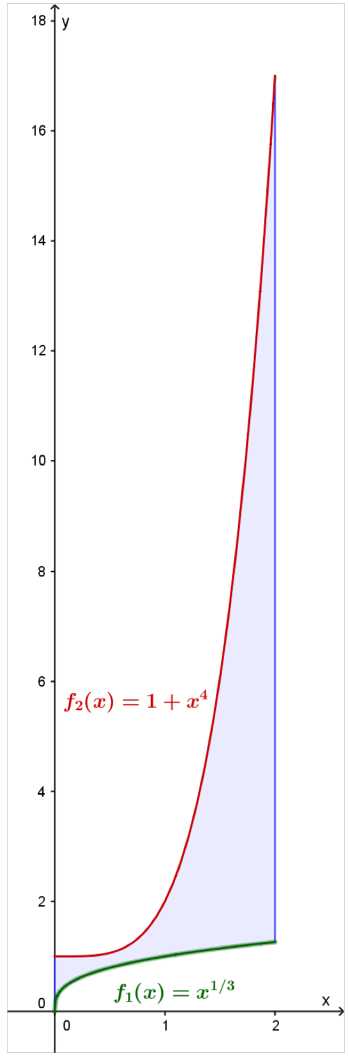
$$F(x) = \int_{x^{\frac{1}{3}}}^{1+x^4} dy = \Big[y \Big]_{\sqrt[3]{x}}^{1-x^4} = 1 + x^4 - x^{1/3}$$

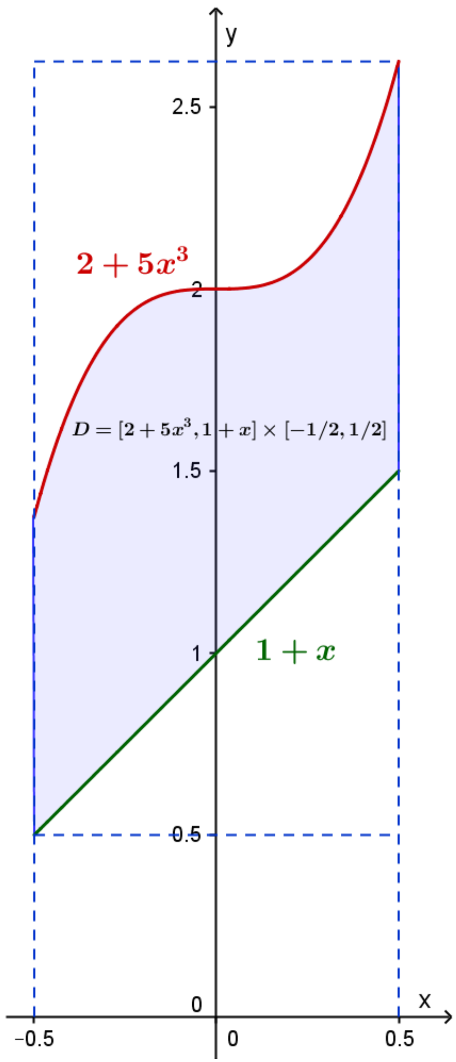
$$S = \int_0^2 (1 + x^4 - x^{1/3}) dx = \left[x + \frac{1}{5}x^5 - \frac{3}{4}x^{4/3} \right]_0^2$$

$$= 2 + \frac{32}{5} - \frac{6}{4}\sqrt[3]{2} = 6{,}51$$



MULTIPLE INTEGRALS – EXAMPLE

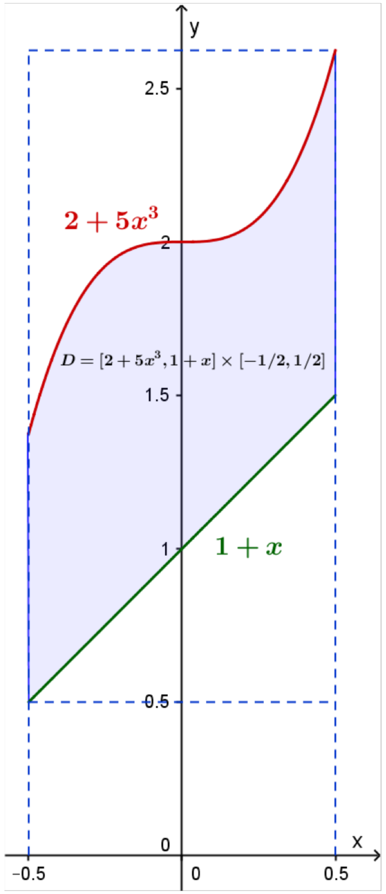
Compute the double integral of $4xy^2$ in the domain $D = [-1/2, 1/2] \times [1 + x, 2 + 5x^3]$



MULTIPLE INTEGRALS– EXAMPLE

Calculate the double integral of $4xy^2$ in

$$D = [-1/2, 1/2]\times[1+x, 2+5x^3]$$



$$\iint_D 4xy^2 dA = 4\iint_D xy^2 dA = 4\mathrm{I}$$

$$\mathrm{I} = \iint_D xy^2 dA = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{1+x}^{2+5x^3} xy^2 dy\, dx$$

MULTIPLE INTEGRALS– EXAMPLE

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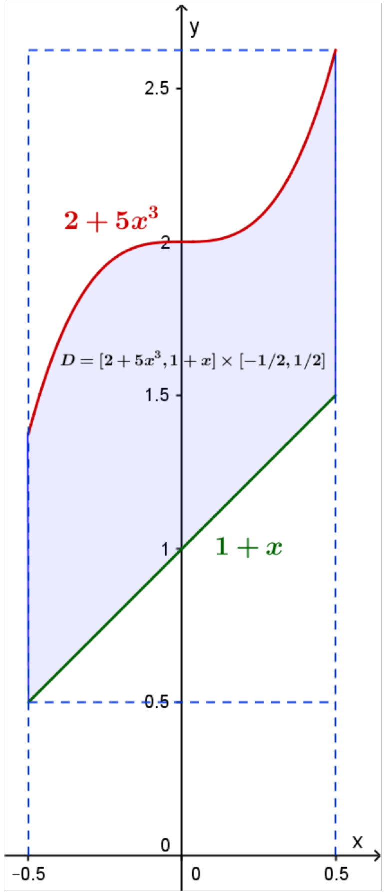
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(x\int_{1+x}^{2+5x^3} y^2 dy\right) dx = \int_{-\frac{1}{2}}^{\frac{1}{2}}(x\, F(x))\, dx$$

$$F(x) = \int_{1+x}^{2+5x^3} y^2 dy$$



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MULTIPLE INTEGRALS– EXAMPLE

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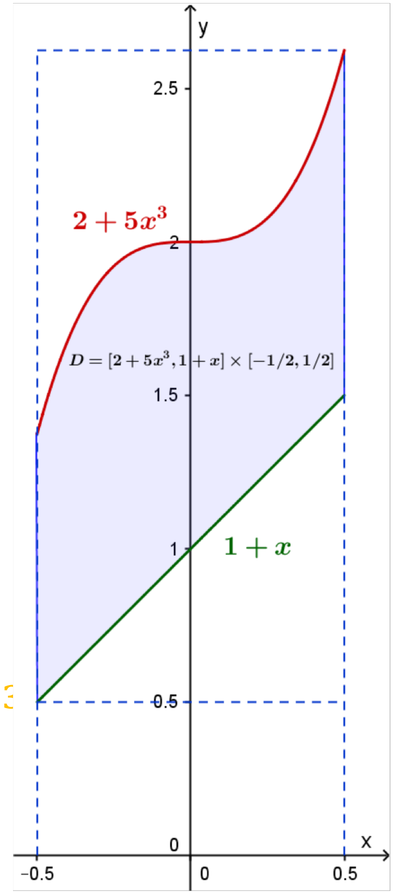
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(x \int_{1+x}^{2+5x^3} y^2 dy \right) dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} (x\, F(x))\, dx$$

$$F(x) = \int_{1+x}^{2+5x^3} y^2 dy = \left[\frac{1}{3} y^3\right]_{1+x}^{2+5x^3} = \left[\frac{1}{3}(2+5x^3)^3 - \frac{1}{3}(1+x)^3\right.$$

$$= \frac{125}{3}x^9 + 50x^6 + \frac{59}{3}x^3 - x^2 - x + \frac{7}{3}$$



MULTIPLE INTEGRALS– EXAMPLE

Calculate the double integral of $4xy^2$ in
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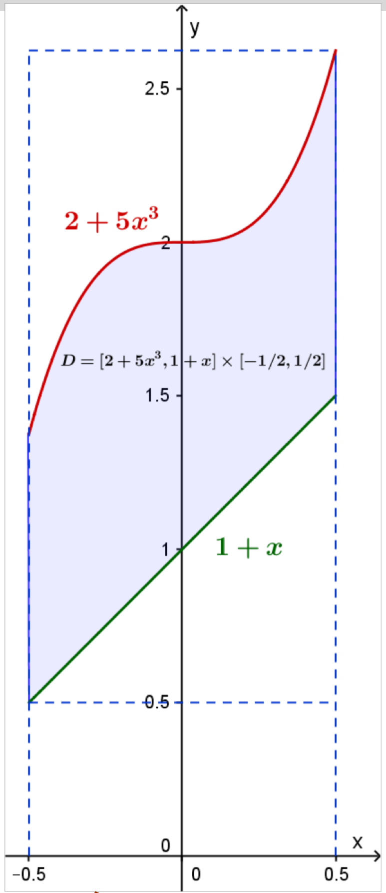


$$\iint_D 4xy^2 dA = 4\iint_D xy^2 dA = 4\mathrm{I}$$

$$\mathrm{I} = \iint_D xy^2 dA = \int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{1+x}^{2+5x^3} xy^2 dy\, dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(x\int_{1+x}^{2+5x^3} y^2 dy\right) dx = \int_{-\frac{1}{2}}^{\frac{1}{2}}(x\,F(x))\,dx$$

$$\mathrm{F(x)} = \frac{125}{3}x^9 + 50x^6 + \frac{59}{3}x^3 - x^2 - x + \frac{7}{3}$$

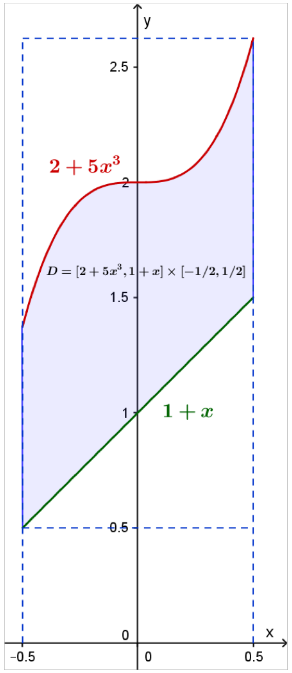
$$= \int_{-1/2}^{1/2}(x(125/3\,x^9 + 50x^6 + 59/3\,x^3 - x^2 - x + 7/3))dx =$$

$$= \int_{-1/2}^{1/2}((125/3)\,x^{10} + 50x^7 + (59/3)x^4 - x^3 - x^2 + (7/3)x)dx$$

MULTIPLE INTEGRALS– EXAMPLE

Calcula la integral doble de $4xy^2$ en el dominio

$$D = [-1/2, 1/2]\times[1+x, 2+5x^3]$$



$$\mathrm{I} = \iint_D xy^2 dA = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{1+x}^{2+5x^3} xy^2 dy\, dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(x \int_{1+x}^{2+5x^3} y^2 dy \right) dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} (x\, F(x))\, dx$$

$$\mathrm{F(x)} = \frac{125}{3}x^9 + 50x^6 + \frac{59}{3}x^3 - x^2 - x + \frac{7}{3}$$

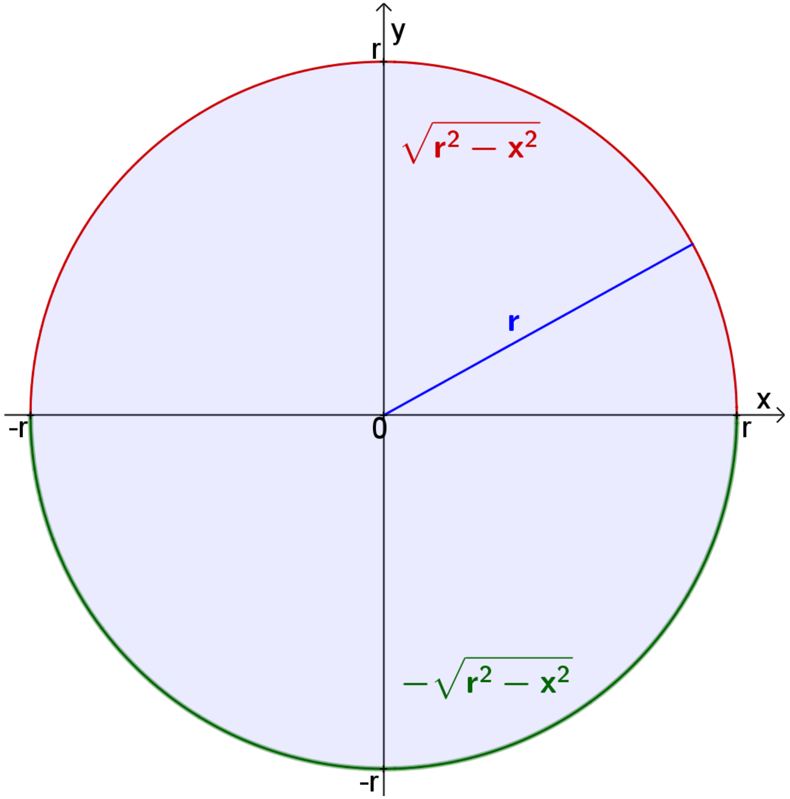
$$= \int_{-1/2}^{1/2} \big(x(125/3\, x^9 + 50x^6 + 59/3\, x^3 - x^2 - x + 7/3)\big) dx =$$

$$= \int_{-1/2}^{1/2} ((125/3)\, x^{10} + 50x^7 + (59/3)x^4 - x^3 - x^2 + (7/3)x) dx =$$

$$= (2/3) \int_0^{1/2} (125x^{10} + 59x^4 - 3x^2) dx = 28081/168960$$

MULTIPLE INTEGRALS – EXAMPLE

Deduce the surface of the circle by means of the double integral in the domain between the positive and negative semicircles of $-r$ to r



MULTIPLE INTEGRALS– EXAMPLE

Infer the Surface of a circle by means of a double integral in the domain bounded by the positive and negartive semi-circumferences between $-r$ and r

$$\text{Semicircumference}(x) = \sqrt{r^2 - x^2}$$

$$\text{Circle} = [-r, r] \times \left[-\sqrt{r^2 - x^2}, \sqrt{r^2 - x^2}\right]$$

$$S = \iint_{\text{Circle}} dA$$

MULTIPLE INTEGRALS– EXAMPLE

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$$\text{Circle} = [-r, r] \times \left[-\sqrt{r^2 - x^2}, \sqrt{r^2 - x^2}\right]$$

$$S = \iint_{Circle} dA = \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dy\, dx$$

MULTIPLE INTEGRALS– EXAMPLE

Infer the Surface of a circle by means of a double integral in the domain bounded by the positive and negartive semi-circumferences between $-r$ and r

$$\text{Semicircunference}(x) = \sqrt{r^2 - x^2}$$

$$\text{Circle} = [-r, r] \times \left[-\sqrt{r^2 - x^2}, \sqrt{r^2 - x^2}\right]$$

$$S = \iint_{\text{Circle}} dA = \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dy\, dx$$

$$= \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dy = \Big[y\Big]_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} = 2\sqrt{r^2 - x^2}$$

MULTIPLE INTEGRALS– EXAMPLE

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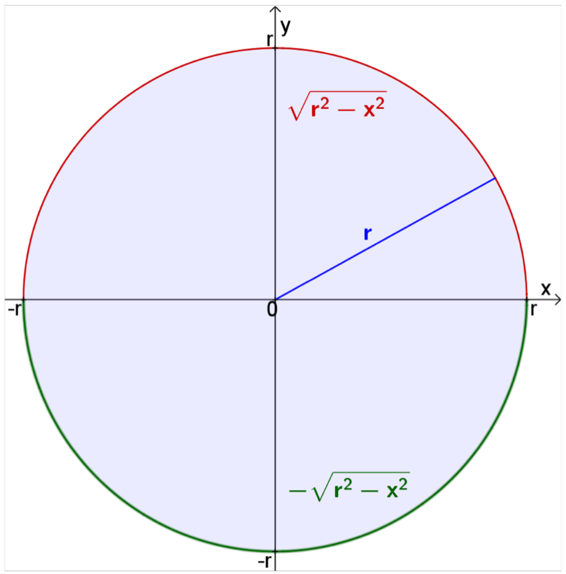

$$S = \int_{-r}^{r} 2\sqrt{r^2 - x^2} dx = 4 \int_{0}^{r} \sqrt{r^2 - x^2} dx = 4I$$

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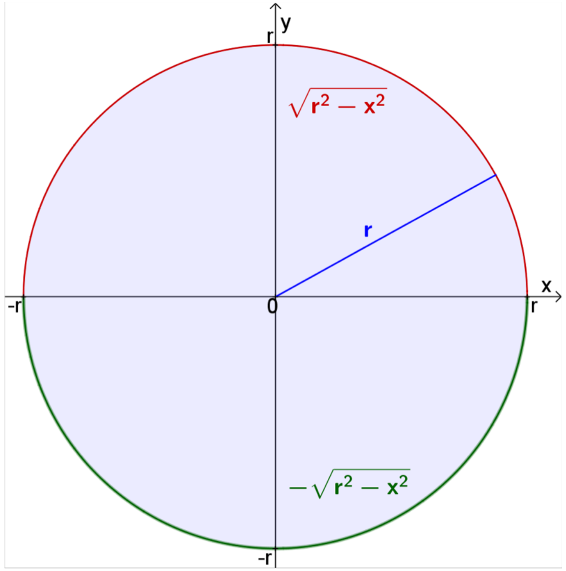

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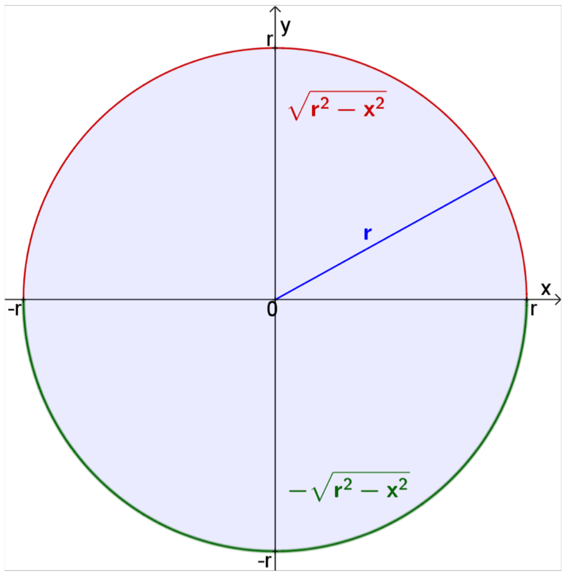
$$I = \int_0^r \sqrt{r^2 - x^2} dx =$$

$$= \left[\frac{x\sqrt{r^2 - x^2}}{2} + \frac{r^2}{2} \text{arcsen}\left(\frac{x}{r}\right)\right]_0^r = \frac{r^2 \pi}{4}$$



MULTIPLE INTEGRALS – EXERCISE

Calculate $\iint_R xdA$ where *R* is the region bounded by $y = 2x, y = x^2$

MULTIPLE INTEGRALS – EXERCISE

Calculate $\iint_R (2x+1)dA$ where *R* is the triangle that has the points at its vertices (-1, 0), (0, 1) and (1, 0)

EXERCISES

1. $\int_0^{+\infty} \frac{dx}{x^2+1}$

2. Calculate the area bounded by the curves $y = x^2, x + y = 2$

3. Calculate the volumen of a cone with radius r and height h considered as a volumen of revolution.

4. $\int_0^1 \int_0^{\pi/2} \; e^y \sin(x)\, dxdy$

5. $\int_0^1 \int_{x^2}^{\sqrt{x}} \; 160xy^3 \quad dydx$

6. $\int_0^1 \int_0^y \; y^2 e^{xy}\, dxdy$

7. Calculate $\iint_R \; dA$ where $\begin{cases} y = x \\ y = 1/x \\ x = 2 \\ y = 0 \end{cases}$

SOLUTIONS

1. $\frac{\pi}{2}$
2. $\frac{9}{2}$
3. $\frac{1}{3}\pi h r^2$
4. $e - 1$
5. 6
6. $\frac{e}{2} - 1$
7. $\frac{1}{2} + \ln(2)$