Lesson 1. Fundamentals of Graphs.

4. Paths and Connectivity

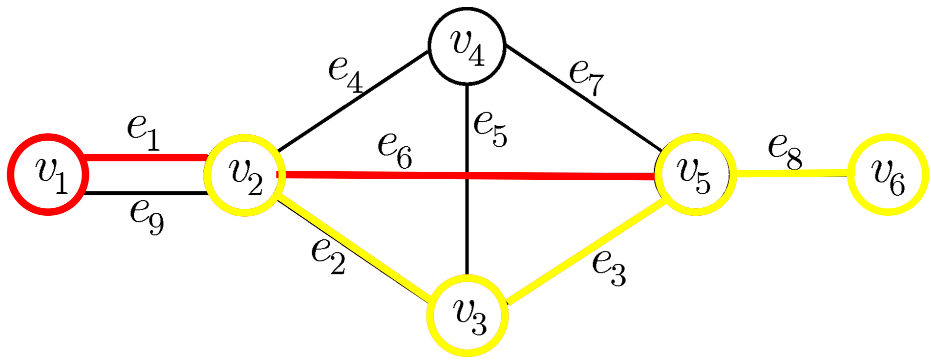
DEFINITIONS:

1. Let x, y be (not necessarily distinct) vertices in an undirected graph G = (V, E). An x-y **walk** in G is a (loop-free) finite alternating sequence

$$x = v_1, e_1, v_2, e_2, v_2, e_2, \ldots v_{n-1}, e_{n-1}, v_n = y$$

of vertices and edges from G, starting at vertex x and ending at vertex y and involving the n-1 edges $e_i = \{v_i, v_{i+1}\}$, where $1 \leq i \leq n-2$.

EXAMPLE:



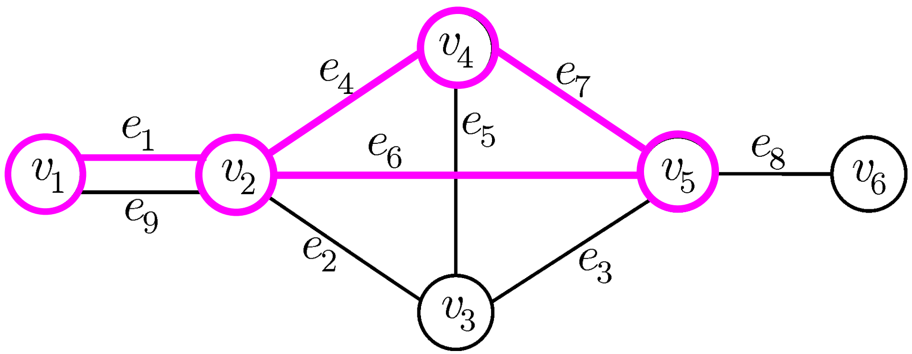
$$\mathbf{C=v_1e_1v_2e_2v_3e_3v_5e_6v_2e_2v_3e_3v_5e_8v_6}$$

2. The **length** of a walk is **the number of edges** in the walk. (When n = 0, there are no edges, x = y, and the walk is called trivial. These walks are not considered very much in our work.)

EXAMPLE: The length of the walk **C** es 7.

3. Any x-y walk where x = y (and n > 1) is called a **closed walk**. Otherwise the walk is called open.

EXAMPLE:



The walk $\mathbf{C_4=v_2e_6v_5e_7v_4e_4v_2e_1v_1e_1v_2}$ is a closed walk.

4. Paths and Connectivity

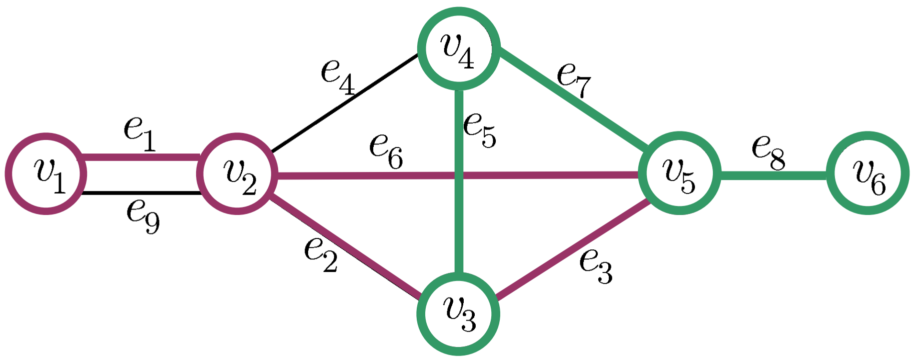
Note that a walk may repeat both vertices and edges.

Consider any x-y walk in an undirected graph G = (V, E).

4. If **no edge** in the x-y walk is repeated, then the walk is called an x-y **trail**.

5. If **no vertex** of the x-y walk occurs more than once, then the walk is called an x-y **path**.
6. When x = y, the term **cycle** is used to describe such a closed path.

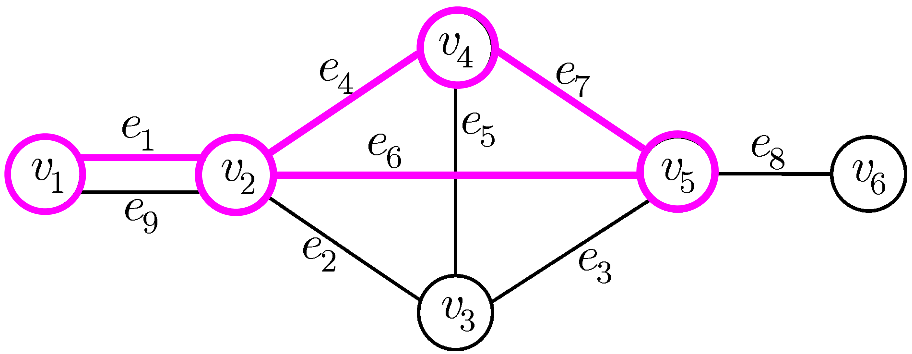
EXAMPLE:



The walk $\mathbf{C_2=v_1e_1v_2e_2v_3e_3v_5e_6v_2}$ is a trail of length 4.

The walk $\mathbf{C_3=v_6e_8v_5e_7v_4e_5v_3}$ is a path.

EXAMPLE:

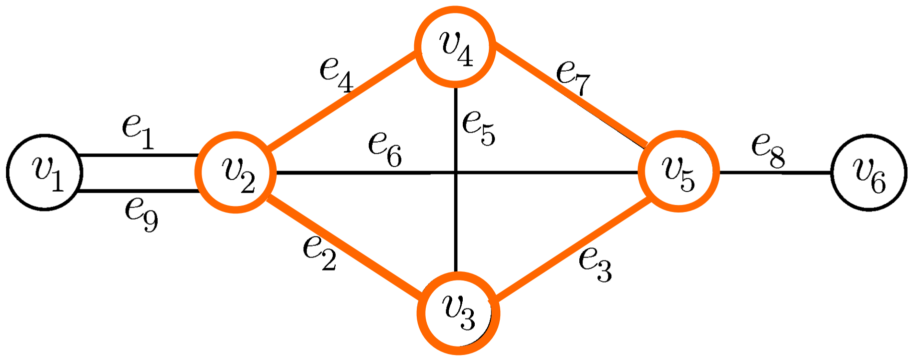


The walk $\mathbf{C_4=v_2e_6v_5e_7v_4e_4v_2e_1v_1e_1v_2}$ is a closed walk.

$\mathbf{e_1}$ is repeated, then $\mathbf{C_4}$ is not a trail.

$\mathbf{v_2}$ is repeated, then $\mathbf{C_4}$ is not a path.

EXAMPLE:



The walk $\mathbf{C=v_2e_4v_4e_7v_5e_3v_3e_2v_2}$ is a cycle.

Remark.-
These concepts are defined similarly for directed graphs with the consideration that the **directions of the arcs must agree with the direction of the path or walk.** In some books directed cycles are called **circuits**.

7. - Let G=(V,A) be an undirected and let u, v two vertices of G.

The **Connectedness** relation in V is defined in the following way:

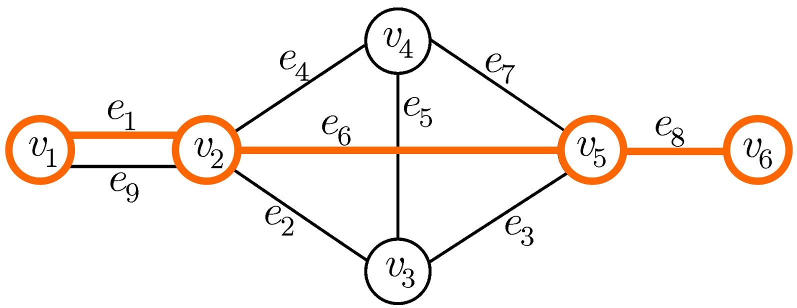
u and v are **connected** if there is a path from u to v.

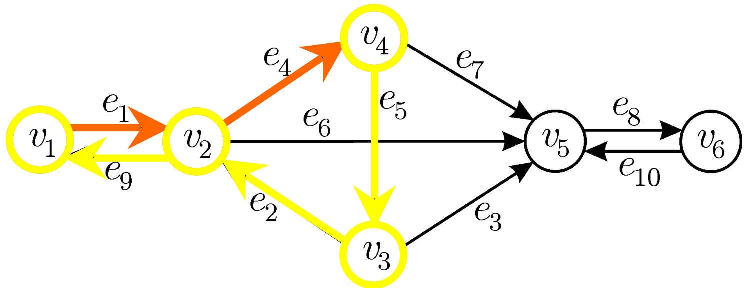
- Let G=(V,A) be an **directed** and let u, v two vertices of G.

The **Connectedness** relation in V is defined in the following way:

u and v are **connected** if there is: a path from u to v, and
a path from v to u.

EXAMPLE:

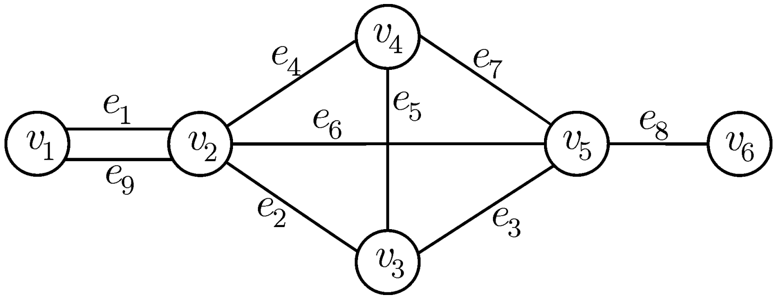




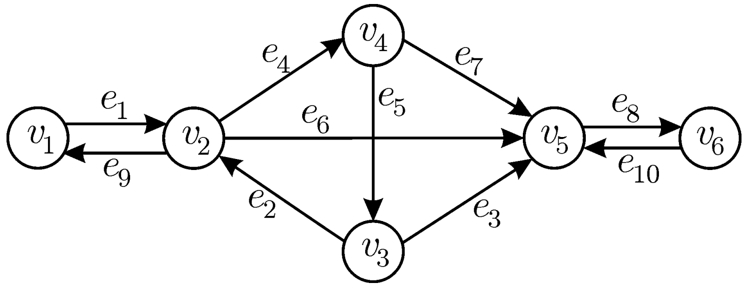
Vertices v_1 and v_6 are connected through the path $C=v_1e_1v_2e_6v_5e_8v_6$.
Every pair of distinct vertices are connected.

Vertices v_1 and v_4 are connected through the paths:
$C_1=v_1e_1v_2e_4v_4$
$C_2=v_4e_5v_3e_2v_2e_9v_1$
Vertices v_4 and v_5 are not connected.

8. A graph (undirected or directed) is called **connected** if every pair of distinct vertices of the graph are connected. A graph that is not connected is called **disconnected**.

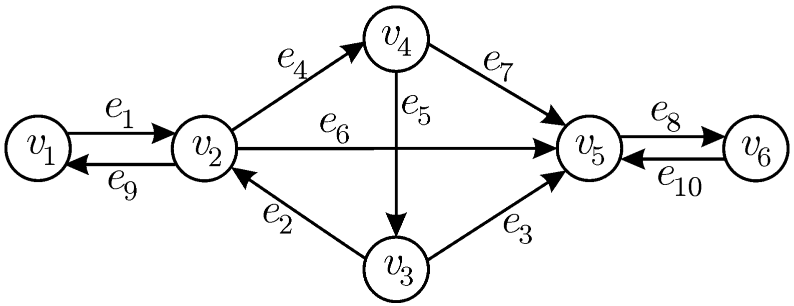


Connected graph



Disconneted graph

9. A directed graph is **weakly connected** if its associated undirected graph is connected.



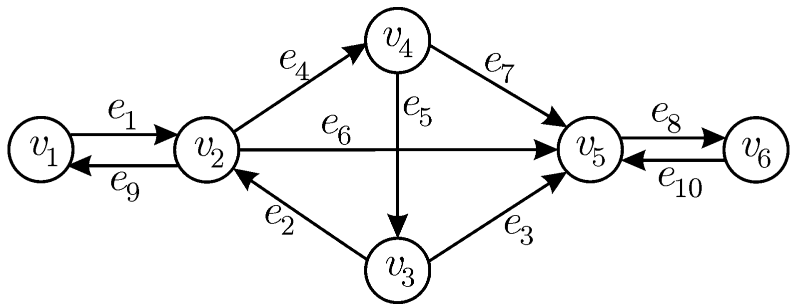
THEOREM

The connectedness relation is an equivalence relation and therefore determines a partition on the set of vertices. The elements of this partition are called **connected components** of the graph.

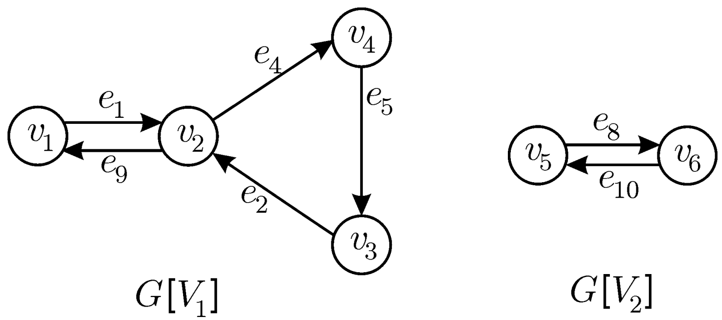
THEOREM

A graph is connected if and only if it has only one connected component.

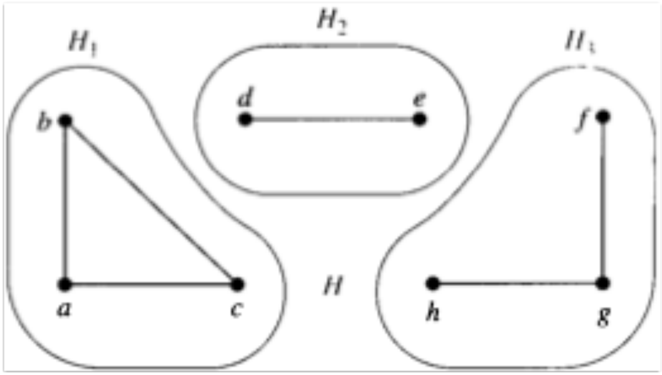
EXAMPLE:



connected components



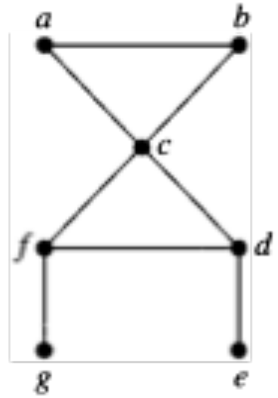
EXAMPLE:



Graph H and its Connected Components H_1, H_2, and H_3

C_1={a,b,c}
C_2={d,e}
C_3={f,g,h}

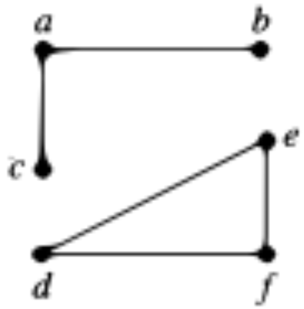
EXAMPLE:



G

- The graph G in Figure is connected, because for every pair of distinct vertices there is a path between them.
- There is only one connected component.

C={a,b,c,d,e,f,g}

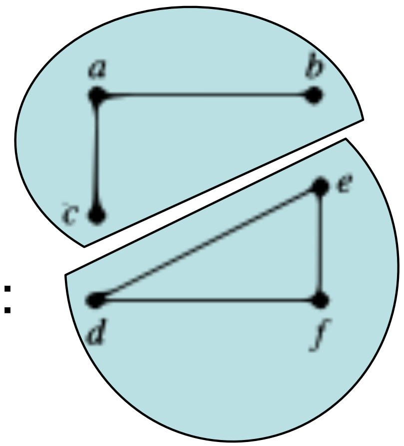


H

- The graph H is not connected. For instance, there is no path in H between vertices a and d.
- There are 2 connected components:

C_1={a,b,c}

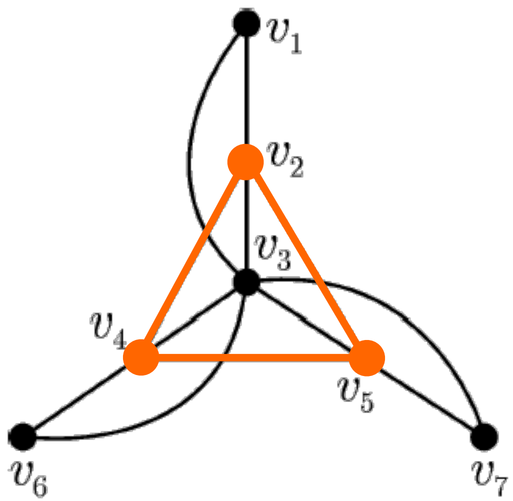
C_2={d,e,f}



H

THEOREM (Undirected Graphs)
A graph is bipartite if and only if it does not contain a cycle of odd length.

EXAMPLE:
The following graph is NOT bipartite because it contains an odd cycle: $v_2v_4v_5$

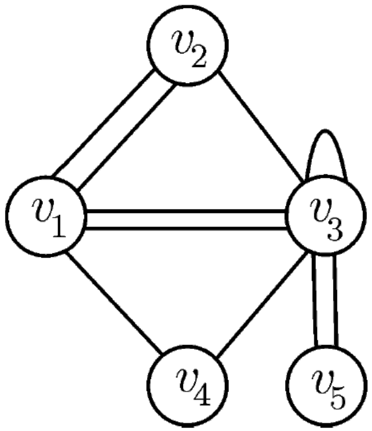


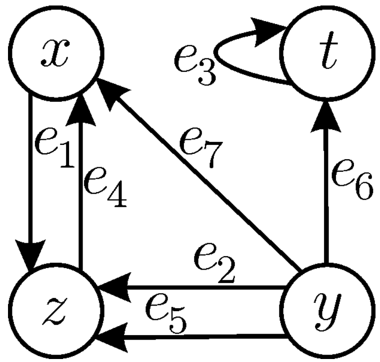
DEFINITION:

Let G be a graph (directed or undirected) with n vertices $\{v_i\}_{i=1}^{n}$. The **adjacency matrix** A of G, with respect to this listing of the vertices, is the n x n matrix A = $[a_{ij}]$ where a_{ij} is the number o edges (arcs) from v_i to v_j. If G is undirected the loops are computed twice.

EXAMPLE:

$$A = \begin{bmatrix} 0 & 2 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 1 & 2 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix}$$





$$A = \begin{array}{cc} \begin{array}{cccc} x & y & z & t \end{array} & \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} x \\ y \\ z \\ t \end{array} \end{array}$$

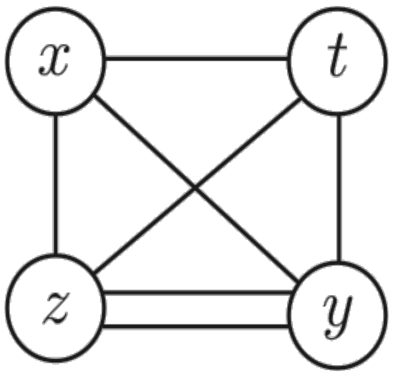
Remarks.

- If G is undirected then A is symmetric.
- If G is undirected, and the vertex v_i has a loop, then a_{ii}=2. If G is directed then a_{ii}=1.

PROPERTIES OF THE ADJACENCY MATRIX:

1. Let G be an undirected graph with adjacency matrix A. Then, the sum of the elements of the row i (or column i) is equal to the degree of the vertex v_i.

EXAMPLE:



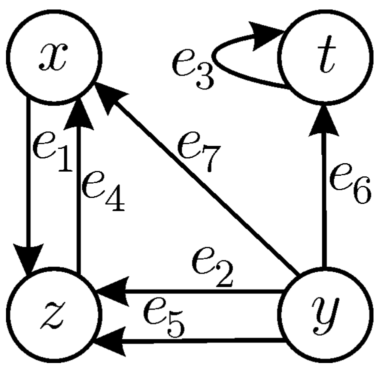
$$A = \begin{array}{c} \begin{array}{cccc} x & y & z & t \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} x \ = 3 \\ y \\ z \\ t \end{array}$$

The degree of the vertex **x** is 3.

5. Matrix representation

2. Let G be a directed graph with adjacency matrix A. Then, the sum of the elements of the **row** i is equal to the **outdegree** of the vertex v_i, and the sum of the elements of the **column** j is equal to the **indegree** of the vertex v_j.

EXAMPLE:



$$A = \begin{array}{c} \begin{array}{cccc} x & y & z & t \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{l} = 1 \\ \\ \\ \\ \end{array}$$

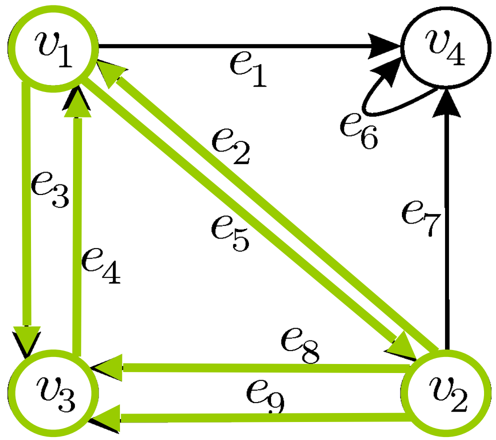
$\downarrow$ 3

$d_{out}(x)=1$

$d_{in}(z)=3$

3. Let G be a graph with adjacency matrix A. Then, the element (i,j) of the matrix A^r, $r \geq 1$, is equal to the number of walks from v_i to v_j of length r.

EXAMPLE:



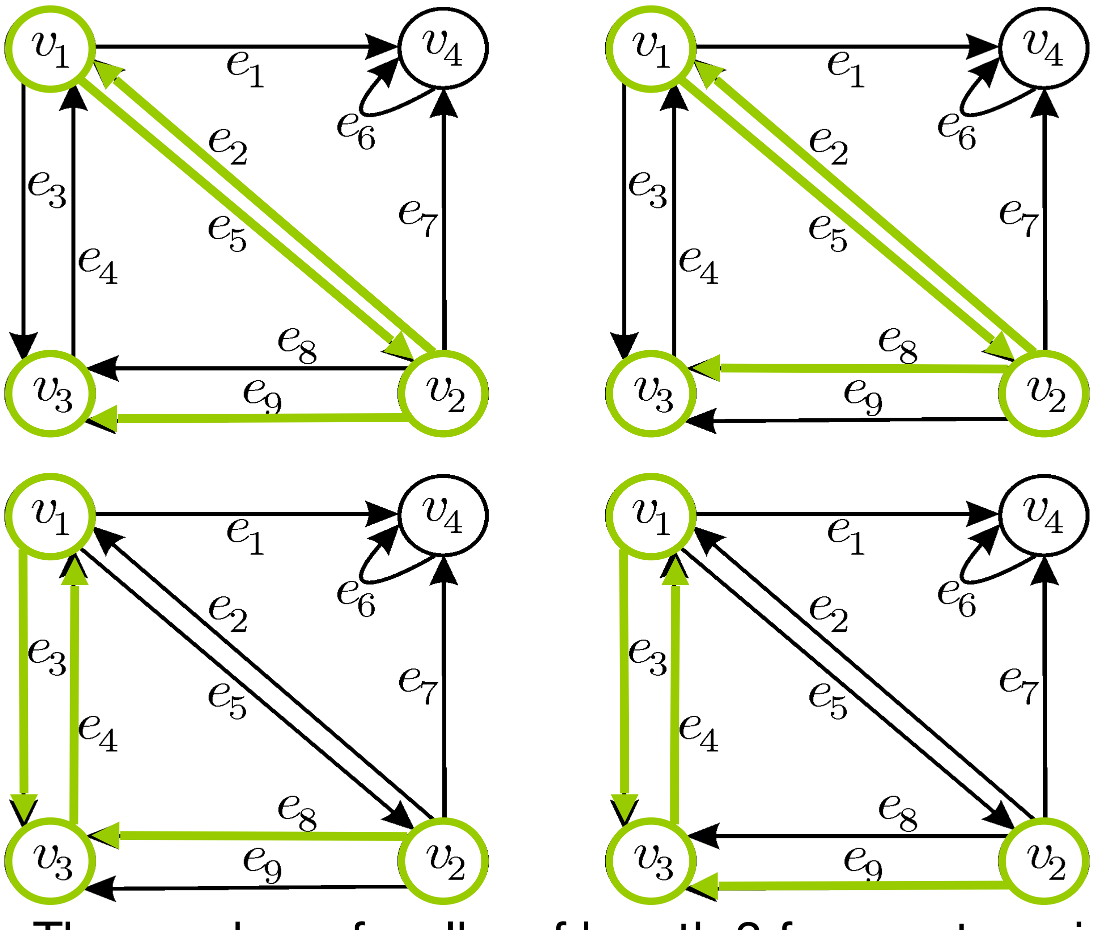
$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & 2 & 2 & 4 \\ 2 & 2 & 4 & 5 \\ 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The number of walks of length 3 from $\mathbf{v_2}$ to $\mathbf{v_3}$ is 4.

3. Let G be a graph with adjacency matrix A. Then, the element (i,j) of the matrix A^r, $r \geq 1$, is equal to the number of walks from v_i to v_j of length r.

EXAMPLE:



$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & 2 & 2 & 4 \\ 2 & 2 & 4 & 5 \\ 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The number of walks of length 3 from $\mathbf{v_2}$ to $\mathbf{v_3}$ is 4.

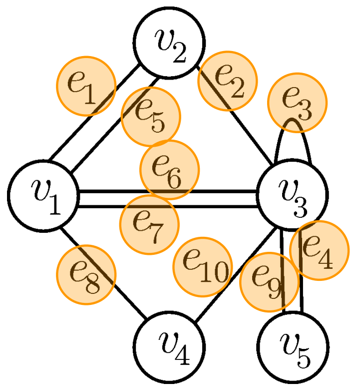
5. Matrix representation

DEFINITIONS:

1. Let G = (V,A) be an undirected graph with n vertices and m edges where $A = \{a_i\}_{i=1}^m$, and $V = \{v_i\}_{i=1}^n$. The **incidence matrix** of G is the n × m matrix

$$M = \left[m_{ij}\right] / \; m_{ij} = \begin{cases} 0 \text{ if } v_i \text{ is not incident with } a_j \\ 1 \text{ if } v_i \text{ is incident with } a_j \\ 2 \text{ if } a_j \text{ is a loop in } v_i \end{cases}$$

<u>EXAMPLE:</u>



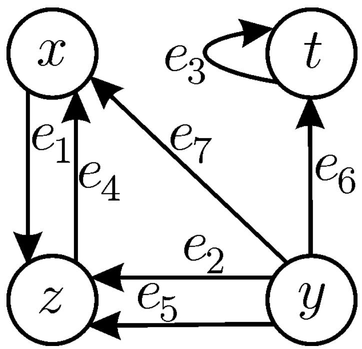
$$\begin{array}{cc} & \begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{array} \\ M = & \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \end{array}$$

5. Matrix representation

2. Let G = (V,A) be an directed graph with n vertices and m arcs where $V = \{v_i\}_{i=1}^{n}$, and $A = \{a_i\}_{i=1}^{m}$. The **incidence matrix** of G is the n × m matrix

$$M = \left[m_{ij} \right] / \; m_{ij} = \begin{cases} 0 \text{ if } v_i \text{ is not incident with } a_j \\ 1 \text{ if } v_i \text{ is the origin vertex of } a_j \\ -1 \text{ if } v_i \text{ is the terminating vertex of } a_j \\ 2 \text{ if } a_j \text{ is a loop in } v_i \end{cases}$$

<u>EXAMPLE:</u>

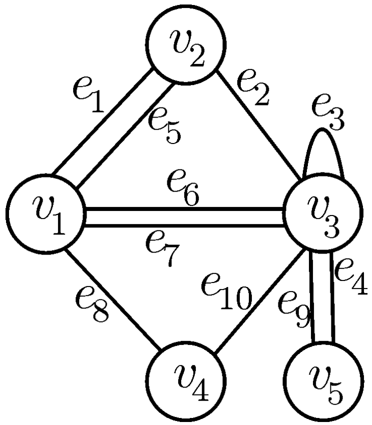


$$M = \begin{array}{c c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 & 0 \end{bmatrix} & \begin{array}{c} x \\ y \\ z \\ t \end{array} \end{array}$$

PROPERTIES OF THE INCIDENCE MATRIX:

1. Let G be an undirected graph with incidence matrix M. Then, the sum of the elements of the row i of M is equal to the degree of the vertex v_i.

EXAMPLE:



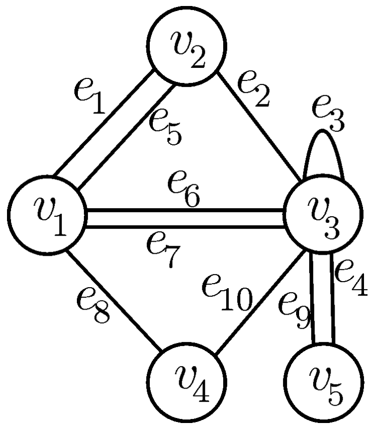
$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{array}{l} =5 \\ =3 \\ =8 \\ =2 \\ =2 \end{array}$$

The degree of the vertices are:

$d(v_1)=5$, $d(v_2)=3$, $d(v_3)=8$, $d(v_4)=2$, $d(v_5)=2$

2. Let G be an undirected graph with incidence matrix M. Then, the sum of the elements of each column of M is equal to 2.

EXAMPLE:

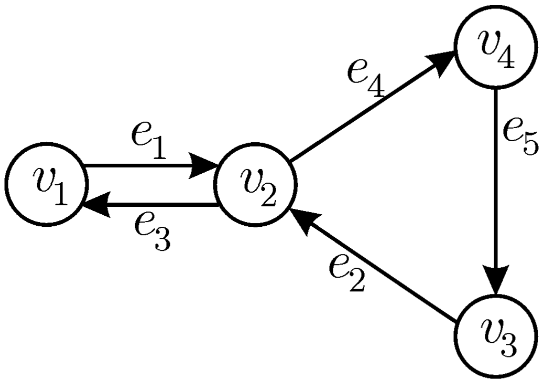


$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

2 2 2 2 2 2 2 2 2 2

3. Let G be a directed graph without loops. Then, the sum of the elements of each column of the incidence matrix is equal to 0.

EXAMPLE:



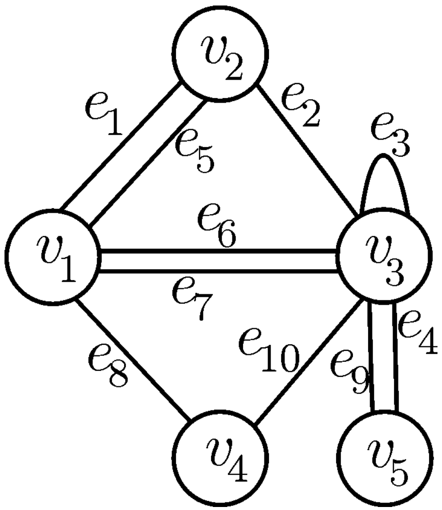
$$M = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

DEFINITIONS:

1. Let G be an undirected graph. Another way to represent a graph is to use **incidence lists**, which list the edges that are incident to each vertex of the graph.

EXAMPLE:



Incidence List

$v_1 : e_1, e_5, e_6, e_7, e_8$

$v_2 : e_1, e_2, e_5$

$v_3 : e_2, e_3, e_4, e_6, e_7, e_9, e_{10}$

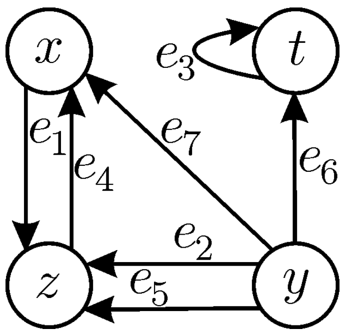
$v_4 : e_8, e_{10}$

$v_5 : e_4, e_9$

5. Matrix representation

2. Let G be a directed graph. The **List of Outward Arcs** of G is a table that lists, for each vertex v, the outward arcs from v. The **List of Inward Arcs** of G is the table that lists, for each vertex v, the inward arcs from v.

EXAMPLE:



Outward	Arcs	Inward	Arcs
x:	e_1	x:	e_4, e_7
y:	e_2, e_5, e_6, e_7	y:	
z:	e_4	z:	e_1, e_2, e_5
t:	e_3	t:	e_3, e_6