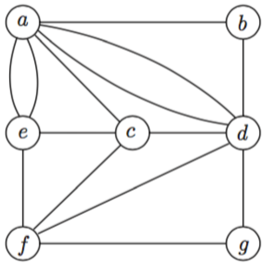
Discrete Mathematics
Practice Class 4
27-02-2024

Problem 1. Consider the graph



(i) Create this graph in Magrada, in the graphic mode. Then, using MaGraDa, analyze is this graph is connected?

(ii) Compute the degree of the vertices using MaGraDa. Then, using the characterization theorem of Eulerian graphs analyze if the graph is Eulerian.

(iii) If the answer to the above question is in the affirmative, then built an Eulerian tour in the graphic mode (Menu **Algorithms**, Option **Fleury**). Write this tour here.

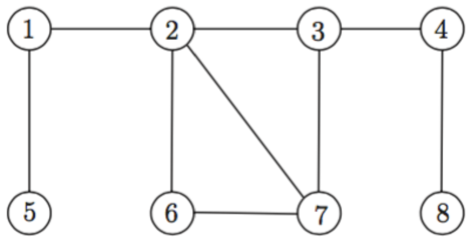
(iv) Delete, in the graphic mode, the edge {c, d}. Draw the resulting graph:

(v) Analyze if the graph is Eulerian.

(vi) Analyze if the graph has an Eulerian trail.

(vii) If the answer to the above question is in the affirmative, then construct such a trail in the graphic mode and write it here.

Problem 2. Consider the graph



(i) If there exist a Hamiltonian cycle, how many edges should have the cycle?

(ii) Compute with MaGraDa the degree of each vertex.

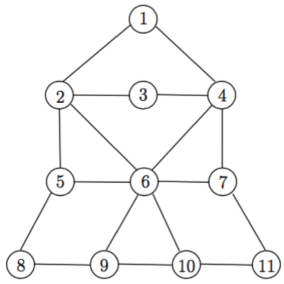
(iii) Applying rule 5, draw the edges that should necessarily be in the cycle, if any.

(iv) For the other edges, using rule 7 explains which of them cannot be in the Hamiltonian cycle.

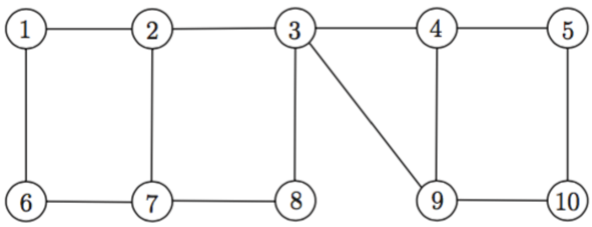
(v) Using the above results and rule 6 explain if the graph has a Hamiltonian cycle or a Hamilton path.

(vi) What alternative rule could have been used to prove that the graph is not Hamiltonian.

Problem 3. In the following graph determine if there exists a Hamilton cycle. If it does, find such a cycle. If it does not, give an argument to show why no such cycle exists.



Problem 4. In the following graph determine if there exists a Hamilton cycle or Hamilton path. If it does, find such a cycle or path.



Reachable Matrix: Warshall's Algorithm

Warshall's algorithm constructs a sequence of $n \times n$ matrices $R_0, R_1, \dots, R_n$ where:

- n is the number of vertices of the graph.
- R_0 is obtained from the adjacency matrix A by replacing the positive elements with ones.
- The reachable matrix R is obtained from R_n changing the diagonal elements of R_n into ones.

Denote the elements of the matrices R_k by

$$R_k = [r_{ij}^{(k)}]_{1\leq i,j\leq n}, \quad k = 0, 1, 2, \dots, n,$$

then Warshall's algorithm computes these elements as

$$r_{ij}^{(k)} = 1 \Longleftrightarrow \begin{cases} r_{ij}^{(k-1)} = 1 \\ \text{ó} \\ r_{ik}^{(k-1)} = r_{kj}^{(k-1)} = 1 \end{cases}$$
$$k = 1, 2, \dots, n.$$

Example. Consider the graph with adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Compute the reachable matrix R using the Warshall's algorithm.

- R_0 is obtained from the adjacency matrix A by replacing the positive elements with ones.

$$R_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now R_1 is computed from R_0 applying algorithm:

- Entry $r^{(1)}_{11}$

i=1
j=1
k=1 $\Rightarrow$ Elements to compare:
$r^{(0)}_{11} = 0$
$r^{(0)}_{11} = 0$ $\Rightarrow$ $r^{(1)}_{11} = 0$

- Entry $r^{(1)}_{22}$

i=2
j=2
k=1 $\Rightarrow$ Elements to compare:
$r^{(0)}_{21} = 1$
$r^{(0)}_{12} = 1$ $\Rightarrow$ $r^{(1)}_{22} = 1$

- Entry $r^{(1)}_{32}$

$$\begin{array}{l} i=3 \\ j=2 \\ k=1 \end{array} \Rightarrow \begin{array}{l} \text{Elements to compare:} \\ r^{(0)}_{31}=0 \\ r^{(0)}_{12}=1 \end{array} \Rightarrow r^{(1)}_{32}=0$$

- …

Remark: Note that building iteration *k* the elements to compare are in **row *k*** and **column *k***.

From $$R_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow R_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From $$R_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow R_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From $$R_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow R_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From $$R_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow R_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The reachable matrix R is obtained from R_n changing the diagonal elements of R_n into ones.

$$R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 5. Consider the graph with adjacency matrix

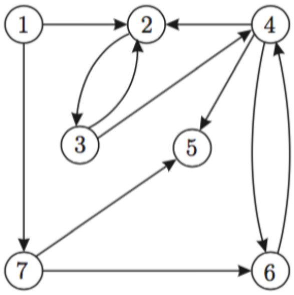
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(i) Compute the reachable matrix R. Check the results with Magrada (Menu **Algorithms**, Option **Warshall.** With option **By steps** the sequence of matrices $\mathbf{R_k}$, k=1,…,n is shown and with option **Final result** only the reachable matrix is given).

(ii) Using the reachable matrix R, compute the access matrix Q.

(iii) Fill in the blanks: Since in the third column of Q there are ones at the rows _____________, this means that vertex________________ reaches vertices________________________.

Problem 6. Consider the graph



(i) Compute the connected components using both studied methods. Check the connected components using Magrada (Menu **Basic calculations**, Option **Connected components**).

(ii) Is this graph connected? Why? Check the answer using Magrada (Menu **Basic calculations,** Option **Graph**, **Connected).**