Warshall's algorithm
Computing the Reachable Matrix

Warshall's algorithm constructs a sequence of $n \times n$ matrices R_0, $R_1, \ldots, R_n$ where:

- n is the number of vertices of the graph.
- R_0 is obtained from the adjacency matrix A by replacing the positive elements with ones.
- The reachable matrix R is obtained from R_n changing the diagonal elements of R_n into ones.

Denote the elements of the matrices R_k by

$$R_k = [r_{ij}^{(k)}]_{1 \le i,j \le n}, \quad k = 0, 1, 2, \ldots, n,$$

then Warshall's algorithm computes these elements as

$$r_{ij}^{(k)} = 1 \Longleftrightarrow \begin{cases} r_{ij}^{(k-1)} = 1 \\ \text{ó} \\ r_{ik}^{(k-1)} = r_{kj}^{(k-1)} = 1 \end{cases}$$

$$k = 1, 2, \ldots, n.$$

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Example. Consider the graph with adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Compute the reachable matrix R using the Warshall's algorithm.

R_0 is obtained from the adjacency matrix A by replacing the positive elements with ones.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$R_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \quad r_{ij}^{(k)} = 1 \Longleftrightarrow \begin{cases} r_{ij}^{(k-1)} = 1 \\ \text{ó} \\ r_{ik}^{(k-1)} = r_{kj}^{(k-1)} = 1 \end{cases}$$

$$k = 1, 2, \ldots, n.$$

$$\begin{bmatrix} & 1 & & \\ 1 & & 1 & \\ & & & 1 \end{bmatrix}$$

Now R_1 is computed from R_0 applying algorithm:

- Entry $r^{(1)}_{11}$

i=1
j=1
k=1

⇨ Elements to compare:
$r^{(0)}_{11} = 0$
$r^{(0)}_{11} = 0$

⇨ $r^{(1)}_{11} = 0$ ⇨

$$\begin{bmatrix} 0 & 1 & & \\ 1 & & 1 & \\ & & & 1 \end{bmatrix}$$

- Entry $r^{(1)}_{22}$

i=2
j=2
k=1

⇨ Elements to compare:
$r^{(0)}_{21} = 1$
$r^{(0)}_{12} = 1$

⇨ $r^{(1)}_{22} = 1$ ⇨

$$\begin{bmatrix} 0 & 1 & & \\ 1 & 1 & 1 & \\ & & & 1 \end{bmatrix}$$

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Remark: Note that building iteration ***k***, the elements to compare are in **row *k*** and **column *k***.

First: Mark row k, column k

Second: copy the ones

Third: Compare ik with kj (draw a cross)

From $R_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ⇨ $R_1 = \begin{bmatrix} & 1 & & \\ 1 & & 1 & \\ & & & \\ & & & 1 \end{bmatrix}$ ⇨ $R_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

From $R_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ⇨ $R_2 = \begin{bmatrix} & 1 & & \\ 1 & 1 & 1 & \\ & & & \\ & & & 1 \end{bmatrix}$ ⇨ $R_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

From $R_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ⇨ $R_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

From $R_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ⇨ $R_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

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- The reachable matrix R is obtained from R_n changing the diagonal elements of R_n into ones.

$$R_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$