How to improve the accuracy of Lattice Boltzmann calculations

Bastien Chopard

May 2008

1 A transformation of the LB dynamics

The LBGK model reads

$$f_i^{out} = f_i + \omega\left(f_i^{eq} - f_i\right) \tag{1}$$

where

$$f_i^{eq} = t_i\rho\left(1 + F_i(\mathbf{u})\right)$$

with

$$F(\mathbf{u}) = \frac{\mathbf{u}\cdot\mathbf{v}_i}{c_s^2} + \frac{1}{2c_s^4}Q_{i\alpha\beta}u_\alpha u_\beta$$

Eq. (1) can be transformed as

$$f_i^{out} - t_i\rho_0 = f_i - t_i\rho_0 + \omega\left(f_i^{eq} - t_i\rho_0 - (f_i - t_i\rho_0)\right) \tag{2}$$

where ρ_0 is a constant corresponding to the average density.

Let us define

$$h_i^{out} = f_i^{out} - t_i\rho_0 \qquad h_i = f_i - t_i\rho_0 \qquad h_i^{eq} = f_i^{eq} - t_i\rho_0$$

Then the LBGK equation becomes

$$\boxed{h_i^{out} = h_i + \omega\left(h_i^{eq} - h_i\right)} \tag{3}$$

The question is now to see whether we can simulate h_i instead of f_i. We have

$$\rho = \sum f_i = \sum h_i + \rho_0\sum t_i = \sum h_i + \rho_0$$

because $\sum t_i = 1$. Similarly, we have

$$\mathbf{u} = \frac{\sum f_i\mathbf{v}_i}{\rho} = \frac{\sum(f_i - \rho_0 t_i)\mathbf{v}_i}{\rho}$$

because $\sum t_i\mathbf{v}_i = 0$. Then

$$\mathbf{u} = \frac{\sum h_i\mathbf{v}_i}{\rho_0 + \sum h_i}$$

In summary the density ρ and the velocity $\mathbf{u}$ can be obtained from the h_i as

$$\boxed{\rho = \rho_0 + \Delta\rho \qquad \Delta\rho = \sum h_i \qquad \mathbf{u} = \frac{\sum h_i\mathbf{v}_i}{\rho_0 + \Delta\rho}} \tag{4}$$

Note that the momentum tensor is also given by a new expression

$$\Pi_{\alpha\beta} = \sum_i f_i v_{i\alpha} v_{i\beta} = \sum_i h_i v_{i\alpha} v_{i\beta} + \sum \rho_0 t_i v_{i\alpha} v_{i\beta} = \rho_0 c_s^2 \delta_{\alpha\beta} + \sum_i h_i v_{i\alpha} v_{i\beta}$$

Finally we can compute h_i^{eq}

$$h_i^{eq} = f_i^{eq} - t_i \rho_0 = t_i(\rho - \rho_0) + t_i \rho F_i(\mathbf{u}) \tag{5}$$

Therefore

$$\boxed{h_i^{eq} = t_i \Delta\rho + t_i(\rho_0 + \Delta\rho) F_i(\mathbf{u})} \tag{6}$$

Thus the LBGK equation can be expressed in terms of the transformed distributions h_i and the value of ρ_0.

Note that f_i is of the order $\rho_0 + Ma$ where $Ma = u/c_s$ is the Mach number which is much smaller than $\rho_0 = \mathcal{O}(1)$. This is the reason of the numerical inaccuracy of eq. (1). On the other hand, h_i is of order $\mathcal{O}(Ma)$ and so are all the terms in h^{eq} and in eq. (3).

2 Incompressible LB

In the case of the incompressible BGK models, the evaluation of h^{eq} is even simpler

$$h_i^{eq} = t_i \sum_j h_j + t_i \rho_0 F_i(\mathbf{u})$$

and $\mathbf{u}$ becomes

$$\mathbf{u} = \frac{\sum h_i \mathbf{v}_i}{\rho_0}$$