2 Governing equations

2.1 Introduction

Waves or spectral wave components in water with limited depth and non-zero mean currents are generally described using several phase and amplitude parameters. Phase parameters are the wavenumber vector $\mathbf{k}$, the wavenumber k, the direction θ and several frequencies. If effects of mean currents on waves are to be considered, a distinction is made between the relative or intrinsic (radian) frequency σ ($= 2\pi f_r$), which is observed in a frame of reference moving with the mean current, and the absolute (radian) frequency ω ($= 2\pi f_a$), which is observed in a fixed frame of reference. The direction θ is by definition perpendicular to the crest of the wave (or spectral component), and equals the direction of $\mathbf{k}$. Equations given here follow the geometrical optics approximation, which is exact in the limit when scales of variation of depths and currents are much larger than those of an individual wave[1]. Diffraction, scattering and interference effects that are neglected by this approximation can be added a posteriori as source terms in the wave action equation. Under this approximation of slowly varying current and depth, the quasi-uniform (linear) wave theory then can be applied locally, giving the following dispersion relation and Doppler-type equation to interrelate the phase parameters

$$\sigma^2 = gk \tanh kd \, , \tag{2.1}$$

$$\omega = \sigma + \mathbf{k} \cdot \mathbf{U} \, , \tag{2.2}$$

where d is the mean water depth and $\mathbf{U}$ is the (depth- and time- averaged over the scales of individual waves) current velocity. The assumption of slowly varying depths and currents implies a large-scale bathymetry, for which wave diffraction can generally be ignored. The usual definition of $\mathbf{k}$ and ω from the phase function of a wave or wave component implies that the number of wave crests is conserved (see, e.g., Phillips, 1977; Mei, 1983)

[1] Even with a factor 5 change in wave height over half a wavelength, the geometrical optics approximation can provide reasonable results as was shown over submarine canyons (Magne et al., 2007)

$$\frac{\partial \mathbf{k}}{\partial t} + \nabla \omega = 0 \, . \tag{2.3}$$

From Eqs. (2.1) through (2.3) the rates of change of the phase parameters can be calculated (e.g., Christoffersen, 1982; Mei, 1983; Tolman, 1990, equations not reproduced here).

For monochromatic waves, the amplitude is described as the amplitude, the wave height, or the wave energy. For irregular wind waves, the (random) variance of the sea surface is described using the surface elevation variance density spectra (in the wave modeling community usually denoted as energy spectra). The variance spectrum F is a function of all independent phase parameters, i.e., $F(\mathbf{k}, \sigma, \omega)$, and furthermore varies in space and time at scales larger than those of individual waves, e.g., $F(\mathbf{k}, \sigma, \omega; \mathbf{x}, t)$. However, it is usually assumed that the individual spectral components satisfy the linear wave theory (locally), so that Eqs. (2.1) and (2.2) interrelate $\mathbf{k}$, σ and ω. Consequently only two independent phase parameters exist, and the local and instantaneous spectrum becomes two-dimensional. Within WAVEWATCH III the basic spectrum is the wavenumber-direction spectrum $F(k, \theta)$, which has been selected because of its invariance characteristics with respect to physics of wave growth and decay for variable water depths. The output of WAVEWATCH III, however, consists of the more traditional frequency-direction spectrum $F(f_r, \theta)$. The different spectra can be calculated from $F(k, \theta)$ using straightforward Jacobian transformations

$$F(f_r, \theta) = \frac{\partial k}{\partial f_r} F(k, \theta) = \frac{2\pi}{c_g} F(k, \theta) \, , \tag{2.4}$$

$$F(f_a, \theta) = \frac{\partial k}{\partial f_a} F(k, \theta) = \frac{2\pi}{c_g} \left(1 + \frac{\mathbf{k} \cdot \mathbf{U}}{k c_g} \right)^{-1} F(k, \theta) \, , \tag{2.5}$$

$$c_g = \frac{\partial \sigma}{\partial k} = n \frac{\sigma}{k} \, , \; n = \frac{1}{2} + \frac{kd}{\sinh 2kd} \, , \tag{2.6}$$

where c_g is the so-called group velocity. From any of these spectra one-dimensional spectra can be generated by integration over directions, whereas integration over the entire spectrum by definition gives the total variance E (in the wave modeling community usually denoted as the wave energy).

In cases without currents, the variance (energy) of a wave package is a conserved quantity. In cases with currents the energy or variance of a spectral component is no longer conserved, due to the work done by current on the mean momentum transfer of waves (Longuet-Higgins and Stewart, 1961, 1962). In a general sense, however, wave action $A \equiv E/\sigma$ is conserved (e.g., Whitham, 1965; Bretherthon and Garrett, 1968). This makes the wave action density spectrum $N(k, \theta) \equiv F(k, \theta)/\sigma$ the spectrum of choice within the model. Wave propagation then is described by

$$\frac{DN}{Dt} = \frac{S}{\sigma} \, , \tag{2.7}$$

where D/Dt represents the total derivative (moving with a wave component) and S represents the net effect of sources and sinks for the spectrum F. Because the left side of Eq. (2.7) generally considers linear propagation without scattering, effects of nonlinear wave propagation (i.e., wave-wave interactions) and partial wave reflections arise in S. Propagation and source terms will be discussed separately in the following sections.

2.2 Propagation

In a numerical model, a Eulerian form of the balance equation (2.7) is needed. This balance equation can either be written in the form of a transport equation (with velocities outside the derivatives), or in a conservation form (with velocities inside the derivatives). The former form is valid for the vector wavenumber spectrum $N(\mathbf{k}; \mathbf{x}, t)$ only, whereas valid equations of the latter form can be derived for arbitrary spectral formulations, as long as the corresponding Jacobian transformation as described above is well behaved (e.g., Tolman and Booij, 1998). Furthermore, the conservation equation conserves total wave energy/action, unlike the transport equation. This is an important feature of an equation when applied in a numerical model. The balance equation for the spectrum $N(k, \theta; \mathbf{x}, t)$ as used in WAVEWATCH III is given as (for convenience of notation, the spectrum is henceforth denoted simply as N):

$$\frac{\partial N}{\partial t} + \nabla_x \cdot \dot{\mathbf{x}} N + \frac{\partial}{\partial k} \dot{k} N + \frac{\partial}{\partial \theta} \dot{\theta} N = \frac{S}{\sigma} \, , \tag{2.8}$$

$$\dot{\mathbf{x}} = \mathbf{c}_g + \mathbf{U} \, , \tag{2.9}$$

$$\dot{k} = -\frac{\partial \sigma}{\partial d} \frac{\partial d}{\partial s} - \mathbf{k} \cdot \frac{\partial \mathbf{U}}{\partial s} \, , \tag{2.10}$$

$$\dot{\theta} = -\frac{1}{k}\left[\frac{\partial \sigma}{\partial d}\frac{\partial d}{\partial m} + \mathbf{k} \cdot \frac{\partial \mathbf{U}}{\partial m}\right] , \tag{2.11}$$

where $\mathbf{c}_g = (c_g \sin\theta, c_g \cos\theta,$ s is a coordinate in the direction θ and m is a coordinate perpendicular to s. Equation (2.8) is valid for Cartesian coordinates. For large-scale applications, this equation is usually transferred to spherical coordinates, defined by longitude λ and latitude ϕ, but maintaining the definition of the local variance (i.e., per unit surface, as in WAMDIG, 1988)

$$\frac{\partial N}{\partial t} + \frac{1}{\cos\phi}\frac{\partial}{\partial \phi}\dot{\phi}N\cos\theta + \frac{\partial}{\partial \lambda}\dot{\lambda}N + \frac{\partial}{\partial k}\dot{k}N + \frac{\partial}{\partial \theta}\dot{\theta}_g N = \frac{S}{\sigma} , \tag{2.12}$$

$$\dot{\phi} = \frac{c_g \cos\theta + U_\phi}{R} , \tag{2.13}$$

$$\dot{\lambda} = \frac{c_g \sin\theta + U_\lambda}{R\cos\phi} , \tag{2.14}$$

$$\dot{\theta}_g = \dot{\theta} - \frac{c_g \tan\phi\cos\theta}{R} , \tag{2.15}$$

where R is the radius of the earth and U_ϕ and U_λ are current components. Equation (2.15) includes a correction term for propagation along great circles, using a Cartesian definition of θ where $\theta = 0$ corresponds to waves traveling from west to east. WAVEWATCH III can be run using either Cartesian or Spherical coordinates. Note that unresolved obstacles such as islands can be included in the equations. In WAVEWATCH III this is done at the level of the numerical scheme, as is discussed in section 3.4.7. Also, depth variations at the scale of the wavelength can be introduced by a scattering source term described in section 2.3.19.

Finally, both Cartesian and spherical coordinates can be discretized in many ways, using quadrangles (rectangular, curvilinear or SMC grids) and triangles. That aspect is treated in chapter 3.

2.3 Source terms

2.3.1 General concepts

In deep water, the net source term S is generally considered to consist of three parts, an atmosphere-wave interaction term S_{in}, which is usually a positive energy input but can also be negative in the case of swell, a nonlinear wave-wave interactions term S_{nl} and a wave-ocean interaction term that generally contains the dissipation S_{ds}. The input term S_{in} is dominated by the exponential wind-wave growth term, and this source term generally describes this dominant process only. For model initialization, and to provide more realistic initial wave growth, a linear input term S_{ln} can also be added in WAVEWATCH III.

In shallow water additional processes have to be considered, most notably wave-bottom interactions S_{bot} (e.g., Shemdin et al., 1978). In extremely shallow water, depth-induced breaking (S_{db}) and triad wave-wave interactions (S_{tr}) also become important. Also available in WAVEWATCH III are source terms for scattering of waves by bottom features (S_{sc}), wave-ice interactions (S_{ice}), reflection off shorelines or floating objects such as icebergs (S_{ref}), which can include sources of infragravity wave energy, and a general purpose slot for additional, user defined source terms (S_{xx}).

This defines the general source terms used in WAVEWATCH III as

$$S = S_{ln} + S_{in} + S_{nl} + S_{ds} + S_{bot} + S_{db} + S_{tr} + S_{sc} + S_{ice} + S_{ref} + S_{xx} \,. \quad (2.16)$$

Other source terms could be easily added. Those source terms are defined for the *energy* spectra. In the model, however, most source terms are directly calculated for the action spectrum. The latter source terms are denoted as $\mathcal{S} \equiv S/\sigma$.

The explicit treatment of the nonlinear interactions defines third-generation wave models. Therefore, the options for the calculation of S_{nl} will be discussed first, starting in section 2.3.2. S_{in} and S_{ds} represent separate processes, but are often interrelated, because the balance of these two source terms governs the integral growth characteristics of the wave energy. Several combinations of these basic source terms are available, and are described in section 2.3.7 and following. The description of linear input starts in section 2.3.12, and section 2.3.13 and following describe available additional processes, mostly related to shallow water and sea ice.

A third-generation wave model effectively integrates the spectrum only up to a cut-off frequency f_{hf} (or wavenumber k_{hf}), that is ideally equal to the highest discretization frequency. In practice the source terms parameterization or the time step used may not allow a proper balance to be obtained, and thus f_{hf} may be taken within the model frequency range. Above the cut-off frequency a parametric tail is applied (e.g., WAMDIG, 1988)

$$F(f_r, \theta) = F(f_{r,hf}, \theta) \left(\frac{f_r}{f_{r,hf}} \right)^{-m} , \tag{2.17}$$

which is easily transformed to any other spectrum using the Jacobian transformations as discussed above. For instance, for the present action spectrum, the parametric tail can be expressed as (assuming deep water for the wave components in the tail)

$$N(k, \theta) = N(k_{hf}, \theta) \left(\frac{f_r}{f_{r,hf}} \right)^{-m-2} , \tag{2.18}$$

the actual values of m and the expressions for $f_{r,hf}$ depend on the source term parameterization used, and will be given below.

Before actual source term parameterizations are described, the definition of the wind requires some attention. In cases with currents, one can either consider the wind to be defined in a fixed frame of reference, or in a frame of reference moving with the current. Both definitions are available in WAVEWATCH III, and can be selected during compilation. The output of the program, however, will always be the wind speed which is not in any way corrected for the current.

The treatment of partial ice coverage (ice concentration) in the source terms follows the concept of a limited air-sea interface. This means that the momentum transferred from the atmosphere to the waves is limited. Therefore, input and dissipation terms are scaled by the fraction of ice concentration. The nonlinear wave-wave interaction term can be used in areas of open water and ice (Polnikov and Lavrenov, 2007). The scaling is implemented so that it is independent of the source term selected.

2.3.2 S_{nl}: Discrete Interaction Approximation (DIA)

Switch:	`NL1`
Origination:	WAM model
Provided by:	H. L. Tolman

Nonlinear wave-wave interactions can be modeled using the discrete interaction approximation (DIA, Hasselmann et al., 1985). This parameterization