On-the-fly steady-state detection for $\mathrm{P}(s,\ \Phi\ \mathrm{U}^{[0,t]}\ \Psi)$, of CSL logic.

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Outline

DTMC & CTMC

Definition

A labeled *DTMC* is a triple $(S, \mathcal{P})$ with:

- S as a finite set of states
- $\mathcal{P} : S \times S \rightarrow [0, 1]$, $\mathcal{P} = (p_{i,j})$ and $\forall i \in S : \sum_{j=1}^{|S|} p_{i,j} = 1$, as a stochastic mtx.
- $L : S \rightarrow 2^{AP}$ as a labeling function

Definition

A labeled *CTMC* is a triple $(S, \mathcal{Q})$ with:

- S as a finite set of states
- $\mathcal{Q} : S \times S \rightarrow \mathcal{R}_{\geq 0}$, $\mathcal{Q} = (q_{i,j})$, as a generator mtx.
- $L : S \rightarrow 2^{AP}$ as a labeling function

Stationary probabilities, DTMC (?)

Definition

The *stationary* or *steady-state* distribution of the DTMC $\mathcal{P}$ is a vector $\overrightarrow{p}$ such that:

$$\overrightarrow{p^*} = lim_{i\to\infty}\overrightarrow{p\,(0)}\mathcal{P}^i \tag{1}$$

where $\overrightarrow{p\,(0)}$ is the initial distribution

Theorem

In an irreducible and aperiodic DTMC, with positive recurrent states, the unique limiting distribution (1) exists and does not depend on the initial distribution $\overrightarrow{p\,(0)}$.

Power Iterations (?)

Definition

For a stochastic matrix $\mathcal{P}$:

$$\overrightarrow{p(i)} = \overrightarrow{p\,(0)}\mathcal{P}^{i},\ \overrightarrow{p\,(0)}\text{ - initial vector}$$

Theorem

If $\mathcal{P}$ is aperiodic and irreducible then the Power Method is guaranteed to converge to some $\overrightarrow{p^{*}}$.

Lemma

For a stochastic matrix $\mathcal{P}$, the number of iterations K needed to satisfy a tolerance criterion ϵ may be approximated by:

$$K = \frac{\log \epsilon}{log|\lambda_2|},\ \lambda_2\text{ - }subd.\ e.v.\ of\ \mathcal{P}$$

Convergence (?)

Tests for convergence

- Absolute: $\|\overrightarrow{p(0,i)} - \overrightarrow{p(0,i+M)}\|_v^\infty \leq \epsilon$, for some M.
- Relative: $\max_j \left(\frac{|\overrightarrow{p(s,i+M)}_j - \overrightarrow{p(s,i)}_j|}{|\overrightarrow{p(s,i+M)}_j|} \right) < \epsilon$
- ...

Warning!

It is best to envisage a battery of convergence tests, all of which must be satisfied before the approximation is accepted as being sufficiently accurate.

Outline

CSL logic (?)

The syntax of CSL logic

$$SF ::= tt|a \in \mathcal{AP}|\neg SF|SF \wedge SF|\mathrm{S}_{\trianglelefteq p}(s,\ SF)|\mathrm{P}_{\trianglelefteq p}(s,\ PF)$$

$$PF ::== \mathrm{X}^{[0,\ t]}\ SF|SF\ \mathrm{U}^{[0,t]}\ SF$$

Where

$\mathcal{AP}$	atomic propositions	s	initial state
SF	state formula	p	probability bound
PF	path formula	t	time bound
$\trianglelefteq$	$\in \{\leq,\ \geq\}$	$Sat\,(SF)$	states satisfying SF

Forward computations (?)

Reduce to transient analysis

Compute $\mathrm{P}(s,\ \Phi\ \mathrm{U}^{[0,t]}\ \Psi)$:

1. Obtain a generator matrix $\mathcal{Q}\left[\neg\Phi \vee \Psi\right]$
2. Compute the transient probabilities vector

$$\overrightarrow{\pi^*\left(s,t\right)} = \overrightarrow{1_s} \cdot e^{\mathcal{Q}[\neg\Phi\vee\Psi]t} \tag{2}$$

3. Compute $\mathrm{P}(s,\ \Phi\ \mathrm{U}^{[0,t]}\ \Psi) = \sum_{j\in Sat(\Psi)} \pi^*\left(s,t\right)_j$

Where

$\overrightarrow{1_s}$ is the initial distribution for the case when we start in the s state.

Backward computations (?)

Change

Instead of computing (2) compute:

$$\overrightarrow{\pi^*(t)} = e^{\mathcal{Q}[\neg\Phi\vee\Psi]t} \cdot \overrightarrow{i_\Psi} \tag{3}$$

Advantages

- $\forall j \in 1,..,N\ :\ \pi^*(t)_j = \mathrm{P}(j,\ \Phi\ \mathrm{U}^{[0,t]}\ \Psi)$
- Better time complexity

Where

$\overrightarrow{i_\Psi}$ is a vector that contains values 1 in places corresponding to $Sat(\Psi)$ states and 0 in others.

Outline

Numerical computations

The Jensen's method

Both (2) and (3) can be computed numerically using the Jensen's method (**?**), also known as Uniformization (**?**).

Uniformization

Rewriting $\vec{p}\mathcal{Q} = \vec{0}$ into

$$\vec{p}\mathcal{P} = \vec{p},\ \mathcal{P} = \frac{\mathcal{Q}}{q} + \mathcal{I}$$

where $q \geq \max_i \sum_{j=1}^{N} q_{i,j}$.

Notice!

Taking

$$q > \max_i \sum_{j=1}^{N} q_{i,j}$$

makes DTMC $\mathcal{P}$ aperiodic

Using uniformization

The Jensen's method

Substitute $\mathcal{Q}$ with $\mathcal{P}$ in tailored representation (4) of equation (2)

$$\overrightarrow{\pi^*(s,t)} = \sum_{i=0}^{\infty} \frac{t^i}{i!} \vec{1_s} \cdot \mathcal{Q}^i \qquad (4)$$

$$\overrightarrow{\pi^*(s,t)} = \sum_{i=0}^{\infty} e^{-qt} \frac{(qt)^i}{i!} \overrightarrow{p(s,i)} \qquad (5)$$

Here we assume $\mathcal{Q} = \mathcal{Q}[\neg\Phi \vee \Psi]$ and $\overrightarrow{p(s,i)} = \vec{1_s} \cdot \mathcal{P}^i$.

Note!

The sum in (5) can be computed using the Fox-Glynn algorithm (**?**).

The Fox-Glynn algorithm (?)

Lemma

Let f be a real-valued function with $\|f\| = \sup_{i \in 0,..,\infty} f(i)$ and $\sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} \gamma(i) \geq 1 - \frac{\epsilon}{2}$. In exact arithmetic,

$$\left|\sum_{i=0}^{\infty} \gamma(i) f(i) - \frac{1}{W} \sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} w(i) f(i)\right| \leq \epsilon \|f\|$$

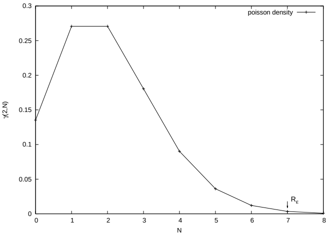
Where

$\gamma(i) = e^{-qt} \frac{(qt)^i}{i!}$

$\alpha \neq 0$, some constant

$w(i) = \alpha\gamma(i)$

$W = w(\mathcal{L}_\epsilon) + \ldots + w(\mathcal{R}_\epsilon)$



Steady-state, backward computations

Steady-state (?)

$\vec{1_s} \cdot \mathcal{P}^i$ in (5) is the power iteration for uniformized DTMC $\mathcal{P}$ and that is where the steady-state detection comes into play.

Backward computations

Notice that all above is applicable to equation (3), see (?), which gives us

$$\overrightarrow{\pi^*(t)} = \sum_{i=0}^{\infty} e^{-qt} \frac{(qt)^i}{i!} \overrightarrow{p(i)}$$

where $\overrightarrow{p(i)} = \mathcal{P}^i \cdot \vec{i_\Psi}$.

Outline

Forward computation, algorithm (?)

Computations with Steady-state detection

1. $(K > \mathcal{R}_\epsilon)$: Steady-state detection has no effect

$$\overrightarrow{\pi(s,t)} = \sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} e^{-qt}\frac{(qt)^i}{i!}\overrightarrow{p(s,i)}$$

2. $(\mathcal{L}_\epsilon \leq K \leq \mathcal{R}_\epsilon)$:

$$\overrightarrow{\pi(s,t)} = \sum_{i=\mathcal{L}_\epsilon}^{K} e^{-qt}\frac{(qt)^i}{i!}\overrightarrow{p(s,i)} + \overrightarrow{p(s,K)}\left(1 - \sum_{i=0}^{K} e^{-qt}\frac{(qt)^i}{i!}\right) \tag{6}$$

3. $(K < \mathcal{L}_\epsilon)$: Take $\overrightarrow{\pi(s,t)} = \overrightarrow{p(s,K)}$

Criteria and problems

Steady-state detection criteria

- If $\|\overrightarrow{p(s,i)} - \overrightarrow{p(s,i+M)}\|_v \leq \frac{\epsilon}{4}$, then $K = i + M$
- Check for K every M iterations

Problems

1. *The steady-state detection is uncertain* - due to the criteria
2. *The error bound is not precise* - as derived under an assumption of knowing *real* steady-state.
3. *The norm $\|.\|_v$ is not defined* - was assumed that $\|\overrightarrow{p(s,i)}\|_v \leq 1$
4. *The weights are not considered* - if the complete Fox-Glynn algorithm is used.

Backward computations, algorithm (?)

Computations with Steady-state detection

1. $(K > \mathcal{R}_\epsilon)$:

$$\ldots \quad \ldots \quad \ldots$$

2. $(\mathcal{L}_\epsilon \leq K \leq \mathcal{R}_\epsilon)$:

$$\overrightarrow{\pi(t)} = \sum_{i=\mathcal{L}_\epsilon}^{K} e^{-qt} \frac{(qt)^i}{i!} \overrightarrow{p(i)} + \overrightarrow{p(K)} \left(1 - \sum_{i=\mathcal{L}_\epsilon}^{K} e^{-qt} \frac{(qt)^i}{i!} \right) \quad (7)$$

3. $(K < \mathcal{L}_\epsilon)$:

$$\ldots \quad \ldots \quad \ldots$$

Criteria and problems

Steady-state detection criteria

- If $\|\overrightarrow{p(i)} - \overrightarrow{p(i+M)}\|_v \leq \frac{\epsilon}{8}$, then $K = i + M$
- Check for K every M iterations

Problems

1. *The error bound for (7) is not correct* - $\forall j \in 1,..,N, \forall i \in 0,..,\infty\ :\ p(i)_j \leq p(i+1)_j$.
2. *An additional error is introduced* - while switching from (6) to (7), $i = 0$ became $i = \mathcal{L}_\epsilon$.
3. *The refinement, done in (**?**), for (6) and (7) is incorrect* - the length of the error interval does not matter.

Outline

Choosing the "right" norm

Facts

- The error estimate, is based on *Geometrical Convergence*
- *G.C.* is proved, using *total variation norm* which, in an N-dimensional space, $\|v\|_v^\infty = \max_{i \in 1,..,N} |v_i|$.
- In a finite dimensional space all norms are equivalent.

Caution!

The $\overrightarrow{i_\Psi}$ vector is not a distribution, $\forall j \in 1,..,N\ :\ 0 \leq p(i)_j \leq 1$.

Example

For N states and Euclidean Norm $\|.\|_v^2$:

$$\|\sum_{i=0}^{\mathcal{L}_\epsilon - 1} \gamma(i)\overrightarrow{p(i)}\|_v^2 \not\leq \frac{\epsilon}{4} \text{ BUT } \|\sum_{i=0}^{\mathcal{L}_\epsilon - 1} \gamma(i)\overrightarrow{p(i)}\|_v^2 \leq \frac{\sqrt{N}}{4}\epsilon$$

The Fox-Glynn error bound refinement

Errors

- $\mathcal{L}_\epsilon$ and $\mathcal{R}_\epsilon$, each, give error $\frac{\epsilon}{4}$
- Normalization $\frac{w(i)}{W}$ gives additional $\frac{\epsilon}{2}$

Lemma

Let f be a real-valued function with $\|f\| = \sup_{i\in 1,..,\infty} f(j)$ and $\sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} \gamma(i) \geq 1 - \frac{\epsilon}{2}$. In exact arithmetic,

$$|\sum_{i=0}^{\infty} \gamma(i) f(i) - \frac{1}{W} \sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} w(i) f(i)| \leq \frac{\epsilon}{2} \|f\|$$

Proof details

Refined steady-state detection error

Forward computations Criterion

1. Steady-state is detected if $\|\overrightarrow{p^*(s)} - \overrightarrow{p(s,K)}\|_v^\infty \leq \frac{\epsilon}{8|Sat(\Psi)|}$
2. Use the Fox-Glynn algorithm with desired error $\frac{\epsilon}{2}$
3. Then the overall error bound for the computed probability $\mathrm{P}(s,\ \Phi\ \mathrm{U}^{[0,t]}\ \Psi)$ will be ϵ

Error bound details

Backward computations Criterion

1. Steady-state is detected if $\|\overrightarrow{p^*} - \overrightarrow{p(K)}\|_v^\infty \leq \frac{\epsilon}{4}$
2. Use the Fox-Glynn algorithm with desired error $\frac{\epsilon}{2}$
3. Then $\forall j \in 1,..,N$ the overall error bound for computed probability $\mathrm{P}(j,\ \Phi\ \mathrm{U}^{[0,t]}\ \Psi)$, will be ϵ

Error bound details

Outline

Making states absorbing I

Definition

For a directed graph, a subgraph is a *Bottom Strongly Connected Component* (BSCC), if it is a maximum strongly connected component such that it has no outgoing edges.

Lemma

If $\mathcal{Q}\,[\neg\Phi \vee \Psi]$ has a BSCC containing at least one Φ state then all its states are Φ states.

Proof details

Where

$\mathcal{Q}$ is a generator matrix, Φ and Ψ are CSL state formulas

Making states absorbing II

Definition

Define $B_\Psi = \{s \in B | (B \textit{ is a BSCC in } \mathcal{Q}[\neg\Phi \vee \Psi]) \wedge (s \in Sat(\Phi) \setminus Sat(\Psi))\}$
Define $\mathcal{Q}^B[\neg\Phi \vee \Psi]$ obtained from $\mathcal{Q}[\neg\Phi \vee \Psi]$ by making all B_Ψ states absorbing.

Definition

Let $\mathrm{P}_\mathcal{Q}(s,\ \Phi)$ be the probability $\mathrm{P}(s,\ \Phi)$ of satisfying CSL state formula Φ in state s, for the CTMC, defined by the generator matrix $\mathcal{Q}$.

Theorem

$\mathrm{P}_\mathcal{Q}(s,\ \Phi\ \mathrm{U}^{[0,t]}\ \Psi) = \mathrm{P}_{\mathcal{Q}[\neg\Phi\vee\Psi]}(s,\ tt\ \mathrm{U}^{[t,t]}\ \Psi) = \mathrm{P}_{\mathcal{Q}^B[\neg\Phi\vee\Psi]}(s,\ tt\ \mathrm{U}^{[t,t]}\ \Psi)$

▸ Proof details

Precise steady-state detection, Forward computations

Theorem

For a uniformized CTMC $\mathcal{P}_B$, obtained from the generator matrix $\mathcal{Q}^B\left[\neg\Phi \vee \Psi\right]$:

$$\forall\delta \geq 0\ :\ \sum_{j \in Sat(\Phi)\setminus(\mathrm{B}_\Psi \cup Sat(\Psi))} p(s,i)_j \leq \delta \Rightarrow \|\overrightarrow{p^*(s)} - \overrightarrow{p(s,i)}\|_v^\infty \leq \delta$$

Where

$\overrightarrow{p^*(s)}$ - the steady-state of $\mathcal{P}_B$ when starting from s

$p(s,i)_j$ - the j'th component of $\overrightarrow{p(s,i)} = \vec{1}_s \cdot \mathcal{P}_B^i$

Precise steady-state detection, Backward computations

Theorem

For a uniformized CTMC $\mathcal{P}_B$, obtained from the generator matrix $\mathcal{Q}^B\left[\neg\Phi \vee \Psi\right]$:

$$\forall \delta \geq 0\ :\ \|\vec{1} - \left(\overrightarrow{p(i)} + \overrightarrow{p^B(i)}\right)\|_v^\infty \leq \delta \Rightarrow \|\overrightarrow{p^*} - \overrightarrow{p(i)}\|_v^\infty \leq \delta$$

Where

$$\overrightarrow{p^B(i)} = \mathcal{P}_B^i \cdot \overrightarrow{i_{\mathrm{B}_\Psi \cup Sat(\neg\Phi)}}$$

$$\overrightarrow{p^*} = \lim_{i\to\infty} \mathcal{P}_B^i \cdot \overrightarrow{i_\Psi}$$

$$\overrightarrow{p(i)} = \mathcal{P}_B^i \cdot \overrightarrow{i_\Psi}$$

Outline

Premature steady-state detection (?)

Tools

Tool Name	Reference	S.s.d. method
Prism v2.1	(?)	*regular*
ETMCC v1.4.2	(?)	*regular*
MRMC v1.0	(?)	*precise*

Example

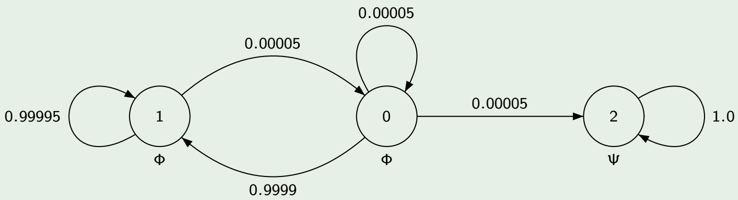
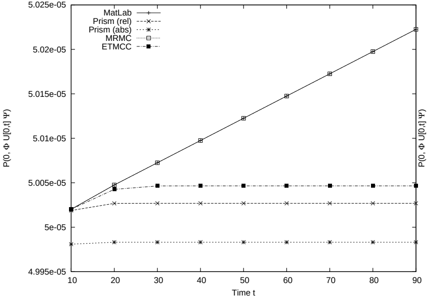


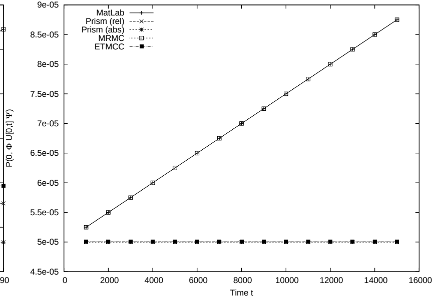
Figure: A slowly convergent CTMC

Computational results

Example

Tool	Error	K	$\mathcal{P}^K \cdot \overrightarrow{i_\Psi}$	$\overrightarrow{p^*}$
Prism v2.1(abs)	10^{-6}	2	$(5.00025 \cdot 10^{-5}, 2.5 \cdot 10^{-9}, 1.0)$	$(1.0, 1.0, 1.0)$
Prism v2.1(rel)	10^{-1}	12	$(5.00275 \cdot 10^{-5}, 2.75 \cdot 10^{-8}, 1.0)$	
ETMCC v1.4.2	10^{-6}	20	$(5.00475 \cdot 10^{-5}, 4.75 \cdot 10^{-8}, 1.0)$	
MRMC v1.0	10^{-6}	—	—	





Outline

Related, ...

..., but unused

(?) - the method, based on Uniformization, to determine the *point availability* and *expected interval availability* of a repairable computer system modeled as a Markov chain with steady-state detection.
Limitation: Results are only applicable for irreducible Markov Chains

(?) - *Phase Type* distribution, has a theorem limiting the time before absorption.
Limitation: Transient states must form an irreducible matrix

Outline

Conclusions

Results

1. The error bound corrections
   - Steady-state detection - fixed multiple problems
   - The Fox-Glynn algorithm - refined original error
   - Uniformization using the Fox-Glynn - added weights influence
2. Precise steady-state detection criteria
   - Forward computations - preserves time complexity, computation time may slightly increase
   - Backward computations - preserves time complexity, computation time may approximately double

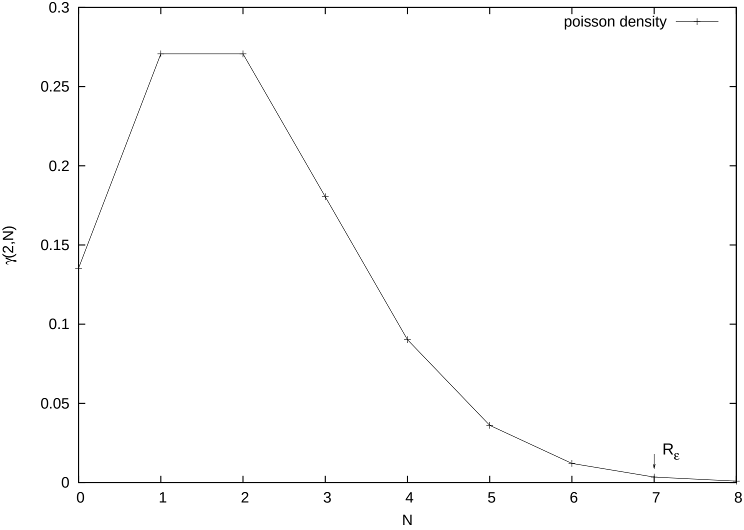
Outline

Appendix Outline

11 Appendix

- Numerical computation of $\mathrm{P}(s,\ \Phi\ \mathrm{U}^{[0,t]}\ \Psi)$
- The Fox-Glynn, steady-state detection, error bound
- Improved steady-state detection
- An example

The Fox-Glynn algorithm (?)



The Fox-Glynn error bound refinement

Proof.

Due to the facts that

$$0 \leq \sum_{i=0}^{\mathcal{L}_\epsilon - 1} \gamma(i) + \sum_{i=\mathcal{R}_\epsilon+1}^{\infty} \gamma(i) \leq \frac{\epsilon}{2}$$

$$-\frac{\epsilon}{2} \leq \sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} (\gamma(i) - \frac{w(i)}{W}) \leq 0$$

$\square$

Return

Refined steady-state detection error

Forward computations Details

Assuming $\forall i \geq K\ :\ \|\overrightarrow{p^*(s)} - \overrightarrow{p(s,i)}\|_v^\infty \leq \delta$ and the Fox-Glynn algorithm's error bound $\frac{\epsilon}{2}$ we have:

1. $(K > \mathcal{R}_\epsilon)$:
$$-\frac{\epsilon}{2} \leq \sum_{j \in Sat(\Psi)} \left(\pi^*(s,t)_j - \pi(s,t)_j\right) \leq \frac{\epsilon}{2}$$
2. $(\mathcal{L}_\epsilon \leq K \leq \mathcal{R}_\epsilon)$:
$$-2\delta|Sat(\Psi)| - \frac{3}{4}\epsilon \leq \sum_{j \in Sat(\Psi)} \left(\pi^*(s,t)_j - \pi(s,t)_j\right) \leq 2\delta|Sat(\Psi)| + \frac{3}{4}\epsilon$$
3. $(K < \mathcal{L}_\epsilon)$:
$$-2\delta|Sat(\Psi)| - \frac{1}{4}\epsilon \leq \sum_{j \in Sat(\Psi)} \left(\pi^*(s,t)_j - \pi(s,t)_j\right) \leq 2\delta|Sat(\Psi)| + \frac{1}{4}\epsilon$$

Return

Refined steady-state detection error

Backward computations Details

Assuming $\forall i \geq K\ :\ \|\overrightarrow{p^*} - \overrightarrow{p(i)}\|_v^\infty \leq \delta$ and the Fox-Glynn algorithm's error bound $\frac{\epsilon}{2}$ we have:

1. $(K > \mathcal{R}_\epsilon)$:
$$-\frac{\epsilon}{2} \leq \pi^*(t)_j - \pi(t)_j \leq \frac{\epsilon}{2}$$
2. $(\mathcal{L}_\epsilon \leq K \leq \mathcal{R}_\epsilon)$:
$$-\delta - \frac{3}{4}\epsilon \leq \pi^*(t)_j - \pi(t)_j \leq \delta + \frac{3}{4}\epsilon$$
3. $(K < \mathcal{L}_\epsilon)$:
$$-\delta - \frac{1}{4}\epsilon \leq \pi^*(t)_j - \pi(t)_j \leq \delta + \frac{1}{4}\epsilon$$

Return

Making states absorbing I

Proof.

The case of a single state BSCC is trivial.
The rest is also trivial, by contradiction.
Let B be a BSCC of $\mathcal{Q}[\neg\Phi \vee \Psi]$ such that it has at least two states, $s_\Phi \in Sat\,(\Phi)$, $s_{\neg\Phi} \in Sat\,(\neg\Phi)$ and s_Φ, $s_{\neg\Phi} \in B$. All $\neg\Phi$ states in $\mathcal{Q}[\neg\Phi \vee \Psi]$ are made absorbing, thus the $s_{\neg\Phi}$ state has only one self-loop transition. This yields that $s_{\neg\Phi} \notin B$. Contradiction. □

Return

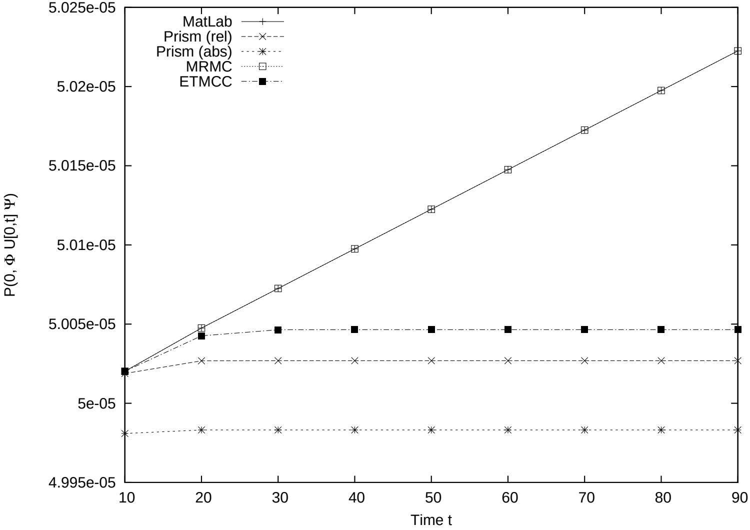
Making states absorbing II

Proof.

The first part $\mathrm{P}_{\mathcal{Q}}(s,\ \Phi\ \mathrm{U}^{[0,t]}\ \Psi) = \mathrm{P}_{\mathcal{Q}[\neg\Phi\vee\Psi]}(s,\ tt\ \mathrm{U}^{[t,t]}\ \Psi)$ was proved in (**?**).
The second part
$\mathrm{P}_{\mathcal{Q}[\neg\Phi\vee\Psi]}(s,\ tt\ \mathrm{U}^{[t,t]}\ \Psi) = \mathrm{P}_{\mathcal{Q}^B[\neg\Phi\vee\Psi]}(s,\ tt\ \mathrm{U}^{[t,t]}\ \Psi)$ is valid due to the fact, that if there is a BSCC consisting of *Sat* (Φ) states then *Sat* (Ψ) states are not reachable from it.
Unless it is a trivial case when a BSCC consists of one state s which satisfies both Φ and Ψ formulas, but in this case it is already made absorbing while obtaining $\mathcal{Q}[\neg\Phi \vee \Psi]$. $\square$

Return

Computational results



Computational results