Safe On-The-Fly Steady-State Detection for Time-Bounded Reachability

Joost-Pieter Katoen[1,2] Ivan S. Zapreev[1,2]

[1]Software Modeling and Verification Group
RWTH Aachen

[2]Formal Methods and Tools Group
University of Twente

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Outline

Motivation

Time-bounded reachability for continuous-time Markov chains

1. Determine the probability to reach a (set of) goal state(s) within a given time span, such that prior to reaching the goal certain states are avoided.
2. Efficient algorithms for time-bounded reachability are at the heart of probabilistic model checkers such as PRISM and ETMCC.
3. For large time spans, on-the-fly steady-state detection is commonly applied.
4. To obtain correct results (up to a given accuracy), it is essential to avoid detecting premature stationarity.

Outline

Transient analysis

Transient probabilities of a CTMC

For a CTMC $(S, \mathcal{Q})$ the state-probability after a delay of t time-units with the initial distribution $\overrightarrow{p(0)}$:

$$\overrightarrow{\pi^*(0, t)} = \overrightarrow{p(0)} \cdot e^{\mathcal{Q} \cdot t}$$

Jensen's method (Uniformization)

- Rewrite $\mathcal{Q} = q \cdot (\mathcal{P}_{unif} - \mathcal{I})$, where $q > \max_{i \in S} |q_{i,i}|$:

$$\overrightarrow{\pi^*(0, t)} = e^{-qt} \cdot \overrightarrow{p(0)} \cdot e^{\mathcal{P}_{unif} \cdot qt}$$

- Rewrite matrix exponent, where $\gamma_i(t) = e^{-qt} \frac{(qt)^i}{i!}$:

$$\overrightarrow{\pi^*(0, t)} = \sum_{i=0}^{\infty} \gamma_i(t) \cdot \overrightarrow{p(0)} \cdot \mathcal{P}^i_{unif} \tag{1}$$

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The Fox-Glynn algorithm (Fox and Glynn, 1988)

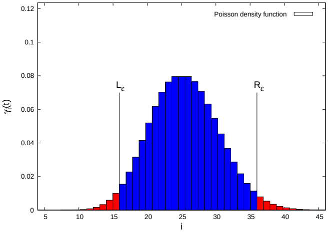
Lemma

For real-valued function f with $\|f\| = \sup_{i\in\mathbb{N}} |f(i)|$ and $\sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} \gamma_i(t) \geq 1 - \frac{\varepsilon}{2}$ it holds:

$$\left| \sum_{i=0}^{\infty} \gamma_i(t) f(i) - \frac{1}{W} \sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} w_i(t) f(i) \right| \leq \varepsilon \cdot \|f\|$$

Where

$\alpha \neq 0$, some constant
$w_i(t) = \alpha\gamma_i(t)$
$W = w(\mathcal{L}_\epsilon) + \ldots + w(\mathcal{R}_\epsilon)$



The Fox-Glynn algorithm (Fox and Glynn, 1988)

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For real-valued function f with $\|f\| = \sup_{i \in \mathbb{N}} |f(i)|$ *and* $\sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} \gamma_i(t) \geq 1 - \frac{\varepsilon}{2}$ *it holds:*

$$\left| \sum_{i=0}^{\infty} \gamma_i(t) f(i) - \frac{1}{W} \sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} w_i(t) f(i) \right| \leq \frac{\varepsilon}{2} \cdot \|f\|$$

if f does not change sign.

Where

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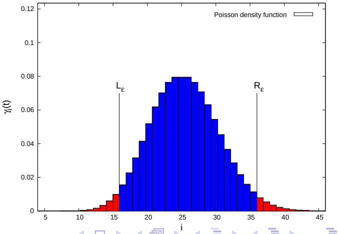
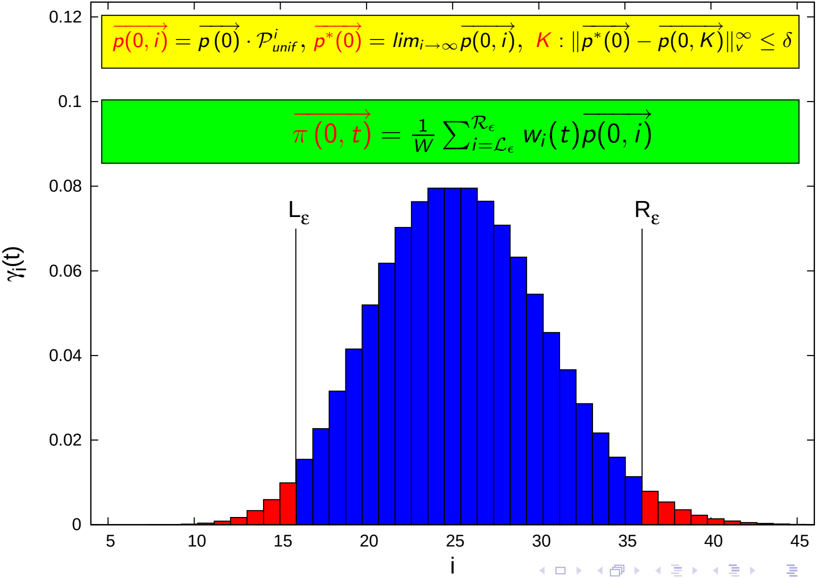


Steady-state detection

$$\overrightarrow{p(0,i)} = \overrightarrow{p(0)} \cdot \mathcal{P}^{i}_{unif}, \; \overrightarrow{p^{*}(0)} = lim_{i\to\infty}\overrightarrow{p(0,i)}, \; K : \|\overrightarrow{p^{*}(0)} - \overrightarrow{p(0,K)}\|^{\infty}_{v} \leq \delta$$

$$\overrightarrow{\pi(0,t)} = \frac{1}{W}\sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} w_i(t)\overrightarrow{p(0,i)}$$



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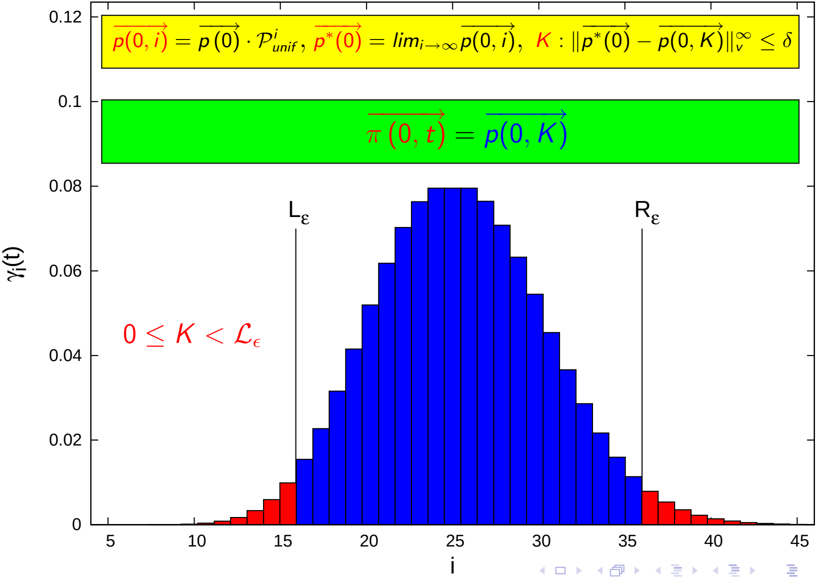

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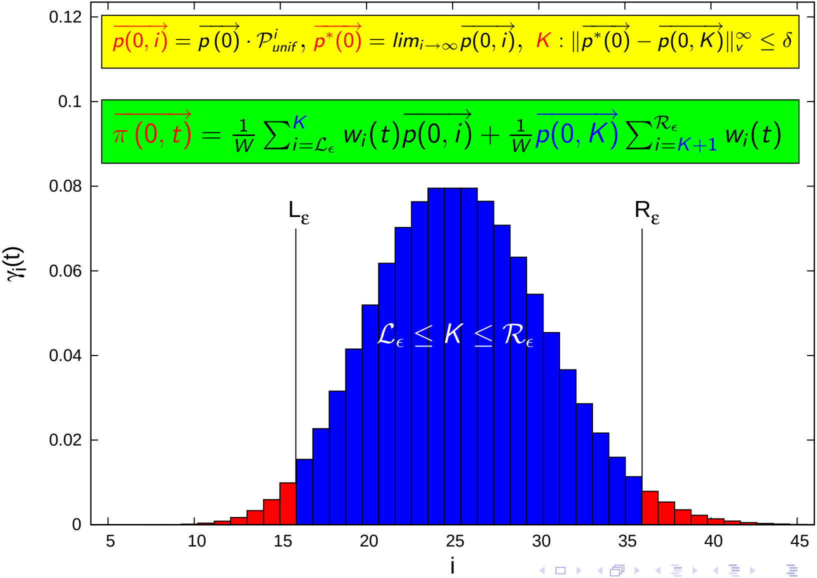

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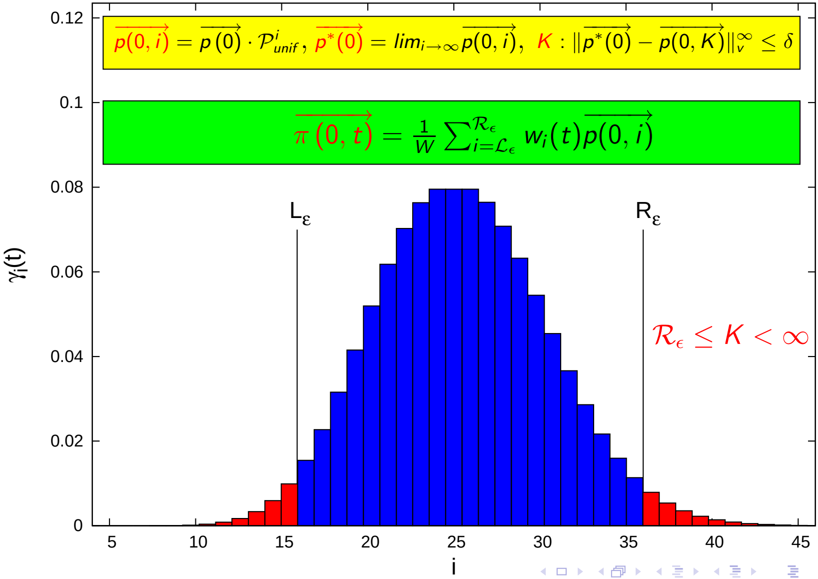


Outline

Time-bounded reachability

Example

Determine states from which goal states may be reached with a probability at least 0.92, within the time interval $[0, 14.5]$, while visiting only allowed states.

$$\mathrm{P}_{\geq 0.92}(\mathcal{A}\ \mathrm{U}^{[0,14.5]}\ \mathcal{G})$$

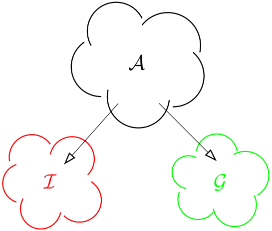
$\mathcal{A}$ - allowed states

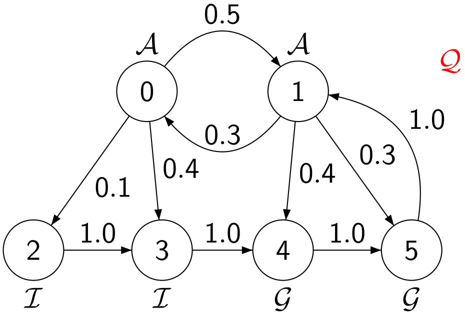
$\mathcal{G}$ - goal states

Definition

For CTMC $(S, \mathcal{Q})$ and $S' \subseteq S$ let CTMC $(S, \mathcal{Q}')$ be obtained by making all states in S' absorbing, i.e., $\mathcal{Q}' = \mathcal{Q}[S']$ where $q'_{i,j} = q_{i,j}$ if $i \notin S'$ and 0 otherwise.

Computing $Prob(s, \mathcal{A}\ \mathrm{U}^{[0,t]}\ \mathcal{G})$





Backward algorithm

1. Determine $\mathcal{Q}[\mathcal{I} \cup \mathcal{G}]$
2. Compute $\overrightarrow{\pi^*(t)} = e^{\mathcal{Q}[\mathcal{I}\cup\mathcal{G}]\cdot t} \cdot \overrightarrow{1_{\mathcal{G}}}$
3. Return $\forall s \in 1, .., N : Prob(s, \mathcal{A}\ \mathrm{U}^{[0,t]}\ \mathcal{G}) = \pi^*(t)_s$

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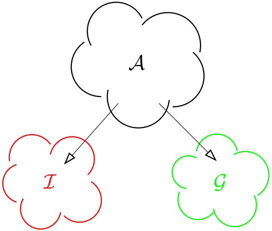

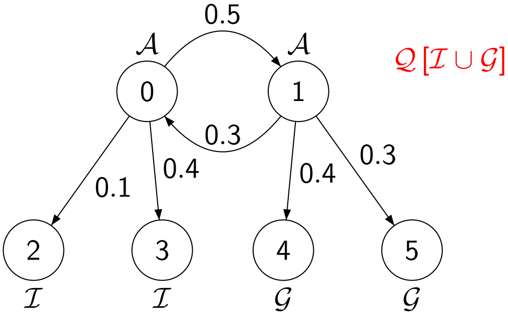

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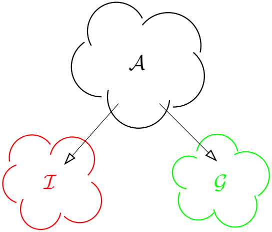

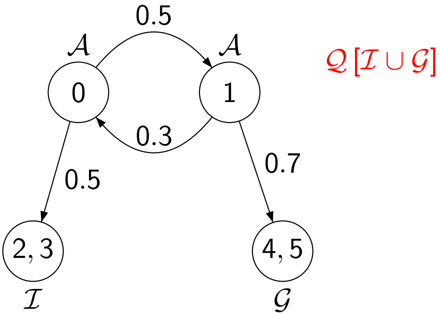

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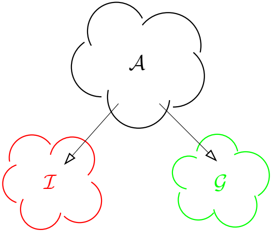

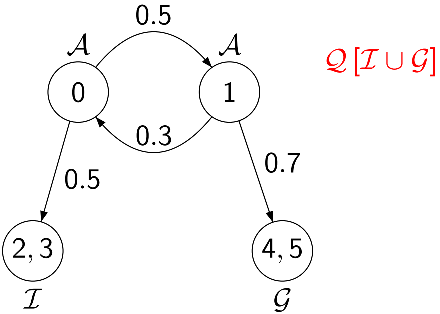

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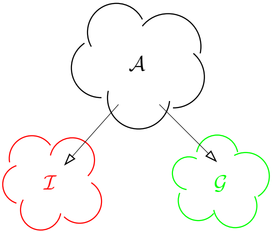

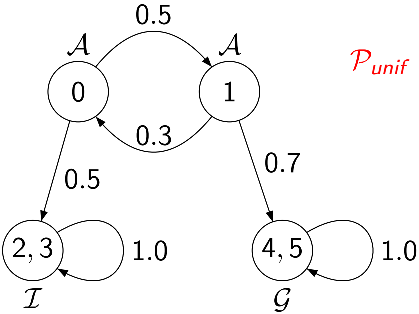

Backward algorithm

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2. Use $\overrightarrow{\pi^*(t)} = \sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} w_i(t) \cdot \mathcal{P}^i_{unif} \cdot \overrightarrow{1_{\mathcal{G}}}$
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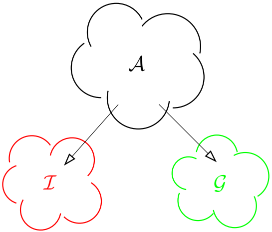

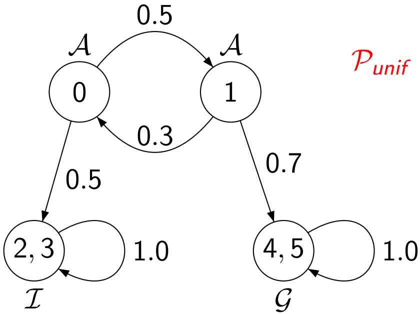

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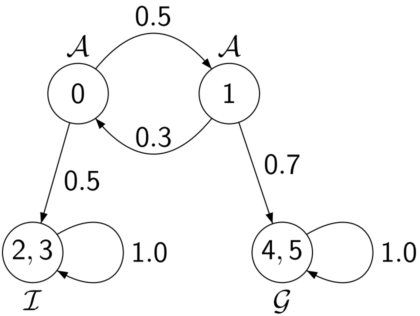
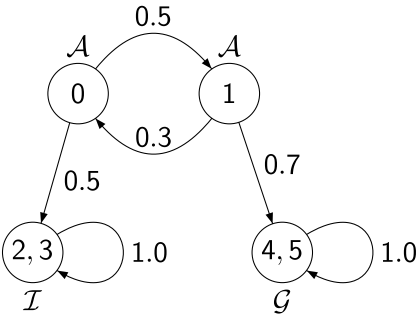
Backward computations



$$\begin{pmatrix} 0.0 & 0.5 & 0.5 & 0.0 \\ 0.3 & 0.0 & 0.0 & 0.7 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix}^{t} \cdot \overrightarrow{1_{\mathcal{G}}} = \begin{pmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 1.0 \end{pmatrix}$$

$t = 0$

Backward computations



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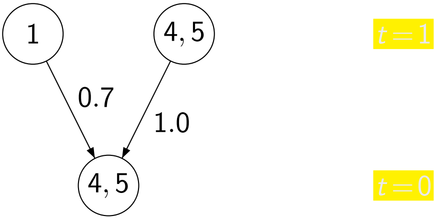

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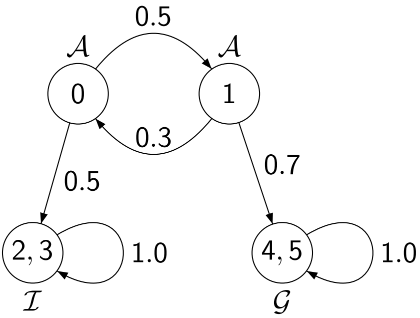

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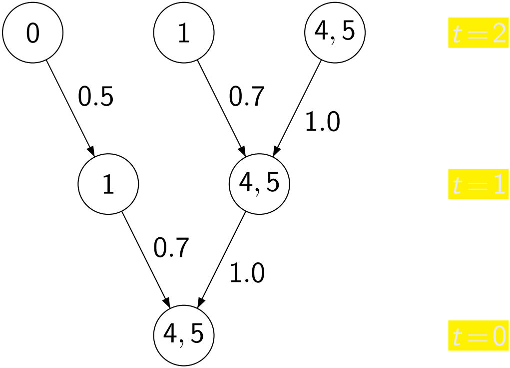

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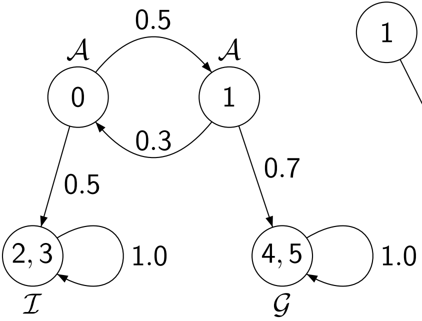

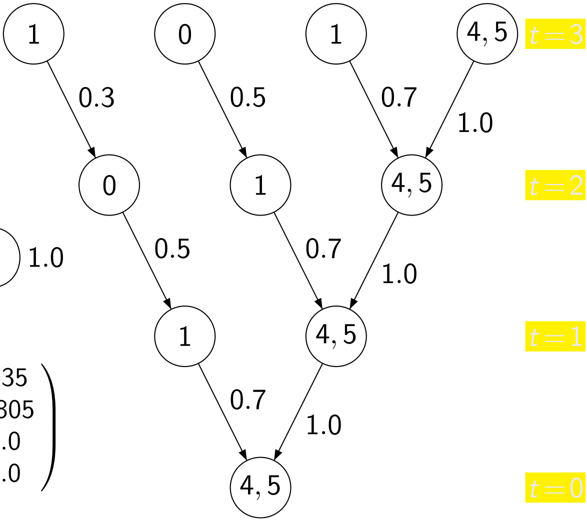

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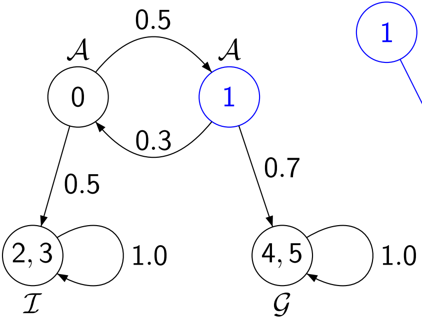

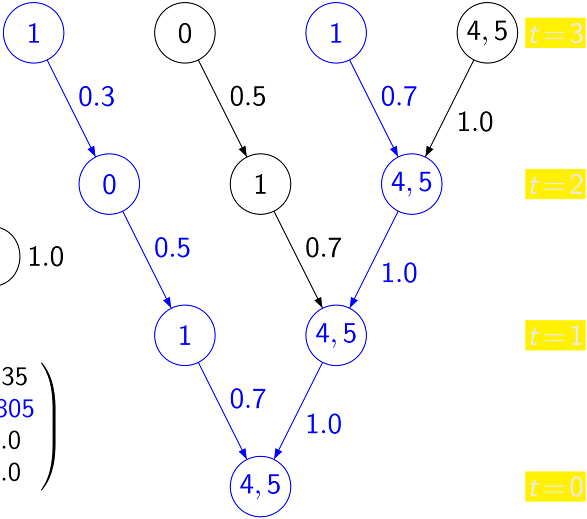

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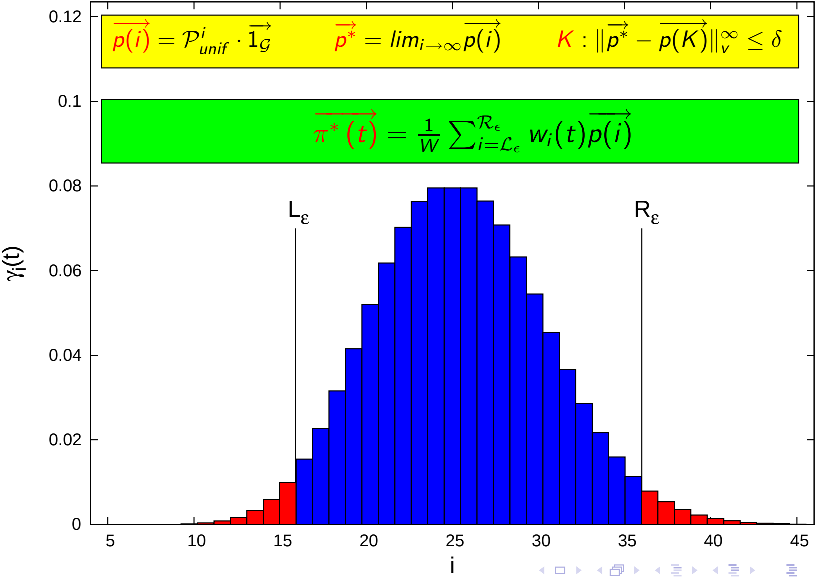
Steady-state detection



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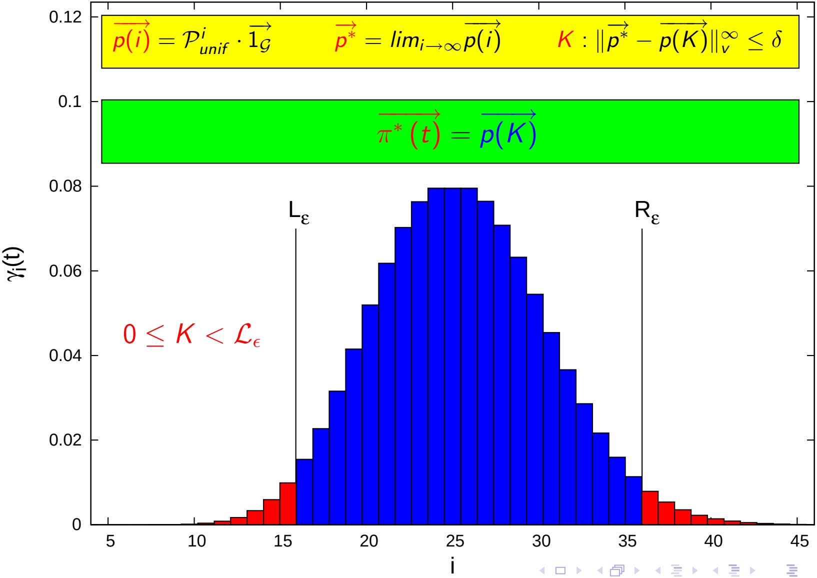

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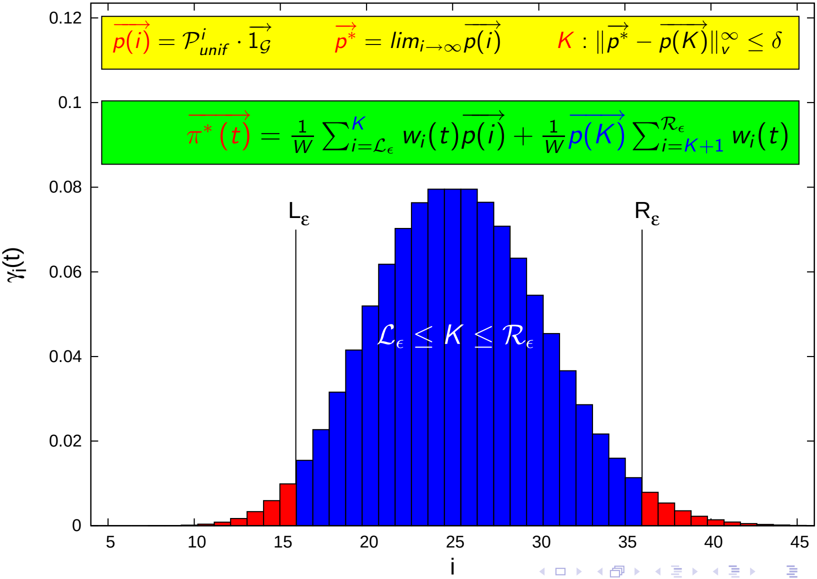

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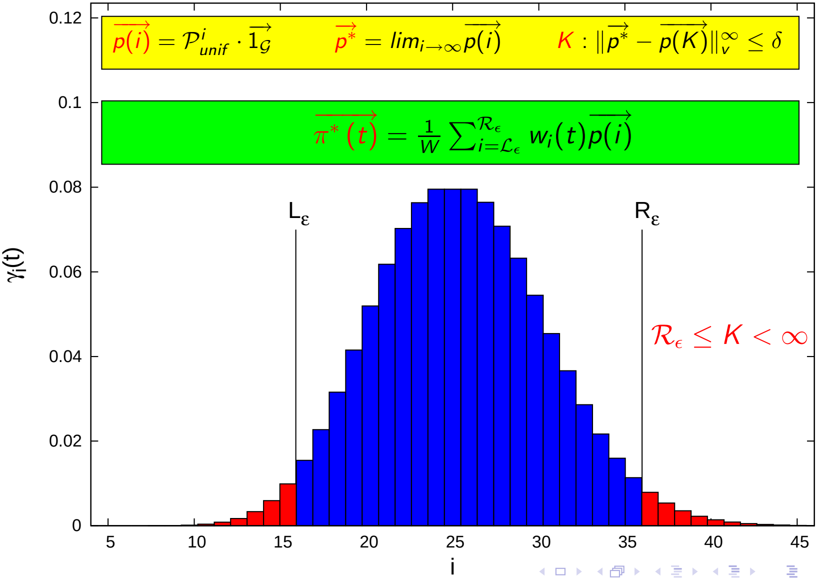

Outline

Refined steady-state detection error

Backward Computations

Let $\exists K : \forall i \geq K : \forall j \in 1,..,N : 0 \leq p_j^* - p(i)_j \leq \delta$.

$$\overrightarrow{\pi(t)} = \begin{cases} \overrightarrow{p(K)} & \text{, if } K < \mathcal{L}_\epsilon \\ \frac{1}{W} \sum_{i=\mathcal{L}_\epsilon}^{K} w_i(t) \overrightarrow{p(i)} + & \\ \overrightarrow{p(K)} \left(1 - \frac{1}{W} \sum_{i=\mathcal{L}_\epsilon}^{K} w_i(t)\right) & \text{, if } \mathcal{L}_\epsilon \leq K \leq \mathcal{R}_\epsilon \\ \frac{1}{W} \sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} w_i(t) \overrightarrow{p(i)} & \text{, if } K > \mathcal{R}_\epsilon \end{cases}$$

Then if $\sum_{i=0}^{\mathcal{L}_\epsilon} \gamma_i(t) \leq \frac{\epsilon}{4}$, $\sum_{i=\mathcal{R}_\epsilon}^{\infty} \gamma_i(t) \leq \frac{\epsilon}{4}$:

$$\boxed{\|\overrightarrow{\pi^*(t)} - \overrightarrow{\pi(t)}\|_v^\infty \leq \delta + \frac{3}{4}\varepsilon}$$

Steady-state detection criteria

Backward

1. Steady-state is detected if $\|\overrightarrow{p^*} - \overrightarrow{p(K)}\|_v^\infty \leq \frac{\varepsilon}{4}$
2. Use the Fox-Glynn algorithm with desired error $\frac{\epsilon}{2}$
3. Then the overall error bound for $Prob(s,\ \mathcal{A}\ \mathrm{U}^{[0,t]}\ \mathcal{G})$, will be ϵ

Comparing the results

Forward computations

Known (Malhotra et. al):

$$\|\overrightarrow{p^*(0)} - \overrightarrow{p(0,K)}\|_v \le \frac{\varepsilon}{4} \qquad \|\overrightarrow{\pi^*(0,t)} - \overrightarrow{\pi(0,t)}\|_v \le \varepsilon$$

New:

$$\|\overrightarrow{p^*(0)} - \overrightarrow{p(0,K)}\|_v^\infty \le \frac{\varepsilon}{8|Ind|} \qquad \left|\sum_{j\in Ind}\left(\pi^*(0,t)_j - \pi(0,t)_j\right)\right| \le \varepsilon$$

Backward computations

Known (Younes et. al):

$$\|\overrightarrow{p^*} - \overrightarrow{p(K)}\|_v \le \frac{\varepsilon}{8} \qquad \forall j \in S : -\frac{\varepsilon}{4} \le \pi^*(t)_j - \pi(t)_j \le \frac{3}{4}\varepsilon$$

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$\|\overrightarrow{p^*(0)} - \overrightarrow{p(0,K)}\|_v^\infty \leq \frac{\varepsilon}{8|Ind|}$ $\quad$ $\left|\sum_{j\in Ind}\left(\pi^*(0,t)_j - \pi(0,t)_j\right)\right| \leq \varepsilon$

Backward computations

Known (Younes et. al):

$\|\overrightarrow{p^*} - \overrightarrow{p(K)}\|_v \leq \frac{\varepsilon}{8}$ $\quad$ $\forall j \in S : -\frac{\varepsilon}{4} \leq \pi^*(t)_j - \pi(t)_j \leq \frac{3}{4}\varepsilon$

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Comparing the results

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Why do our results differ?

The major reasons

1. Improval of the Fox-Glynn error bound
2. Consideration of the error imposed by the weights $w_i(t)$
3. Refinement of the error-bound derivation for steady-state detection
4. Restriction to l^{∞}-norm

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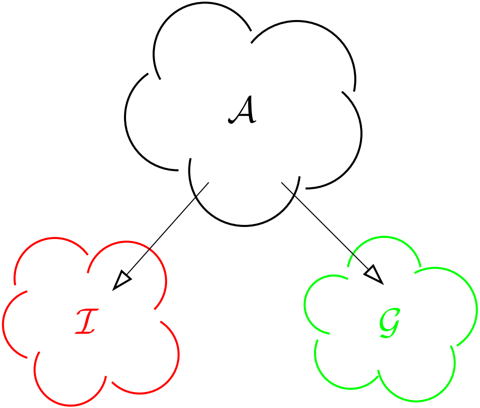
Outline

Making states absorbing, for $\mathcal{A}\ \mathrm{U}^{[0,t]}\ \mathcal{G}$



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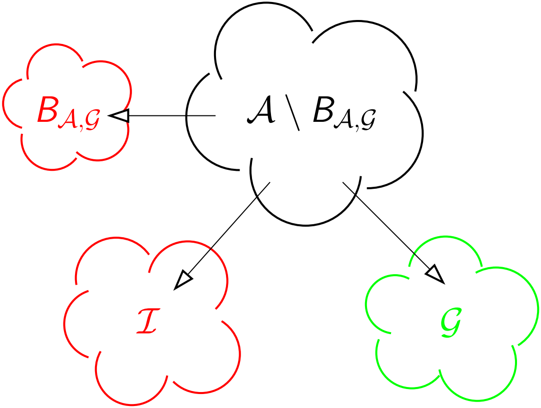

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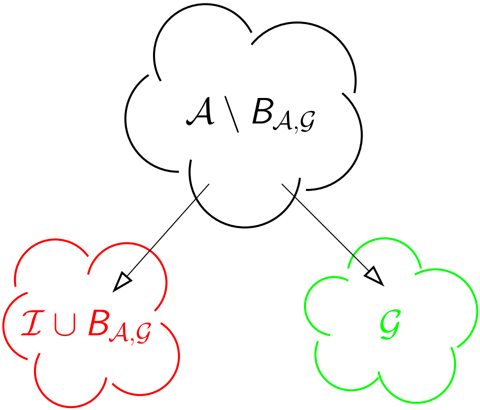

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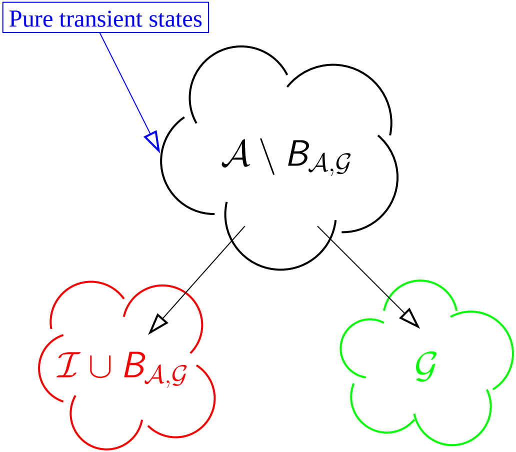


Precise steady-state detection, Backward computations

Theorem

For the stochastic matrix $\mathcal{P}_B$ obtained after uniformizing CTMC $(S, \mathcal{Q}^B)$, for any K and $\delta > 0$ the following holds:

$$\| \overrightarrow{1} - \left(\overrightarrow{p(K)} + \overrightarrow{p^B(K)} \right) \|_v^\infty \leq \delta \Rightarrow \forall i \geq K : \| \overrightarrow{p^*} - \overrightarrow{p(i)} \|_v^\infty \leq \delta$$

Where

$$\begin{aligned} \overrightarrow{p(i)} &= \mathcal{P}_B^i \cdot \overrightarrow{1_\mathcal{G}} \\ \overrightarrow{p^B(i)} &= \mathcal{P}_B^i \cdot \overrightarrow{i_{B_{\mathcal{A},\mathcal{G}} \cup \mathcal{I}}} \\ \overrightarrow{p^*} &= \lim_{i \to \infty} \mathcal{P}_B^i \cdot \overrightarrow{1_\mathcal{G}} \end{aligned}$$

Outline

1. Motivation
2. On-the-fly steady-state detection
3. Time-bounded reachability
4. Results
5. Detecting steady state
6. **Experiments**
7. Conclusions

Premature steady-state detection

Tools

Tool Name	Reference	S.s.d. method
Prism v2.1	(Kwiatkowska et al., 2004)	*regular*
ETMCC v1.4.2	(Hermanns et al., 2003)	*regular*
MRMC v1.0	(Katoen et al., 2005)	*precise*

Example

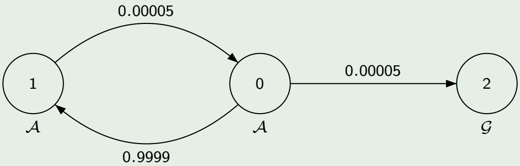
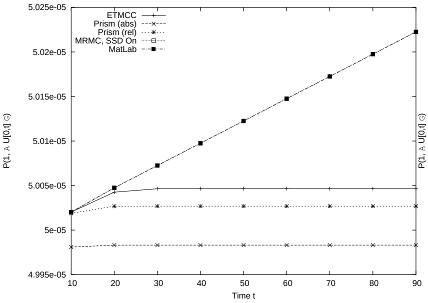


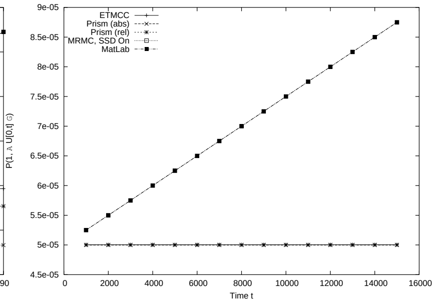
Figure: A slowly convergent CTMC

Computational results

Example

Tool	Error	K	$\mathcal{P}^K \cdot \overrightarrow{1_{\mathcal{G}}}$	$\overrightarrow{p^*}$
Prism v2.1(abs)	10^{-6}	2	$(5.00025 \cdot 10^{-5}, 2.5 \cdot 10^{-9}, 1.0)$	$(1.0, 1.0, 1.0)$
Prism v2.1(rel)	10^{-1}	12	$(5.00275 \cdot 10^{-5}, 2.75 \cdot 10^{-8}, 1.0)$	
ETMCC v1.4.2	10^{-6}	20	$(5.00475 \cdot 10^{-5}, 4.75 \cdot 10^{-8}, 1.0)$	
MRMC v1.0	10^{-6}	—	—	





Workstation cluster (Haverkort et al., 2000)

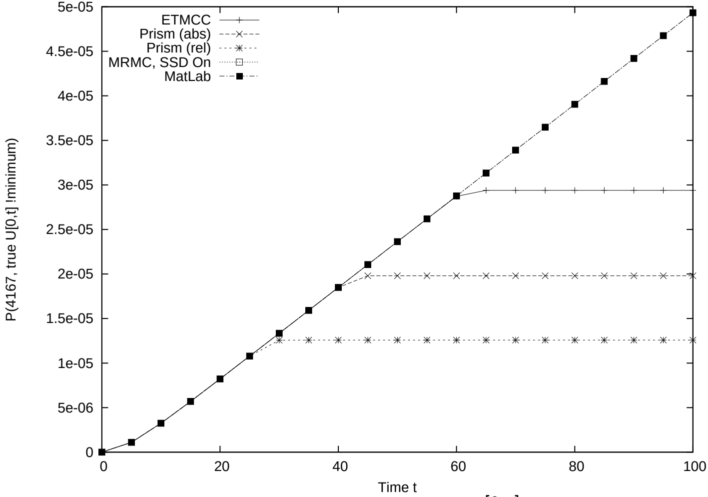


Figure: Results for $Prob(4167,\ true\ \mathrm{U}^{[0,t]}\ !minimum)$

IEEE 802.11 protocol (Massink et al., 2004)

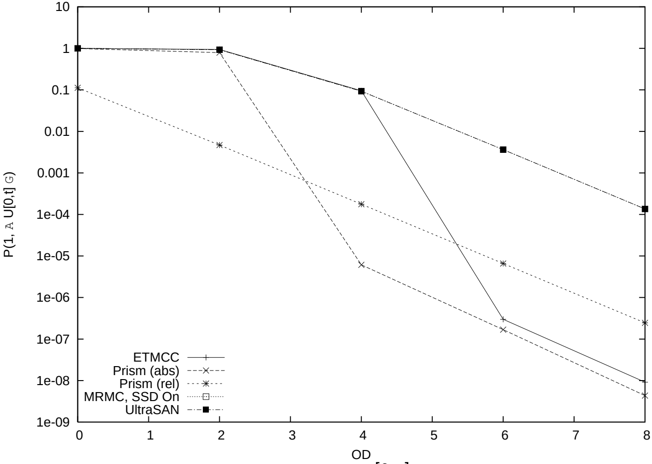


Figure: Results for *Prob*(0, *true* $\mathrm{U}^{[0,t]}$ *break*), for various *OD*

Computation time

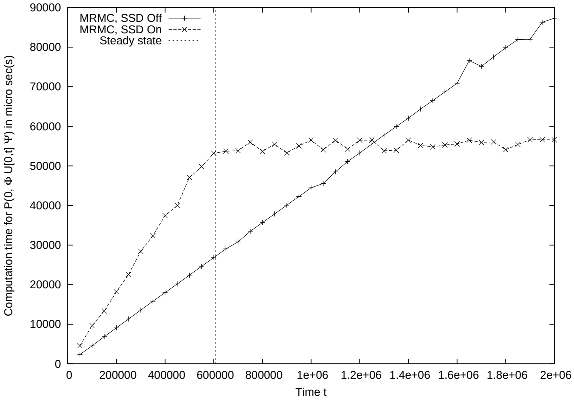


Figure: Time required to compute $Prob(0,\ \Phi\ \mathrm{U}^{[0,t]}\ \Psi)$ probabilities

Outline

Conclusions

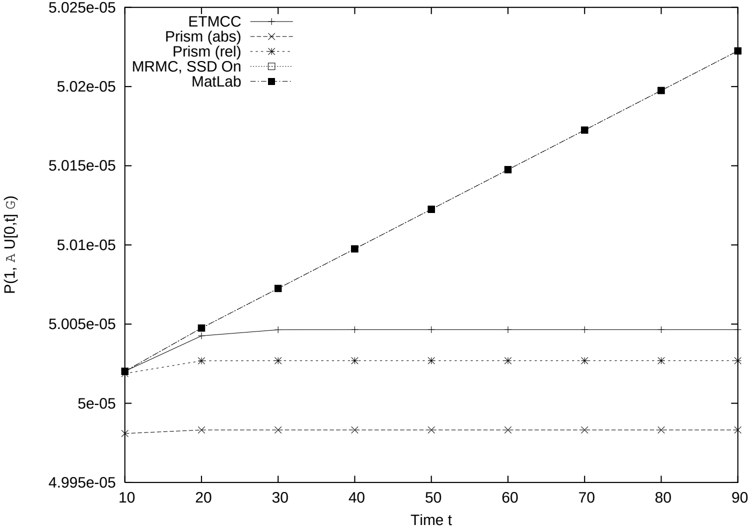
Results

1. The error bound corrections
   - Steady-state detection - fixed multiple problems
   - The Fox-Glynn algorithm - partial error-bound refinement
   - Uniformization using the Fox-Glynn - added weights influence
2. Precise steady-state detection criteria
   - Forward computations - preserves time complexity, computation time may slightly increase
   - Backward computations - preserves time complexity, computation time may approximately double

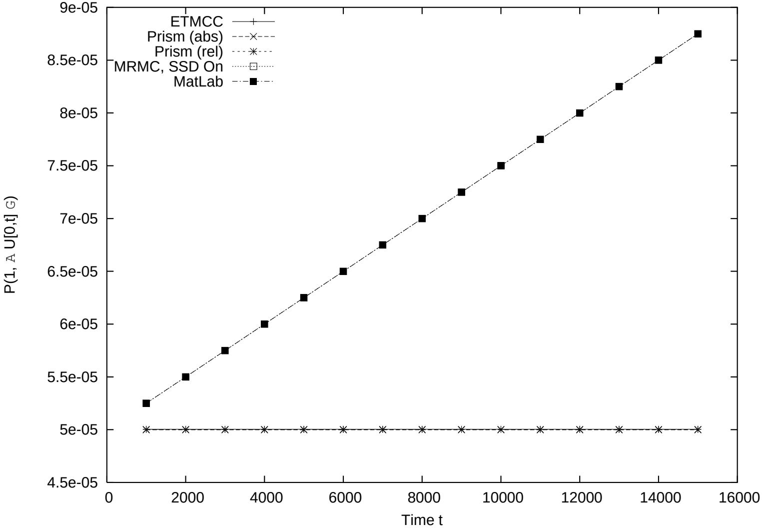
(Katoen and Zapreev, 2006)

For more details see our QEST'06 paper.

Computational results



Computational results



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