On-the-fly steady-state detection for time-bounded reachability in CTMCs

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Outline

DTMC & CTMC

Definition

A Discrete-time Markov Chain (*DTMC*) is a tuple $(S, \mathcal{P})$ with S as a finite set of states, $\mathcal{P} : S \times S \rightarrow [0, 1]$, where $\mathcal{P} = (p_{i,j})$ and $\forall i \in S : \sum_{j=1}^{|S|} p_{i,j} = 1$, as a state-transition probability matrix.

Definition

A Continuous-time Markov Chain (*CTMC*) is a tuple $(S, \mathcal{Q})$ with S as a finite set of states, $\mathcal{Q} : S \times S \rightarrow \mathcal{R}_{\geq 0}$, where $\mathcal{Q} = (q_{i,j})$ and $\forall i \in S : q_{i,i} = -\sum_{j,\, i \neq j} q_{i,j}$, as a generator matrix. Here $\forall i \neq j$, $q_{i,j}$ defines the rate of going from state i to j.

Stationary probabilities, DTMC (Haverkort, 1998)

Definition

The *stationary* or *steady-state* probability of a DTMC is a vector $\overrightarrow{p^*(0)}$ such that:

$$\overrightarrow{p^*(0)} = lim_{n\rightarrow\infty} \overrightarrow{p(0)} \cdot \mathcal{P}^n \tag{1}$$

where $\overrightarrow{p(0)}$ is the initial distribution.

Theorem

An irreducible and aperiodic DTMC with positive recurrent states, has a unique limiting distribution (1) which does not depend on the initial distribution $\overrightarrow{p(0)}$.

Power Iterations (Stewart, 1994)

Definition

For a stochastic matrix $\mathcal{P}$:

$$\overrightarrow{p(0,i)} = \overrightarrow{p\,(0)} \cdot \mathcal{P}^i,\ \overrightarrow{p\,(0)} \text{ - initial vector}$$

Theorem

If $\mathcal{P}$ is aperiodic and irreducible then the Power Method is guaranteed to converge to some $\overrightarrow{p^*}$.

Lemma

For a stochastic matrix $\mathcal{P}$, the number of iterations K needed to satisfy a tolerance criterion ϵ may be approximated by:

$$K = \frac{\log \epsilon}{\log|\lambda_2|},\ \lambda_2 \text{ - } \textit{subd. e.v. of } \mathcal{P}$$

Convergence (Stewart, 1994)

Tests for convergence

- Absolute: $\|\overrightarrow{p(0,i)} - \overrightarrow{p(0,i+M)}\|_v^\infty \leq \epsilon$, for some M.
- Relative: $\max_j \left(\frac{|\overrightarrow{p(s,i+M)}_j - \overrightarrow{p(s,i)}_j|}{|\overrightarrow{p(s,i+M)}_j|} \right) < \epsilon$
- ...

Warning!

It is best to envisage a battery of convergence tests, all of which must be satisfied before the approximation is accepted as being sufficiently accurate.

Outline

CSL logic (Baier et al., 2003)

The syntax of CSL logic

$$SF ::= tt|a \in \mathcal{AP}|\neg SF|SF \wedge SF|\mathrm{S}_{\trianglelefteq p}(s,\ SF)|\mathrm{P}_{\trianglelefteq p}(s,\ PF)$$

$$PF ::== \mathrm{X}^{[0,\ t]}\ SF|SF\ \mathrm{U}^{[0,t]}\ SF$$

Where

$\mathcal{AP}$	atomic propositions	s	initial state
SF	state formula	p	probability bound
PF	path formula	t	time bound
$\trianglelefteq$	$\in \{\leq,\ \geq\}$	$Sat\,(SF)$	states satisfying SF

Forward computations (Baier et al., 2003)

Reduce to transient analysis

Compute $Prob(s, \Phi \, \mathrm{U}^{[0,t]} \, \Psi)$:

1. Obtain a generator matrix $\mathcal{Q}[\neg\Phi \vee \Psi]$
2. Compute the transient probabilities vector

$$\overrightarrow{\pi^{*}(s,t)} = \overrightarrow{1_s} \cdot e^{\mathcal{Q}[\neg\Phi \vee \Psi]t} \tag{2}$$

3. Compute $Prob(s, \Phi \, \mathrm{U}^{[0,t]} \, \Psi) = \sum_{j \in Sat(\Psi)} \pi^{*}(s,t)_j$

Where

$\overrightarrow{1_s}$ is the initial distribution for the case when we start in the s state.

Backward computations (Katoen et al., 2001)

Change

Instead of computing (2) compute:

$$\overrightarrow{\pi^*(t)} = e^{\mathcal{Q}[\neg\Phi\vee\Psi]t} \cdot \overrightarrow{i_\Psi} \qquad (3)$$

Advantages

- $\forall s \in 1,..,N \;:\; \pi^*(t)_s = Prob(s,\ \Phi\ \mathrm{U}^{[0,t]}\ \Psi)$
- Better time complexity

Where

$\overrightarrow{i_\Psi}$ characteristic vector of the set $Sat(\Psi)$.

Outline

Numerical computations

The Jensen's method

Both (2) and (3) can be computed numerically using the Jensen's method (Baier et al., 2003), also known as Uniformization (Stewart, 1978).

Uniformization

Rewriting $\vec{p} \cdot \mathcal{Q} = \vec{0}$ into

$$\vec{p} \cdot \mathcal{P} = \vec{p},\ \mathcal{P} = \frac{\mathcal{Q}}{q} + \mathcal{I}$$

where $q \geq \max_i \sum_{j=1}^{N} q_{i,j}$.

Notice!

Taking

$$q > \max_i \sum_{j=1}^{N} q_{i,j}$$

makes DTMC $\mathcal{P}$ aperiodic

Using uniformization

The Jensen's method

Substitute $\mathcal{Q}$ with $\mathcal{P}$ in tailored representation (4) of equation (2)

$$\overrightarrow{\pi^*(s,t)} = \sum_{i=0}^{\infty} \frac{t^i}{i!} \vec{1_s} \cdot \mathcal{Q}^i \qquad (4)$$

$$\overrightarrow{\pi^*(s,t)} = \sum_{i=0}^{\infty} e^{-qt} \frac{(qt)^i}{i!} \overrightarrow{p(s,i)} \qquad (5)$$

Here we assume $\mathcal{Q} = \mathcal{Q}[\neg\Phi \vee \Psi]$ and $\overrightarrow{p(s,i)} = \vec{1_s} \cdot \mathcal{P}^i$.

Note!

The sum in (5) can be computed using the Fox-Glynn algorithm (Fox and Glynn, 1988).

The Fox-Glynn algorithm (Fox and Glynn, 1988)

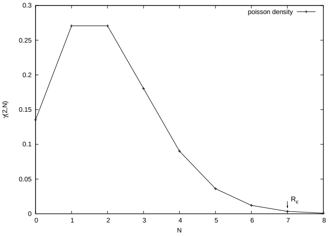
Lemma

Let f be a real-valued function with $\|f\| = \sup_{i \in 0,..,\infty} |f(i)|$ and $\sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} \gamma_i(t) \geq 1 - \frac{\epsilon}{2}$. In exact arithmetic,

$$|\sum_{i=0}^{\infty} \gamma_i(t) f(i) - \frac{1}{W} \sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} w_i(t) f(i)| \leq \epsilon \cdot \|f\| \qquad (6)$$

Where

$\gamma_i(t) = e^{-qt} \frac{(qt)^i}{i!}$
$\alpha \neq 0$, some constant
$w_i(t) = \alpha \gamma_i(t)$
$W = w(\mathcal{L}_\epsilon) + \ldots + w(\mathcal{R}_\epsilon)$



Steady-state, backward computations

Steady-state (Reibman and Trivedi, 1988)

$\vec{1_s} \cdot \mathcal{P}^i$ in (5) is the power iteration for uniformized DTMC $\mathcal{P}$ and that is where the steady-state detection comes into play.

Backward computations

Notice that all above is applicable to equation (3), see (Katoen et al., 2001), which gives us

$$\overrightarrow{\pi^*(t)} = \sum_{i=0}^{\infty} e^{-qt} \frac{(qt)^i}{i!} \overrightarrow{p(i)}$$

where $\overrightarrow{p(i)} = \mathcal{P}^i \cdot \overrightarrow{i_\Psi}$.

Outline

Forward computation, algorithm (Malhotra et al., 1994)

Computations with Steady-state detection

Let $\exists K \,:\, \|\overrightarrow{p^*(s)} - \overrightarrow{p(s,K)}\|_v \leq \delta$, $\sum_{i=0}^{\mathcal{L}_\epsilon} \gamma_i(t) \leq \frac{\epsilon}{2}$, and $\sum_{i=\mathcal{R}_\epsilon}^{\infty} \gamma_i(t) \leq \frac{\epsilon}{2}$.

$$\overrightarrow{\pi^*(s,t)} = \sum_{i=0}^{\infty} \gamma_i(t) \overrightarrow{p(s,i)}$$

$$\overrightarrow{\pi(s,t)} = \begin{cases} \overrightarrow{p(s,K)}, \text{ if } K < \mathcal{L}_\epsilon \\ \sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} \gamma_i(t) \overrightarrow{p(s,i)}, \text{ if } K > \mathcal{R}_\epsilon \\ \sum_{i=\mathcal{L}_\epsilon}^{K} \gamma_i(t) \overrightarrow{p(s,i)} + \overrightarrow{p(s,K)} \left(1 - \sum_{i=0}^{K} \gamma_i(t)\right), \text{ else} \end{cases} \tag{7}$$

Then:

$$\|\overrightarrow{\pi^*(s,t)} - \overrightarrow{\pi(s,t)}\|_v \leq 2\delta + \frac{\epsilon}{2}$$

Criteria

A corollary

$$\|\overrightarrow{p^*(s)} - \overrightarrow{p(s,K)}\|_v \leq \frac{\epsilon}{4} \text{ implies } \|\overrightarrow{\pi^*(s,t)} - \overrightarrow{\pi(s,t)}\|_v^\infty \leq \epsilon$$

The actual steady-state detection criteria

- Take $K = i + M$ if

$$\|\overrightarrow{p(s,i)} - \overrightarrow{p(s,K)}\|_v \leq \frac{\epsilon}{4}$$

- Check for K every M iterations

The known problems

Problems

1. *The steady-state detection is uncertain* - due to the criteria
2. *The error bound is not precise* - as derived under an assumption of knowing *real* steady-state.
3. *The norm $\|.\|_v$ is not defined* - was assumed that:

$$\|\overrightarrow{p(s,i)}\|_v \leq 1$$

4. *The weights are not considered* - if the complete Fox-Glynn algorithm is used.

Backward computations, algorithm (Younes et al., 2004)

Computations with Steady-state detection

Let $\exists K\ :\ \|\overrightarrow{p^*} - \overrightarrow{p(K)}\|_v \leq \delta$, $\sum_{i=0}^{\mathcal{L}_\epsilon} \gamma_i(t) \leq \frac{\epsilon}{2}$, and $\sum_{i=\mathcal{R}_\epsilon}^{\infty} \gamma_i(t) \leq \frac{\epsilon}{2}$.

$$\overrightarrow{\pi^*(t)} = \sum_{i=0}^{\infty} \gamma_i(t)\overrightarrow{p(i)}$$

$$\overrightarrow{\pi(t)} = \begin{cases} \overrightarrow{p(K)}, \text{ if } K < \mathcal{L}_\epsilon \\ \sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} \gamma_i(t)\overrightarrow{p(i)}, \text{ if } K > \mathcal{R}_\epsilon \\ \sum_{i=\mathcal{L}_\epsilon}^{K} \gamma_i(t)\overrightarrow{p(i)} + \overrightarrow{p(K)}\left(1 - \sum_{i=\mathcal{L}_\epsilon}^{K} \gamma_i(t)\right), \text{ else} \end{cases} \tag{8}$$

Then:

$$\|\overrightarrow{\pi^*(t)} - \overrightarrow{\pi(t)}\|_v \leq 2\delta + \frac{\epsilon}{2}$$

Criteria

A corollary

$$\|\overrightarrow{p^*} - \overrightarrow{p(K)}\|_v \leq \frac{\epsilon}{8} \text{ implies } \|\overrightarrow{\pi^*(t)} - \overrightarrow{\pi(t)}\|_v^\infty \leq \epsilon$$

The actual steady-state detection criteria

- Take $K = i + M$ if

$$\|\overrightarrow{p(i)} - \overrightarrow{p(K)}\|_v \leq \frac{\epsilon}{8}$$

- Check for K every M iterations

The known problems

Problems

1. *The error bound for (8) can be refined* - $\forall j \in 1, .., N, \forall i \in 0, .., \infty\ :\ p(i)_j \leq p(i+1)_j$.
2. *An additional error is introduced* - while switching from (7) to (8), $i = 0$ became $i = \mathcal{L}_\epsilon$.
3. *The refinement, done in (Younes et al., 2004), for (7) and (8) is incorrect* - the length of the error interval does not matter.

Outline

Choosing the "right" norm

Facts

- The error estimate, is based on *Geometrical Convergence*
- *G.C.* is proved, using *total variation norm* which, in an N-dimensional space, $\|v\|_v^\infty = \max_{i \in 1,..,N} |v_i|$.
- In a finite dimensional space all norms are equivalent.

Caution!

The $\overrightarrow{i_\Psi}$ vector is not a distribution, $\forall j \in 1, .., N \ : \ 0 \leq p(i)_j \leq 1$.

Example

For N states and Euclidean Norm $\|.\|_v^2$:

$$\|\sum_{i=0}^{\mathcal{L}_\epsilon - 1} \gamma_i(t)\overrightarrow{p(i)}\|_v^2 \not\leq \frac{\epsilon}{4} \text{ BUT } \|\sum_{i=0}^{\mathcal{L}_\epsilon - 1} \gamma_i(t)\overrightarrow{p(i)}\|_v^2 \leq \frac{\sqrt{N}}{4}\epsilon$$

The Fox-Glynn error-bound refinement

Errors

- $\mathcal{L}_\epsilon$ and $\mathcal{R}_\epsilon$, each, give error $\frac{\epsilon}{4}$
- Normalization $\frac{w_i(t)}{W}$ gives additional $\frac{\epsilon}{2}$

Lemma

Let f be a real-valued function with $\|f\| = \sup_{i\in 0,..,\infty} |f(i)|$, f does not change sign, and $\sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} \gamma_i(t) \geq 1 - \frac{\epsilon}{2}$. In exact arithmetic,

$$|\sum_{i=0}^{\infty} \gamma_i(t) f(i) - \frac{1}{W} \sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} w_i(t) f(i)| \leq \frac{\epsilon}{2} \cdot \|f\| \tag{9}$$

▸ Proof details

Refined steady-state detection error

Forward Computations

Let $\exists K : \|\overrightarrow{p^*(s)} - \overrightarrow{p(s,K)}\|_v^\infty \leq \delta$, $\sum_{i=0}^{\mathcal{L}_\epsilon} \gamma_i(t) \leq \frac{\epsilon}{4}$, $\sum_{i=\mathcal{R}_\epsilon}^{\infty} \gamma_i(t) \leq \frac{\epsilon}{4}$, and $Ind \in 2^N$ is a set of indexes.

$$\overrightarrow{\pi^*(s,t)} = \sum_{i=0}^{\infty} \gamma_i(t)\overrightarrow{p(s,i)}$$

$$\overrightarrow{\pi(s,t)} = \begin{cases} \overrightarrow{p(s,K)}, \text{ if } K < \mathcal{L}_\epsilon \\ \frac{1}{W}\sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} w_i(t)\overrightarrow{p(s,i)}, \text{ if } K > \mathcal{R}_\epsilon \\ \frac{1}{W}\sum_{i=\mathcal{L}_\epsilon}^{K} w_i(t)\overrightarrow{p(s,i)} + \overrightarrow{p(s,K)}\left(1 - \frac{1}{W}\sum_{i=\mathcal{L}_\epsilon}^{K} w_i(t)\right), \text{ else} \end{cases}$$

Then:

$$\left|\sum_{j \in Ind}\left(\pi^*(s,t)_j - \pi(s,t)_j\right)\right| \leq 2\delta|Ind| + \frac{3}{4}\epsilon$$

Refined steady-state detection error

Backward Computations

Let $\exists K\ :\ \|\overrightarrow{p^*} - \overrightarrow{p(K)}\|_v^\infty \leq \delta$, $\sum_{i=0}^{\mathcal{L}_\epsilon} \gamma_i(t) \leq \frac{\epsilon}{4}$, $\sum_{i=\mathcal{R}_\epsilon}^{\infty} \gamma_i(t) \leq \frac{\epsilon}{4}$.

$$\overrightarrow{\pi^*(t)} = \sum_{i=0}^{\infty} \gamma_i(t)\overrightarrow{p(i)}$$

$$\overrightarrow{\pi(t)} = \begin{cases} \overrightarrow{p(K)}, \text{ if } K < \mathcal{L}_\epsilon \\ \frac{1}{W}\sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} w_i(t)\overrightarrow{p(i)}, \text{ if } K > \mathcal{R}_\epsilon \\ \frac{1}{W}\sum_{i=\mathcal{L}_\epsilon}^{K} w_i(t)\overrightarrow{p(i)} + \overrightarrow{p(K)}\left(1 - \frac{1}{W}\sum_{i=\mathcal{L}_\epsilon}^{K} w_i(t)\right), \text{ else} \end{cases}$$

Then:

$$\|\overrightarrow{\pi^*(t)} - \overrightarrow{\pi(t)}\|_v^\infty \leq \delta + \frac{3}{4}\epsilon$$

Steadt-state detection criteria

Forward computations Criterion

1. Steady-state is detected if $\|\overrightarrow{p^*(s)} - \overrightarrow{p(s,K)}\|_v^\infty \leq \frac{\epsilon}{8|Sat(\Psi)|}$
2. Use the Fox-Glynn algorithm with desired error $\frac{\epsilon}{2}$
3. Then the overall error bound for the computed probability $Prob(s,\ \Phi\ \mathrm{U}^{[0,t]}\ \Psi)$ will be ϵ

Error bound details

Backward computations Criterion

1. Steady-state is detected if $\|\overrightarrow{p^*} - \overrightarrow{p(K)}\|_v^\infty \leq \frac{\epsilon}{4}$
2. Use the Fox-Glynn algorithm with desired error $\frac{\epsilon}{2}$
3. Then $\forall j \in 1, .., N$ the overall error bound for computed probability $Prob(j,\ \Phi\ \mathrm{U}^{[0,t]}\ \Psi)$, will be ϵ

Error bound details

Outline

Making states absorbing I

Definition

For a directed graph a subgraph is a *Bottom Strongly Connected Component* (BSCC) if it is a maximum strongly connected component such that it has no edges to outside its vertices.

Lemma

If $\mathcal{Q}[\neg\Phi \vee \Psi]$ has a BSCC containing at least one Φ state then all its states are Φ states.

Proof details

Where

$\mathcal{Q}$ is a generator matrix, Φ and Ψ are CSL state formulas

Making states absorbing II

Definition

Define $B_{\Phi,\Psi} = \{s \in B \cap (Sat(\Phi) \setminus Sat(\Psi)) \,|\, B \textit{ is a BSCC in } \mathcal{Q}[\neg\Phi \vee \Psi]\}$
Define $\mathcal{Q}^B[\neg\Phi \vee \Psi]$ obtained from $\mathcal{Q}[\neg\Phi \vee \Psi]$ by making all $B_{\Phi,\Psi}$ states absorbing.

Definition

Let $\mathrm{P}_{\mathcal{Q}}(s, \Phi)$ be the probability $Prob(s, \Phi)$ of satisfying CSL state formula Φ in state s, for the CTMC, defined by the generator matrix $\mathcal{Q}$.

Theorem

$$\mathrm{P}_{\mathcal{Q}}(s, \Phi\, \mathrm{U}^{[0,t]}\, \Psi) = \mathrm{P}_{\mathcal{Q}[\neg\Phi\vee\Psi]}(s, tt\, \mathrm{U}^{[t,t]}\, \Psi) = \mathrm{P}_{\mathcal{Q}^B[\neg\Phi\vee\Psi]}(s, tt\, \mathrm{U}^{[t,t]}\, \Psi)$$

▸ Proof details

Precise steady-state detection, Forward computations

Theorem

For a uniformized CTMC $\mathcal{P}_B$, obtained from the generator matrix $\mathcal{Q}^B\left[\neg\Phi \vee \Psi\right]$:

$$\forall \delta \geq 0 \;:\; \sum_{j \in Sat(\Phi) \setminus (\mathrm{B}_\Psi \cup Sat(\Psi))} p(s,i)_j \leq \delta \Rightarrow \|\overrightarrow{p^*(s)} - \overrightarrow{p(s,i)}\|_v^\infty \leq \delta$$

Where

$\overrightarrow{p^*(s)}$ - the steady-state of $\mathcal{P}_B$ when starting from s

$p(s,i)_j$ - the j'th component of $\overrightarrow{p(s,i)} = \vec{1_s} \cdot \mathcal{P}_B^i$

Precise steady-state detection, Backward computations

Theorem

For a uniformized CTMC $\mathcal{P}_B$, obtained from the generator matrix $\mathcal{Q}^B\left[\neg\Phi \vee \Psi\right]$:

$$\forall \delta \geq 0 \;:\; \|\vec{1} - \left(\overrightarrow{p(i)} + \overrightarrow{p^B(i)}\right)\|_v^\infty \leq \delta \Rightarrow \|\overrightarrow{p^*} - \overrightarrow{p(i)}\|_v^\infty \leq \delta$$

Where

$$\overrightarrow{p^B(i)} = \mathcal{P}_B^i \cdot \overrightarrow{i_{\mathrm{B}_\Psi \cup Sat(\neg\Phi)}}$$

$$\overrightarrow{p^*} = \lim_{i\to\infty} \mathcal{P}_B^i \cdot \overrightarrow{i_\Psi}$$

$$\overrightarrow{p(i)} = \mathcal{P}_B^i \cdot \overrightarrow{i_\Psi}$$

Outline

Premature steady-state detection (Massink et al., 2004)

Tools

Tool Name	Reference	S.s.d. method
Prism v2.1	(Kwiatkowska et al., 2004)	*regular*
ETMCC v1.4.2	(Hermanns et al., 2003)	*regular*
MRMC v1.0	(Katoen et al., 2005)	*precise*

Example

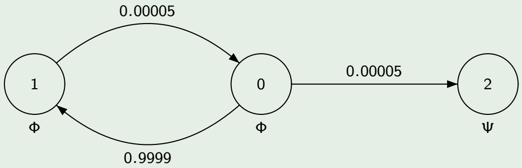
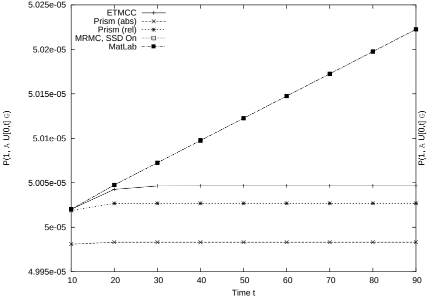


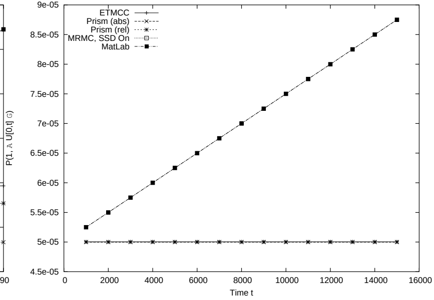
Figure: A slowly convergent CTMC

Computational results

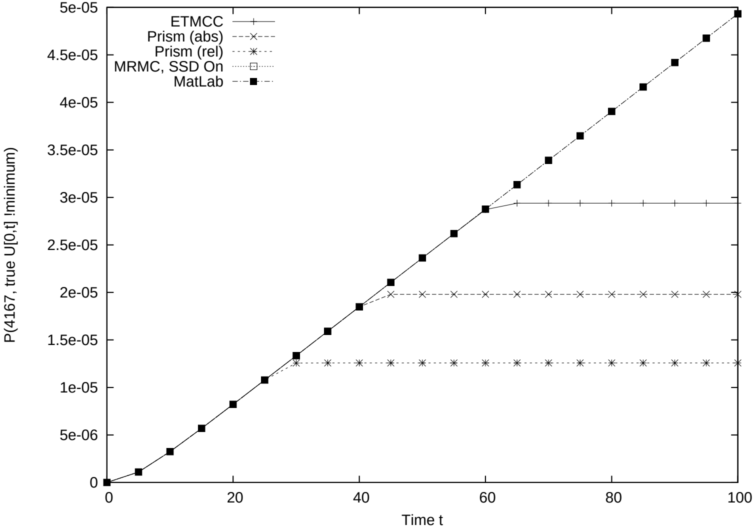
Example

Tool	Error	K	$\mathcal{P}^K \cdot \overrightarrow{i_\Psi}$	$\overrightarrow{p^*}$
Prism v2.1(abs)	10^{-6}	2	$(5.00025 \cdot 10^{-5}, 2.5 \cdot 10^{-9}, 1.0)$	$(1.0, 1.0, 1.0)$
Prism v2.1(rel)	10^{-1}	12	$(5.00275 \cdot 10^{-5}, 2.75 \cdot 10^{-8}, 1.0)$	
ETMCC v1.4.2	10^{-6}	20	$(5.00475 \cdot 10^{-5}, 4.75 \cdot 10^{-8}, 1.0)$	
MRMC v1.0	10^{-6}	—	—	

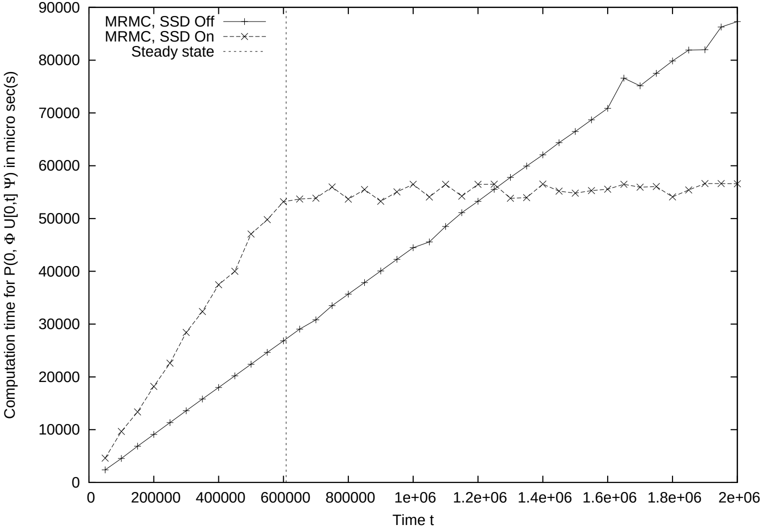




Workstation cluster (Haverkort et al., 2000)



Computation time



Outline

Related, ...

..., but unused

(Sericola, 1999) - the method, based on Uniformization, to determine the *point availability* and *expected interval availability* of a repairable computer system modeled as a Markov chain with steady-state detection.
Limitation: Results are only applicable for irreducible Markov Chains

(Neuts, 1981) - *Phase Type* distribution, has a theorem limiting the time before absorption.
Limitation: Transient states must form an irreducible matrix

Outline

Conclusions

Results

1. The error bound corrections
   - Steady-state detection - fixed multiple problems
   - The Fox-Glynn algorithm - partial error-bound refinement
   - Uniformization using the Fox-Glynn - added weights influence
2. Precise steady-state detection criteria
   - Forward computations - preserves time complexity, computation time may slightly increase
   - Backward computations - preserves time complexity, computation time may approximately double

Outline

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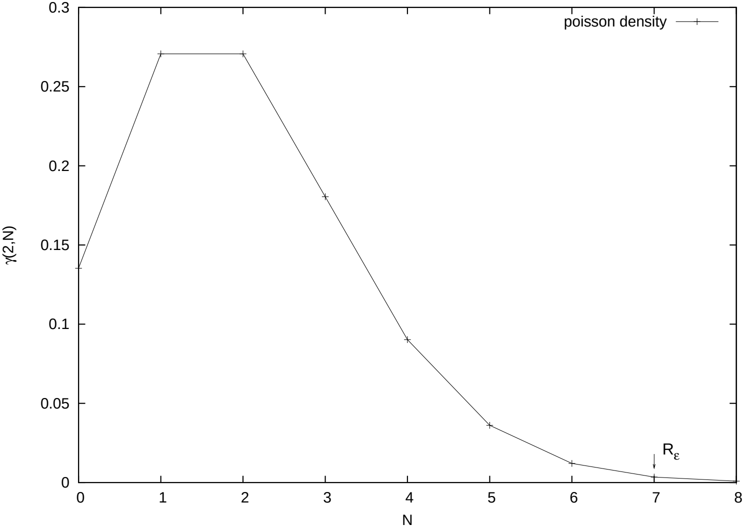
Appendix Outline

12 Appendix

- Numerical computation of $Prob(s, \Phi\ \mathrm{U}^{[0,t]}\ \Psi)$
- The Fox-Glynn, steady-state detection, error bound
- Improved steady-state detection
- An example

The Fox-Glynn algorithm (Fox and Glynn, 1988)



The Fox-Glynn error-bound refinement

Proof.

Due to the facts that

$$0 \leq \sum_{i=0}^{\mathcal{L}_\epsilon - 1} \gamma_i(t) + \sum_{i=\mathcal{R}_\epsilon + 1}^{\infty} \gamma_i(t) \leq \frac{\epsilon}{2}$$

$$-\frac{\epsilon}{2} \leq \sum_{i=\mathcal{L}_\epsilon}^{\mathcal{R}_\epsilon} (\gamma_i(t) - \frac{w_i(t)}{W}) \leq 0$$

$$\forall i \in 0, .., \infty \ : \ 0 \leq f(i) \leq \|f\|, \ f \text{ is non-negative}$$

$$\forall i \in 0, .., \infty \ : \ -\|f\| \leq f(i) \leq 0, \ f \text{ is non-positive}$$

$\square$

Return

Steadt-state detection criteria

Forward computations Details

Assuming $\forall i \geq K \ : \ \|\overrightarrow{p^*(s)} - \overrightarrow{p(s,i)}\|_v^\infty \leq \delta$ and the Fox-Glynn algorithm's error bound $\frac{\epsilon}{2}$ we have:

1. $(K > \mathcal{R}_\epsilon)$:

$$-\frac{\epsilon}{2} \leq \sum_{j \in Sat(\Psi)} \left(\pi^* \left(s, t\right)_j - \pi \left(s, t\right)_j \right) \leq \frac{\epsilon}{2}$$

2. $(\mathcal{L}_\epsilon \leq K \leq \mathcal{R}_\epsilon)$:

$$-2\delta |Sat\left(\Psi\right)| - \frac{3}{4}\epsilon \leq \sum_{j \in Sat(\Psi)} \left(\pi^* \left(s, t\right)_j - \pi \left(s, t\right)_j \right) \leq 2\delta |Sat\left(\Psi\right)| + \frac{3}{4}\epsilon$$

3. $(K < \mathcal{L}_\epsilon)$:

$$-2\delta |Sat\left(\Psi\right)| - \frac{1}{4}\epsilon \leq \sum_{j \in Sat(\Psi)} \left(\pi^* \left(s, t\right)_j - \pi \left(s, t\right)_j \right) \leq 2\delta |Sat\left(\Psi\right)| + \frac{1}{4}\epsilon$$

Return

Steadt-state detection criteria

Backward computations Details

Assuming $\forall i \geq K \ : \ \|\overrightarrow{p^*} - \overrightarrow{p(i)}\|_v^\infty \leq \delta$ and the Fox-Glynn algorithm's error bound $\frac{\epsilon}{2}$ we have:

1. $(K > \mathcal{R}_\epsilon)$:
$$-\frac{\epsilon}{2} \leq \pi^*(t)_j - \pi(t)_j \leq \frac{\epsilon}{2}$$
2. $(\mathcal{L}_\epsilon \leq K \leq \mathcal{R}_\epsilon)$:
$$-\delta - \frac{3}{4}\epsilon \leq \pi^*(t)_j - \pi(t)_j \leq \delta + \frac{3}{4}\epsilon$$
3. $(K < \mathcal{L}_\epsilon)$:
$$-\delta - \frac{1}{4}\epsilon \leq \pi^*(t)_j - \pi(t)_j \leq \delta + \frac{1}{4}\epsilon$$

Return

Making states absorbing I

Proof.

The case of a single state BSCC is trivial.
The rest is also trivial, by contradiction.
Let B be a BSCC of $\mathcal{Q}[\neg\Phi \vee \Psi]$ such that it has at least two states, $s_\Phi \in Sat(\Phi)$, $s_{\neg\Phi} \in Sat(\neg\Phi)$ and $s_\Phi, s_{\neg\Phi} \in B$. All $\neg\Phi$ states in $\mathcal{Q}[\neg\Phi \vee \Psi]$ are made absorbing, thus the $s_{\neg\Phi}$ state has only one self-loop transition. This yields that $s_{\neg\Phi} \notin B$. Contradiction. $\square$

Return

Making states absorbing II

Proof.

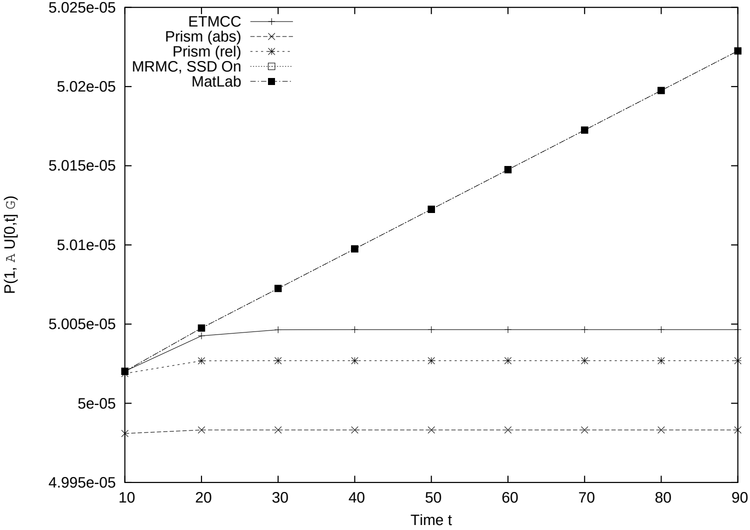
The first part $\mathrm{P}_{\mathcal{Q}}(s, \Phi \, \mathrm{U}^{[0,t]} \, \Psi) = \mathrm{P}_{\mathcal{Q}[\neg\Phi\vee\Psi]}(s, \, tt \, \mathrm{U}^{[t,t]} \, \Psi)$ was proved in (Baier et al., 2003).

The second part $\mathrm{P}_{\mathcal{Q}[\neg\Phi\vee\Psi]}(s, \, tt \, \mathrm{U}^{[t,t]} \, \Psi) = \mathrm{P}_{\mathcal{Q}^{B}[\neg\Phi\vee\Psi]}(s, \, tt \, \mathrm{U}^{[t,t]} \, \Psi)$ is valid due to the fact, that if there is a BSCC consisting of $Sat\,(\Phi)$ states then $Sat\,(\Psi)$ states are not reachable from it.

Unless it is a trivial case when a BSCC consists of one state s which satisfies both Φ and Ψ formulas, but in this case it is already made absorbing while obtaining $\mathcal{Q}[\neg\Phi \vee \Psi]$. $\square$

Return

Computational results



Computational results