1. Determine the values $\int_1^2 e^x \sin(4x)dx$ with $h=0.1$ by

a. Use the composite trapezoidal rule

b. Use the composite Simpsons' method

c. Use the composite midpoint rule

$0.1 = \frac{2-1}{2n}$ $\quad n = \frac{1}{0.2} = 5$

$0.2n = 1$ $\quad X_{2n} = b = 2$

a. $I = \int_1^2 e^x \sin(4x)\,dx$, $n = \frac{2-1}{0.1} = 10$ $\quad X_0 = 1$

$x_0 = 1,\ x_1 = 1.1,\ x_2 = 1.2 \ \dots\ x_{10} = 2$

$f(x) = e^x \sin(4x)$

$\Rightarrow \frac{h}{2}[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)] - \frac{b-a}{12} h^2 f''(\xi)$

$\Rightarrow \frac{0.1}{2}[e\sin(4) + 2(e^{1.1}\sin 4.4 + e^{1.2}\sin 4.8 + e^{1.3}\sin 5.2 + e^{1.4}\sin 5.6 + e^{1.5}\sin 6$

$+ e^{1.6}\sin 6.4 + e^{1.7}\sin 6.8 + e^{1.8}\sin 7.2 + e^{1.9}\sin 7.6)) + e^2\sin 8]$

$- \frac{2-1}{12} \times 0.1^2 \times (8e^x\cos 4x - 15e^x\sin 4x)(\xi)$

$\Rightarrow 0.51567 - \frac{1}{12} \times 0.1^2 \times 23e^2 = 0.3941$ #

b. $I = \frac{h}{3}[f(x_0) + 4\sum_{odd\ i} f(x_i) + 2\sum_{even\ i} f(x_i) + f(x_n)] - \frac{b-a}{180} h^4 f^{iv}(\xi)$

$= \frac{0.1}{3}[e\sin 4 + 4(e^{1.1}\sin 4.4 + e^{1.3}\sin 5.2 + e^{1.5}\sin 6 + e^{1.7}\sin 6.8 + e^{1.9}\sin 7.6)$

$+ 2(e^{1.2}\sin 4.8 + e^{1.4}\sin 5.6 + e^{1.6}\sin 6.4 + e^{1.8}\sin 7.2) + e^2\sin 8] - \frac{1}{12} \times 0.1^4 f^{iv}(\xi)$

$= 0.5147 - \frac{1}{180} \times 0.1^4 \times 401e^4 = 0.383$ #

$|f^{iv}(x)| = |-240\cos(4x) + 161\sin 4x| \leq |240\cos x| + |161\sin 4x| = 401e^x$

$C_1 \Rightarrow 2h[f(x_1)+f(x_3)+\dots f(x_{2n-1})] + \frac{b-a}{6}h^2 f''(\xi)$

$= 2\times 0.1[e^{1.1}\sin 4.4 + e^{1.3}\sin 5.2 + e^{1.5}\sin 6 + e^{1.7}\sin 6.8 + e^{1.9}\sin 7.6]$

$+\frac{1}{6}\times 0.1^2 \times 23 e^2 = 0.366$ #

2. Approximate $\int_1^{1.5} x^2 \ln x dx$ using Gaussian Quadrature with $n=3$ and $n=4$. Then compare the result to the exact value of the integral.

$\int_a^b f(x)\,dx = \frac{b-a}{2}\sum_{i=1}^{n} C_i f[\frac{a+b}{2}+\frac{b-a}{2}\eta_i]$

$\Rightarrow \int_1^{1.5} f(x)\,dx = \frac{1.5-1}{2}[C_1 f(1.25+0.25\eta_1) + C_2 f(1.25+0.25\eta_2)$

$+ C_3 f(1.25+0.25\eta_3)]$

$f(x) = x^2 \ln x$

$n=3 \Rightarrow x_1 = -\sqrt{\frac{3}{5}}, \quad x_2 = 0, \quad x_3 = \sqrt{\frac{3}{5}}$

$C_1 = \frac{5}{9} = C_3, \quad C_2 = \frac{8}{9}$

$\Rightarrow 0.25[\frac{5}{9}f(\underbrace{1.25+0.25\times -\sqrt{\frac{3}{5}}}_{1.056}) + \frac{8}{9}f(\underbrace{1.25+0.25\cdot 0}_{1.25})$

$+\frac{5}{9}f(\underbrace{1.25+0.25\times\sqrt{\frac{3}{5}}}_{1.444})] = 0.192$ #

$n=4 \Rightarrow x_1 = -x_4 = -0.861, \quad x_2 = -x_3 = 0.34$

$C_1 = C_4 = 0.348, \quad C_2 = C_3 = 0.652$

$\Rightarrow 0.25[\ 0.348 \cdot f(1.25+0.25\overset{\nearrow 1.03475}{\times}-0.861) + 0.652 f(1.25+0.25\times 0.34)\overset{1.335}{}$

$+\ 0.652 \cdot f(1.25+0.25\times -0.34)_{1.165} + 0.348 \cdot f(1.25+0.25\times 0.861)_{1.46525}]$

$= 0.1923$ #

3. Approximate $\int_0^{\pi/4}\int_{\sin x}^{\cos x}(2y\sin x+\cos^2 x)dydx$ using

a. Simpson's rule for $n=4$ and $m=4$

b. Gaussian Quadrature, $n=3$ and $m=3$

c. Compare these results with the exact value.

a.

① 对y積 ⇒ 对x積

$y_0 = \sin x$

$y_1 = \sin x + \frac{\cos x - \sin x}{4}$

$y_2 = \sin x + \frac{\cos x - \sin x}{2}$

$y_3 = \sin x + \frac{3(\cos x - \sin x)}{4}$

$y_4 = \cos x$

$F(x) = \int_{\sin x}^{\cos x}(2y\sin x + \cos^2 x)\,dy$

$I = \int_0^{\pi/4} F(x)\,dx \quad , \quad h_y = \frac{\cos x - \sin x}{4}$

$f(y_i) = 2y_i \sin x + \cos^2 x$

$F(x) = \frac{h_y}{3}[f(y_0) + 4f(y_1) + 2f(y_2) + 4f(y_3) + f(y_4)]$

$= \frac{\cos x - \sin x}{12}[2\sin^2 x + \cos^2 x$

⋮

用程式 $\Rightarrow 0.6032$

b. 0.5118 (用程式)

c. 0.404

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Trapezoid Rule (n=4, m=4): 0.6031695455915869
Gaussian Quadrature (n=3, m=3): 0.511865539945296
Exact Value: 0.4041197515454243
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4.

① $[0,1]$ 分 4 段

算 $f(x_i)$

$$\int_0^1 x^{-\frac{1}{4}}\sin x\,dx \approx \frac{1}{12}\left[f(0)+4f(0.25)+2f(0.5)+4f(0.75)+f(1)\right] \approx 0.557$$

(b) $t=\frac{1}{x}$, $dx=\frac{-1}{t^2}dt$

$x=1 \rightarrow t=1$, $x\rightarrow\infty$, $t\rightarrow 0$

$$\int_1^{\infty} x^{-4}\sin x\,dx \rightarrow \int_{t=0}^{1} t^{-\frac{5}{4}}\sin\left(\frac{1}{t}\right)dt$$

$\epsilon = 0.001$

$$\int^1 t^{-\frac{5}{4}}\sin\left(\frac{1}{t}\right)dt$$

$$\approx \frac{1-\epsilon}{12}\left[g(t_0)+4g(t_1)+2g(t_2)+4g(t_3)+g(t_4)\right]$$

$$g(t) = t^{-5/4}\sin\left(\frac{1}{t}\right)$$

$$t_k = \epsilon + k \cdot \frac{1-\epsilon}{4}$$

$$\Rightarrow \frac{h}{3} \left[f(t_0) + 2\sum_{\substack{i=2 \\ i=2}}^{n} f(t) + 4\sum_{i=1}^{n} f(t_{2i-1}) + f(t)_{n} \right]$$

$$\approx 0.129$$

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