BASED ON OUR RECENT ARTICLES:

A Survey on Mixed-Integer Programming Techniques in Bilevel Optimization
EURO Journal on Computational Optimization. 2021. DOI: 10.1016/j.ejco.2021.100007
Jointly with Thomas Kleinert, Martine Labbé, and Martin Schmidt

A Survey on Bilevel Optimization Under Uncertainty
Jointly with Yasmine Beck and Martin Schmidt, Optimization Online, 2022

A Brief Introduction to Robust Bilevel Optimization
Jointly with Yasmine Beck and Martin Schmidt,
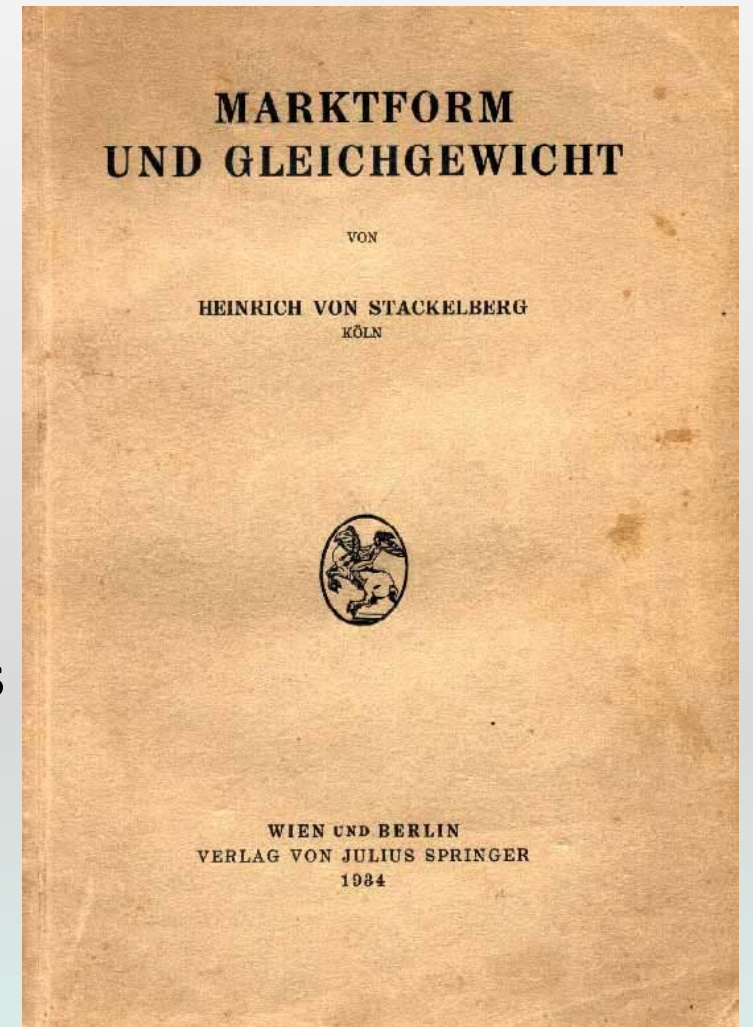**Views-and-News of the SIAM Activity Group on Optimization, to appear 2022**

BILEVEL OPTIMIZATION

WITH DETERMINISTIC DATA

STACKELBERG GAMES

- **Introduced in economy by H. v. Stackelberg in 1934**
- **Two-player sequential game: LEADER and FOLLOWER**
- **The LEADER moves before the FOLLOWER**
- **Perfect information: the leader has a perfect knowledge of the followers strategy**
- **The follower observes leader's action and acts rationally**
- **Rationality: agents act optimally, maximizing their payoffs**
- **BILEVEL OPTIMIZATION: Bracken & McGill (1973), Candler & Norton (1977)**

MARKTFORM
UND GLEICHGEWICHT

VON

HEINRICH VON STACKELBERG
KÖLN

WIEN UND BERLIN
VERLAG VON JULIUS SPRINGER
1934

APPLICATIONS: PRICING

Pricing: operator sets tariffs, and then customers choose the cheapest alternative

- **Tariff-setting, toll optimization (Labbé et al., 1998; Brotcorne et al., 2001; Labbé & Violin, 2016)**

- **Network Design and Pricing (Brotcorne et al., 2008)**
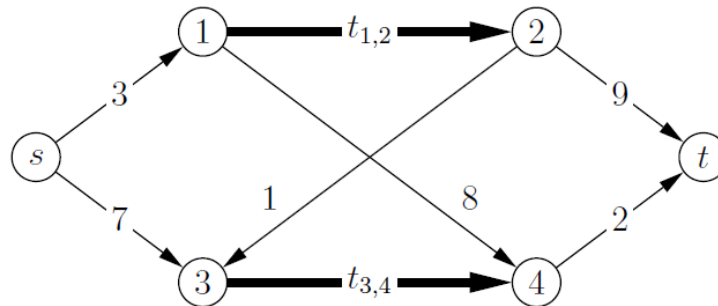
- **Survey (van Hoesel, 2008)**

Figure 1: 1-commodity network with two tariff arcs.

A DETERMINISTIC BILEVEL PROBLEM

$$\text{``}\min_{x \in X}\text{''} \quad F(x, y) \tag{1a}$$

$$\text{s.t.} \quad G(x, y) \geq 0, \tag{1b}$$

$$y \in S(x), \tag{1c}$$

upper level

$S(x)$: the set of optimal solutions of the x-parameterized problem

$$\min_{y \in Y} \quad f(x, y) \tag{2a}$$

$$\text{s.t.} \quad g(x, y) \geq 0. \tag{2b}$$

lower level

- **Both levels may involve integer decision variables. Functions can be non-linear, non-convex…**

- **(1) could be ill-posed (if LL solution is not unique). "min" to be replaced by**

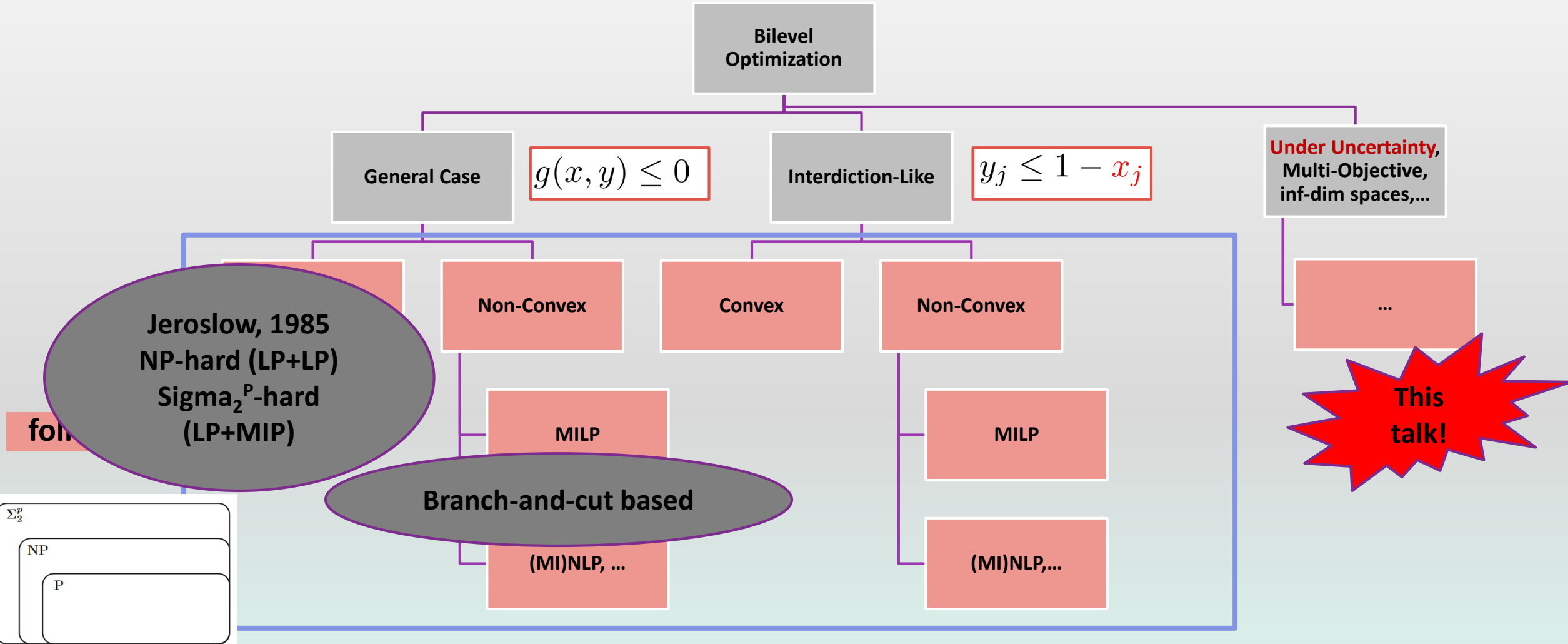$$\min_{x \in \bar{X}} \quad \min_{y \in S(x)} \quad F(x, y)$$

🙂 Optimistic!

$$\min_{x \in \bar{X}} \quad \max_{y \in S(x)} \quad F(x, y)$$

😦 Pessimistic!

OVERVIEW OF BILEVEL OPTIMIZATION PROBLEMS

THIS TALK

- **From deterministic bilevel optimization to bilevel optimization under uncertainty**

- **Sources of uncertainty**
  - **Data uncertainty**
  - **Decision uncertainty**

- **Timing for the data uncertainty**
  - **Here-and-now follower**
  - **Wait-and-see follower**

- **Challenges & opportunities**

SOURCES OF UNCERTAINTY

UNCERTAINTY: SINGLE-LEVEL VS BILEVEL

Single-level optimization:

$$\min_x \{c^\top x : Ax \geq b\}$$

- **"Only" subject to data uncertainty in A,b,c**
- **Stochastic optimization**
- **Robust optimization**
- **Distributionally robust, etc**

Bilevel optimization:

$$\text{``}\min_{x \in X}\text{''} \quad F(x,y) \tag{1a}$$

$$\text{s.t.} \quad G(x,y) \geq 0, \tag{1b}$$

$$y \in S(x), \tag{1c}$$

$S(x)$: optimal solutions of the x-parameterized problem

$$\min_{y \in Y} \quad f(x,y) \tag{2a}$$

$$\text{s.t.} \quad g(x,y) \geq 0. \tag{2b}$$

- **Subject to: data uncertainty**

- **But also: decision uncertainty. The leader is not sure about the reaction of the follower, or the follower is not certain about the observed leader's decision.**

TIMING OF UNCERTAINTY

WAIT-AND-SEE FOLLOWER

$$\text{leader } x \quad \curvearrowright \quad \text{uncertainty } u \quad \curvearrowright \quad \text{follower } y = y(x,u).$$

The leader is uncertain about the optimization parameters of the follower
Example: the leader solves a robust optimization problem

$$\text{``}\min_{x \in X} \max_{u \in \mathcal{U}}\text{''} \quad F(x,y) \quad \text{s.t.} \quad y \in S(x,u),$$

$$S(x,u) := \arg\min_{y \in Y} \quad f(x,u,y) \quad \text{s.t.} \quad g(x,u,y) \geq 0.$$

Example: the leader is risk-neutral wrt data uncertainty (discrete scenario set)

Optimistic or pessimistic leader

$$\text{``}\min_{x \in X}\text{''} \sum_{u \in \mathcal{U}} p_u F(x,y(x,u)) \quad \text{s.t.} \quad y(x,u) \in S(x,u), u \in \mathcal{U}$$

$$S(x,u) := \arg\min_{y \in Y} \quad f(x,u,y) \quad \text{s.t.} \quad g(x,u,y) \geq 0.$$

HERE-AND-NOW FOLLOWER

$$\text{leader } x \quad \curvearrowright \quad \text{follower } y = y(x) \quad \curvearrowright \quad \text{uncertainty } u.$$

The follower solves the problem under data uncertainty (stochastic, robust,…).

Optimistic vs pessimistic leader

For example: optimistic leader, the robust follower hedges against uncertainty in the objective function

$$\min_{x \in X} \; \min_{y \in S(x)} \; F(x, y)$$

$$S(x) := \arg\min_{y' \in Y} \left\{ \max_{u \in \mathcal{U}} f(x, u, y') \colon g(x, y') \geq 0 \right\}.$$
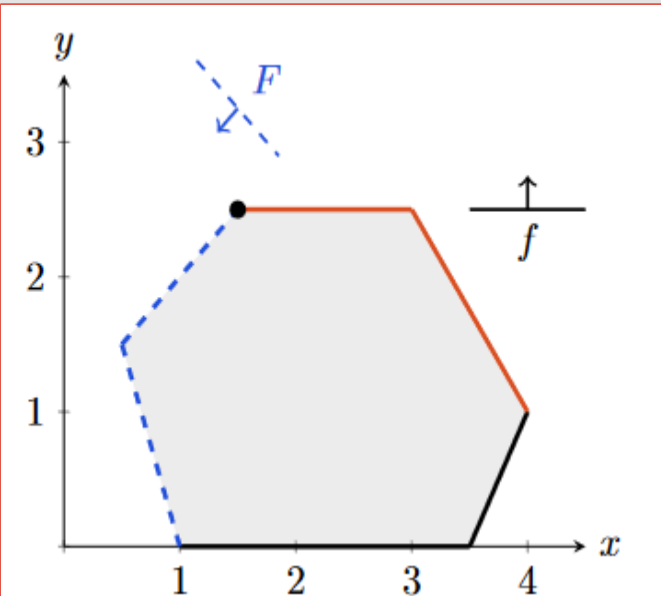
A SMALL EXAMPLE

Deterministic bilevel

$$\underset{x\in\mathbb{R}}{\text{``min''}}\quad F(x,y)=x+y$$

$$\text{s.t.}\quad x-y\geq -1,$$
$$3x+y\geq 3,$$
$$y\in S(x),$$

$$S(x):=\underset{y\in\mathbb{R}}{\arg\min}\quad f(x,y)=-0.1y$$

$$\text{s.t.}\quad -2x+y\geq -7,$$
$$-3x-2y\geq -14,$$
$$0\leq y\leq 2.5.$$

(a)

Here-and-now follower

$$\mathcal{U}:=\{u\in\mathbb{R}:|u|\leq 0.5\}$$

$$S(x):=\underset{y\in\mathbb{R}}{\arg\min}\,\underset{u\in\mathcal{U}}{\max}\quad \tilde{f}(x,u,y)=(-0.1+u)y$$

$$\text{s.t.}\quad -2x+y\geq -7,$$
$$-3x-2y\geq -14,$$
$$0\leq y\leq 2.5.$$
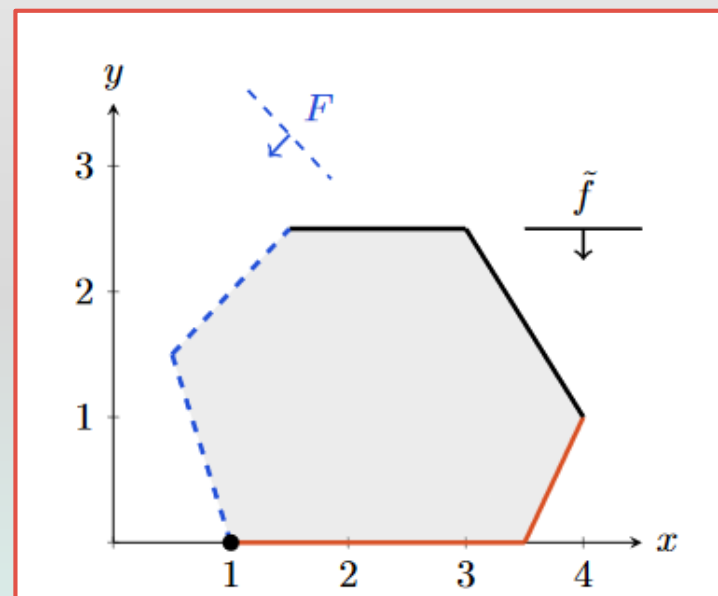
(b)

Wait-and-see follower

$$\underset{x\in\mathbb{R}}{\text{``min}}\,\underset{u\in\mathcal{U}}{\text{max''}}\quad F(x,y)=x+y$$

$$\text{s.t.}\quad x-y\geq -1,$$
$$3x+y\geq 3,$$
$$y\in S(x,u),$$

$$S(x,u):=\underset{y\in\mathbb{R}}{\arg\min}\quad (-0.1+u)y$$

$$\text{s.t.}\quad -2x+y\geq -7,$$
$$-3x-2y\geq -14,$$
$$0\leq y\leq 2.5.$$

$$\underset{x}{\min}\quad \hat{F}(x)\quad \text{s.t.}\quad 1.5\leq x\leq 4$$

$$\hat{F}(x)=\begin{cases} x+2.5, & 1.5\leq x\leq 3,\\ -0.5x+7, & 3\leq x\leq 4.\end{cases}$$

$$u\in[-0.5,0.1)\Rightarrow (a)$$
$$u\in(0.1,0.5]\Rightarrow (b)$$
$$u=0.1\Rightarrow \text{ all LL sol feasible}$$

CHALLENGES

PROBLEM COMPLEXITY

Robust single-level LPs:

- Interval, ball, ellipsoidal, polyhedral or Gamma-uncertainty preserve "tractability" of their deterministic counterpart (Ben-Tal & Nemirovski, Bertsimas & Sim)

$$\min \quad c^T x$$
$$\text{s.t.} \quad (a + u)^T x \leq b \text{ for all } u \in \mathcal{U}$$

Robust bilevel optimization:

- **Here-and-now follower**: tractability of the lower-level remains preserved for these uncertainty types

- Continuous convex lower level: KKT-based, strong duality-based reformulations still possible

- Discrete lower level: branch-and-cut still possible

- Major challenge: much larger in size, parallelization

Robust bilevel optimization:

- **Wait-and-see follower**: the problems may climb up in the complexity hierarchy!

ROBUST BILEVEL OPTIMIZATION

Deterministic bilevel

$$\text{``}\max_{x \in X}\text{''} \quad d^T y$$
$$\text{s.t.} \ \ y \in S(x)$$
$$S(x) := \arg\max\{u^T y : Ay \le Bx + b\}$$
$$X \subseteq \{0,1\}^{n_x}$$

NP-hard

Robust bilevel: Wait-and-see follower

$$\text{``}\max_{x \in X} \quad \min_{u \in U}\text{''} \quad d^T y$$
$$\text{s.t.} \ \ y \in S(x, u)$$
$$S(x, u) := \arg\max\{u^T y : Ay \le Bx + b\}$$
$$X \subseteq \{0,1\}^{n_x}$$

$$\mathcal{U} := [u_1^-, u_1^+] \times \cdots \times [u_{n_y}^-, u_{n_y}^+]$$

Under interval uncertainty, the robust counterpart is Sigma$_2^P$-hard
The "adversarial problem" (inner min) is NP-hard

Buchheim, Henke, Hommelsheim:
On the complexity of robust bilevel optimization with uncertain follower's objective. OR Letters 49(5): 703-707 (2021)

OPPORTUNITIES

BILEVEL STOCHASTIC MIP

Discrete scenario set

$$\min_{x \in X, y} \quad c^T x + \sum_{u \in \mathcal{U}} p_u \quad d_L^T \, y(x, u)$$

$$\text{s.t. } y(x, u) \in \arg\min_{y \in Y}\{d_F^T \, y : Ay \le B_u x + b_u\}$$

$$X \subseteq \{0, 1\}^{n_x}, Y \subseteq \{0, 1\}^{n_y}$$

Value function:
$$\Phi(x, u) = \min_{y \in Y}\{d_F^T \, y : Ay \le B_u x + b_u\}$$

Value-function reformulation (optimistic)

$$\min_{x \in X, y} \quad c^T x + \sum_{u \in \mathcal{U}} p_u \quad d_L^T \, y_u$$

$$\text{s.t.} \quad d_F^T \, y_u \le \Phi(x, u), \quad u \in \mathcal{U}$$

$$Ay_u \le B_u x + b_u, \quad u \in \mathcal{U}$$

$$y_u \in Y, \quad u \in \mathcal{U}$$

$$X \subseteq \{0, 1\}^{n_x}, Y \subseteq \{0, 1\}^{n_y}$$

Single-leader, multiple independent followers

Leverage on the existing branch-and-cut methods (Fischetti et al, 2017; Tahernejad et al, 2020)

S. Bolusani, S. Coniglio, T. K. Ralphs, and S. Tahernejad, "A Unified Framework for Multistage Mixed Integer Linear Optimization," in *Bilevel optimization: advances and next challenges*, S. Dempe and A. Zemkoho, Eds., 2020, p. 513–560.

BILEVEL GAMMA-ROBUST MIP

- Bilevel Knapsack Interdiction

Upper Level

$$\min_{x \in \{0,1\}^n} \quad d^\top y$$

$$\text{s.t.} \quad v^\top x \le B$$

$$y \in S(x)$$

Lower Level

$$\max_{y \in \{0,1\}^n} \quad d^\top y$$

$$\text{s.t.} \quad w^\top y \le C$$

$$y_i \le 1 - x_i \quad \forall i$$

- Players share common set of items
- Leader interdicts usage of certain items
- Deterministic interdiction cuts (Fischetti et al. 2019)
- Γ-robust variant
  - (Ext) or (MS) at lower level
  - (Scenario) interdiction cuts as generalization of deterministic cuts

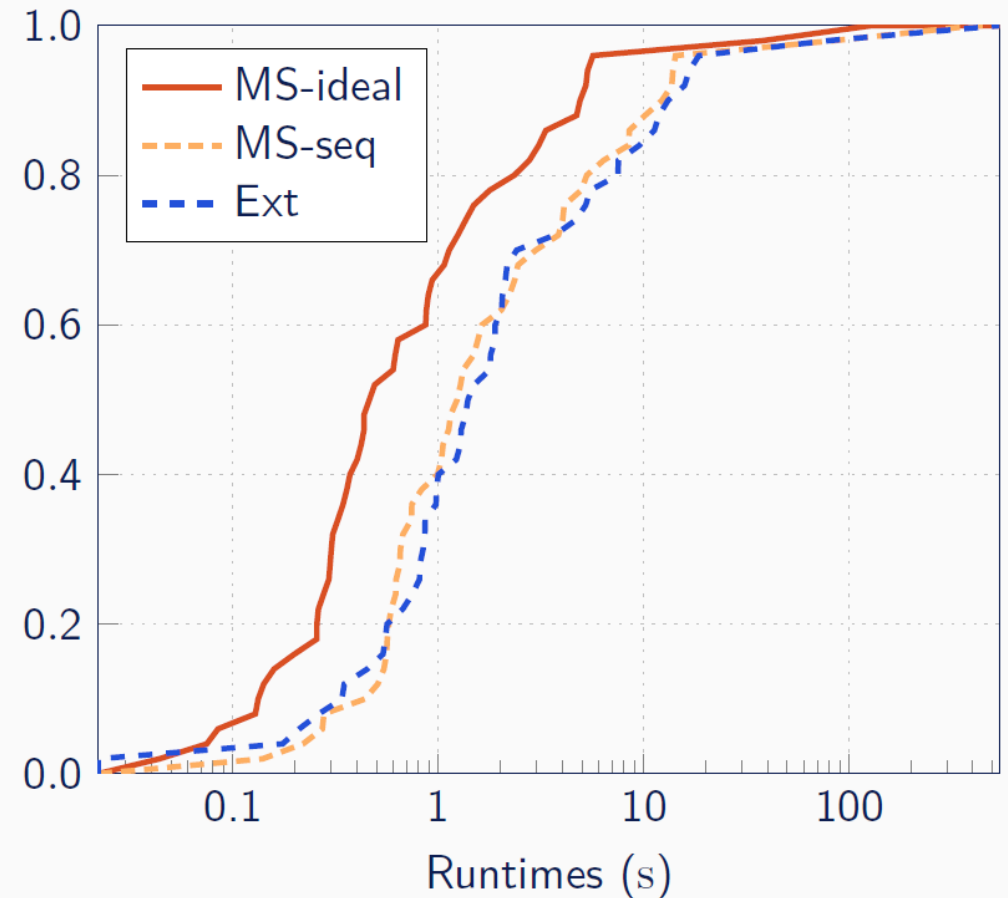$$\max_{y} \quad d^\top y - \max_{\{S \subseteq N : |S| \le \Gamma\}} \sum_{i \in S} \Delta d_i y_i$$

- Yasmine Beck, I. L., Martin Schmidt Exact Methods for Discrete Γ-Robust Interdiction Problems with an Application to the Bilevel Knapsack Problem, Optimization Online, 2022
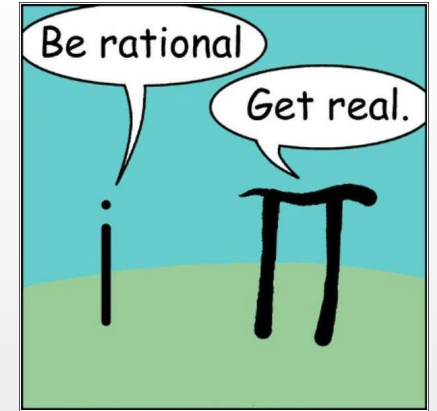
BILEVEL GAMMA-ROBUST MIP

(MS): Multi-scenario formulation: single-leader, multiple independent followers

$$\Phi(x) = \max_{\ell \in \{\Gamma,\dots,n+1\}} \Phi^\ell(x) = \max_{\ell \in \{\Gamma,\dots,n+1\}} \left\{ -\Gamma \Delta d_\ell + \max_{y \in Y(x)} \left\{ \tilde{d}(\ell)^\top y \right\} \right\}$$

(Ext): Extended formulation: dualize the inner max term

CRITICISM… AND OUTLOOK

PERFECT INFORMATION AND RATIONALITY OF DECISION MAKERS…

TOWARDS BOUNDED RATIONALITY, LIMITED OBSERVABILITY AND MORE

DECISION UNCERTAINTY: EXAMPLES

- **Near-optimal robust bilevel models** (Besancon et al, 2019): Leader hedges against sub-optimal follower reactions

- If the level of cooperation/confrontation of the follower is unknown → **intermediate cases, between the optimistic and the pessimistic one** (Aboussoror & Loridan, 1995; Mallozzi & Morgan, 1996).

- **Limited observability:** the follower cannot perfectly observe the decision of the leader and hedges against all possible leader decisions given the noisy observation (Bagwell, 1995; van Damme & Hurkens,1997; Beck & Schmidt:2021).

- **Limited intellectual or computational resources:** the follower resorts to heuristic approaches and the leader may be uncertain w.r.t. which heuristic is used (Zare et al, 2020).

CONCLUSIONS

- Connections between bilevel and robust/stochastic optimization still to be better understood

- When can we retain the tractability of the deterministic bilevel counterpart?

- When can we solve uncertain bilevel problems through a serious of deterministic ones?

- When do the bilevel problems under uncertainty become significantly harder?

- How can we better exploit the existing computational frameworks for deterministic bilevel optimization? (decomposition, SAA, scenario aggregation…)

- Data uncertainty vs Decision uncertainty, which paradigm to follow?

THANK YOU FOR YOUR ATTENTION

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