SVM Classifier

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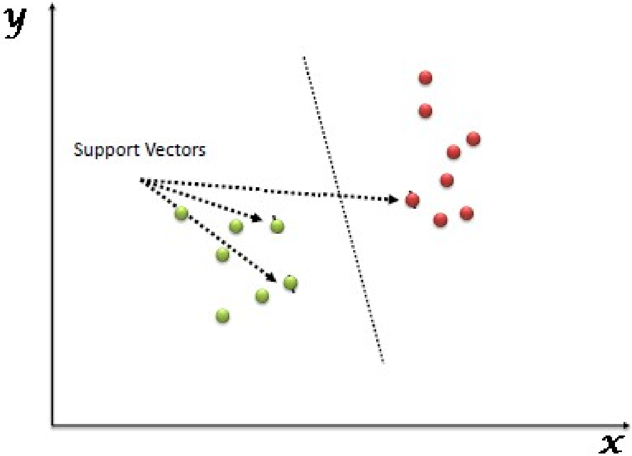
Outline

Introduction

In the SVM algorithm, we plot each data item as a point in n-dimensional space (where n is a number of features you have) with the value of each feature being the value of a particular coordinate. Then, we perform classification by finding the hyper-plane that differentiates the two classes very well.

Introduction



Introduction

The hyperplane is our decision boundary. Everything on one side belongs to one class, and everything on the other side belongs to a different class.

Formulation

We'd like to find the point closest to the separating hyperplane and make sure this is as far away from the separating line as possible. This is known as margin. We want to have the greatest possible margin, because if we made a mistake or trained our classifier on limited data, we'd want it to be as robust as possible. The equation of the hyperplane is of the form

$$y(x) = w^T \phi(x) + b$$

Where $\phi(x)$ denotes a feature-space transformation function and b bias parameter We want to find the best hyperplane (that maximises the distance between the closest data points and the hyperplane) that satisfies $y(x_n) > 0$ for points having $t_n = +1$ and $y(x_n) < 0$ for points having $t_n = -1$, so that $t_n y(x_n) > 0$ for all training data points.

Formulation

The perpendicular distance of a point x from a hyperplane defined by $y(x) = 0$ where $y(x)$ is given by $|y(x)|/||w||$
Furthermore, we are only interested in solutions for which all data points are correctly classified, so that $t_n y(x_n) > 0$ for all n.
Thus the distance of a point x_n to the decision surface is given by

$$\frac{t_n y(x_n)}{||w||} = \frac{t_n(w^T(x_n) + b)}{||w||}$$

The margin is given by the perpendicular distance to the closest point x_n from the data set, and we wish to optimize the parameters w and b in order to maximize this distance. Thus the maximum margin solution is found by solving

$$arg\ max_{w,b} \left\{ \frac{1}{||w||} min_n [t_n(w^T \phi(x_n) + b)] \right\}$$

Formulation

Direct solution of this optimization problem would be very complex, and so we shall convert it into an equivalent problem that is much easier to solve.

$$\widetilde{L}(\alpha) = \sum_{n=1}^{N} \alpha_n - \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} \alpha_n \alpha_m t_n t_m k(X_n, X_m)$$

with respect to a subject to the constraints

$$\alpha_n \geq 0, n = 1, ..., N,$$

$$\sum_{n=1}^{N} \alpha_n t_n = 0.$$

Here k is the kernel function representing the dot product in the target dimesion, in our case the original dimension so $k(x, x') = \phi(x)^T \phi(x')$.

Algorithm

To solve the above optimisation problem multiple optimisers exist but a principal algorithm SMO introduced as a faster optimization algorithm for SVM that works by optimizing pairs of alpha's ans updating bs accordingly until no further optimization is possible.

Outline

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        - Optimizing α_i and *alpha*$_j$
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The Simplified SMO Algorithm

Algorithm

the SMO algorithm selects two α parameters, α_i and α_j and optimizes the objective value jointly for both these α's. Finally it adjusts the b parameter based on the new α's. This process is repeated until the α's converge. We now describe these three steps in greater detail.

Selecting α Parameters

SMO

We simply iterate over all α_i, i = 1, . . . m. If α_i does not fulfill the KKT conditions to within some numerical tolerance, we select α_j at random from the remaining $m - 1\alpha_i$'s and attempt to jointly optimize α_i and α_j. If none of the α's are changed after a few iteration over all the α_i's, then the algorithm terminates.

Optimizing α_i and *alpha*$_j$

SMO

Having chosen the Lagrange multipliers α_i and α_j to optimize, we first compute constraints on the values of these parameters, then we solve the constrained maximization problem. First we want to find bounds L and H such that $L \leq \alpha_j \leq H$ must hold in order for α_j to satisfy the constraint that $0 \leq \alpha_j \leq C$. It can be shown that these are given by the following:

- If $y^{(i)} \neq y^{(j)}, L = max(0, \alpha_j - \alpha_i), \quad H = min(C, C + \alpha_j - \alpha_i)(1)$
- If $y^{(i)} = y^{(j)}, L = max(0, \alpha_i + \alpha_j - C), \quad H = min(C, \alpha_i + \alpha_j)(2)$

Optimizing α_i and *alpha*$_j$

SMO

Now we want to find α_j so as to maximize the objective function. If this value ends up lying outside the bounds L and H, we simply clip the value of α_j to lie within this range. It can be shown that the optimal α_j is given by:

$$\alpha_j := \alpha_j - \frac{y(j)(E_i - E_j)}{\eta} (3)$$

where

$$E_k = f(x^{(k)}) - y^{(k)} (4)$$

$$\eta = 2k(x^{(i)}, x^{(j)}) - k(x^{(i)}, x^{(i)}) - k(x^{(j)}, x^{(j)}) (5)$$

Optimizing α_i and *alpha*$_j$

SMO

You can think of E_k as the error between the SVM output on the kth example and the true label $y^{(k)}$. Next we clip α_j to lie within the range [L, H]

$$\alpha_j := \begin{cases} H \text{ if } \alpha_j > H \\ \alpha_j \text{ if } L \leq \alpha_j \leq H \\ L \text{ if } \alpha_j < L \end{cases} \quad (6)$$

Finally, having solved for *alpha*$_j$ we want to find the value for α_i. This is given by

$$\alpha_i := \alpha_i + y^{(i)} y^{(j)} (\alpha_j^{(old)} - \alpha_j) \quad (7)$$

Computing the b threshold
SMO

After optimizing α_i and α_j, we select the threshold b such that the KKT conditions are satisfied for the ith and jth examples. If, after optimization, α_i is not at the bounds (i.e., $0 < \alpha_i < C$), then the following threshold b1 is valid, since it forces the SVM to output $y^{(i)}$ when the input is $x^{(i)}$

$$b_1 = b - E_i - y^{(i)}(\alpha_i - \alpha_i^{(old)})k(x^{(i)}, x^{(i)}) - y^{(j)}(\alpha_j - \alpha_j^{(old)})k(x^{(i)}, x^{(j)}) \quad (8)$$

Similarly, the following threshold b_2 is valid if $0 < \alpha_j < C$

$$b_2 = b - E_j - y^{(i)}(\alpha_i - \alpha_i^{(old)})k(x^{(i)}, x^{(j)}) - y^{(j)}(\alpha_j - \alpha_j^{(old)})k(x^{(j)}, x^{(j)}) \quad (9)$$

Computing the b threshold

SMO

If both $0 < \alpha_i < C$ and $0 < \alpha_j < C$ then both these thresholds are valid, and they will be equal. If both new α's are at the bounds (i.e., $\alpha_i = 0$ or $\alpha_i = C$ and $\alpha_j = 0$ or $\alpha_j = C$) then all the thresholds between b_1 and b_2 satisfy the KKT conditions, we let $b := (b_1 + b_2)/2$.

Pseudo-Code
SMO

In this section we present pseudo-code for the simplified SMO algorithm.

Algorithm: Simplified SMO

Input:

C: regularization parameter

tol: numerical tolerance

max_passes: max $\#$ of times to iterate over α's without changing

$(x^{(1)}, y^{(1)}), ..., (x^{(m)}, y^{(m)})$: training data

Output:

$\alpha \in \mathbb{R}^m$ Lagrange multipliers for solution

$b \in \mathbb{R}$ threshold for solution

Pseudo-Code

SMO

```
Initialize α_i = 0, ∀i, b = 0.
Initialize passes = 0.
while(passes < max_passes)
  num changed alphas = 0.
  for i = 1, . . . m,
    Calculate E_i = f(x^(i)) − y^(i)
        if((y^(i) E_i < −tol && α_i < C) || (y^(i) E_i > tol && α_i > 0))
      Select j ≠ i randomly.
      Calculate E_j = f(x^(j)) − y^(j)
      Save old α's: α^(old)_i = α_i, α^(old)_j = α_j.
      Compute L and H
      if(L == H)
        continue to next i.
      Compute η
      if(η >= 0)
        continue to next i.
```

Pseudo-Code for Simplified SMO

Algorithm

```
      Compute and clip new value for αj
      if(|αj − αj^(old)| < 10^−5)
        continue to next i.
      Determine value for αi.
      Compute b1 and b2
      Compute b
      num_changed_alphas := num_changed_alphas + 1.
    endif
  endfor
  if (num_changed_alphas == 0)
    passes := passes + 1
  else
    passes := 0
endwhile
```