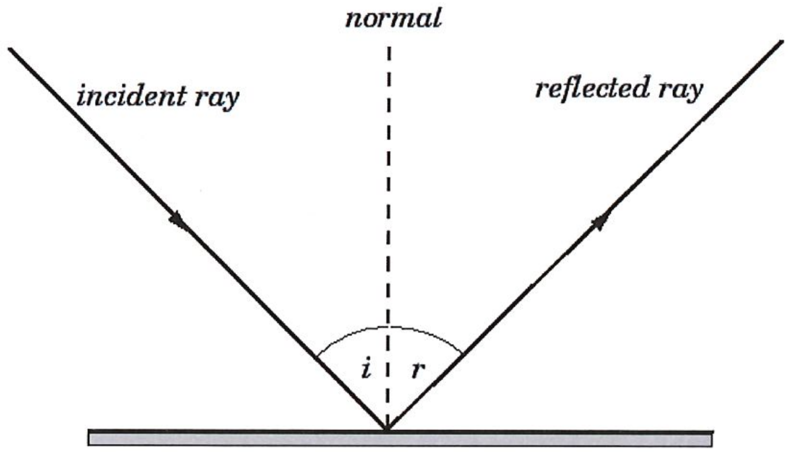
Jared Fowler

Physics M20CL – Thursday 1pm

Theory

Incident Ray

A ray of light that strikes a surface. The angle between this ray and the perpendicular or normal to the surface is the angle of incidence. The reflected ray corresponding to a given incident ray, is the ray that represents the light reflected by the surface (Wikipedia).



Law of Reflection

The law of reflection governs the reflection of light-rays off smooth conducting surfaces, and states that the incident ray, the reflected ray, and the normal to the surface of the surface all lie in the same plane. Furthermore, the angle of reflection 'r' is equal to the angle of incidence 'i' (http://farside.ph.utexas.edu).

Relationship Between Prism Apex Angle and Angles of Reflection

A prism apex incident with a wide collimated beam of light will reflect half the beam from each face adjacent to the vertex. The angle between the two reflected beams, 'z' as seen in Figure 0.1, is twice the prism angle 'A'.

Proof:

By the Law of Reflection, the angle of incidence with respect to the normal is equal to the angle of reflection with respect to the normal.

Figure 0.1

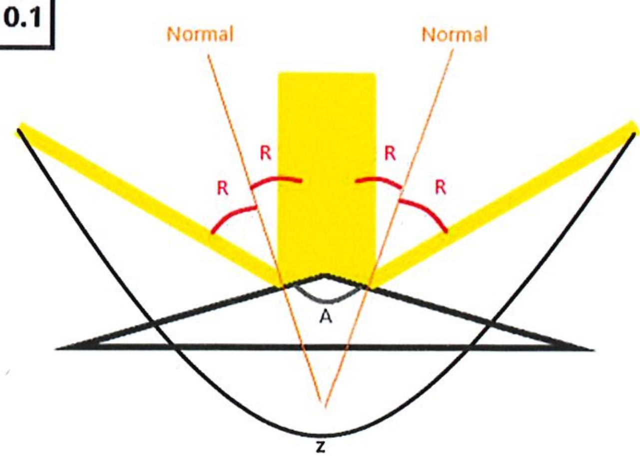
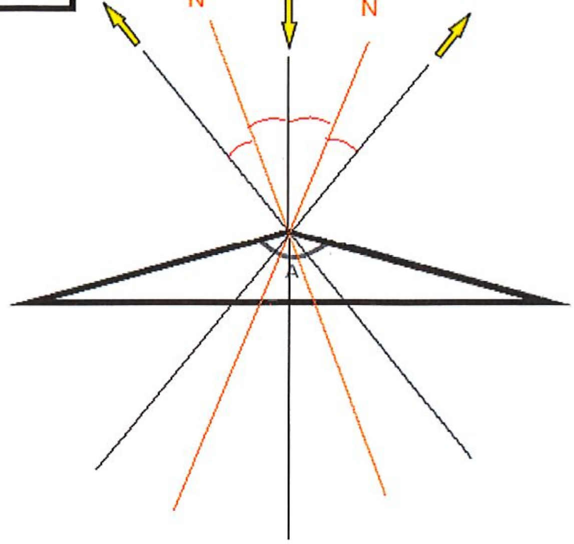


Figure 0.2



Modify the diagram by shifting all vertices to overlay the prism apex. Propagate the rays of incidence and reflection through the prism for reference, as seen in Figure 0.2.

Propagate the apex vectors for reference. Use the rule of vertical angles to label matching angles.

The following properties are gathered from Figure 0.3:

$$90° = 3R + G$$

$$90° = R + G + B$$

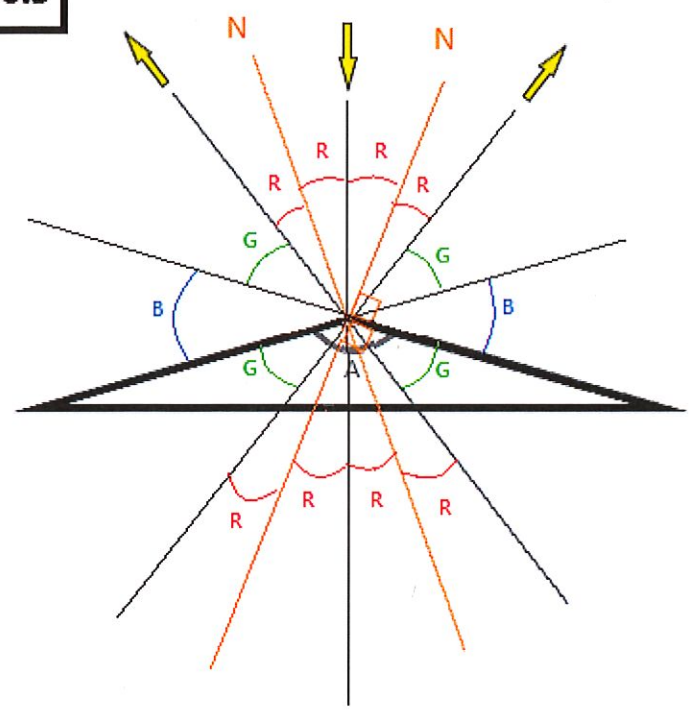
This can only be true if $B = 2R$. Therefore, the total angle between the two rays of reflection is:

$$4G + 8R$$

And the apex angle is equal to:

$$2G + 4R$$

Figure 0.3



Note that $2(A) = 2(2G + 4R) = 4G + 8R$. Therefore, the angle between the two reflected beams is twice the prism apex angle.

Data and Analysis

The angular difference of each side of the light beam as seen through the telescope was measured and taken as the absolute angle error. $\boldsymbol{\delta\theta = \delta\varphi = 0.42° = 0.0073\ rads}$

Measured Values (referring to Figure 2.1):

$$\theta_1 = 120.66°$$

$$\theta_2 = 241.68°$$

$$\theta_2 - \theta_1 = 2A$$

$$A = \frac{\theta_2 - \theta_1}{2} = \frac{241.68° - 120.66°}{2} = \mathbf{60.51°}$$

Figure 2.1

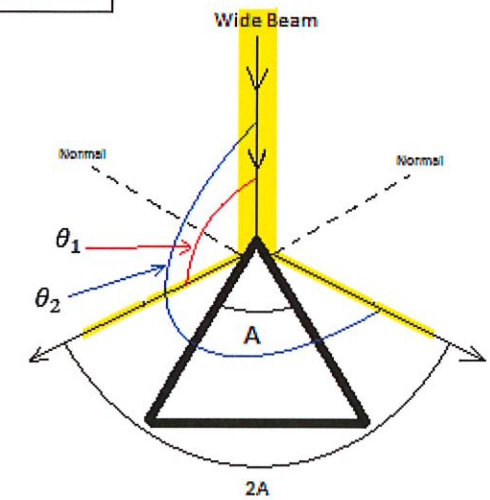
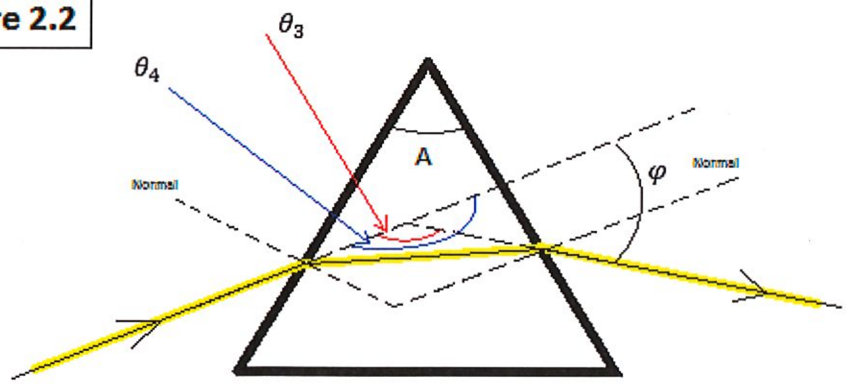


Figure 2.2



Measured Values (referring to Figure 2.2):

$$\theta_3 = 133.25°$$

$$\theta_4 = 181.50°$$

$$\varphi = \theta_4 - \theta_3 = 181.50° - 133.25° = \mathbf{48.25°}$$

Index of Refraction

$$n = \left(\frac{\sin\left(\frac{\varphi + A}{2}\right)}{\sin\left(\frac{A}{2}\right)} \right)$$ Theoretical Equation

$$n = \left(\frac{\sin\left(\frac{48.25° + 60.51°}{2}\right)}{\sin\left(\frac{60.51°}{2}\right)} \right) = \mathbf{1.61}$$ Index of Refraction

Error Propagation – Index of Refraction with Minimum Angle of Deviation

$$n = \left(\frac{\sin\left(\frac{\varphi + A}{2}\right)}{\sin\left(\frac{A}{2}\right)} \right)$$ Theoretical Equation

$$\frac{\partial n}{\partial \varphi} = \left(\frac{\cos\left(\frac{\varphi + A}{2}\right)}{2\sin\left(\frac{A}{2}\right)} \right)$$ Take partial derivative for φ

Take partial derivative for A.

$$n = \sin\left(\frac{\varphi + A}{2}\right) * \csc\left(\frac{A}{2}\right)$$ Rewrite base equation.

Product Rule: Use product rule.

$$\frac{d}{dx}(u, v) = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$Where\ u = \csc\left(\frac{A}{2}\right)\ and\ v = \sin\left(\frac{\varphi + A}{2}\right)$$

$$\frac{\partial n}{\partial A} = \left(\sin\left(\frac{\varphi + A}{2}\right) * \left(-\frac{1}{2}\csc\left(\frac{A}{2}\right)\cot\left(\frac{A}{2}\right)\right)\right) + \left(\csc\left(\frac{A}{2}\right) * \left(\frac{1}{2}\cos\left(\frac{\varphi + A}{2}\right)\right)\right)$$

$$\frac{\partial n}{\partial A} = \frac{1}{2}\csc\left(\frac{A}{2}\right)\left(\cos\left(\frac{\varphi + A}{2}\right) - \sin\left(\frac{\varphi + A}{2}\right)\cot\left(\frac{A}{2}\right)\right)$$

$$\delta n = \left\{ \left[\delta\varphi\left(\frac{\partial n}{\partial \varphi}\right)\right]^2 + \left[\delta A\left(\frac{\partial n}{\partial A}\right)\right]^2 \right\}^{\frac{1}{2}}$$ Definition for absolute error.

$$\frac{\delta n}{n} = \left\{ \left[\frac{\delta\varphi\left(\frac{\partial n}{\partial \varphi}\right)}{n}\right]^2 + \left[\frac{\delta A\left(\frac{\partial n}{\partial A}\right)}{n}\right]^2 \right\}^{\frac{1}{2}}$$ Definition for relative error.

$$\frac{\delta n}{n} = \left\{ \left[\frac{\delta\varphi\left(\frac{\cos\left(\frac{\varphi + A}{2}\right)}{2\sin\left(\frac{A}{2}\right)}\right)}{\left(\frac{\sin\left(\frac{\varphi + A}{2}\right)}{\sin\left(\frac{A}{2}\right)}\right)}\right]^2 + \left[\frac{\delta A\left(\frac{1}{2}\csc\left(\frac{A}{2}\right)\left(\cos\left(\frac{\varphi + A}{2}\right) - \sin\left(\frac{\varphi + A}{2}\right)\cot\left(\frac{A}{2}\right)\right)\right)}{\left(\frac{\sin\left(\frac{\varphi + A}{2}\right)}{\sin\left(\frac{A}{2}\right)}\right)}\right]^2 \right\}^{\frac{1}{2}}$$ Substitute

$$\frac{\delta n}{n} = \left\{ \left[\frac{\delta\varphi\cot\left(\frac{\varphi + A}{2}\right)}{2}\right]^2 + \left[\frac{\delta A\left(\cot\left(\frac{\varphi + A}{2}\right) - \cot\left(\frac{A}{2}\right)\right)}{2}\right]^2 \right\}^{\frac{1}{2}}$$ Simplify

$$\frac{\delta n}{n} = \left\{ \left[\frac{(0.0073)\cot\left(\frac{48.25^\circ + 60.51^\circ}{2}\right)}{2} \right]^2 + \left[\frac{(0.0073)\left(\cot\left(\frac{48.25^\circ + 60.51^\circ}{2}\right) - \cot\left(\frac{60.51^\circ}{2}\right)\right)}{2} \right]^2 \right\}^{\frac{1}{2}} = 0.0052 \rightarrow 1\% \; error$$

Percent Discrepancies

$$\% \, Discrepancy = \left| \frac{Value_{Theoretical} - Value_{Experimental}}{Value_{Theoretical}} \right| * 100$$

Definition of percent discrepancy.

$$\% \, Discrepancy = \left| \frac{60.00 - 60.51}{60.00} \right| * 100 = 1\%$$

Apex Angle of Prism (Theoretical: 60°)

$$\% \, Discrepancy = \left| \frac{1.60 - 1.61}{1.60} \right| * 100 = 1\%$$

Index of Refraction (Theoretical: 1.60)

Conclusion

The experiments yielded favorable results consistent with the glass prism's theoretical apex angle ($\mathbf{60^\circ}$) and index of refraction (**1.60**). The apex angle was experimentally found to be $\mathbf{60.51^\circ \pm 1\%}$ and the index of refraction, calculated with the experimentally found apex angle and minimum angle of deviation, was found to be $\mathbf{1.61 \pm 1\%}$. The percent discrepancies of these results are both **1%**, which is equal to the allowed margins of error.

The margins of error were found using an error propagation technique. The absolute error ($\boldsymbol{\delta\theta} = \mathbf{0.42^\circ}$) for all measured angles was taken as the angular difference measured on the spectrometer from one vertical edge of the light beam to the other (beam's width).

While error-prone human observation was present in the experiment, the spectrometer provided very precise measurements and the absolute angular error was relatively small compared to the measured angles. Perhaps the most erroneous part of the experiment was attempting to observe the minimum angle of deviation while rotating the prism. Regardless, the absolute angular error sufficiently accounted for this observational error.

The experiment could be improved by having a precise theoretical value for the prism's index of refraction, as the value given in lab was only an astute estimation.

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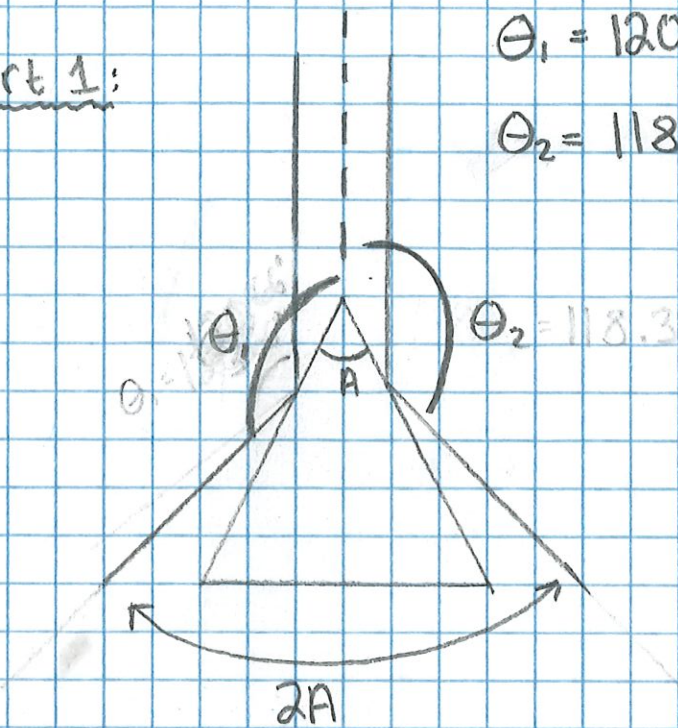
Data Sheet

Note: Width of beam was found to have an absolute error of 0.42°.

Part 1:

$\theta_1 = 120.66°$

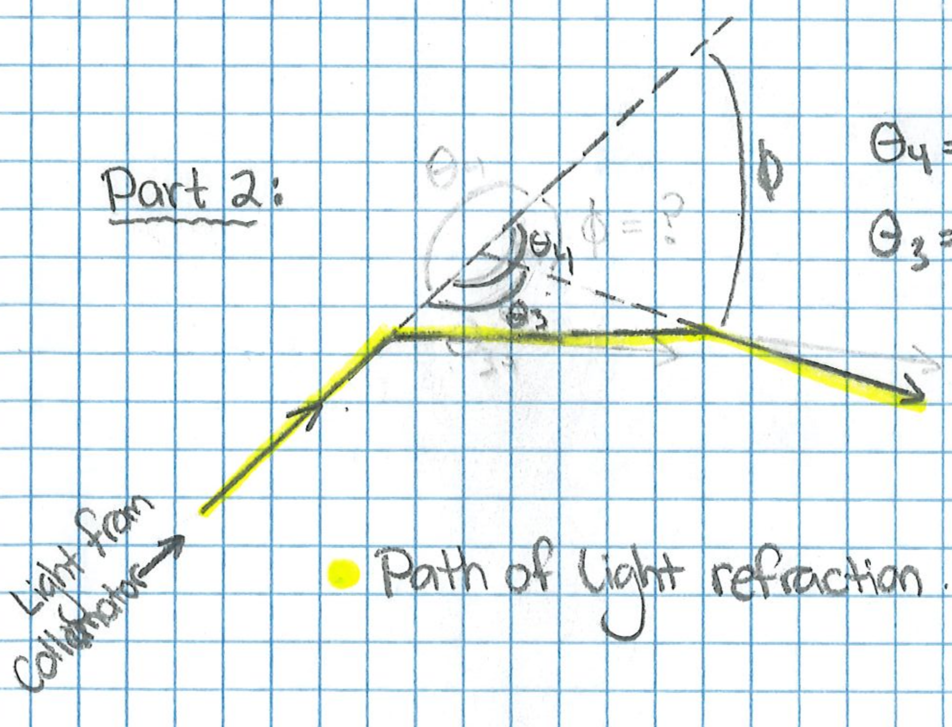
$\theta_2 = 118.33°$



Part 2:

$\theta_4 = 181.50°$

$\theta_3 = 133.25°$



$\rightarrow \phi = \theta_4 - \theta_3 = 48.25°$

● Path of light refraction.

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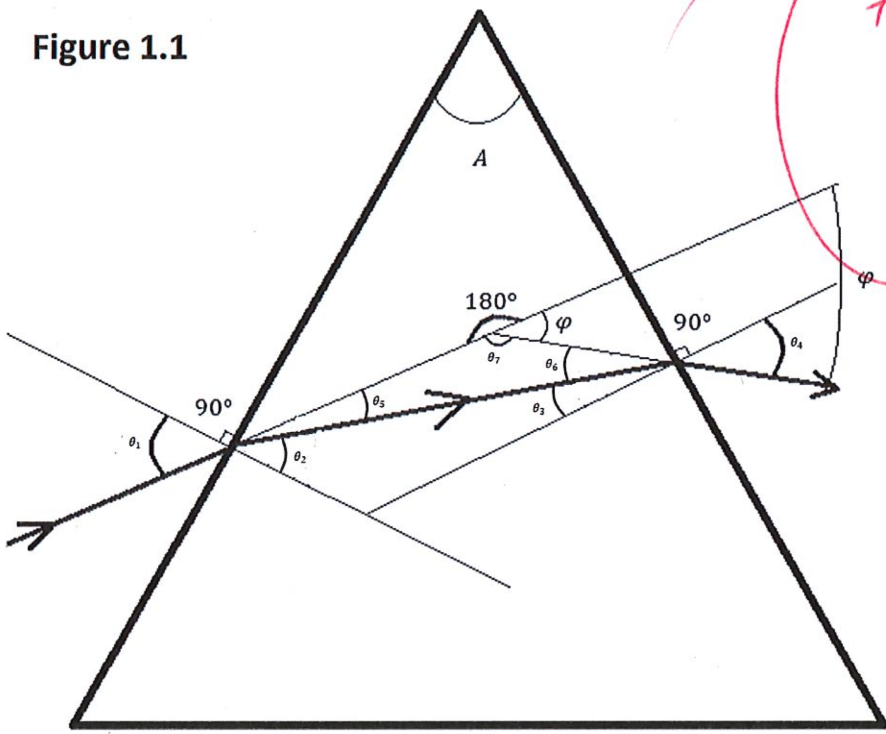
Derivation – Index of Refraction and the Minimum Angle of Deviation Relationship

+1

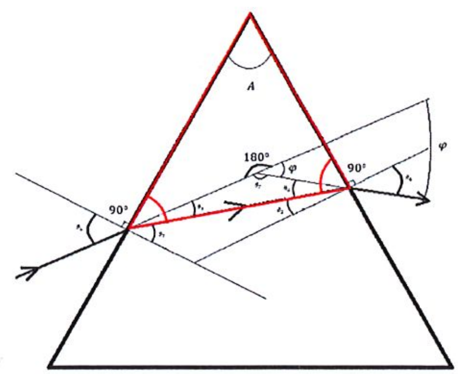
Figure 1.1



Use the sum of angles in a triangle is equal to 180 degrees rule to get the relationship between the apex and the two internal angles of refraction.

$$(90° - \theta_2) + (90° - \theta_3) + A = 180°$$

$$\boldsymbol{\theta_2 + \theta_3 = A} \quad \boldsymbol{(EQ\ 1.1)}$$



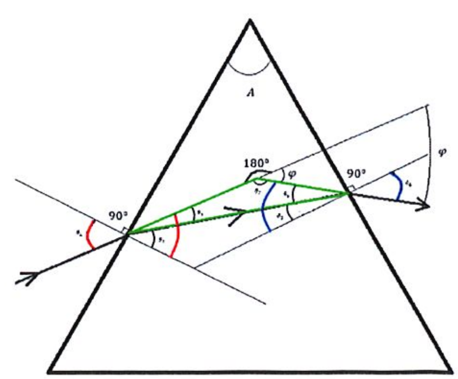
Notice the vertically opposite angles marked in red and blue. These indicates that:

$$\theta_1 = \theta_2 + \theta_5 \quad and \quad \theta_4 = \theta_3 + \theta_6$$

Use the sum of angles in a triangle is equal to 180 degrees rule to find the relationship between the apex, external angles of refraction, and the angle of deviation.

$$180° - \theta_7 = \varphi = (\theta_1 - \theta_2) + (\theta_4 - \theta_3)$$
$$\varphi = \theta_1 + \theta_4 - (\theta_2 + \theta_3) \quad (Use\ EQ\ 1.1)$$

$$\boldsymbol{\varphi = \theta_1 + \theta_4 - A} \quad \boldsymbol{(EQ\ 1.2)}$$



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$$n_1 sin\theta_1 = n_2 sin\theta_2$$
$$n_3 sin\theta_3 = n_4 sin\theta_4$$

Snell's law, where n_1 and n_4 are equal to 1. Also, n_2 and n_3 are equal. Simplify.

$$\boldsymbol{sin\theta_1 = n\,sin\theta_2\ \ and\ \ sin\theta_4 = n\,sin\theta_3}$$

(EQ 1.3) and **(EQ 1.4)**

$$\frac{d\varphi}{d\theta_1} = 0$$

Minimum deviation definition

$$\frac{d\theta_4}{d\theta_1} = (\varphi - \theta_1 + A)\frac{d}{d\theta_1} = -1$$

Solve for the derivative of θ_4 with respect to θ_1. Use EQ 1.2

$$\boldsymbol{\frac{d\theta_4}{d\theta_1} = -1}$$

(EQ 1.5)

$$\frac{d}{d\theta_1}[n\,sin\theta_3 = sin\theta_4] \to n\,cos\theta_3\frac{d\theta_3}{d\theta_1} = cos\theta_4\frac{d\theta_4}{d\theta_1}$$

Differentiate EQ 1.4 with respect to θ_1.
Then substitute in EQ 1.5.

$$\boldsymbol{\frac{d\theta_3}{d\theta_1} = -\frac{cos\theta_4}{n\,cos\theta_3}}$$

(EQ 1.6)

$$\frac{d}{d\theta_1}[\theta_2 + \theta_3 = A] \to \boldsymbol{\frac{d\theta_3}{d\theta_1} = -\frac{d\theta_2}{d\theta_1}}$$

Differentiate EQ 1.1 with respect to θ_1.
(EQ 1.7)

$$cos\theta_4 = n\,cos\theta_3\frac{d\theta_2}{d\theta_1} \to cos\theta_2\left[cos\theta_4 = n\,cos\theta_3\frac{d\theta_2}{d\theta_1}\right]$$

Substitute in EQ 1.7 and simplify.
Then multiply each side by $cos\theta_2$

$$\boldsymbol{cos\theta_2 cos\theta_4 = n\,cos\theta_2 cos\theta_3\frac{d\theta_2}{d\theta_1}}$$

(EQ 1.8)

$$\frac{d}{d\theta_1}[n\,sin\theta_2 = sin\theta_1] \to cos\theta_3\left[n\,cos\theta_2\frac{d\theta_2}{d\theta_1} = cos\theta_1\right]$$

Differentiate EQ 1.3 with respect to θ_1.
Then multiply each side by $cos\theta_3$

$$\boldsymbol{cos\theta_3 cos\theta_1 = n\,cos\theta_3 cos\theta_2\frac{d\theta_2}{d\theta_1}}$$

(EQ 1.9)

$$cos\theta_3 cos\theta_1 - cos\theta_2 cos\theta_4 = n\,cos\theta_2 cos\theta_3\frac{d\theta_2}{d\theta_1} - n\,cos\theta_3 cos\theta_2\frac{d\theta_2}{d\theta_1}$$

Subtract EQ 1.8 from EQ 1.9

$$cos\theta_3 cos\theta_1 = cos\theta_2 cos\theta_4 \to [cos\theta_3 cos\theta_1 = cos\theta_2 cos\theta_4]^2$$
$$\cos^2\theta_3\cos^2\theta_1 = \cos^2\theta_2\cos^2\theta_4$$
$$(1-\sin^2\theta_3)(1-\sin^2\theta_1) = (1-\sin^2\theta_2)(1-\sin^2\theta_4)$$

Simplify. Square the equation and use the trig-identity: $sin^2x + cos^2x = 1$

$$(1-\sin^2\theta_3)(1-n^2\sin^2\theta_2) = (1-\sin^2\theta_2)(1-n^2\sin^2\theta_3)$$
$$1 - n^2\sin^2\theta_2 - \sin^2\theta_3 + n^2\sin^2\theta_3\sin^2\theta_2$$
$$= 1 - n^2\sin^2\theta_3 - \sin^2\theta_2 + n^2\sin^2\theta_3\sin^2\theta_2$$
$$n^2\sin^2\theta_2 + \sin^2\theta_3 = n^2\sin^2\theta_3 + \sin^2\theta_2$$
$$n^2\sin^2\theta_2 - \sin^2\theta_2 = n^2\sin^2\theta_3 - \sin^2\theta_3$$
$$\sin^2\theta_2\,(n^2-1) = \sin^2\theta_3\,(n^2-1)$$

Substitute in EQ 1.3 and 1.4.
Simplify.

$$\sin^2\theta_2 = \sin^2\theta_3$$

(EQ 1.10)

$$\boldsymbol{|\sin\theta_2| = |\sin\theta_3|}$$

$$\boldsymbol{\theta_2 = \theta_3 = \frac{1}{2}A}$$
$$\boldsymbol{\theta_2 = \theta_3 \to\ \theta_1 = \theta_4}$$

Solution needs to be valid for both EQ 1.1 and EQ 1.10. $\frac{A}{2}$ is a valid solution. Revisiting the vertically opposite angles rule indicates that $\boldsymbol{\theta_2 = \theta_3 \to\ \theta_1 = \theta_4}$

$\varphi = \theta_1 + \theta_4 - A \rightarrow \varphi = 2\theta_1 - A \rightarrow \theta_1 = \dfrac{\varphi + A}{2}$	Substitute into EQ 1.2 and solve for θ_1. **(EQ 1.11)**
$sin\theta_1 = n\, sin\dfrac{A}{2}$	Substitute into EQ 1.3 where $\theta_2 = \frac{A}{2}$ **(EQ 1.12)**
$sin\dfrac{\varphi + A}{2} = n\, sin\dfrac{A}{2}$	Combine EQ 1.12 with EQ 1.11
$n = \dfrac{\left(sin\dfrac{\varphi + A}{2}\right)}{\sin\left(\dfrac{A}{2}\right)}$	**(EQ 1.13)**