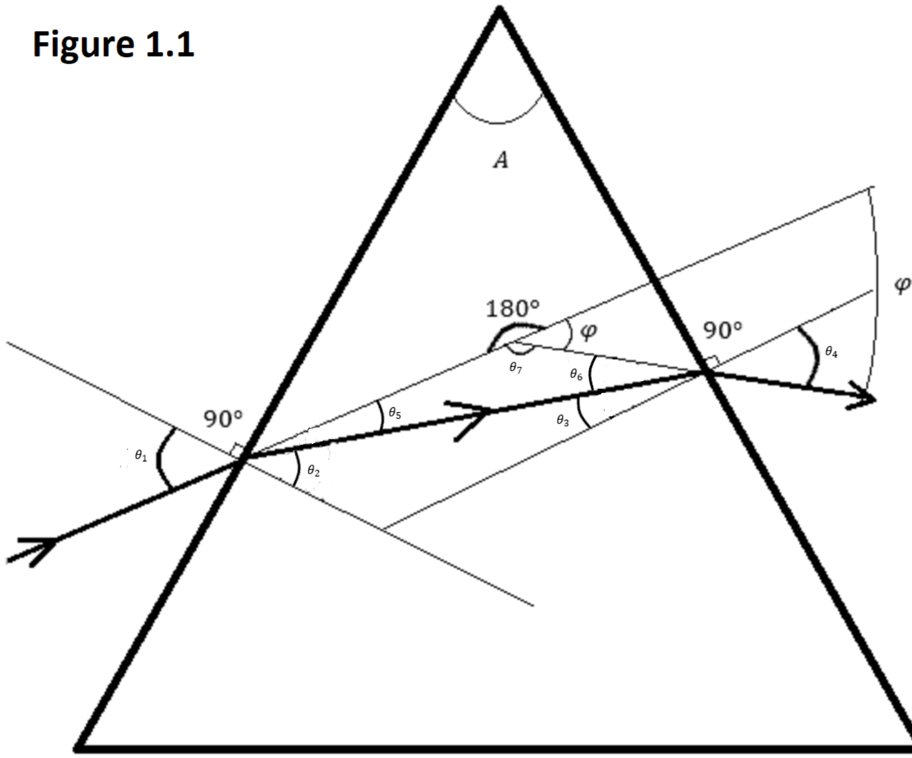
Jared Fowler

Derivation – Index of Refraction and the Minimum Angle of Deviation Relationship

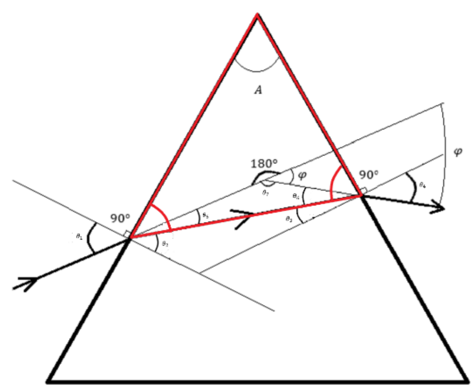
Figure 1.1



Use the sum of angles in a triangle is equal to 180 degrees rule to get the relationship between the apex and the two internal angles of refraction.

$$(90° - \theta_2) + (90° - \theta_3) + A = 180°$$

$$\boldsymbol{\theta_2 + \theta_3 = A} \quad (\boldsymbol{EQ\ 1.1})$$



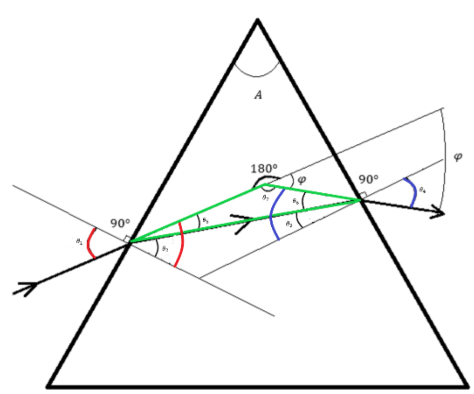
Notice the vertically opposite angles marked in red and blue. These indicates that:

$$\theta_1 = \theta_2 + \theta_5 \ \ and \ \ \theta_4 = \theta_3 + \theta_6$$

Use the sum of angles in a triangle is equal to 180 degrees rule to find the relationship between the apex, external angles of refraction, and the angle of deviation.

$$180° - \theta_7 = \varphi = (\theta_1 - \theta_2) + (\theta_4 - \theta_3)$$
$$\varphi = \theta_1 + \theta_4 - (\theta_2 + \theta_3) \quad (Use\ EQ\ 1.1)$$

$$\boldsymbol{\varphi = \theta_1 + \theta_4 - A} \quad (\boldsymbol{EQ\ 1.2})$$



$$n_1 sin\theta_1 = n_2 sin\theta_2$$
$$n_3 sin\theta_3 = n_4 sin\theta_4$$

Snell's law, where n_1 and n_4 are equal to 1. Also, n_2 and n_3 are equal. Simplify.

$$\boldsymbol{sin\theta_1 = n\, sin\theta_2} \;\; and \;\; \boldsymbol{sin\theta_4 = n\, sin\theta_3}$$

(EQ 1.3) and **(EQ 1.4)**

$$\frac{d\varphi}{d\theta_1} = 0$$

Minimum deviation definition

$$\frac{d\theta_4}{d\theta_1} = (\varphi - \theta_1 + A)\frac{d}{d\theta_1} = -1$$

Solve for the derivative of θ_4 with respect to θ_1. Use EQ 1.2

$$\boldsymbol{\frac{d\theta_4}{d\theta_1} = -1}$$

(EQ 1.5)

$$\frac{d}{d\theta_1}[n\, sin\theta_3 = sin\theta_4] \rightarrow n\, cos\theta_3 \frac{d\theta_3}{d\theta_1} = cos\theta_4 \frac{d\theta_4}{d\theta_1}$$

Differentiate EQ 1.4 with respect to θ_1.
Then substitute in EQ 1.5.

$$\boldsymbol{\frac{d\theta_3}{d\theta_1} = -\frac{cos\theta_4}{n\, cos\theta_3}}$$

(EQ 1.6)

$$\frac{d}{d\theta_1}[\theta_2 + \theta_3 = A] \rightarrow \boldsymbol{\frac{d\theta_3}{d\theta_1} = -\frac{d\theta_2}{d\theta_1}}$$

Differentiate EQ 1.1 with respect to θ_1.
(EQ 1.7)

$$cos\theta_4 = n\, cos\theta_3 \frac{d\theta_2}{d\theta_1} \rightarrow cos\theta_2 \left[cos\theta_4 = n\, cos\theta_3 \frac{d\theta_2}{d\theta_1}\right]$$

Substitute in EQ 1.7 and simplify.
Then multiply each side by $cos\theta_2$

$$\boldsymbol{cos\theta_2 cos\theta_4 = n\, cos\theta_2 cos\theta_3 \frac{d\theta_2}{d\theta_1}}$$

(EQ 1.8)

$$\frac{d}{d\theta_1}[n\, sin\theta_2 = sin\theta_1] \rightarrow cos\theta_3 \left[n\, cos\theta_2 \frac{d\theta_2}{d\theta_1} = cos\theta_1\right]$$

Differentiate EQ 1.3 with respect to θ_1.
Then multiply each side by $cos\theta_3$

$$\boldsymbol{cos\theta_3 cos\theta_1 = n\, cos\theta_3 cos\theta_2 \frac{d\theta_2}{d\theta_1}}$$

(EQ 1.9)

$$cos\theta_3 cos\theta_1 - cos\theta_2 cos\theta_4 = n\, cos\theta_2 cos\theta_3 \frac{d\theta_2}{d\theta_1} - n\, cos\theta_3 cos\theta_2 \frac{d\theta_2}{d\theta_1}$$

Subtract EQ 1.8 from EQ 1.9

$$cos\theta_3 cos\theta_1 = cos\theta_2 cos\theta_4 \rightarrow [cos\theta_3 cos\theta_1 = cos\theta_2 cos\theta_4]^2$$
$$\cos^2\theta_3 \cos^2\theta_1 = \cos^2\theta_2 \cos^2\theta_4$$
$$(1 - \sin^2\theta_3)(1 - \sin^2\theta_1) = (1 - \sin^2\theta_2)(1 - \sin^2\theta_4)$$

Simplify. Square the equation and use the trig-identity: $sin^2x + cos^2x = 1$

$$(1 - \sin^2\theta_3)(1 - n^2\sin^2\theta_2) = (1 - \sin^2\theta_2)(1 - n^2\sin^2\theta_3)$$
$$1 - n^2\sin^2\theta_2 - \sin^2\theta_3 + n^2\sin^2\theta_3\sin^2\theta_2$$
$$= 1 - n^2\sin^2\theta_3 - \sin^2\theta_2 + n^2\sin^2\theta_3\sin^2\theta_2$$
$$n^2\sin^2\theta_2 + \sin^2\theta_3 = n^2\sin^2\theta_3 + \sin^2\theta_2$$
$$n^2\sin^2\theta_2 - \sin^2\theta_2 = n^2\sin^2\theta_3 - \sin^2\theta_3$$
$$\sin^2\theta_2\,(n^2 - 1) = \sin^2\theta_3\,(n^2 - 1)$$

Substitute in EQ 1.3 and 1.4.
Simplify.

$$\sin^2\theta_2 = \sin^2\theta_3$$

(EQ 1.10)

$$\boldsymbol{|\sin\theta_2| = |\sin\theta_3|}$$

$$\boldsymbol{\theta_2 = \theta_3 = \frac{1}{2}A}$$
$$\boldsymbol{\theta_2 = \theta_3 \rightarrow \;\theta_1 = \theta_4}$$

Solution needs to be valid for both EQ 1.1 and EQ 1.10. $\frac{A}{2}$ is a valid solution. Revisiting the vertically opposite angles rule indicates that $\boldsymbol{\theta_2 = \theta_3 \rightarrow \;\theta_1 = \theta_4}$

$$\varphi = \theta_1 + \theta_4 - A \rightarrow \varphi = 2\theta_1 - A \rightarrow \boldsymbol{\theta_1 = \frac{\varphi + A}{2}}$$

Substitute into EQ 1.2 and solve for θ_1.
(EQ 1.11)

$$\boldsymbol{sin\theta_1 = n\, sin\frac{A}{2}}$$

Substitute into EQ 1.3 where $\theta_2 = \frac{A}{2}$
(EQ 1.12)

$$sin\frac{\varphi + A}{2} = n\, sin\frac{A}{2}$$

Combine EQ 1.12 with EQ 1.11

$$\boldsymbol{n = \frac{\left(sin\frac{\varphi + A}{2}\right)}{\sin\left(\frac{A}{2}\right)}}$$

(EQ 1.13)